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Cohen-Macaulayness in graded rings associated to ideals

By

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1. Introduction

Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let $d = \dim A$ and assume the field $A/\mathfrak{m}$ is infinite. For a given ideal I in A $(I \neq A)$ we define

$$R(I) = \sum_{n \geq 0} I^n t^n \subseteq A[t] \qquad \text{and} \qquad G(I) = R(I)/IR(I)$$

(here t is an indeterminate over A) and respectively call $R(I)$ and $G(I)$ the Rees algebra and the associated graded ring of I. The purpose of this paper is to find any practical conditions under which the graded algebras $R(I)$ and $G(I)$ are Cohen-Macaulay and/or Gorenstein rings. And, because Cohen-Macaulayness and Gorensteinness in $R(I)$ are now known to be fairly determined by the corresponding ring-theoretic properties of $G(I)$ (see, for examples, [GS], [I], [TI], [GNi], [V], and [L]), in this paper we devote our attention to the problem how to check Cohen-Macaulayness or Gorensteinness in the graded rings $G(I)$. We shall develop our study along the notion, due to [HH1], *analytic deviation* $\operatorname{ad}(I)$ *of* I. Actually, for the ideals I having $\operatorname{ad}(I) \leq 2$ Huckaba and Huneke [HH1] and [HH2] have already studied Cohen-Macaulayness in graded rings $R(I)$ and $G(I)$ and the readers may consult [GNa1] and [GNa2] about Gorensteinness in them. This paper succeeds the researches [HH1], [HH2], [GNa1], and [GNa2]. Here we shall generalize their results for ideals of $\operatorname{ad}(I) \geq 3$.

To state the results precisely, we set up the following notation. Let I $(\neq A)$ be an ideal in A of $\operatorname{ht}_A I = s$ and put $\lambda(I) = \dim A/\mathfrak{m} \otimes_A G(I)$, that we call the analytic spread of I. We generally have

$$s \leq \lambda(I) \leq d - \inf_{n \geq 1} \operatorname{depth} A/I^n$$

([B]). So the difference $\operatorname{ad}(I) = \lambda(I) - s$ is called the analytic deviation. Let J be another ideal in A. We say that J is a reduction of J if $J \subseteq I$ and $I^{n+1} = JI^n$ for all $n \gg 0$. A reduction is called minimal if it is minimal among reductions. As is well-known, a reduction J of I is minimal if and if J is generated by $\lambda(I)$ elements ([NR]). For each reduction J of I let $r_J(I) = \min\{n \geq 0 \mid I^{n+1} = JI^n\}$ and call it the reduction number of I with respect to J. We put $r(I) = \min r_J(I)$ where J runs over minimal reductions.

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Assume that A is a Cohen-Macaulay ring and that our ideal I is generically a complete intersection in A, that is for any $Q \in \mathrm{Min}_A A/I$ the ideal I_Q is generated by an A_Q-regular sequence of length s. In this situation, if $\mathrm{ad}(I) = 0$, I is a complete intersection in A ([CN]) and we certainly have $R(I)$ and $G(I)$ are Cohen-Macaulay rings. The study of the case $\mathrm{ad}(I) > 0$ was started from Huckaba and Huneke [HH1], in which they showed that when $\mathrm{ad}(I) = 1$ and $r(I) \leq 1$ the graded ring $G(I)$ is Cohen-Macaulay if and only if $\mathrm{depth}\, A/I \geq d - s - 1$. Assuming that A is a Gorenstein ring, A/I is Cohen-Macaulay, and that I_Q is a complete intersection in A_Q for all prime ideals $Q \in V(I)$ with $\mathrm{ht}_{A/I} Q/I \leq 1$, Huckaba and Huneke [HH2] proved also that $R(I)$ and $G(I)$ are Cohen-Macaulay rings if $\mathrm{ad}(I) = 2$ and $r(I) \leq 1$. Here we are going to generalize these results for ideals I of $\mathrm{ad}(I) \geq 3$. But, of course, to get these generalizations, we need more assumptions on I than Huckaba and Huneke did in the case $\mathrm{ad}(I) \leq 2$. In this paper we put on I two conditions appropriate to our study; firstly, inspired by the assumptions in [HH1] and [HH2], we assume as is in [AH] that our ideal I contains a special reduction (see (2.1) below for the definition of special reduction). As was proved in [U], this assumption is equivalent to saying that the ideal I satisfies condition $G_{\lambda(I)}$ in the sense of Artin and Nagata [AN], that is the ideal I_Q is generated by at most h elements for all prime ideals $Q \in V(I)$ with $\mathrm{ht}_A Q = h < \lambda(I)$. We note here that even in the case $\mathrm{ad}(I) = 2$ this assumption is somewhat weaker than that in [HH2], where Huckaba and Huneke assumed I_Q is a complete intersection in A_Q for all prime ideals $Q \in V(I)$ with $ht_{A/I} Q/I \leq 1$. Secondly, we assume some estimation on $\mathrm{depth}\,(A/I^n)_Q$, the depth of local rings $(A/I^n)_Q$, for prime ideals $Q \in V(I)$ and integers n with $1 \leq n \leq \mathrm{ad}(I)$. This condition was first studied in [N], where the third author Nishida established criteria for the equality of symbolic powers $I^{(n)}$ and ordinary powers I^n of I for all $n \geq 1$. We need some results in [N] which play a key role also in the present research.

Now let us state the main results of this paper.

Theorem (1.1). *Let A be a Cohen-Macaulay ring of* $\dim A = d$. *Let I be an ideal in A of* $\mathrm{ht}_A I = s$. *Assume that I contains a special reduction J with $r_J(I) \leq \mathrm{ad}(I)$ and that I satisfies the following inequalities*; $\mathrm{depth}\,(A/I^n)_Q \geq \min\{\mathrm{ad}(I) - n, \mathrm{ht}_A Q - s - n\}$ *and* $\mathrm{depth}\, A/I^n \geq d - s - n + 1$ *for all prime ideals $Q \in V(I)$ and for all integers n with $1 \leq n \leq \mathrm{ad}(I)$. Then*

(1) *$G(I)$ is a Cohen-Macaulay ring of $a(G(I)) = -s$.*

(2) *$G(I)$ is a Gorenstein ring if A is a Gorenstein ring.*

Here $a(G(I))$ denotes the *a-invariant* of $G(I)$ ([GW, (3.1.4)]). The equality $a(G(I)) = -s$ in Theorem (1.1) (1) follows also from [AH, 5.10] and [T, 2.5], once we know the ring $G(I)$ is Cohen-Macaulay. And, as an immediate consequence of Theorem (1.1), by [TI, (1.1)] and [I, (3.1)] we get the following result on Cohen-Macaulayness and Gorensteinness in $R(I)$.

Corollary (1.2). *Let A and I be as in Theorem* (1.1). *Then*

(1) *$R(I)$ is a Cohen-Macaulay ring if $s > 0$.*

(2) *Suppose $s \geq 2$. Then $R(I)$ is a Gorenstein ring if and only if A is a Gorenstein ring and $s = 2$.*

When A is a Gorenstein ring, in assuming the ring A/I is Cohen-Macaulay, we can weaken the assumption on the estimations of $\operatorname{depth}(A/I^n)_Q$ and get the following criterion of the ring $G(I)$ being Gorenstein in terms of reduction numbers.

Theorem (1.3). *Let A be a Gorenstein ring of $\dim A = d$. Let I be an ideal in A with $\operatorname{ht}_A I = s$ and $\operatorname{ad}(I) > 0$. Suppose that A/I is a Cohen-Macaulay ring and that I contains a special reduction J. Then*

(1) *$r_J(I) \leq \operatorname{ad}(I) - 1$ if $G(I)$ is a Gorenstein ring.*

(2) *Assume that the inequalities $\operatorname{depth}(A/I^n)_Q \geq \min\{\operatorname{ad}(I) - 1 - n, \operatorname{ht}_A Q - s - n\}$ and $\operatorname{depth} A/I^n \geq d - s - n$ hold for all prime ideals $Q \in V(I)$ and for all integers n with $1 \leq n \leq \operatorname{ad}(I) - 1$. Then $G(I)$ is a Gorenstein ring if and only if $r_J(I) \leq \operatorname{ad}(I) - 1$.*

And similarly as is in (1.2) we get

Corollary (1.4). *Let A and I be as in (1.3) and assume that I satisfies the inequalities stated in (1.3) (2). Then $R(I)$ is a Gorenstein ring if $s = 2$ and $r_J(I) \leq \operatorname{ad}(I) - 1$.*

For a Gorenstein ring A and its ideal I which has a special reduction J, we assume that A/I is Cohen-Macaulay and that $\operatorname{depth}(A/I^n)_Q \geq \min\{\operatorname{ad}(I) - n, \operatorname{ht}_A Q - s - n\}$ for all prime ideals $Q \in V(I)$ and for all integers n with $1 \leq n \leq \operatorname{ad}(I)$. Then once we know the inequality $r_J(I) \leq \operatorname{ad}(I) - 1$, to see whether $G(I)$ is a Gorenstein ring or not it suffices by Theorem (1.3) to check if the inequalities $\operatorname{depth} A/I^n \geq d - s - n$ hold for $1 \leq n \leq \operatorname{ad}(I) - 1$. However, if we do have the inequality $r_J(I) \leq \operatorname{ad}(I)$ only, without knowing whether $r_J(I) \leq \operatorname{ad}(I) - 1$ or not, then we are not able to directly apply Theorem (1.3). And to apply Theorem (1.1), we need the stronger estimation $\operatorname{depth} A/I^n \geq d - s - n + 1$ for $1 \leq n \leq \operatorname{ad}(I)$. Of course once we get it, then $G(I)$ is a Gorenstein ring and Theorem (1.3) (1) yields the sharper estimation $r_J(I) \leq \operatorname{ad}(I) - 1$ on reduction numbers. For this reason it seems to us that among the standard hypotheses in Theorem (1.1) the assumption $\operatorname{depth} A/I^n \geq d - s - n + 1$ is somewhat superfluous. And, as we will show in Theorems (1.5) and (1.6) below for the special case $\operatorname{ad}(I) \leq 3$, Theorem (1.1) (1) might be true if hold the inequalities $\operatorname{depth} A/I^n \geq d - s - n$ for $1 \leq n \leq \operatorname{ad}(I)$ instead of those $\operatorname{depth} A/I^n \geq d - s - n + 1$, provided that A/I is a Cohen-Macaulay ring.

Theorem (1.5). *Let I be an ideal with $\operatorname{ht}_A I = s$ and $\operatorname{ad}(I) = 2$ in a Cohen-Macaulay ring A of $\dim A = d$. Assume that A/I is Cohen-Macaulay and that I contains a special reduction J with $r_J(I) \leq 2$. Then the following conditions are equivalent.*

(1) *$G(I)$ is a Cohen-Macaulay ring.*

(2) depth $A/I^2 \geq d - s - 2$.

Hence $R(I)$ is a Cohen-Macaulay ring if $s > 0$ and depth $A/I^2 \geq d - s - 2$.

Theorem (1.6). *Let I be an ideal with $\mathrm{ht}_A I = s$ and $\mathrm{ad}(I) = 3$ in a Gorenstein ring A of $\dim A = d$. Assume that A/I is Cohen-Macaulay and that I contains a special reduction J with $r_J(I) \leq 2$. Then the following conditions are equivalent.*

(1) *$G(I)$ is a Cohen-Macaulay ring.*

(2) depth $A/I^2 \geq d - s - 3$.

Here we note that Theorem (1.5) is already shown by [GNa3] on the additional assumptions that A is a Gorenstein ring and that the ideal I_Q is generated by an A_Q-regular sequence for all prime ideals $Q \in V(I)$ with $\mathrm{ht}_{A/I}\, Q/I \leq 1$.

As for Gorensteinness in $G(I)$ when $\mathrm{ad}(I) = 3$ and $r_J(I) \leq 1$ we are able to add the following criterion.

