J. Math. Kyoto Univ. (JMKYAZ)
36-3 (1996) 481-498

Nonsymmetric Ornstein-Uhlenbeck semigroup as second quantized operator

By

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0. Introduction

This work deals with properties of the semigroup related to the Ornstein-Uhlenbeck operator

$$L\phi(x) = \tfrac{1}{2} \operatorname{Tr} QD^2\phi(x) + \langle Ax, D\phi(x)\rangle \tag{1}$$

in a real separable Hilbert space H. We assume that A is a generator of C_0-semigroup $S(t)$, $t \geq 0$, of bounded operators on H, Q is bounded, selfadjoint and nonnegative. By $D\phi$ we denote the Fréchet derivative of a function $\phi: H \to \mathbf{R}$. Notice that $L\phi(x)$ is well-defined for every $x \in H$, at least for appropriately chosen cylindrical functions (see [CG1] for details). In this paper we require that

(A1a) $$\int_0^\infty \operatorname{tr} S(u)QS^*(u)\,du < \infty.$$

If (A1a) is satisfied then we can define on H the family of Gaussian measures μ_t, $t \geq 0$, and μ with the mean zero and the covariance operators

$$Q_t = \int_0^t S(u)QS^*(u)\,du$$

and

$$Q_\infty = \int_0^\infty S(u)QS^*(u)\,du$$

respectively. For simplicity of presentation we assume that

(A1b) $$\ker Q_\infty = \{0\}.$$

Let

$$R_t\phi(x) = \int_H \phi(S(t)x + y)\mu_t(dy).$$

Then the family of operators R_t, $t \geq 0$, forms a strongly continuous semigroup

Communicated by Prof. K. Ueno, August 17, 1995

of contractions on $L^p(H,\mu)$ for every $p \geq 1$. Moreover, it can be shown [CG1] that L has a unique extension to a generator of C_0-semigroup on $L^p(H,\mu)$ $(p \geq 1)$ which coincides with R_t. The semigroup R_t can be identified as the transition semigroup corresponding to the Ornstein-Uhlenbeck process on H:

$$Z^x(t) = S(t)x + \int_0^t S(t-s)dW(s),$$

where W is a Wiener process on H with the covariance operator Q that is $R_t\phi(x) = E\phi(Z^x(t))$ (see [DZ2] for details).

In this paper we investigate hypercontractivity, compactness and space-time regularity of the semigroup R_t and give necessary and sufficient conditions for each case. The results obtained are important for the investigation of uniqueness and ergodic properties of the semigroups related to more general operators of the form

$$L_F\phi(x) = \tfrac{1}{2}\operatorname{tr} QD^2\phi(x) + \langle Ax + F(x), D\phi(x)\rangle.$$

The question of hypercontractivity of the Ornstein-Uhlenbeck semigroup goes back to the seminal papers of Nelson [N] and Gross [G1] and has been investigated mainly for symmetric semigroups or more generally for the class of semigroups possessing Sobolev generators [G2]. Essentially, hypercontractivity in infinite dimensions is known in the symmetric case. We are not aware though of any such result for the general Ornstein-Uhlenbeck semigroup. The generator L of the nonsymmetric semigroup R_t need not be a Sobolev generator, even in finite dimensions, and hence the methods developed in [G1] are not applicable. However, we show that hypercontraction property holds in this case as well. Hyperboundedness of the infinite dimensional nonsymmetric Ornstein-Uhlenbeck semigroup in stronger norms has been obtained recently in [CG2] and under stronger conditions then ours in [F1].

The first results on compactness of the Ornstein-Uhlenbeck semigroup in infinite dimensions have been obtained in [DZ3] and extended in [CG1] under the assumption that all the measures $\mu_t * \delta_{S(t)x}$ are equivalent. In both papers the main tool to show this was the smoothing property of the semigroup R_t and compactness of the appropriate Sobolev imbedding. In this paper we give necessary and sufficient conditions for compactness of R_t as a mapping from $L^p(H,\mu)$ into $L^q(H,\mu)$ and moreover we use a completely different method. The Hilbert-Schmidt property of the semigroup R_t has been shown for the first time in [F1] under more restrictive conditions and with more general assumptions in [CG2]. In this paper we give necessary and sufficient condition for the Hilbert-Schmidt property with a simpler proof.

