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A note on residually transcendental prolongations with uniqueness property

By

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1. Introduction

Throughout $K(x)$ is a simple transcendental extension of a field K, and v is a (Krull) valuation of K with value group G_v and residue field k_v. Let w be a valuation of $K(x)$ extending v whose residue field is a transcendental (to be abbreviated as tr.) extension of k_v; such a valuation w is called a residually transcendental prolongation of v. We say that w has uniqueness property if there exists $t \in K(x)\backslash K$ such that (i) w coincides with the Gaussian valuation v^t of the field $K(t)$ defined on $K[t]$ by $v^t\left(\sum_i a_i t^i\right) = \min_i v(a_i)$; (ii) w is the only valuation of $K(x)$ which extends v^t.

In 1990, Matignon and Ohm [3, Cor. 3.3.1, Remark 3.4] proved that if (K, v) is henselian or of rank 1, then each residually transcendental prolongation w of v to $K(x)$ has uniqueness property. Alexandru, Popescu and Zaharescu have shown that such prolongations w of v have uniqueness property provided the completion $(\hat{K}, \hat{v})$ of (K, v) is henselian and each finite simple extension of $\hat{K}$ is defectless (cf. [1, Theorem 4.5]). The converse problem is dealt with here. We prove:

Theorem 1.1. *Let v be a valuation of any rank of a field K. Each residually transcendental prolongation of v to $K(x)$ has uniqueness property if and only if the completion of (K, v) is henselian.*

2. Definition, notation and some preliminary results

Recall that for a finite extension $(K_1, v_1)/(K, v)$ of valued fields, the henselian defect is defined to be $[K_1^h : K^h]/ef$, where "h" stands for henselisation with respect to the underlying valuation and e, f for the index of ramification and the residual degree of v_1/v. We shall denote this defect by $\mathrm{def}^h\,(K_1, v_1)/(K, v)$ or by $\mathrm{def}^h\,(v_1/v)$.

The proof of the following already known lemma is omitted (cf. [4, p. 306, Lemma]).

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Lemma 2.1. *Let $(K, v) \subseteq (K_1, v_1)$ be a finite extension of valued fields, t an indeterminate and v^t, v_1^t be the Gaussian valuations of $K(t)$, $K_1(t)$ respectively extending v, v_1. Then* $\operatorname{def}^h(v_1/v) = \operatorname{def}^h(v_1^t/v^t)$.

Notation. Let v be a valuation of K with value group G_v and residue field k_v. Let w be a residually transcendental prolongation of v to $K(x)$ having value group G_w and residue field k_w. For any ξ in the valuation ring of w, ξ^* will stand for its w-residue, i.e., the image of ξ under the canonical homomorphism from the valuation ring of w onto k_w. We shall denote by E, I, R (more precisely by $E(w/v)$ etc.) the numbers defined by

$E = \min\{[K(x): K(\xi)] \mid w(\xi) \geq 0, \xi^* \text{ tr. over } k_v\}$,

$I = [G_w : G_v]$,

$R = [\Delta : k_v]$, where Δ is the algebraic closure of k_v in k_w.

Let t be an element in the valuation ring of w with t^* tr. over k_v; this is the same as saying that w coincides with the Gaussian valuation v^t on $K(t)$ (cf. [2, § 10.1; Prop. 3]). Such an element t will be called residually transcendental (with respect to w/v). We shall denote by $D^h(w/v)$ (or briefly by D^h) the henselian defect of the finite extension $(K(x), w)/(K(t), v^t)$; in view of [3, Thm. 2.2] $D^h(w/v)$ is independent of the choice of the residually tr. element t.

Fix completions $(\hat{K}, \hat{v}) \subseteq (K(x)^\wedge, \hat{w})$ of v and w. Let w_c denote the valuation of $\hat{K}(x)$ obtained by restricting $\hat{w}$. Since residue field does not change on taking completion, the residue field of w_c must be k_w. So w_c is a residually tr. prolongation of $\hat{v}$. As in [3, Lemma 2.2.2], it can be easily shown that

$$E(w_c/\hat{v}) = E(w/v). \tag{1}$$

The following results of Matignon and Ohm (whose proofs are omitted) are quoted for future reference (cf. [3, Cor. 2.3.2, 3.3.1]).