Theorem (1.7). *Let I be an ideal with $ht_A I = s$ and $\mathrm{ad}(I) = 3$ in a Gorenstein ring A of $\dim A = d$. Assume that A/I is Cohen-Macaulay and that I contains a special reduction J with $r_J(I) \leq 1$. Then the following conditions are equivalent.*

(1) *$G(I)$ is a Gorenstein ring.*

(2) depth $A/I^2 \geq d - s - 2$.

Suppose A is Gorenstein, A/I is Cohen-Macaulay, and I has a special reduction J. Then we have by (1.3) and (1.6) that $r_J(I) \leq 2$ and depth $A/I^2 \geq d - s - 3$, if $\mathrm{ad}(I) = 3$ and if $G(I)$ is a Gorenstein ring. The criterion of $G(I)$ being Gorenstein in the case $\mathrm{ad}(I) = 3$ is settled by (1.3) and (1.7), if either depth $A/I^2 \geq d - s - 2$ or $r_J(I) \leq 1$. However, when depth $A/I^2 = d - s - 3$ and $r_J(I) = 2$, the authors do not know any similar practical criteria as in (1.3) and (1.7). There are, of course, examples in that case. Let $A = k[[X_{ij} | i = 1, 2, 1 \leq j \leq 5]]$ be a formal power series ring in 10 variables over an infinite field k and let I be the ideal in A generated by the maximal minors of the 2 by 5 generic matrix $X = [X_{ij}]$. Then I is a perfect ideal of height 4 and $\lambda(I) = 7$. Hence $\mathrm{ad}(I) = 3$. As A/I is an isolated singularity, any minimal reduction J of I is special (cf. (2.3)). We have $r_J(I) = 2$, depth $A/I^2 = 3$ $(= d - s - 3)$, and $G(I)$ is a Gorenstein ring.

Let us now briefly explain how to organize this paper. In Section 2 we pick up from [N] some results on special reductions of ideals, which we need for the rest of this paper. We prove Theorem (1.1) (resp. Theorem (1.3)) in Section 3 (resp. Section 4). Section 5 is devoted to study the case where $\mathrm{ad}(I) \leq 3$. As a consequence, we prove Theorems (1.5), (1.6), and (1.7).

Throughout this paper $(A, \mathfrak{m})$ is a Noetherian local ring and $d = \dim A$. We always assume the residue class field $A/\mathfrak{m}$ is infinte. $H^i_{\mathfrak{m}}(*)$ $(i \in \mathbf{Z})$ stand for local cohomology functors. For each finitely generated A-module M, $\mu_A(M)$ denotes the number of elements in a minimal system of generators for M.

2. Auxiliary results on special reductions

In this section we shall summarize some auxiliary results on special reductions, which we need this paper. Let I be an ideal in A, $s = \mathrm{ht}_A I$, and let $\ell = \lambda(I)$. Hence $\mathrm{ad}(I) = \ell - s$. First of all let us recall the definition of special reductions.

Definition (2.1) ([AH, 5.1]). Let J be a minimal reduction of I. Then J is said to be a special reduction of I, if I contains a system of generators $a_1, a_2, \ldots, a_\ell$ for J which satisfy the equality $I_Q = (a_1, a_2, \ldots, a_h)A_Q$ for all prime ideals $Q \in V(I)$ with $\mathrm{ht}_A Q = h < \ell$.

Lemma (2.2) [(U)]. *The following conditions are equivalent.*

(1) *I has a special reduction.*

(2) *For all prime ideals $Q \in V(I)$ with $\mathrm{ht}_A Q = h < \ell$ the ideals I_Q is generated by at most h elements.*

Proof. See [U, 1.4] or [N, (2.2)].

Corollary (2.3) ([AH]). *Suppose that for any prime ideals $Q \in V(I)$ with $\mathrm{ht}_A Q < \ell$, the ideal I_Q is generated by an A_Q-regular sequence of length s. Then every minimal reduction of I is special.*

Proof. See [AH, 6.4] or [N, (2.5)].

For the rest of this section we assume that A is a Cohen-Macaulay ring and that I contains a special reduction J. Let $r = r_J(I)$. We choose a system of generators $a_1, a_2, \ldots, a_\ell$ for J which satisfy the equality $I_Q = (a_1, a_2, \ldots, a_h)A_Q$ for all prime ideals $Q \in V(I)$ with $\mathrm{ht}_A Q = h < \ell$. Let $J_i = (a_1, a_2, \ldots, a_i)A$ for $0 \leq i \leq \ell$. Then we may further assume that the system $a_1, a_2, \ldots, a_\ell$ satisfies all the conditions stated in the next lemma.

Lemma (2.4) ([N]). (1) *The element a_i does not belong to Q for any $1 \leq i \leq \ell$ and for any prime ideal $Q \in \left[\mathrm{Ass}\, A \cup \left(\bigcup_{m \geq 1} \mathrm{Ass}_A\, A/J_{i-1}^m\right)\right] \Big\backslash V(I)$. If $\ell > s$, then for all prime ideals $Q \in \mathrm{Min}_A\, A/I \cap \mathrm{Supp}_A\, I$ the element a_{s+1} forms part of a minimal system of generators for the ideal I_Q.*

(2) *$a_1, a_2, \ldots, a_s$ forms an A-regular sequence.*

(3) *$[(0) : a_i] \cap I = (0)$ for all $1 \leq i \leq \ell$.*

(4) *$[J_s : a_{s+1}] \cap I = J_s$ if $\ell > s$. Hence if $\ell > s = 0$, then $(0) : a_1 = (0) : I$, $[(0) : I] \cap I = (0)$, and a_1 is $A/[(0) : I]$-regular.*

Proof. See [N, Proof of (3.2), (3.3), (3.4), and (3.13)] for assertions (1), (2), and (3). To get the assertion (4), let $Q \in \mathrm{Ass}_A\, A/J_s$. Then if $I \subseteq Q$, by (2.1) we have $I_Q = J_{sQ}$ as $\mathrm{ht}_A Q = s < \ell$. Hence $[J_s : a_{s+1}]_Q \cap I_Q = J_{sQ}$, which does also hold if $I \nsubseteq Q$, because $a_{s+1} \notin Q$ by (1). Thus we get $[J_s : a_{s+1}] \cap I = J_s$. Suppose $\ell > s = 0$ and let $x \in (0) : a_1$. Then because $xI \subseteq [(0) : a_1] \cap I$ and because $[(0) : a_1] \cap I = (0)$ by (3), we have $x \in (0) : I$. Thus $(0) : a_1 = (0) : I$ and $[(0) : I] \cap I =$

(0). Let $a_1 x \in (0):I$ with $x \in A$. Then we get $a_1 x = 0$ as $a_1 x \in [(0):I] \cap I$, whence $x \in (0):a_1 = (0):I$. Thus a_1 is $A/[(0):I]$-regular.

We note one more result in [N].

Lemma (2.5) ([N]). *Assume the inequality* $\operatorname{depth}(A/I^k)_Q \geq \min\{\ell - s - k, \operatorname{ht}_A Q - s - k\}$ *holds for all prime ideals* $Q \in V(I)$ *and for all integers* k *with* $1 \leq k \leq \ell - s$. *Let* m, n, *and* i *be integers such that* $m \geq 1$, $0 \leq n \leq \ell - s - 1$, *and* $0 \leq i \leq n + s$. *Then we have the following.*

(1) $[J_i^m : a_{i+1}] \cap J_{i+1}^{m-1} I^{n+1} = J_i^m I^n$.

(2) $J_{i+1}^m I^{n+1}/J_i^m I^{n+1} \cong J_{i+1}^{m-1} I^{n+1}/J_i^m I^n$.

Proof. See [N, Proof of (3.5)].

Now our goal of this section is to show that under the extra conditions on I cited in section 1 the sequence $a_1 t, a_2 t, \ldots, a_s t$ is $G(I)$-regular. So we begin with the following lemma.

Proposition (2.6). *Assume that* $r \leq \ell - s$ *and that the inequality* $\operatorname{depth}(A/I^n)_Q \geq \min\{\ell - s - n, \operatorname{ht}_A Q - s - n\}$ *holds for all prime ideals* $Q \in V(I)$ *and for all integers* n *with* $1 \leq n \leq \ell - s$. *Then* $J_i \cap I^{m+1} = J_i I^m$ *for* $0 \leq i \leq \ell$ *and all* $m \geq i - s$.

Proof. We will prove the assertion by descending induction on i. Let $i = \ell$. Then as $r \leq \ell - s \leq m$, $I^{m+1} = J_\ell I^m$ whence $J_\ell \cap I^{m+1} = J_\ell I^m$ for all $m \geq \ell - s$. Now let $i < \ell$ and assume $J_{i+1} \cap I^{m+1} = J_{i+1} I^m$ for any $m \geq i + 1 - s$. We shall show by induction on m that $J_i \cap I^{m+1} = J_i I^m$ for all $m \geq i - s$. If $i - s \geq 0$, then by (2.5)(1) we get $J_i \cap I^{i-s+1} \subseteq [J_i : a_{i+1}] \cap I^{i-s+1} = J_i I^{i-s}$. Hence $J_i \cap I^{i-s+1} = J_i I^{i-s}$. And $J_i \cap I^{i-s+1} = J_i I^{i-s}$ if $i < s$. Thus we have the equality $J_i \cap I^{m+1} = J_i I^m$ whenever $m = i - s$. Now assume $m \geq i - s + 1$ and $J_i \cap I^m = J_i I^{m-1}$. Then the inductive hypothesis on i says $J_{i+1} \cap I^{m+1} = J_{i+1} I^m$. So we have

$$
\begin{aligned}
J_i \cap I^{m+1} &= J_i \cap (J_{i+1} \cap I^{m+1}) \\
&= J_i \cap J_{i+1} I^m \\
&= J_i \cap (J_i I^m + a_{i+1} I^m) \\
&= J_i I^m + J_i \cap a_{i+1} I^m \\
&= J_i I^m + a_{i+1}([J_i : a_{i+1}] \cap I^{i-s+1} \cap I^m] \quad (\text{note } m \geq i - s + 1).
\end{aligned}
$$

As $[J_i : a_{i+1}] \cap I^{i-s+1} \subseteq J_i$ by (2.5)(1), we get $J_i \cap I^{m+1} \subseteq J_i I^m + a_{i+1}(J_i \cap I^m)$. Then the hypothesis on m guarantees $J_i \cap I^{m+1} \subseteq J_i I^m + a_{i+1} J_i I^{m-1} = J_i I^m$ as required.

Corollary (2.7). *Assume that* $r \leq \ell - s$ *and that the inequality* $\operatorname{depth}(A/I^n)_Q \geq \min\{\ell - s - n, \operatorname{ht}_A Q - s - n\}$ *holds for all prime ideals* $Q \in V(I)$ *and for all integers* n *with* $1 \leq n \leq \ell - s$. *Then the sequence* $a_1 t, a_2 t, \ldots, a_s t$ *is* $G(I)$*-regular.*

Proof. The assertion directly follows from [VV, 2.7], because $a_1, a_2, \ldots, a_s$ forms by (2.4)(2) an A-regular sequence and because by (2.6) $J_s \cap I^{m+1} = J_s I^m$ for all $m \geq 0$.