Necessary and sufficient conditions for the semigroup R_t to transform bounded measurable function into Fréchet differentiable C^∞ functions were given in [DZ1]. In the present setup this property does not hold. However, defining Sobolev spaces as in the Malliavin calculus we show that R_t transforms elements of arbitrary negative Sobolev space into infinitely smooth functions. We give also explicit estimates on the directional derivatives of the function $R_t\phi$. The same estimates are shown to hold for the adjoint semigroup R_t^* also. Finally we obtain necessary and sufficient conditions for the semigroup R_t to be differentiable. This result is optimal in some sense. Namely, it can be shown [F2] that the semigroup R_t is not analytic in general, even in finite dimensions.

Let $H_0 = Q_\infty^{1/2}(H)$ be the Reproducing Kernel Hilbert Space of the measure μ. If (A1) holds then H_0 endowed with the norm $\|x\|_0 = \|Q_\infty^{-1/2}x\|$ is continuously and densely imbedded into H. In Section 3 we show that if (A1) is satisfied then the space H_0 is invariant for the semigroup $S(t)$:

$$S(t)(H_0) \subset H_0$$

for every $t \geq 0$. This property is equivalent to the boundedness of the operator

$$S_0(t) = Q_\infty^{-1/2} S(t) Q_\infty^{1/2}.$$

If the controllability condition $\operatorname{im} S(t) \subset \operatorname{im} Q_t^{1/2}$ introduced by DaPrato and Zabczyk [DZ1] holds for $t > 0$ then the semigroup $S_0(t)$ is of Hilbert-Schmidt type and the following condition holds:

$$\text{(A2)} \qquad \operatorname{im} Q_t^{1/2} = \operatorname{im} Q_\infty^{1/2}$$

which we will need also in many cases.

In order to analyze the Ornstein-Uhlenbeck semigroup we show that R_t can be represented as the so-called second quantization of the operator $S_0^*(t)$, the property well known for the case $A = -I$. To the best of our knowledge such a representation for the nonsymmetric case has not been presented before even in the finite dimensional case. The second quantization operator via the Mehler formula has been discussed recently in [FP] within the framework of locally convex spaces. It has been applied to show tightness of capacities related to the symmetric Ornstein-Uhlenbeck semigroup. In our paper, motivated by applications to stochastic evolution equations, we restrict ourselves to the hilbertian case only and provide more detailed result about the nonsymmetric semigroup R_t. First, we investigate the suitable properties of the abstract second quantization operator. Some of them are new and also of interest. Then the properties of R_t are obtained as rather simple consequences of the representation of R_t as the second quantization of $S_0^*(t)$.

In Section 1 below we recall basic properties of the second quantization operator and prove a version of the Mehler formula. Compactness, Hilbert-Schmidt property and smoothing properties of the second quantization operator are proved in Section 2. The main results are presented in Section 3.

1. The operator of second quantization: basic properties

Let μ be a Gaussian measure on H with covariance operator C. The Reproducing Kernel Hilbert Space of the measure μ will be denoted by $H_0 = C^{1/2}(H)$.

If $h \in H_0$ then we can define a linear function

$$\phi_h(x) = \langle C^{-1/2}h, x \rangle.$$

If $h \notin H_0$ then there exists a sequence $h_n \in H_0$ converging to h and we denote by ϕ_h a unique limit in $L^2(H, \mu)$ of the sequence ϕ_{h_n}. Note that

$$\int_H \phi_h(x)\phi_k(x)\mu(dx) = \langle h, k \rangle. \tag{2}$$

Let $\mathscr{H}_{\le n}$ denote the closed subspace of $L^2(H, \mu)$ spanned by all products $\phi_{h_1} \dots \phi_{h_m}$ of order $m \le n$ of the functions $\phi_{h_1}, \dots, \phi_{h_m}$ and let $\mathscr{H}_n$ be the orthogonal complement of $\mathscr{H}_{\le n-1}$ in $\mathscr{H}_{\le n}$. Then the Ito-Wiener decomposition says that

$$L^2(H, \mu) = \bigoplus_{n=0}^{\infty} \mathscr{H}_n,$$

where $\mathscr{H}_0$ is the space generated by constants. Denoting by I_n the orthogonal projection of $L^2(H, \mu)$ onto $\mathscr{H}_n$ one can easily show that (see for example [S])