Theorem 2.2. *Let v be a valuation of K and w be a residually transcendental prolongation of v to $K(x)$. With E, I, R, D^h as above, we have:*

(i) *w has uniqueness property if and only if $E = IRD^h$ holds for w/v.*

(ii) *If (K, v) is henselian, then w has uniqueness property.*

3. Proof of Theorem 1.1

Suppose first that (K, v) has henselian completion. Let w be a residually tr. prolongation of v to $K(x)$. Fix a completion $(K(x)^\wedge, \hat{w})$ of $(K(x), w)$ and a completion $(\hat{K}, \hat{v})$ of (K, v) contained in this completion. Since $(\hat{K}, \hat{v})$ is henselian, there exists a henselisation (K^h, v^h) of (K, v) which is contained in $(\hat{K}, \hat{v})$ as a valued subfield. Let w_c, w_h denote the valuations obtained by restricting $\hat{w}$ to $\hat{K}(x)$, $K^h(x)$ respectively. Then clearly $w_c/\hat{v}$ and w_h/v^h are residually tr. prolongations, and

$$I(w_h/v^h) = I(w/v), \qquad R(w_h/v^h) = R(w/v). \tag{2}$$

In view of the fact that $K \subseteq K^h \subseteq \hat{K}$, we have

$$E(w_c/\hat{v}) \leq E(w_h/v^h) \leq E(w/v)$$

which together with (1) gives

$$E(w_h/v^h) = E(w/v). \tag{3}$$

Since (K^h, v^h) is henselian, by Theorem 2.2(ii) w^h has uniqueness property and consequently w has this property in view of (2), (3) and Theorem 2.2(i).

To prove the converse, assume that a completion $(\hat{K}, \hat{v})$ of (K, v) is not henselian. We shall construct a residually tr. prolongation w of v to $K(x)$ satisfying $E(w/v) > 1$ and

$$I(w/v) = R(w/v) = D^h(w/v) = 1.$$

In view of Theorem 2.2(i), such a prolongation w does not satisfy uniqueness property.

Fix a prolongation $\bar{\hat{v}}$ of $\hat{v}$ to an algebraic closure $\bar{\hat{K}}$ of $\hat{K}$ and a henselisation (K^h, v^h) of (K, v) contained in $(\bar{\hat{K}}, \bar{\hat{v}})$ as a valued subfield. As the completion of a henselian field is henselian (see [2, Exercises §8 Ex. 14(a)]) and $(\hat{K}, \hat{v})$ is assumed to be non-henselian, it follows that K^h is not contained in $\hat{K}$. Let β be an element of K^h which is not in $\hat{K}$. Then the set $\{v^h(\beta - a) | a \in K\}$ is bounded above, i.e., there exists $\delta = v(d)$ in the value group of v such that

$$v^h(\beta - a) < \delta \qquad \text{for all } a \text{ in } K. \tag{4}$$

Let v_1 denote the valuation of $K_1 = K(\beta)$ obtained by restricting v^h and w_1 the valuation of $K_1(x)$ defined on $K_1[x]$ by

$$w_1\left(\sum_i a_i(x - \beta)^i\right) = \min_i (v(a_i) + i\delta), \qquad a_i \in K_1. \tag{5}$$

As in the proof of [2, § 10.1, Prop. 2], one can easily see that the residue field of w_1 is the simple tr. extension of $k_{v_1}(x_1^*)$ of the residue field k_{v_1} of v_1, where x_1^* is the w_1-residue of $x_1 = (x - \beta)/d$. Since (K^h, v^h) is an immediate extension of (K, v), so is (K_1, v_1). It is now clear that if w is the valuation obtained by restricting w_1 to $K(x)$, then

$$I(w/v) = R(w/v) = 1.$$

Claim is that $E(w/v) > 1$. Suppose not, then there exist a, b in K such that the w-residue of $(x - a)/b$ is tr. over k_v. By virtue of (5) and (4), we have

$$w_1(x - a) = \min(\delta, v_1(\beta - a)) < \delta,$$

which implies that

$$\left(\frac{x - a}{b}\right)^* = \left(\frac{x - \beta}{b}\right)^* + \left(\frac{\beta - a}{b}\right)^* = \left(\frac{\beta - a}{b}\right)^*$$

is algebraic over k_v. This contradiction proves the claim.

It only remains to be shown that $D^h(w/v) = 1$. Let $t \in K(x)$ be a residually tr. element with respect to w/v and v^t, v_1^t be the Gaussian valuations of $K(t)$, $K_1(t)$ respectively. Since $K_1 \subseteq K^h$, the henselisations of (K_1, v_1) and (K, v) coincide; in particular $\mathrm{def}^h(v_1/v) = 1$. Therefore by Lemma 2.1,

$$\mathrm{def}^h(v_1^t/v^t) = 1. \tag{6}$$

Keeping in view the first assertion of [3, Thm 2.2] and the fact that the generator $x_1 = (x - \beta)/d$ of $K_1(x)/K_1$ is residually tr. with respect to w_1/v_1, we see that

$$\mathrm{def}^h((K_1(x), w_1)/(K_1(t), v_1^t)) = \mathrm{def}^h((K_1(x), w_1)/(K_1(x_1), w_1)) = 1. \tag{7}$$

If follows from (6), (7) and the multiplicative property of henselian defect that $\mathrm{def}^h(w/v^t) = 1$ as desired.

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