Lemma (2.8). *Let A be a Gorenstein ring. Assume that A/I is Cohen-Macaulay and that the inequality* $\operatorname{depth}(A/I^n)_Q \geq \min\{\ell - s - 1 - n, \operatorname{ht}_A Q - s - n\}$ *holds for all prime ideals $Q \in V(I)$ and for all integers n with $1 \leq n \leq \ell - s - 1$. Then $[J_i : a_{i+1}] \cap I^{i-s} = J_i I^{i-s-1}$ for $s < i < \ell$.*

Proof. It suffices to show $[J_i : a_{i+1}]_Q \cap I_Q^{i-s} \subseteq J_i I_Q^{i-s-1}$ for all Q in $\operatorname{Ass}_A A/J_i I^{i-s-1}$. We have nothing to prove if $I \nsubseteq Q$, because $a_{i+1} \notin Q$. Hence we may assume $I \subseteq Q$. Then $\operatorname{ht}_A Q \leq i < \ell$ by [N, (3.11)] applied to the case $N = \alpha = \ell - s - 1$. Hence $J_{iQ} = I_Q$ by (2.1) and we have

$$\begin{aligned}[J_i : a_{i+1}]_Q \cap I_Q^{i-s} &= [I_Q : a_{i+1}] \cap I_Q^{i-s} \\ &= I_Q^{i-s} \quad (\text{note } i - s > 0) \\ &= J_i I_Q^{i-s-1}.\end{aligned}$$

Proposition (2.9). *Let A be a Gorenstein ring. Assume that A/I is Cohen-Macaulay and that $r \leq \ell - s - 1$. Assume the inequality* $\operatorname{depth}(A/I^n)_Q \geq \min\{\ell - s - 1 - n, \operatorname{ht}_A Q - s - n\}$ *holds for all prime ideals $Q \in V(I)$ and for all integers n with $1 \leq n \leq \ell - s - 1$. Let i be an integer satisfying $s \leq i \leq \ell$. Then $J_i \cap I^{m+1} = J_i I^m$ for $s \leq i \leq \ell$ and for all $m \geq i - s - 1$.*

Proof. Let us prove the claim by descending induction on i. The assertion is true if $i = \ell$ because $I^{m+1} = J_\ell I^m$ for all $m \geq \ell - s - 1$ $(\geq r)$. Let i be an integer satisfying $s \leq i < \ell$ and assume that $J_{i+1} \cap I^{m+1} = J_{i+1} I^m$ for all $m \geq i - s$. We shall show, by induction on m, that $J_i \cap I^{m+1} = J_i I^m$ for $m \geq i - s - 1$.

Firstly, suppose $i \geq s + 1$. Then by (2.8) we have $J_i \cap I^{i-s} \subseteq [J_i : a_{i+1}] \cap I^{i-s} = J_i I^{i-s-1}$ whence $J_i \cap I^{i-s} = J_i I^{i-s-1}$. Let $m \geq i - s$ and assume $J_i \cap I^m = J_i I^{m-1}$. Then as $J_i \cap I^{m+1} \subseteq J_{i+1} \cap I^{m+1} = J_{i+1} I^m$, we have $J_i \cap I^{m+1} = J_i \cap (J_i I^m + a_{i+1} I^m)$ whence $J_i \cap I^{m+1} = J_i I^m + a_{i+1}([J_i : a_{i+1}] \cap I^{i-s} \cap I^m)$ (note $m \geq i - s$). Thus by (2.8) we get $J_i \cap I^{m+1} = J_i I^m + a_{i+1}(J_i \cap I^m)$ and the hypothesis on m yields $J_i \cap I^{m+1} = J_i I^m + a_{i+1} J_i I^{m-1} = J_i I^m$.

Now consider the case $i = s$. We must show $J_s \cap I^{m+1} = J_s I^m$ for all $m \geq -1$. We may assume $m \geq 1$ and $J_s \cap I^m = J_s I^{m-1}$. Then as $J_s \cap I^{m+1} \subseteq J_{s+1} \cap I^{m+1} = J_{s+1} I^m$, we have $J_s \cap I^{m+1} = J_s I^m + a_{s+1}([J_s : a_{s+1}] \cap I^m)$. Hence as $[J_s : a_{s+1}] \cap I = J_s$ by (2.4)(4), we get $[J_s : a_{s+1}] \cap I^m = J_s \cap I^m = J_s I^{m-1}$ so the equality $J_s \cap I^{m+1} = J_s I^m$ follows.

Corollary (2.10). *Let A be a Gorenstein ring. Assume that A/I is Cohen-Macaulay and that $r \leq \ell - s - 1$. Assume the inequality* $\operatorname{depth}(A/I^n)_Q \geq \min\{\ell - s - 1 - n, \operatorname{ht}_A Q - s - n\}$ *holds for all prime ideals $Q \in V(I)$ and for all integers n with $1 \leq n \leq \ell - s - 1$. Then $a_1 t, a_2 t, \ldots, a_s t$ forms a $G(I)$-regular sequence.*

Proof. The assertion follows from [VV, 2.7] as by (2.9) $J_s \cap I^{n+1} = J_s I^n$ for all $n \geq 0$.

3. Proof of Theorem (1.1)

Throughout this section we assume that A is a Cohen-Macaulay ring and that I is an ideal in A, which contains a special reduction J with $r_J(I) = r \leq \ell - s$, where $\mathrm{ht}_A I = s$ and $\lambda(I) = \ell$. Then we can choose a system of generators $a_1, a_2, \ldots, a_\ell$ for J so that the conditions stated in (2.4) are all fulfilled. Further we assume the inequalities $\operatorname{depth}(A/I^n)_Q \geq \min\{\ell - s - n, \mathrm{ht}_A Q - s - n\}$ and $\operatorname{depth} A/I^n \geq d - s - n + 1$ hold for all prime ideals $Q \in V(I)$ and for all integers n with $1 \leq n \leq \ell - s$. Let $G = G(I)$ and $R = R(I)$. Then by (2.7) the sequence $a_1t, a_2t, \ldots, a_st$ is G-regular.

The purpose of this section is to prove Theorem (1.1). We begin with the following lemma, which enables us to reduce the problem to the case where $s = 0$.

Lemma (3.1). *Suppose $s > 0$ and let $\bar{A} = A/a_1A$, $\bar{I} = I\bar{A}$, and $\bar{J} = J\bar{A}$. Then*

(1) *$\bar{A}$ is a Cohen-Macaulay ring, $ht_{\bar{A}} \bar{I} = s - 1$, and $\lambda(\bar{I}) = \ell - 1$.*

(2) *$\bar{J}$ is a special reduction of $\bar{I}$ and $r_{\bar{J}}(\bar{I}) \leq \mathrm{ad}(\bar{I})$.*

(3) *The inequalities $\operatorname{depth}(\bar{A}/\bar{I}^n)_{\bar{Q}} \geq \min\{\mathrm{ad}(\bar{I}) - n, \mathrm{ht}_{\bar{A}} \bar{Q} - \mathrm{ht}_{\bar{A}} \bar{I} - n\}$ and $\operatorname{depth} \bar{A}/\bar{I}^n \geq \dim \bar{A} - \mathrm{ht}_{\bar{A}} \bar{I} - n + 1$ hold for all prime ideals $\bar{Q} \in V(\bar{I})$ and for all integers n with $1 \leq n \leq \mathrm{ad}(\bar{I})$.*

(4) *G is a Cohen-Macaulay (resp. Gorenstein) ring if and only if $G(\bar{I})$ is a Cohen-Macaulay (resp. Gorenstein) ring. When this is the case, one has the equality $a(G) = a(G(\bar{I})) - 1$.*

Proof. $\bar{A}$ is a Cohen-Macaulay ring with $\mathrm{ht}_{\bar{A}} \bar{I} = s - 1$, because a_1 is chosen to be A-regular. As a_1t is by (2.7) G-regular, we get by [VV, 1.1] an isomorphism $G(\bar{I}) \cong G/a_1tG$ of A-algebras. Hence the assertion (4) follows (see [GW, (3.1.6)] for the equality $a(G) = a(G(\bar{I})) - 1$). Further, because $G(\bar{I})/\mathfrak{m}G(\bar{I}) \cong G/(\mathfrak{m}G + a_1tG)$ and because a_1t forms part of a linear system of parameters of the $A/\mathfrak{m}$-algebra $G/\mathfrak{m}G$, we get the equality $\lambda(\bar{I}) = \ell - 1$. Hence $\mathrm{ad}(\bar{I}) = \ell - s$. Because $\bar{I}^{n+1} = \bar{J} \cdot \bar{I}^n$ if $I^{n+1} = JI^n$, the ideal $\bar{J}$ is a minimal reduction of $\bar{I}$ with $r_{\bar{J}}(\bar{I}) \leq r$ and $r_{\bar{J}}(\bar{I}) \leq \mathrm{ad}(\bar{I})$. Take $Q \in V(I)$ with $\mathrm{ht}_A Q = h$ and assume that $\mathrm{ht}_{\bar{A}} \bar{Q} < \lambda(\bar{I}) = \ell - 1$, where $\bar{Q} = Q/a_1A$. Then as $h = \mathrm{ht}_{\bar{A}} \bar{Q} + 1 < \ell$, we get by (2.1) $I_Q = (a_1, a_2, \ldots, a_h)A_Q$ whence $\bar{I}_{\bar{Q}} = (a_2, \ldots, a_h)\bar{A}_{\bar{Q}}$. Thus $\bar{J}$ is a special reduction of $\bar{I}$. To see the assertion (3), let n be an integer with $1 \leq n \leq \ell - s$. Then $a_1A \cap I^n = a_1I^{n-1}$ by (2.6). So we have $a_1A/(a_1A \cap I^n) = a_1A/a_1I^{n-1} \cong A/I^{n-1}$. Hence the exact sequence

$$0 \to A/I^{n-1} \to A/I^n \to \bar{A}/\bar{I}^n \to 0$$

follows. As $\operatorname{depth} A/I^n \geq d - s - n + 1$ and $\operatorname{depth} A/I^{n-1} \geq d - s - n + 2$, it follows from Depth Lemma that $\operatorname{depth} \bar{A}/\bar{I}^n \geq d - s - n + 1 = \dim \bar{A} - \mathrm{ht}_{\bar{A}} \bar{I} - n + 1$. And the rest of the assertion (3) follows from the exact sequence above via localization.

In what follows, till(3.4) we maintain the assumption that $s = 0$ and $\mathrm{ad}(I) = \lambda(I) = \ell$.

Lemma (3.2). *For integers i and n satisfying $0 \le i \le n \le \ell$,*

$$\operatorname{depth} A/J_iI^n \ge \begin{cases} d-i & \text{if } n = i\,, \\ d-n+1 & \text{if } n > i\,. \end{cases}$$

Proof. We shall prove the assertion by induction on i. We may assume that $i > 0$ and that our assertion is true for $i-1$. Take an integer n with $i \le n \le \ell$, then because $J_iI^n/J_{i-1}I^n \cong I^n/J_{i-1}I^{n-1}$ by (2.5) (2), we get the exact sequence

(a) $$0 \to I^n/J_{i-1}I^{n-1} \to A/J_{i-1}I^n \to A/J_iI^n \to 0$$

of A-modules. We consider the canonical exact sequence

(b) $$0 \to I^n/J_{i-1}I^{n-1} \to A/J_{i-1}I^{n-1} \to A/I^n \to 0$$

as well. Then if $n > i$, the inductive hypothesis on i says depth $A/J_{i-1}I^{n-1} \ge d-n+2$ and depth $A/J_{i-1}I^n \ge d-n+1$, while depth $A/I^n \ge d-n+1$ by our standard assumption of this section. Hence thanks to Depth Lemma, by (b) we have $\operatorname{depth}_A I^n/J_{i-1}I^{n-1} \ge d-n+2$. So by (a) we find depth $A/J_iI^n \ge d-n+1$. If $n = i$, the inductive hypothesis on i says depth $A/J_{i-1}I^{i-1} \ge d-i+1$ and depth $A/J_{i-1}I^i \ge d-i+1$. Hence the inequality depth $A/J_iI^i \ge d-i$ similarly follows from exact sequences (a) and (b) above.

We put $G^{(i)} = G/(a_1t, a_2t, \ldots, a_it)G$ for $0 \le i \le \ell$. Let $[G^{(i)}]_n$ ($n \in \mathbf{Z}$) stand for the homogeneous component of degree n in the graded G-module $G^{(i)}$. Notice that $[G^{(i)}]_n = I^n/(J_iI^{n-1} + I^{n+1})$. Let $U^{(i)} = \sum_{n \ge i+1} [G^{(i)}]_n$.