$$\langle I_n(\phi_{h_1} \dots \phi_{h_n}), I_n(\phi_{k_1} \dots \phi_{k_n}) \rangle = \sum_{\sigma} \langle h_1, k_{\sigma(1)} \rangle \dots \langle h_n, k_{\sigma(n)} \rangle,$$

where the sum is taken over all permutations of the set $\{1, \dots, n\}$. In particular

$$\int_H |I_m(\phi_{h_1}^{k_1} \dots \phi_{h_n}^{k_n})(x)|^2 \mu(dx) = k_1! \dots k_n! \|h_1\|^{2k_1} \dots \|h_n\|^{2k_n} \tag{3}$$

for $m = \sum_{j=1}^n k_j$ and every collection of orthogonal vectors $h_1, \dots h_n$. Let $\mathscr{A}$ denote the set of all infinite sequences of nonnegative integers $\alpha = (\alpha_n)$ such that

$$|\alpha| = \sum_{n=1}^{\infty} \alpha_n < \infty.$$

For an arbitrary complete orthonormal system (CONS) $\{e_k: k \ge 1\}$ in H and $\alpha \in \mathscr{A}$ we define the vector

$$f_\alpha = \prod_{j=1}^{\infty} \frac{1}{\sqrt{\alpha_j!}} I_{\alpha_j}(\phi_{e_j}^{\alpha_j}).$$

Then from (2) and (3) the system $\{f_\alpha: \alpha \in \mathscr{A}\}$ is an orthonormal system in $L^2(H, \mu)$ and because the space of polynomials $\mathscr{P}$ (see Section 2 for definition) is dense in $L^2(H, \mu)$ this system is complete. Moreover, $\{f_\alpha: \alpha \in \mathscr{A}, |\alpha| = n\}$ is a complete orthonormal system in $\mathscr{H}_n$. Note that for a fixed unit vector $e \in H$ $f_{n,0,\dots}$, $n \ge 0$, are usual Hermite polynomials.

Let $F_n: \mathscr{H}_n \to \mathscr{H}_n$ be a sequence of bounded operators. Then the equality

$$F\phi = \sum_{n=0}^{\infty} F_n I_n(\phi)$$

defines the possibly unbounded operator in $L^2(H, \mu)$. For bounded operators T_1 and T_2 on H we define the operator

$$T_1 \circ T_2^{\circ(n-1)} I_n(\phi_{h_1} \dots \phi_{h_n}) = I_n(\phi_{T_1 h_1} \phi_{T_2 h_2} \dots \phi_{T_2 h_n}) + \dots + I_n(\phi_{T_2 h_1} \dots \phi_{T_2 h_{n-1}} \phi_{T_1 h_n})$$

acting in $\mathscr{H}_n$. Finally, if T is a bounded operator on H then we define an operator $\Gamma_n(T)\colon \mathscr{H}_n \to \mathscr{H}_n$ for $n \geq 1$ by the formula

$$\Gamma_n(T) I_n(\phi_{h_1} \dots \phi_{h_n}) = I_n(\phi_{Th_1} \dots \phi_{Th_n}).$$

For $n = 0$ we put $\Gamma_0(T)1 = 1$. The following simple lemma will be useful in the sequel.

Lemma 1. a) *The operator F defined above is bounded on $L^2(H, \mu)$ if and only if*

$$\sup_{n \geq 1} \|F_n\| < \infty$$

and in that case

$$\|F\| = \sup_{n \geq 1} \|F_n\|.$$

b) *The operator $T_1 \circ T_2^{\circ(n-1)}$ is bounded on $\mathscr{H}_n$ and*

$$\|T_1 \circ T_2^{\circ(n-1)}\| \leq n \|T_1\| \, \|T_2\|^{n-1}.$$

c) $\|\Gamma_n(T)\| = \|T\|^n$.

We are going to recall now the definition and basic properties of the so-called second quantization operator. For more detailed discussion see for example [S]. Let $T\colon H \to H$ be a contraction and let us define the operator $\Gamma(T)$ on the algebraic sum $\sum_{n=0}^{\infty} \mathscr{H}_n$ by the formula

$$\Gamma(T)\phi = \sum_{n=0}^{\infty} \Gamma_n(T) I_n(\phi).$$

Then by Lemma 1 $\Gamma(T)$ has a unique extension to the contraction on $L^2(H, \mu)$.