Lemma (3.3). *Take an integer i satisfying $0 \le i < \ell$. Then*
(1) $[U^{(i)}]_{i+1} \ne (0)$.
(2) $\operatorname{depth}_A [U^{(i)}]_{i+1} \ge d-i-1$.
(3) *$a_{i+1}t$ is $U^{(i)}$-regular.*

Proof. Suppose $[U^{(i)}]_{i+1} = (0)$. Then $I^{i+1} = J_iI^i + I^{i+2}$ and $I^{i+1} = J_iI^i$. Thus by definition $J = J_i$. This is absurd because $i < \ell = \lambda(I)$. Since depth $A/J_iI^i \ge d-i$ by (3.2) and since depth $A/I^{i+1} \ge d-i$ by our standard assumption, we get $\operatorname{depth}_A I^{i+1}/J_iI^i \ge d-i$ thanks to Depth Lemma applied to the sequence $0 \to I^{i+1}/J_iI^i \to A/J_iI^i \to A/I^{i+1} \to 0$. Similarly, as depth $A/J_iI^{i+1} \ge d-i$ by (3.2) and as depth $A/I^{i+2} \ge d-i-1$ by our standard assumption (note that $I^{\ell+1} = JI^\ell$ and depth $A/I^{\ell+1} \ge d-\ell$ by (3.2)), we get $\operatorname{depth}_A I^{i+2}/J_iI^{i+1} \ge d-i$ by virtue of the exact sequence $0 \to I^{i+2}/J_iI^{i+1} \to A/J_iI^{i+1} \to A/I^{i+2} \to 0$. Now let $\varepsilon\colon I^{i+1}/J_iI^i \to [U^{(i)}]_{i+1}$ be the canonical epimorphism and put $K = \operatorname{Ker} \varepsilon$. Then $K \cong I^{i+2}/(J_iI^i \cap I^{i+2}) \cong I^{i+2}/J_iI^{i+1}$, because $J_i \cap I^{i+2} = J_iI^{i+1}$ by (2.6). So we have the exact sequence $0 \to I^{i+2}/J_iI^{i+1} \to I^{i+1}/J_iI^i \to [U^{(i)}]_{i+1} \to 0$. Thus $\operatorname{depth}_A [U^{(i)}]_{i+1} \ge d-i-1$, because $\operatorname{depth}_A I^{i+2}/J_iI^{i+1} \ge d-i$ and $\operatorname{depth}_A I^{i+1}/J_iI^i \ge d-i$ as we have shown above. For the assertion (3), let $x \in I^n$ with $n \ge i+1$ and assume $(a_{i+1}t)(xt^n) \equiv 0 \bmod IR + (a_1t, \ldots, a_it)R$. We will show $xt^n \in IR + (a_1t, \ldots, a_it)R$. Firstly, recall that $J_{i+1} \cap I^{n+2} = J_{i+1}I^{n+1}$ (see (2.6)). Then as $a_{i+1}x \in (J_iI^n + I^{n+2}) \cap$

J_{i+1}, we get $a_{i+1}x \in J_i I^n + a_{i+1} I^{n+1}$. Choose $y \in I^{n+1}$ so that $a_{i+1}(x-y) \in J_i I^n$. Then as $x - y \in [J_i : a_{i+1}] \cap I^{i+1}$ (note $n \geq i+1$), by (2.5)(1) we have $x - y \in J_i$. Hence by (2.6) $x - y \in J_i \cap I^n = J_i I^{n-1}$. So $x \in J_i I^{n-1} + I^{n+1}$, and $xt^n \in IR + (a_1 t, \dots, a_i t)R$. Thus $a_{i+1}t$ is $U^{(i)}$-regular.

Let $G_+ = \sum_{n \geq 1} G_n$ and let $\mathfrak{M} = \mathfrak{m}G + G_+$ stand for the graded maximal ideal in G. Let $H^j_{\mathfrak{M}}(*)$ ($j \in \mathbf{Z}$) denote local cohomology functors. For each graded G-module M we put $\operatorname{Soc} M = (0) :_M \mathfrak{M}$ and call it the socle of M. We denote $\operatorname{depth}_{G_{\mathfrak{M}}} M_{\mathfrak{M}}$ simply by $\operatorname{depth}_G M$ when M is finitely generated. Then M is a Cohen-Macaulay G-module if and only if $\dim_G M = \operatorname{depth}_G M$ (cf. [GW, (1.1.3)]).

Proposition (3.4). *Take an integer i satisfying $0 \leq i < \ell$. Then*
(1) *$U^{(i)}$ is a Cohen-Macaulay G-module of dimension $d - i$.*
(2) *$\operatorname{Soc} H^{d-i}_{\mathfrak{M}}(U^{(i)})$ is concentrated in degree i.*

Proof. Firstly, by descending induction on i we will show that $\operatorname{depth}_G U^{(i)} \geq d - i$ for all $0 \leq i \leq \ell$. As $r \leq \ell$, $U^{(\ell)} = (0)$. So we have nothing to prove for $i = \ell$. Let $i < \ell$ and assume that $\operatorname{depth}_G U^{(i+1)} \geq d - i - 1$. We put $\overline{U}^{(i)} = U^{(i)}/a_{i+1}tU^{(i)}$. Then as $[\overline{U}^{(i)}]_n = [U^{(i+1)}]_n$ *for all* $n \geq i+2$, we get the exact sequence of graded G-modules

$$\text{(a)} \qquad 0 \to U^{(i+1)} \to \overline{U}^{(i)} \to W^{(i)} \to 0\,,$$

where $W^{(i)}$ is concentrated in degree $i+1$ and $[W^{(i)}]_{i+1} = [U^{(i)}]_{i+1}$. Recall that $H^j_{\mathfrak{M}}(W^{(i)}) = [H^j_{\mathfrak{M}}(W^{(i)})]_{i+1} = H^j_{\mathfrak{m}}([U^{(i)}]_{i+1})$ for all $j \in \mathbf{Z}$ (see [GH, 2.2]), and we get $\operatorname{depth}_G W^{(i)} \geq d - i - 1$ by (3.3)(2). Hence as $\operatorname{depth}_G U^{(i+1)} \geq d - i - 1$, by the sequence (a) $\operatorname{depth}_G \overline{U}^{(i)} \geq d - i - 1$. So we have $\operatorname{depth}_G U^{(i)} \geq d - i$, because $\overline{U}^{(i)} = U^{(i)}/a_{i+1}tU^{(i)}$ and because $a_{i+1}t$ is $U^{(i)}$-regular by (3.3)(3). Thus $\operatorname{depth}_G U^{(i)} \geq d - i$ for all $0 \leq i < \ell$. We particularly have $\dim_G U^{(0)} = \operatorname{depth}_G U^{(0)} = d$. Here notice that $\dim_G U^{(i+1)} \leq \dim_G \overline{U}^{(i)}$ (see the exact sequence (a)). Then as $\dim_G \overline{U}^{(i)} = \dim_G U^{(i)} - 1$ if $0 \leq i < \ell$, we get $\dim_G U^{(i+1)} \leq \dim_G U^{(i)} - 1$, and $\dim_G U^{(i)} \leq d - i$ for all $0 \leq i < \ell$. Hence $U^{(i)}$ is a Cohen-Macaulay G-module of dimension $d - i$.

Secondly, we will prove $\operatorname{Soc} H^{d-i}_{\mathfrak{M}}(U^{(i)}) = [\operatorname{Soc} H^{d-i}_{\mathfrak{M}}(U^{(i)})]_i$, for all $0 \leq i \leq \ell$, by descending induction on i. As $U^{(\ell)} = (0)$, this is obviously true for $i = \ell$. Let $i < \ell$ and assume $\operatorname{Soc} H^{d-i-1}_{\mathfrak{M}}(U^{(i+1)})$ is concentrated in degree $i+1$. We apply local cohomology functors $H^j_{\mathfrak{M}}(*)$ to the sequence (a). Then as $\operatorname{depth}_G W^{(i)} \geq d - i - 1$, we get the exact sequence

$$\text{(b)} \qquad 0 \to H^{d-i-1}_{\mathfrak{M}}(U^{(i+1)}) \to H^{d-i-1}_{\mathfrak{M}}(\overline{U}^{(i)}) \to H^{d-i-1}_{\mathfrak{M}}(W^{(i)})\,.$$

As $H^{d-i-1}_{\mathfrak{M}}(W^{(i)}) = [H^{d-i-1}_{\mathfrak{M}}(W^{(i)}]_{i+1}$ by [GH, 2.2] and as $\operatorname{Soc} H^{d-i-1}_{\mathfrak{M}}(U^{(i+1)})$ is concentrated in degree $i+1$, by the sequence (b) $\operatorname{Soc} H^{d-i-1}_{\mathfrak{M}}(\overline{U}^{(i)})$ is also concentrated in degree $i+1$. We now look at the exact sequence $0 \to U^{(i)}(-1) \xrightarrow{a_{i+1}t} U^{(i)} \to \overline{U}^{(i)} \to 0$ and apply local cohomology functors $H^j_{\mathfrak{M}}(*)$ to it. Then as $U^{(i)}$ is a Cohen-Macaulay G-module of dimension $d - i$, we get the short exact sequence

$$\text{(c)} \qquad 0 \to H_{\mathfrak{M}}^{d-i-1}(\overline{U}^{(i)}) \to [H_{\mathfrak{M}}^{d-i}(U^{(i)})](-1) \xrightarrow{a_{i+1}t} H_{\mathfrak{M}}^{d-i}(U^{(i)}) \to 0\,.$$

And as $a_{i+1}tG \subseteq \mathfrak{M}$, applying the functor $\mathrm{Hom}_G(G/\mathfrak{M}, *)$ to the sequence (c), we get the isomorphism $\mathrm{Soc}\, H_{\mathfrak{M}}^{d-i-1}(\overline{U}^{(i)}) \cong [\mathrm{Soc}\, H_{\mathfrak{M}}^{d-i}(U^{(i)})](-1)$ of graded G-modules. Thus $\mathrm{Soc}\, H_{\mathfrak{M}}^{d-i}(U^{(i)})$ is concentrated in degree i, because $\mathrm{Soc}\, H_{\mathfrak{M}}^{d-i-1}(\overline{U}^{(i)})$ is concentrated in degree $i+1$.

For the next result we only assume $s \geq 0$.

Corollary (3.5). (1) *G is a Cohen-Macaulay ring.*
(2) $\mathrm{Soc}\, H_{\mathfrak{M}}^{d}(G) = [\mathrm{Soc}\, H_{\mathfrak{M}}^{d}(G)]_{-s}$.
(3) $a(G) = -s$.

Proof. First we consider the case $s = 0$. If $\ell = 0$, then $I = J = (0)$ as $r \leq \ell$, whence $G = A$ and we have nothing to prove. Suppose $\ell > 0$. Note that $U^{(0)} = G_+$ and we have $\mathrm{depth}_G\, G_+ = d$ by (3.4)(1). Let us identify $G/G_+ = A/I$. Then as $\mathrm{depth}_G\, G_+ = d$ and $\mathrm{depth}\, A/I = d$ (recall that the inequality $\mathrm{depth}\, A/I^n \geq d - n + 1$ holds for all integers n with $1 \leq n \leq \ell$, which is one of our standard assumptions of this section), we get $\mathrm{depth}\, G = d$ by the exact sequence

$$\text{(a)} \qquad 0 \to G_+ \to G \to A/I \to 0\,.$$

Hence G is a Cohen-Macaulay ring. Apply local cohomology functors $H_{\mathfrak{M}}^{j}(*)$ to (a) and look at the resulting short exact sequence

$$\text{(b)} \qquad 0 \to H_{\mathfrak{M}}^{d}(G_+) \to H_{\mathfrak{M}}^{d}(G) \to H_{\mathfrak{M}}^{d}(A/I) \to 0$$

of local cohomology modules. Now recall $H_{\mathfrak{M}}^{d}(A/I) = [H_{\mathfrak{M}}^{d}(A/I)]_0 \cong H_{\mathfrak{m}}^{d}(A/I)$ (see [GH, 2.2]). Then as $\mathrm{Soc}\, H_{\mathfrak{M}}^{d}(G_+)$ is by (3.4)(2) concentrated in degree 0, we see by the sequence (b) that $\mathrm{Soc}\, H_{\mathfrak{M}}^{d}(G)$ is concentrated in degree 0 too. Thus $\mathrm{Soc}\, H_{\mathfrak{M}}^{d}(G) = [\mathrm{Soc}\, H_{\mathfrak{M}}^{d}(G)]_0$ and $a(G) = 0$. Let us now consider the case $s > 0$ and put $\bar{A} = A/a_1A$ and $\bar{I} = I\bar{A}$. Then passing to the ring $G(\bar{I})$, thanks to (3.1) the assertion (1) readily follows by induction on s, while the exact sequence $0 \to G(-1) \xrightarrow{a_1 t} G \to G(\bar{I}) \to 0$ guarantees the isomorphism $\mathrm{Soc}\, H_{\mathfrak{M}}^{d-1}(G(\bar{I})) \cong [\mathrm{Soc}\, H_{\mathfrak{M}}^{d}(G)](-1)$ on socles. Hence the induction on s works also to get the assertion (2). The assertion (3) now follows from the assertion (2).

We are now ready to prove Theorem (1.1).

Proof of Theorem (1.1). (1) See (3.5)(1).

(2) Let K_G stand for the graded canonical module of G. Then as K_G is, by (3.5)(2), generated by elements of degree s, we see by [HSV, 2.3] that K_G is a cyclic G-module (notice that [HSV, 2.3] is true whenever G is Cohen-Macaulay). Thus G is a Gorenstein ring.

Let us close this section with a proof of Corollary (1.2).

Proof of Corollary (1.2). (1) This follows from [TI, 1.1], because G is by (1.1) a Cohen-Macaulay ring of $a(G) = -s$.

(2) This follows from [I, 3.1], because G is by (1.1) a Gorenstein ring of $a(G) = -s$.

4. Proof of Theorem (1.3)

In this section we assume that A is a Gorenstein ring and that I is an ideal in A, which contains a special reduction J, with $\mathrm{ht}_A I = s$ and $\lambda(I) = \ell$. We also assume that A/I is a Cohen-Macaulay ring and that $\mathrm{ad}(I) = \ell - s \geq 1$. We choose a system of generators $a_1, a_2, \ldots, a_\ell$ for J so that the conditions stated in (2.4) are fulfilled. We put $G = G(I)$, $R = R(I)$, and $\mathfrak{M} = \mathfrak{m}G + G_+$.

The purpose of this section is to prove Theorem (1.3). Take an integer i with $s + 1 \leq i \leq \ell$. Then we have $\mathrm{ht}_A(I + [J_{i-1} : I]) \geq i$, because $J_{i-1}A_Q = I_Q$ by (2.1) for all prime ideals $Q \in V(I)$ with $\mathrm{ht}_A Q < i$. We can therefore choose a system of generators $x_{s+1}, \ldots, x_\ell, x_{\ell+1}, \ldots, x_d$ for the ring A/I so that $x_i \in J_{i-1} : I$ for all integers i with $s + 1 \leq i \leq \ell$. We put $\mathfrak{a} = (a_1t, a_2t, \ldots, a_st)G + (x_{s+1} + a_{s+1}t, x_{s+2} + a_{s+2}t, \ldots, x_\ell + a_\ell t)G + (x_{\ell+1}, x_{\ell+2}, \ldots, x_d)G$. Then we have

Lemma (4.1) ([AH, 5.6]). (1) $\mathfrak{M} = \sqrt{\mathfrak{a}}$

(2) *G is a Cohen-Macaulay ring if and only if the sequence $a_1t, a_2t, \ldots, a_st, x_{s+1} + a_{s+1}t, x_{s+2} + a_{s+2}t, \ldots, x_\ell + a_\ell t, x_{\ell+1}, x_{\ell+2}, \ldots, x_d$ is G-regular.*

The next lemma enables us to reduce the problem to the case where $s = 0$.

Lemma (4.2). *Assume that $s \geq 1$ and that a_1t is G-regular. Let $\bar{A} = A/a_1A$, $\bar{I} = I\bar{A}$, and $\bar{J} = J\bar{A}$. Then*

(1) *$\bar{A}$ is a Gorenstein ring, $ht_{\bar{A}}\, \bar{I} = s - 1$, and $\lambda(\bar{I}) = \ell - 1$. Hence* $\mathrm{ad}(\bar{I}) = \mathrm{ad}(I)$.

(2) *$\bar{J}$ is a special reduction of $\bar{I}$ and $r_{\bar{J}}(\bar{I}) = r_J(I)$.*

(3) *Assume further that the inequalities* $\mathrm{depth}\,(A/I^n)_Q \geq \min\{\mathrm{ad}(I) - n - 1, \mathrm{ht}_A Q - \mathrm{ht}_A I - n\}$ *and* $\mathrm{depth}\, A/I^n \geq d - \mathrm{ht}_A I - n$ *hold for all prime ideals $Q \in V(I)$ and for all integers n with $1 \leq n \leq \mathrm{ad}(I) - 1$. Then one has the inequalities* $\mathrm{depth}\,(\bar{A}/\bar{I}^n)_{\bar{Q}} \geq \min\{\mathrm{ad}(\bar{I}) - n - 1, \mathrm{ht}_{\bar{A}}\, \bar{Q} - ht_{\bar{A}}\, \bar{I} - n\}$ *and* $\mathrm{depth}\, \bar{A}/\bar{I}^n \geq \dim \bar{A} - \mathrm{ht}_{\bar{A}}\, \bar{I} - n$ *for all prime ideals $\bar{Q} \in V(\bar{I})$ and for all integers n with $1 \leq n \leq \mathrm{ad}(\bar{I}) - 1$.*

(4) *G is a Gorenstein ring if and only if $G(\bar{I})$ is a Gorenstein ring.*

Proof. Let $n = r_{\bar{J}}(\bar{I})$. Then as $I^{n+1} \subseteq JI^n + a_1A$, we get $I^{n+1} = JI^n + a_1A \cap I^{n+1}$, while $a_1A \cap I^{n+1} = a_1I^n$ because a_1t is G-regular. Hence $I^{n+1} = JI^n$. So we have $r_{\bar{J}}(\bar{I}) = r_J(I)$. Consult Proof of Lemma (3.1) for the other assertions.

For the rest of this section we assume that $s = 0$. Hence $\mathrm{ad}(I) = \ell$. We put $B = A/[(0) : I]$. But B is a Cohen-Macaulay ring of $\dim B = d$ (see [PS, 1.3]). Let $K_{A/I}$ and K_B respectively denote the canonical modules of A/I and B. Then as A is a Gorenstein ring and as $\dim A/I = \dim B = d$, by [HK, 5.20] we have isomorphisms $K_{A/I} \cong (0) : I$ and $K_B \cong (0) : [(0) : I]$. Note that $I = (0) : [(0) : I]$ because $(0) :_A K_{A/I} = I$ by [HK, 6.7] (recall that A/I is Cohen-Macaulay) and we get $K_B \cong I$, while $IB \cong I$ as $[(0):I] \cap I = (0)$ by (2.4)(4). Thus we have the assertion (1) in the next lemma. See [HK, 6.13] for the proof of assertions (2) and (3).

Lemma (4.3). (1) $K_B \cong IB$.
(2) $\mathrm{ht}_B\, IB = 1$.
(3) *B/IB is a Gorenstein ring.*

Put $T = G(IB)$. Let $\varphi: G \to T$ be the canonical epimorphism and let $K = \mathrm{Ker}\, \varphi$. Then $K_n \cong [I^n \cap ((0):I)]/[I^{n+1} \cap ((0):I)]$ $(n \in \mathbf{Z})$. Hence $K_n = (0)$ if $n \geq 1$, because $[(0):I] \cap I = (0)$. Then we have $K = K_0 \subseteq \mathfrak{m}G$ and $K_0 \cong (0):I \cong K_{A/I}$ whence $\mathrm{depth}_G\, K = d$. We note

Lemma (4.4). (1) *$\lambda(IB) = \ell$ and $\mathrm{ad}\,(IB) = \ell - 1$.*
(2) *JB is a special reduction of IB and $r_{JB}(IB) = r_J(I)$.*
(3) *Assume that the inequalities $\mathrm{depth}\,(A/I^n)_P \geq \min\{\mathrm{ad}\,(I) - n - 1, \mathrm{ht}_A\, P - n\}$ and $\mathrm{depth}\, A/I^n \geq d - n$ hold for all prime ideals $P \in V(I)$ and for all integers n with $1 \leq n \leq \mathrm{ad}\,(I) - 1$. Then one has the inequalities $\mathrm{depth}\,(B/I^nB)_Q \geq \min\{\mathrm{ad}\,(IB) - n, \mathrm{ht}_B\, Q - \mathrm{ht}_B\, IB - n\}$ and $\mathrm{depth}\, B/I^nB \geq \dim B - \mathrm{ht}_B\, IB - n + 1$ for all prime ideals $Q \in V(IB)$ and for all integers n with $1 \leq n \leq \mathrm{ad}\,(IB)$.*

Proof. As $K = K_0 \subseteq \mathfrak{m}G$, the map $A/\mathfrak{m} \otimes_A \varphi$: $A/\mathfrak{m} \otimes_A G \to A/\mathfrak{m} \otimes_A T$ is an isomorphism. Thus $\lambda(IB) = \ell$, and JB is a minimal reduction of IB. We have $\mathrm{ad}\,(IB) = \ell - 1$ because $\mathrm{ht}_B\, IB = 1$ by (4.3)(2). Take a prime ideal $Q \in V(IB)$ with $\mathrm{ht}_B\, Q = h < \ell$ and choose $P \in V(I + [(0):I])$ so that $Q = P/[(0):I]$. Then $\mathrm{ht}_A\, P = \mathrm{ht}_B\, Q$ $(= h < \ell)$. Hence by (2.1) we have $IA_P = (a_1, a_2, \ldots, a_h)A_P$, and $IB_Q = (a_1, a_2, \ldots, a_h)B_Q$. Thus JB is a special reduction of IB. Let $n = r_{JB}(IB)$. Then as $I^{n+1} \subseteq JI^n + [(0):I]$, we have $I^{n+1} = JI^n + I^{n+1} \cap [(0):I]$. Hence $I^{n+1} = JI^n$ because $[(0):I] \cap I = (0)$ by (2.4)(4). Thus $r_{JB}(IB) = r_J(I)$. To see the assertion (3), take an integer n with $1 \leq n \leq \ell - 1$. We look at the exact sequence

$$\text{(a)} \qquad 0 \to (0):I \to A/I^n \to B/I^nB \to 0\,,$$

which follows from the facts that $B/I^nB = A/(I^n + [(0):I])$ and $[(0):I] \cap I = (0)$. Notice that $\mathrm{depth}\, A/I^n \geq d - n$ and that $\mathrm{depth}_A\,(0):I = d$ (recall $(0):I = K_{A/I}$). Then by the sequence (a) we find that $\mathrm{depth}\, B/I^nB \geq d - n = \dim B - \mathrm{ht}_B\, IB - n + 1$. Thanks to Depth Lemma, the rest of the inequalities follow similarly as above via exact the sequence (a) after localization.