If T is a bounded linear operator on H then the operator $C^{1/2} T C^{-1/2}$ is bounded on the space $\operatorname{im}(C^{1/2})$ and therefore it can be extended in a unique way to a μ-measurable linear transformation T_C on H such that

$$\int_H \|T_C x\|^2 \mu(dx) = \operatorname{tr}(C^{1/2} T T^* C^{1/2}).$$

Therefore if additionally T is a contraction and ϕ is a bounded Borel function on H then the formula

$$M_T \phi(x) = \int_H \phi((T^*)_C x + (\sqrt{I - T^*T})_C y) \mu(dy) \qquad \text{for } \mu\text{-a.a. } x,$$

which is sometimes called the generalized Mehler formula, defines a bounded measurable function on H.

Proposition 1. *For arbitrary contraction T on H and every $p \geq 1$ the operator M_T is a contraction on $L^p(H, \mu)$. Moreover $M_T = \Gamma(T)$ on $L^2(H, \mu)$.*

Proof. Note that

$$M_T\phi(x) = \int_H \phi((T^*)_C x + z)\nu_1(dz),$$

where ν_1 is a centered Gaussian measure on H with the covariance operator $C^{1/2}(I - T^*T)C^{1/2}$ and hence $M_T\phi(x)$ is well defined for $\phi \in L^p(H, \mu)$ and μ-a.a. x. As a consequence we find that

$$\begin{aligned}\int_H \left| \int_H \phi((T^*)_C x + (\sqrt{I - T^*T})_C y)\mu(dy) \right|^p \mu(dx) &\leq \int_H |\phi(z)|^p \nu_1 * \nu_2(dz) \\ &= \int_H |\phi(z)|^p \mu(dz),\end{aligned}$$

where ν_2 is a centered Gaussian measure with the covariance operator $C^{1/2}T^*TC^{1/2}$. This shows that M_T is a contraction on $L^p(H, \mu)$. To prove the last statement of the proposition let

$$E_h(x) = e^{\phi_h(x) - 1/2\,\|h\|^2}.$$

Then the set $\{E_h: h \in \operatorname{im}(C^{1/2})\}$ is linearly dense in $L^2(H, \mu)$ and for $h \in \operatorname{im}(C^{1/2})$

$$\begin{aligned} &M_T E_h(x) \\ &= \int_H \exp\left(-\frac{1}{2}\|h\|^2 + \langle C^{-1/2}h, (T^*)_C x + (\sqrt{I - T^*T})_C y\rangle \right)\mu(dy) \\ &= \exp\left(-\frac{1}{2}\|h\|^2 + \langle C^{-1/2}h, (T^*)_C x\rangle \right) \int_H \exp(\langle C^{-1/2}h, (\sqrt{I - T^*T})_C y\rangle)\mu(dy) \\ &= \exp\left(-\frac{1}{2}\|h\|^2 + \langle C^{-1/2}h, (T^*)_C x\rangle \right) \exp\left(\frac{1}{2}\langle (I - T^*T)h, h\rangle \right) = E_{Th}(x). \end{aligned}$$

On the other hand, taking into account that (see [S])

$$E_h = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(\phi_h^n)$$

we obtain

$$\Gamma(T)E_h = \Gamma(T) \sum_{n=0}^{\infty} \frac{1}{n!} I_n(\phi_h^n) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(\phi_{Th}^n) = E_{Th}$$

and that concludes the proof.

In Lemma 2 below we collect some basic properties of the operator Γ.