Assume now that G is Cohen-Macaulay ring. Let K_G stand for the graded canonical module of G and put $E = \mathrm{Ext}^1_G\,(T, K_G)$. Take the K_G-dual of the sequence

$$\text{(4.5)} \qquad 0 \to K \to G \xrightarrow{\varphi} T \to 0$$

and we get the exact sequence

$$\text{(4.6)} \qquad 0 \to K_T \to K_G \to \mathrm{Hom}_G\,(K, K_G) \to E \to 0$$

of graded G-modules. Because $\mathrm{Hom}_G(K_{A/I}, K_G) \cong A/I$ by [HK, 6.1] and because $K = K_0 \cong K_{A/I}$, we get $\mathrm{Hom}_G(K, K_G) \cong A/I$ in which A/I is considered to be a graded G-module concentrated in degree 0. Hence from (4.6) we have the exact sequence

$$0 \to K_T \to K_G \to A/I \to E \to 0\,. \tag{4.7}$$

If G is furthermore a Gorenstein ring with $a = a(G)$, identifying $K_G = G(a)$, we get by (4.7) the exact sequence

$$0 \to K_T \to G(a) \to A/I \to E \to 0 \tag{4.8}$$

of graded G-modules. Here we note the following

Proposition (4.9). *Suppose that G is a Gorenstein ring. Then*

(1) $a(G) = 0$.

(2) *T is a Cohen-Macaulay ring of* $a(T) = -1$.

(3) $K_T \cong G_+$.

Proof. Assume that $a = a(G) < 0$. Then considering the homogeneous components of degree 0 in the exact sequence (4.8), we find $A/I \cong E$. So $(0):I \subseteq I$ because $[(0):I]E = (0)$. Hence $(0):I = (0)$ by (2.4)(4), which is impossible because $\mathrm{ht}_A I = s = 0$. Now suppose that $a \geq 1$. Then considering the homogeneous components of degree $-a$ in (4.8), we find $[K_T]_{-a} \cong G_0 = A/I$. Hence $(0):I \subseteq I$ because $[(0):I]T = (0)$. This also cannot happen. Thus $a = 0$ and we get the exact sequence of graded G-modules.

$$0 \to K_T \to G \to A/I \to E \to 0\,. \tag{4.10}$$

We have $\mathrm{depth}_G T \geq d - 1$ by (4.5) because $\mathrm{depth}\, G = \mathrm{depth}_G K = d$. Hence to see Cohen-Macaulayness in T, it suffices to show $H^{d-1}_{\mathfrak{M}}(T) = (0)$, or equivalently, $E = (0)$ (cf. [HK, 5.12]). Assume the contrary and choose a prime ideal $Q \in \mathrm{Supp}_A E$ so that $\dim_A E = \dim A/Q$. Then as E is a factor module of A/I by a single element (look at the homogeneous components of degree 0 in the exact sequence (4.10)), we have $\dim_A E \geq \dim A/I - 1 = d - 1$. Hence $\mathrm{ht}_A Q \leq 1$. As $(I + [(0):I])E = (0)$, we get $I + [(0):I] \subseteq Q$. Thus $\mathrm{ht}_A Q = 1$ (recall that $\mathrm{ht}_A (I + [(0):I]) \geq 1$) and QB is a prime ideal in B containing IB. Now if $\ell = 1$, then we have $r_J(I) = 0$ by [GNa1, (2.11)]. So $I = a_1 A$. And if $\ell \geq 2$, we have $I_Q = a_1 A_Q$ by (2.1). Hence $IB_Q = a_1 B_Q$ in any case. As $\mathrm{ht}_B IB = 1$ by (4.3)(2), we find $\dim B_Q = 1$ and a_1 is B_Q-regular. Thus $T_Q \cong (B_Q/a_1 B_Q)[t]$ is a polynomial ring in one variable t over $B_Q/a_1 B_Q$. Hence T_Q is a Cohen-Macaulay ring with $\dim T_Q = \dim G_Q = 1$. So we have $E_Q = \mathrm{Ext}^1_{G_Q}(T_Q, K_{G_Q}) = (0)$ by [HK, 6.1]. This contradicts the choice of Q. Thus T is a Cohen-Macaulay ring and $E = (0)$. Hence by (4.10) we get the exact sequence of graded G-modules

$$0 \to K_T \to G \to A/I \to 0\,. \tag{4.11}$$

Now look at the homogeneous components $0 \to [K_T]_0 \to G_0 \xrightarrow{\varepsilon} A/I \to 0$ of degree

0 in the exact sequence (4.11). Then as $G_0 = A/I$, the map ε has to be an isomorphism. So we have $[K_T]_0 = (0)$. Thus by (4.11) we get $K_T \cong G_+$ and $[K_T]_1 \cong I/I^2 \neq (0)$. Hence $a(T) = -1$.

If G is a Gorenstein ring, T is by (4.9)(2) a Cohen-Macaulay ring. As $\mathrm{ht}_B\, IB = 1$ by (4.3)(2), we get $a(T) = \max\{r_{JB}(IB) - \lambda(IB), -1\}$ by [AH, 5.10] and [T, 2.5]. Hence $r_{JB}(IB) < \lambda(IB)$ because $a(T) = -1$ by (4.9)(2). Hence by (4.4) we get the following

Corollary (4.12). *Suppose that G is a Gorenstein ring. Then $r_J(I) \leq \ell - 1$.*

We close this section by proving Theorem (1.3).

Proof of Theorem (1.3). (1) If $s > 0$, by (4.1)(2) a_1t is G-regular. Hence by (4.2) the inequality $r_J(I) \leq \mathrm{ad}\,(I) - 1$ readily follows from (4.12) by induction on s.

(2) By (2.10) the sequence $a_1t, a_2t, \ldots, a_st$ is G-regular. Hence passing to the ring $G(I/(a_1, a_2, \ldots, a_s)A)$ ($\cong G/(a_1t, a_2t, \ldots, a_st)G$, cf. [VV, 1.1] and thanks to (4.2), we may assume without loss of generality that $s = 0$. Let us maintain the same notation as we have settled in this section. Firstly, note that by (4.4) the hypotheses in Theorem (1.1) are all fulfilled for the ideal IB in the ring B. Hence by (3.5) T is a Cohen-Macaulay ring and the graded canonical module K_T of T is generated by elements of degree -1 (recall that $\mathrm{ht}_B\, IB = 1$ by (4.3)(2)). Therefore by [HSV, 2.4] we get $K_T = \mathrm{gr}_{IB}(K_B)(-1)$ where $\mathrm{gr}_{IB}(K_B)$ denotes the graded module associated to the filtration $\{I^nK_B\}_{n\geq 0}$ of K_B. As $K_B = IB$ by (4.3)(1), we also get $\mathrm{gr}_{IB}(K_B)(-1) = \mathrm{gr}_{IB}(IB)(-1) = T_+$. Thus $K_T = T_+$. We consider the exact sequence (4.5). Recall that $K = K_0$. Then $G_+ = T_+$, whence $K_T = G_+$. Further, by the sequence (4.5) G is a Cohen-Macaulay ring, because both $K = K_{A/I}$ and T are Cohen-Macaulay G-modules of dimension d. Now take the K_G-dual of the canonical exact sequence $0 \to G_+ \to G \to A/I \to 0$. Then because $\mathrm{Hom}_G(G_+, K_G) = \mathrm{Hom}_G(K_T, K_G) = T$ ([HK, 6.1]) and because $\mathrm{Hom}_G(A/I, K_G) = K_{A/I}$ (here A/I is considered to be a graded G-module concentrated in degree 0), we get the exact sequence $0 \to K_{A/I} \to K_G \to T \to 0$. Thus K_G is generated by elements of degree 0. On the other hand, in the exact sequence (4.7) we get $E = (0)$ because T is a Cohen-Macaulay ring. So we have the exact sequence $0 \to K_T \to K_G \to A/I \to 0$ of graded G-modules. Hence because $a(T) = -1$, we get $[K_G]_0 = A/I$. Thus K_G is cyclic and G is a Gorenstein ring.

5. The case where ad $(I) \leq 3$

The purpose of this section is to prove Theorems (1.5), (1.6), and (1.7). We assume that I is an ideal in a Cohen-Macaulay ring A of $\dim A = d$, which contains a special reduction J, with $s = \mathrm{ht}_A\, I$ and $\ell = \lambda(I)$. We also assume that A/I is a Cohen-Macaulay ring, $\mathrm{ad}\,(I) \leq 3$, and $r_J(I) \leq 2$. We choose a system of generators $a_1, a_2, \ldots, a_\ell$ for J so that the conditions stated in (2.4) are fulfilled. We put $G = G(I)$, $R = R(I)$, and $\mathfrak{M} = \mathfrak{m}G + G_+$. Here we note that if $\mathrm{ad}\,(I) = 2$

(resp. $\mathrm{ad}(I) = 3$), one naturally has the inequality $\operatorname{depth}(A/I^n)_Q \geq \min\{\mathrm{ad}(I) - n, \mathrm{ht}_A Q - s - n\}$ for all prime ideals $Q \in V(I)$ and for all integers n with $1 \leq n \leq \mathrm{ad}(I)$ (resp. $\operatorname{depth}(A/I^n)_Q \geq \min\{\mathrm{ad}(I) - n - 1, \mathrm{ht}_A Q - s - n\}$ for all prime ideals $Q \in V(I)$ and for all integers n with $1 \leq n \leq \mathrm{ad}(I) - 1$). Hence as $r_J(I) \leq 2$ by our standard assumption of this section, the results obtained in section 2 are applicable. In particular, the sequence $a_1t, a_2t, \ldots, a_st$ is G-regular by (2.7) (resp. (2.10)), if $\mathrm{ad}(I) = 2$ (resp. if $\mathrm{ad}(I) = 3$ and A is a Gorenstein ring); thus passing to the ring $G(I/(a_1, a_2, \ldots, a_s)A) = G/(a_1t, a_2t, \ldots, a_st)G$ and thanks to (3.1) (resp. (4.2)), in order to prove Theorem (1.5) (resp. Theorems (1.6) and (1.7)) we may assume without loss of generality that $s = 0$.

For the rest of this section we assume $s = 0$ and $\mathrm{ad}(I) = \ell$. We begin with the following lemma, in which the first assertion is fairly well-known but let us note a proof for completeness.

Lemma (5.1). (1) $\operatorname{depth} A/I^n \geq d - \ell$ *for all* $n \geq 1$ *if* G *is a Cohen-Macaulay ring.*

(2) *Let* $d \geq 3$, $s = 0$, *and* $\ell = 2$. *Suppose* $\operatorname{depth} A/I^2 \geq 1$. *Then* $\operatorname{depth} A/I^n \geq 1$ *for all* $n \geq 1$.

Proof. (1) We have $\mathrm{grade}_G \mathfrak{m}G = \inf_{n \geq 1} \operatorname{depth} A/I^n$ ([B]), while $\mathrm{grade}_G \mathfrak{m}G = \mathrm{ht}_G \mathfrak{m}G = d - \ell$ as G is a Cohen-Macaulay ring. Hence $\operatorname{depth} A/I^n \geq d - \ell$ for all $n \geq 1$.

(2) We may assume $n \geq 3$. Hence $I^n = J^{n-2}I^2$ as $n > r_J(I)$. By induction on m we will show $\operatorname{depth} A/J^mI^2 \geq 1$ for all $m \geq 0$. As $\operatorname{depth} A/I^2 \geq 1$, we may assume $m \geq 1$ and $\operatorname{depth} A/J^{m-1}I^2 \geq 1$. First, notice that $I \cong a_1^mI$ and $I^2 \cong a_1^mI^2$ because of the isomorphisms $a_1^kI \cong a_1^{k-1}I$ and $a_1^kI^2 \cong a_1^{k-1}I^2$ $(k \geq 1)$ given by (2.5)(2). We consider the following six exact sequences

(a) $0 \to I \to A \to A/I \to 0$,

(b) $0 \to I \to A \to A/a_1^mI \to 0$ (recall $a_1^mI \cong I$),

(c) $0 \to I^2 \to A \to A/I^2 \to 0$,

(d) $0 \to I^2 \to A \to A/a_1^mI^2 \to 0$ (recall $a_1^mI^2 \cong I^2$),

(e) $0 \to J^{m-1}I^2/a_1^mI \to A/a_1^mI \to A/J^{m-1}I^2 \to 0$, and

(f) $0 \to J^{m-1}I^2/a_1^mI \to A/a_1^mI^2 \to A/J^mI^2 \to 0$,

where the last one follows from the isomorphism $J^mI^2/a_1^mI^2 \cong J^{m-1}I^2/a_1^mI$ in (2.5)(2). Then as $\operatorname{depth} A/I = \operatorname{depth} A = d$, by the sequence (a) we get $\operatorname{depth}_A I = d$ so that by (b) the inequality $\operatorname{depth} A/a_1^mI \geq d - 1 \geq 2$. Hence as $\operatorname{depth} A/J^{m-1}I^2 \geq 1$ by our hypothesis on m, we get by the sequence (e) that $\operatorname{depth}_A J^{m-1}I^2/a_1^mI \geq 2$ too. Similarly, by sequences (c) and (d) we find $\operatorname{depth} A/a_1^mI^2 \geq 1$. Thus by the sequence (f) we conclude $\operatorname{depth} A/J^mI^2 \geq 1$, because $\operatorname{depth}_A J^{m-1}I^2/a_1^mI \geq 2$ and $\operatorname{depth} A/a_1^mI^2 \geq 1$ as we have shown above.

For a moment assume that depth $A/I^n \geq 1$ for all $n \geq 1$. Then by Burch's inequality ([B]) $s \leq \ell \leq d - \inf_{n \geq 1}$ depth A/I^n, we get $\ell < d$. Let $\mathscr{F}_i = \{Q \in V(I) | \operatorname{ht}_A Q = i$ and $Q \in \operatorname{Supp}_A I/J_{i-1}\}$ for each $1 \leq i \leq \ell$. Then $\mathscr{F}_i \subset \operatorname{Min}_A I/(J_{i-1} + I^2)$ (see (2.1)). Hence $\mathscr{F} = \bigcup_{1 \leq i \leq \ell} \mathscr{F}_i$ is a finite set and $\mathfrak{m} \notin \mathscr{F}$ as $\ell < d$. As $\mathfrak{m} \notin \bigcup_{n \geq 1} \operatorname{Ass}_A A/I^n$ and as the set $\bigcup_{n \geq 1} \operatorname{Ass}_A A/I^n$ is also finite ([Br]), we may choose an element x of $\mathfrak{m}$ so that $x \notin Q$ for any $Q \in \left(\bigcup_{n \geq 1} \operatorname{Ass}_A A/I^n\right) \cup \mathscr{F}$. Let $\bar{A} = A/xA$, $\bar{I} = I\bar{A}$, and $\bar{J} = J\bar{A}$. Then as x is A-regular, $\bar{A}$ is a Cohen-Macaulay ring of $\dim \bar{A} = d - 1$. We furthermore have the following, which we later need to reduce the problem also to the case where $d = \ell$.

Lemma (5.2). (1) depth $\bar{A}/\bar{I}^n =$ depth $A/I^n - 1$ *for all* $n \geq 1$. *In particular* $\bar{A}/\bar{I}$ *is a Cohen-Macaulay ring.*

(2) $\operatorname{ht}_{\bar{A}} \bar{I} = 0$ *and* $\lambda(\bar{I}) = \ell$.

(3) $\bar{J}$ *is a special reduction of* $\bar{I}$ *with* $r_{\bar{J}}(\bar{I}) \leq 2$.

(4) G *is a Cohen-Macaulay (resp. Gorenstein) ring if and only if* $G(\bar{I})$ *is a Cohen-Macaulay (resp. Gorenstein) ring.*

Proof. The assertion (1) follows from the fact that x is A/I^n-regular for all $n \geq 1$. Since $\dim \bar{A}/\bar{I} = d - 1 = \dim \bar{A}$, we have $\operatorname{ht}_{\bar{A}} \bar{I} = 0$. As x is G-regular, we get by [VV, 1.1] an isomorphism $G(\bar{I}) \cong G/xG$ of A-algebras. Hence the assertion (4) and the equality $\lambda(\bar{I}) = \ell$ follow. As the ideal $\bar{J}$ is a reduction of $\bar{I}$ with $\mu_{\bar{A}}(\bar{J}) \leq \ell$, $\bar{J}$ is a minimal reduction of $\bar{I}$ with $r_{\bar{J}}(\bar{I}) \leq 2$. Let $\bar{Q} \in V(\bar{I})$ with $\operatorname{ht}_{\bar{A}} \bar{Q} < \ell$, and choose a prime ideal $Q \in V(I + xA)$ so that $Q/xA = \bar{Q}$. Let $i = \operatorname{ht}_A Q$. Then $1 \leq i \leq \ell$ as $\operatorname{ht}_A Q = \operatorname{ht}_{\bar{A}} \bar{Q} + 1$. As $x \in Q$, $Q \notin \operatorname{Supp}_A I/J_{i-1}$ so that we have $I_Q = J_{i-1}A_Q$ whence $\bar{I}_{\bar{Q}} = J_{i-1}\bar{A}_{\bar{Q}}$. Thus $\bar{J}$ is a special reduction of $\bar{I}$.

Now let us note a proof of Theorem (1.5).

Proof of Theorem (1.5). The last assertion follows from [TI, 1.1], since $a(G) = -s$ by [AH, 5.10] and [T, 2.5]. To see the equivalence of assertions (1) and (2) we may assume $s = 0$. Hence $\ell = 2$.

(1) $\Rightarrow$ (2) This follows from (5.1)(1).

(2) $\Rightarrow$ (1) If $d \geq 3$, then by (5.1)(2) we get depth $A/I^n \geq 1$ for all $n \geq 1$. Hence by (5.2) we may furthermore assume $d = 2$. First of all, we choose an element $x \in (0):I$ so that x is A/I-regular (this choice is possible, since $\operatorname{ht}_A (I + [(0):I]) \geq 1$ by (2.1) and since A/I is a Cohen-Macaulay ring of $\dim A = 2$). Hence depth $A/(xA + I) = 1$. We now recall depth $A/a_1 I \geq 1$ (see Proof of (5.1) (2)). And we choose an element $y \in \mathfrak{m}$ so that y is regular on both of $A/(xA + I)$ and $A/a_1 I$. In what follows, we will show that the sequence $x + a_1 t$, $y + a_2 t$ is G-regular. Let $f = \alpha_0 + \alpha_1 t + \cdots + \alpha_n t^n \in R$ with $\alpha_i \in I^i$ and assume $(x + a_1 t)f \in IR$. Then as $x \in (0):I$, we get $(x + a_1 t)f = x\alpha_0 + \sum_{i \geq 1} a_1 \alpha_{i-1} t^i \equiv 0 \bmod IR$. As $x\alpha_0 \in I$, we have $\alpha_0 \in I$ since x is regular on A/I. For $i \geq 2$, we have $a_1 \alpha_{i-1} \in a_1 A \cap I^{i+1} = a_1 I^i$ by (2.6). We write $a_1 \alpha_{i-1} = a_1 \xi$ with $\xi \in I^i$. Then as $a_1(\alpha_{i-1} -$

$\xi) = 0$, we get $\alpha_{i-1} - \xi \in [(0):a_1] \cap I = (0)$ (see (2.4)(4)). Hence $\alpha_{i-1} \in I^i$ for all $i \geq 2$ and so we have $f \in IR$. Thus $x + a_1 t$ is G-regular.

Let us check $y + a_2 t$ is regular on $G/(x + a_1 t)G$. Let $L = (x + a_1 t)R + IR$. Let $g = \beta_0 + \beta_1 t + \cdots + \beta_n t^n \in R$ with $n \geq 2$ and $\beta_i \in I^i$ and assume $(y + a_2 t)g \in L$. Choose $f = \alpha_0 + \alpha_1 t + \cdots + \alpha_m t^m \in R$ with $\alpha_i \in I^i$ and $m \geq n$ so that $(y + a_2 t)g \equiv (x + a_1 t)f \bmod IR$. Then as $(x + a_1 t)f = x\alpha_0 + \sum_{i \geq 1} a_1 \alpha_{i-1} t^i$, we get $y\beta_0 + \sum_{1 \leq i \leq n} (a_2 \beta_{i-1} + y\beta_i)t^i + a_2 \beta_n t^{n+1} \equiv x\alpha_0 + \sum_{i \geq 1} a_1 \alpha_{i-1} t^i \bmod IR$. Hence as $a_2 \beta_n \equiv a_1 \alpha_n \bmod I^{n+2}$, we have $a_2 \beta_n - a_1 \alpha_n \in (a_1, a_2) \cap I^{n+2} = (a_1, a_2)I^{n+1}$ by (2.6). Choose $\xi \in I^{n+1}$ so that $a_2(\beta_n - \xi) \in a_1 A$. Then as $\beta_n - \xi \in [a_1 A : a_2] \cap I^n$ and as $[a_1 A : a_2] \cap I^2 \subseteq a_1 A$ by (2.5)(1), we have by (2.6) that $\beta_n - \xi \in a_1 I^{n-1}$. Write $\beta_n \equiv a_1 \eta \bmod I^{n+1}$ with $\eta \in I^{n-1}$. Then as $\beta_n t^n \equiv (a_1 t)(\eta t^{n-1}) = (x + a_1 t)\eta t^{n-1} \bmod IR$ (recall $x \in (0):I$), we get $\beta_n t^n \in L$ and $(y + a_2 t)(g - \beta_n t^n) \in L$. Thus repeating this procedure we find $\beta_i t^i \in L$ for all $2 \leq i \leq n$ and $(y + a_2 t)(\beta_0 + \beta_1 t) \in L$. We then have

(a) $$y\beta_0 \equiv x\alpha_0 \bmod I\,,$$

(b) $$y\beta_1 + a_2 \beta_0 \equiv a_1 \alpha_0 \bmod I^2\,, \qquad \text{and}$$

(c) $$a_2 \beta_1 \equiv a_1 \alpha_1 \bmod I^3\,.$$

As x, y forms an A/I-regular sequence, by (a) we may write $\beta_0 \equiv xu \bmod I$ and $\alpha_0 \equiv yu \bmod I$ for some $u \in A$. Then as $y\beta_1 + a_2 xu \equiv a_1 yu \bmod I^2$ by (b) and as $xI = (0)$, we find $y\beta_1 - a_1 yu \in I^2$. On the otherhand by (c) and (2.6) we have $a_2 \beta_1 - a_1 \alpha_1 \in (a_1, a_2) \cap I^3 = (a_1, a_2)I^2$ whence $a_2(\beta_1 - \rho) \in a_1 A$ for some $\rho \in I^2$. Thus $a_2(y(\beta_1 - \rho) - ya_1 u) \in a_1 A$. Therefore we find $y(\beta_1 - \rho - a_1 u) = (y\beta_1 - a_1 yu) - y\rho \in [a_1 A : a_2] \cap I^2 = a_1 I$ by (2.5)(1). As y is a regular on $A/\alpha_1 I$ by its choice, we get $\beta_1 - \rho - a_1 u \in a_1 I$. Hence we have $\beta_1 \equiv a_1 u \bmod I^2$. As $\beta_0 \equiv xu \bmod I$, we get $\beta_0 + \beta_1 t \equiv (x + a_1 t)u \bmod IR$ so that $\beta_0 + \beta_1 t \in L$. Thus the sequence $x + a_1 t$, $y + a_2 t$ is G-regular whence $\operatorname{depth} G = 2$ so that G is a Cohen-Macaulay ring. This completes the proof of (1.5).

To prove Theorem (1.7) we need the following

Lemma (5.3). *Let A be a homomorphic image of a Gorenstein ring. Let $d = 2$, $s = 0$, and $\ell = 2$. Assume $r_J(I) \leq 1$. Then* $\operatorname{depth}_A [K_G] \geq 1$.

Proof. Let $B = A/[0:I]$ and $T = G(IB)$. Then $\dim T = \dim B = 2$. The ring G is Cohen-Macaulay by virtue of (1.5). The element a_1 is B-regular by (2.4)(4) whence $\operatorname{depth} B > 0$. We begin with the following.