Lemma 2 ([S], chapter 1). *Assume that T, T_1, T_2 are contractions on H. Then the operator $\Gamma(T)$: $L^2(H, \mu) \to L^2(H, \mu)$ enjoys the following properties:*

a) $\Gamma(I_H) = I_{L^2(H,\mu)}$.
b) $\Gamma(T_2 T_1) = \Gamma(T_2)\Gamma(T_1)$.
c) $\Gamma^*(T) = \Gamma(T^*)$.
d) $\Gamma(T)1 = 1$.
e) *The operator* $\Gamma(T)$ *is positivity preserving: if* $\phi \geq 0$ μ*-a.s. then* $\Gamma(T)\phi(x) \geq 0$ μ*-a.s.*
f) *The operator* $\Gamma(T)$ *has an extension (restriction) to a positive contraction on every* $L^p(H, \mu)$ *for* $p \geq 1$.
g) *For every* $p \geq 1$ *and*

$$q_0 = 1 + \frac{p-1}{\|T\|^2} \tag{4}$$

we have

$$\|\Gamma(T)\|_{L^p \to L^{q_0}} = 1$$

and if $q > q_0$ *then*

$$\|\Gamma(T)\|_{L^p \to L^q} = \infty.$$

Proof. The properties a)–d) are obvious, e) and f) follow easily from Proposition 1 and the proof of g) can be found in chapter 1 of [S].

2. The operator of second quantization: compactness and smoothing properties

Let T be a selfadjoint contraction with a complete orthonormal set of eigenvectors $\{v_k : k \geq 1\}$ and the corresponding sequence of eigenvalues $1 \geq t_1 \geq t_2 \geq \cdots \geq 0$ (multiplicities taken into account). Then the operator $\Gamma(T)$ is also selfadjoint with the complete orthogonal set of eigenvectors

$$\left\{ \prod_{j=1}^{\infty} I_{\alpha_j}(\phi_{v_j}^{\alpha_j}) : \alpha \in \mathscr{A} \right\}$$

and the corresponding set of eigenvalues

$$\left\{ t_\alpha = \prod_{j=1}^{\infty} t_j^{\alpha_j} : \alpha \in \mathscr{A} \right\}. \tag{5}$$

Proposition 2. (a) *Let* $p, q \geq 1$ *and* $T \neq 0$. *The operator* $\Gamma(T)$: $L^p(H, \mu) \to L^q(H, \mu)$ *is compact if and only if* T *is a compact strict contraction and* $q < q_0$, *where* q_0 *is given by* (4).

(b) *The operator* $\Gamma(T)$ *is of Hilbert-Schmidt type on* $L^2(H, \mu)$ *if and only if* T *is a strict contraction of Hilbert-Schmidt type and in that case*

$$\|\Gamma(T)\|_{HS} = \frac{1}{\sqrt{\det(I - T^*T)}}.$$

Proof. a) By Lemma 2 the operator $\Gamma(T)$: $L^p(H, \mu) \to L^{q_0}(H, \mu)$ is bounded.

First we shall show compactness for $p = q = 2$. Let T be a compact strict contraction with the polar decomposition $T = U|T|$. Then the same property holds for $|T|$ and therefore by b) of Lemma 2 we can assume that T is selfadjoint and nonnegative. Then all eignvalues of T are nonnegative and less than 1 and hence for any given $s > 0$ there is only a finite number of products (5) greater than s. Therefore zero can be the only accumulation point of the spectrum of $\Gamma(T)$. If we fix the eigenvalue $t_{\alpha_0} < 1$, say, then by the similar argument there can be at most the finite number of eigenvalues $t_\alpha = t_{\alpha_0}$. Moreover, we also obtain from (5) that 1 is an eigenvalue of $\Gamma(T)$ of multiplicity one (for $\alpha = 0$ only). Then compactness of $\Gamma(T)\colon L^2(H, \mu) \to L^2(H, \mu)$ follows.

Therefore, because $\Gamma(T)\colon L^p(H, \mu) \to L^p(H, \mu)$ is bounded for every $p \geq 1$, we find by interpolation [T] that $\Gamma(T)\colon L^p(H, \mu) \to L^p(H, \mu)$ is compact for all $p \geq 2$. Repeating this argument for $\Gamma(T^*)$ we obtain compactness for all $p > 1$.

Finally, to prove compactness of $\Gamma(T)\colon L^p(H, \mu) \to L^q(H, \mu)$ note that

$$\Gamma(T) = \Gamma(U)\Gamma(|T|^{1-\varepsilon})\Gamma(|T|^{\varepsilon}), \tag{6}$$

with the operator

$$\Gamma(U)\colon L^p(H, \mu) \to L^p(H, \mu)$$

bounded, the operator

$$\Gamma(|T|^{1-\varepsilon})\colon L^p(H, \mu) \to L^{q_0(\varepsilon)}$$

bounded for

$$q_0(\varepsilon) = 1 + \frac{p-1}{\|T\|^{2(1-\varepsilon)}}$$

and the operator

$$\Gamma(|T|^{\varepsilon})\colon L^p(H, \mu) \to L^p(H, \mu)$$

compact. Because

$$\lim_{\varepsilon \to 0+} q_0(\varepsilon) = q_0$$

it follows that $\Gamma(T)$ is compact from L^p to L^q for every $q < q_0$.