Claim 1. *$a_1 t$ is T-regular.*

Proof of Claim 1. Let $\alpha \in I^n$ with $n \geq 0$ and assume $(a_1 t)(\alpha t^n) \equiv 0 \bmod [(0):I] + IR$. Then as $a_1 \alpha \in a_1 A \cap I^{n+2} = a_1 I^{n+1}$ by (2.6), letting $a_1 \alpha = a_1 \xi$ with $\xi \in I^{n+1}$, we get $\alpha - \xi \in (0):a_1 = (0):I$ by (2.4)(4). Hence $\alpha \in I^{n+1} + [(0):I]$ so that we have $\alpha t^n \in [(0):I] + IR$.

Let $C = B/a_1B$ and $S = G(IC)$. Then $\dim C = 1$ as a_1 is B-regular. We get $S = T/a_1tT$ by Claim 1. Let $X = (0):_S a_2t$ and $Y = S/a_2tS$. Then we have

CLAIM 2. *$X_n = (0)$ and $Y_n = (0)$ for all $n \geq 2$.*

Proof of Claim 2. Let $n \geq 2$ be an integer. Then $(IC)^2 = a_2IC$ as $r_J(I) \leq 1$ whence $S_n \subseteq a_2tS$ so that $Y_n = (0)$. Let $\alpha \in I^n$ and assume $(a_2t)(\alpha t^n) \equiv 0 \mod [(0):I] + IR + a_1tR$. Then $a_2\alpha \in I^{n+2} + a_1A$. As $I^{n+2} = (a_1, a_2)I^{n+1}$, we have $a_2\alpha \in a_1A + a_2I^{n+1}$. Write $a_2(\alpha - \xi) \in a_1A$ with $\xi \in I^{n+1}$. Then as $\alpha - \xi \in [a_1A : a_2] \cap I^n = a_1I^{n-1}$ by (2.5)(1) and (2.6), we get $\alpha \in I^{n+1} + a_1I^{n-1}$. Thus $\alpha t^n \in [(0):I] + IR + a_1tR$. Hence $X_n = (0)$.

CLAIM 3. $a(S) \leq 0$.

Proof of Claim 3. Split the sequence $0 \to X(-1) \to S(-1) \xrightarrow{a_2t} S \to Y \to 0$ into the following two exact sequences $0 \to X(-1) \to S(-1) \to a_2tS \to 0$ and $0 \to a_2tS \to S \to Y \to 0$ of graded S-modules and apply functors $H^i_{\mathfrak{M}}(*)$ to them. Then we get exact sequences $[H^1_{\mathfrak{M}}(X)](-1) \to [H^1_{\mathfrak{M}}(S)](-1) \to H^1_{\mathfrak{M}}(a_2tS)$ and $H^0_{\mathfrak{M}}(Y) \to H^1_{\mathfrak{M}}(a_2tS) \to H^1_{\mathfrak{M}}(S)$ of local cohomology modules. Let $a = a(S)$ and look at the homogeneous components of degree $a + 1$. Then we get the diagram

$$
\begin{array}{ccccc}
 & & & & [H^0_{\mathfrak{M}}(Y)]_{a+1} \\
 & & & & \downarrow \rho \\
[H^1_{\mathfrak{M}}(X)]_a & \to & [H^1_{\mathfrak{M}}(S)]_a & \xrightarrow{\sigma} & [H^1_{\mathfrak{M}}(a_2tS)]_{a+1} \\
 & & & & \downarrow \tau \\
 & & & & [H^1_{\mathfrak{M}}(S)]_{a+1}
\end{array}
$$

with exact row and column. We have $[H^1_{\mathfrak{M}}(S)]_a \neq (0)$ and $[H^1_{\mathfrak{M}}(S)]_{a+1} = (0)$ (recall $\dim S = \dim C = 1$). Hence the map ρ is an epimorphism. Therefore, if $\sigma \neq 0$, we have $(0) \neq [H^0_{\mathfrak{M}}(Y)]_{a+1} \subseteq Y_{a+1}$ whence $a \leq 0$ by Claim 2. Assume $\sigma = 0$. Then as $[H^1_{\mathfrak{M}}(S)]_a \neq (0)$, we get $[H^1_{\mathfrak{M}}(X)]_a \neq (0)$ whence $H^1_{\mathfrak{m}}(X_a) \neq (0)$ as $[H^1_{\mathfrak{M}}(X)]_a = H^1_{\mathfrak{m}}(X_a)$ by [GH, 2.2]. Hence $a \leq 1$ by Claim 2. Assume now $a = 1$ and choose a prime ideal $Q \in \operatorname{Supp}_A X_1$. Then as $X_1 \subseteq S_1$ and $IS_1 = (0)$, we have $I \subseteq Q$. If $\operatorname{ht}_A Q \leq 1$ $(< \ell = 2)$, then we get $I_Q = a_1A_Q$ by (2.1) whence $IC_Q = (0)$ so that we have $S_{1Q} = (0)$. This is impossible as $X_{1Q} \neq (0)$. Hence we have $Q = \mathfrak{m}$ as $\dim A = 2$. Thus $\dim_A X_1 = 0$ and $H^1_{\mathfrak{m}}(X_1) = (0)$. This contradicts the fact that $H^1_{\mathfrak{m}}(X_a) \neq (0)$. Thus $a \leq 0$.

CLAIM 4. $a(T) < 0$.

Proof of Claim 4. By Claim 1 we have a_1t to be T-regular. Apply functors $H^i_{\mathfrak{M}}(*)$ to the sequence $0 \to T(-1) \xrightarrow{a_1t} T \to S \to 0$. Then we get the exact sequence $H^1_{\mathfrak{M}}(S) \to [H^2_{\mathfrak{M}}(T)](-1) \xrightarrow{a_1t} H^2_{\mathfrak{M}}(T)$ of local cohomology modules. Let $a = a(T)$ and look at the homogeneous components $[H^1_{\mathfrak{M}}(S)]_{a+1} \to [H^2_{\mathfrak{M}}(T)]_a \to [H^2_{\mathfrak{M}}(T)]_{a+1}$ of degree $a + 1$. Then as $[H^2_{\mathfrak{M}}(T)]_{a+1} = (0)$ and as $[H^2_{\mathfrak{M}}(T)]_a \neq (0)$, we get

$[H^1_{\mathfrak{M}}(S)]_{a+1} \neq (0)$. Hence $a+1 \leq a(S)$ so that we have $a < 0$ as $a(S) \leq 0$ by Claim 3.

Now let $\varphi: G \to T$ be the canonical epimorphism and $K = \operatorname{Ker} \varphi$. Then $K_n \cong (I^n \cap [(0):I])/(I^{n+1} \cap [(0):I])$ whence $K_n = (0)$ if $n \geq 1$ and $K_0 \cong (0):I$ (recall $[(0):I] \cap I = (0)$ by (2.4)(4)). We consider the exact sequence $0 \to K \to G \to T \to 0$ and take the K_G-dual of it. Then we get the exact sequence $0 \to K_T \to K_G \to \operatorname{Hom}_G(K, K_G)$ of graded G-modules. As $[K_T]_0 = (0)$ by Claim 4, we have the embedding $[K_G]_0 \subseteq \operatorname{Hom}_G(K, K_G)]_0$. Now recall that $\operatorname{depth}_A K = \operatorname{depth}_A [(0):I] = 2$ (use the exact sequence $0 \to (0):I \to A \to B \to 0$ and the fact that depth $B > 0$ as well). Choose an element $x \in \mathfrak{m}$ so that x is K-regular and let $\overline{K} = K/xK$. Then as $\overline{K}$ is a Cohen-Macaulay G-module of $\dim_G \overline{K} = 1$, we have $\operatorname{Hom}_G(\overline{K}, K_G) = (0)$ by [HK, 6.1] so that x is a nonzerodivisor on $\operatorname{Hom}_G(K, K_G)$. Hence x is a nonzerodivisor on $[K_G]_0$ too. Thus we get $\operatorname{depth}_A [K_G]_0 \geq 1$ as claimed. This complete the proof of (5.3).

We are now closing this section by proving theorems (1.6) and (1.7).

Proof of Theorem (1.6). (1)$\Rightarrow$(2) See (5.1)(1).

(2)$\Rightarrow$(1) We may assume $s = 0$. Let $B = A/[(0):I]$. Then B is a Cohen-Macaulay ring of $\dim B = d$ ([PS, 1.3]), while we have by (4.3) and (4.4) that $\operatorname{ht}_B IB = 1$, $\lambda(IB) = 3$, B/IB is Cohen-Macaulay, and JB is a special reduction of IB with $r_{JB}(IB) = r_J(I) \leq 2$. Hence the hypotheses in Theorem (1.5) are satisfied for the ideal IB in B. Recall the exact sequences

(a) $$0 \to I^2 \to A \to A/I^2 \to 0 \qquad \text{and}$$

(b) $$0 \to I^2 \to B \to B/I^2B \to 0\,,$$

in which the latter one follows from the equality that $[(0):I] \cap (0)$. Then as depth $A/I^2 \geq d-3$, by the sequence (a) we have $\operatorname{depth}_A I^2 \geq d-2$ whence by (b) we get depth $B/I^2B \geq d-3$. Thus by (1.5) $T = G(IB)$ is a Cohen-Macaulay ring of $\dim T = d$. Now let $\varphi: G \to T$ be the canonical eqimorphism and look at the exact sequence $0 \to K \to G \xrightarrow{\varphi} T \to 0$ with $K = \operatorname{Ker} \varphi$. Then as $\operatorname{depth}_G K = d$ (recall $K = K_0 = K_{A/I}$, cf. the remark just after (4.3)), we get depth $G = d$. Thus G is a Cohen-Macaulay ring.

Proof of Theorem (1.7). (2)$\Rightarrow$(1) This follows from (1.3)(2).

(1)$\Rightarrow$(2) We may assume $s = 0$ and $\ell = 3$. If $d \geq 4$, then by (5.1)(1) we get depth $A/I^n \geq 1$ for all $n \geq 1$. Hence passing to the ring $G(\bar{I})$, we may assume by (5.2) that $d = \ell = 3$. We must show depth $A/I^2 \geq 1$. For this it is enough to see $\operatorname{depth}_A I/I^2 \geq 1$. Let $B = A/[(0):I]$, $C = B/a_1B$, $T = G(IB)$, and $S = G(IC)$. Then B is a Cohen-Macaulay ring of $\dim B = 3$ ([PS]). By (4.4) the ideal JB is a special reduction of IB with $r_{JB}(IB) \leq 1$ and $\lambda(IB) = 3$. By (4.3) we have $\operatorname{ht}_B IB = 1$ and B/IB is a Cohen-Macaulay ring of $\dim B/IB = 2$. And, C is a Cohen-Macaulay ring of $\dim C = 2$ as a_1 is B-regular (cf. (2.4)(4)), whence $\operatorname{ht}_C IC = 0$ (note $\dim C/IC = \dim B/IB = 2$), so that the proof of (3.1) works to get, passing to the above data on B, that $\lambda(IC) = 2$ and IC contains JC as a special reduction with $r_{JC}(IC) \leq 1$. Hence the hypotheses in Lemma (5.3) are satisfied for the ideal IC in C. Let us now notice by (4.9) T is a Cohen-Macaulay ring of $a(T) = -1$ and by (2.7) a_1t is T-regular. Hence $K_S \cong [K_T/a_1tK_T](1)$ ([GW, (2.2.10)] as $S \cong T/a_1tT$ ([VV, 1.1]). We have $[K_S]_0 \cong [K_T]_1$ as $[K_T]_0 = (0)$ (recall $a(T) = -1$), while $G_1 \cong [K_T]_1$ by the sequence (4.11) and $\text{depth}_A [K_S]_0 \geq 1$ by (5.3). Hence we get $\text{depth}_A I/I^2 \geq 1$, which completes the proof of Theorem (1.7).

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