To prove necessity let $\Gamma(T)\colon L^p(H, \mu) \to L^1(H, \mu)$ be compact. Then every sequence of the elements of the set $\{\Gamma(T)\phi_h\colon \|h\| \leq 1\}$ contains a convergent in $L^1(H, \mu)$ subsequence $\Gamma(T)\phi_{h_n}$, say. It follows that

$$\lim_{n,m \to \infty} \|\Gamma(T)(\phi_{h_n} - \phi_{h_m})\|_1 = \sqrt{\frac{2}{\pi}} \lim_{n,m \to \infty} \|T(h_n - h_m)\| = 0$$

and hence T is compact and so is $|T|$. If 1 is an eigenvalue of $|T|$ then by (5) there is a non-zero eigenvalue of $\Gamma(|T|)$ of infinite multiplicity. But, by Lemma 2b), $\Gamma(|T|)$ is compact and we have a contradiction. Therefore $\|T\| < 1$. Finally, assume that $\Gamma(T)\colon L^p(H, \mu) \to L^{q_0}(H, \mu)$ is compact, where $T \neq 0$ and $p \geq 1$. Then from the previous reasoning T is compact and it is enough to consider T selfadjoint and nonnegative. Let h be a norm one eigenvector such that $Th = \|T\|h$ and define

$$f_n(x) = \exp\left(n\phi_h - \frac{p}{2}n^2\right) = \exp\left(\frac{1-p}{2}n^2\right)E_{nh}(x).$$

Then $\|f_n\|_p = 1$ for every $n \geq 1$ and by Proposition 1

$$\Gamma(T)f_n(x) = \exp\left(\frac{1-p}{2}n^2\right)E_{n\|T\|h}(x)$$

which implies

$$\|\Gamma(T)f_n\|_{q_0} = \exp\left(\frac{1-p}{2}n^2\right)\exp\left(\frac{q_0-1}{2}n^2\|T\|^2\right) = 1. \tag{7}$$

If $g \neq 0$ and $|\phi_g(x)| < \infty$ then

$$\lim_{n\to\infty} E_{ng}(x) = 0.$$

Therefore this convergence holds for μ-a.a. x but in view of (7) it is a contradiction with compactness of $\Gamma(T)$. The proof of (a) is finished.

b) As in a) it is enough to prove b) for T selfadjoint and positive. If T is of Hilbert-Schmidt type then

$$\sum_{\alpha\in\mathscr{A}} \|\Gamma(T)f_\alpha\|^2 = \sum_{\alpha\in\mathscr{A}}\prod_{j=1}^{\infty} t_j^{2\alpha_j} = \sum_{n=0}^{\infty}\sum_{|\alpha|=n}\prod_{j=1}^{\infty} t_j^{2\alpha_j}$$

$$= \sum_{j=1}^{\infty}\sum_{n=0}^{\infty} t_j^{2n} = \prod_{j=1}^{\infty}\frac{1}{1-t_j^2}$$

provided $\|T\| < 1$ and the first part of (b) follows. Conversely, if $\Gamma(T)$ is a Hilbert-Schmidt operator then the same property holds for $\Gamma(|T|)$ hence for $\Gamma_1(|T|)$ as well. We find that

$$\sum_{j=1}^{\infty} \|\Gamma_1(|T|)v_j\|^2 = \sum_{j=1}^{\infty} t_j^2 < \infty$$

and in view of (a) $\|T\| < 1$.

Remark 1. Note that $\Gamma(0)$ is the expectation and hence $\Gamma(0)\colon L^1(H,\mu) \to L^\infty(H,\mu)$ is compact. If $T \neq 0$ then, by Proposition 2(a) $\Gamma(T)\colon L^1(H,\mu) \to L^1(H,\mu)$ is not compact.

We shall discuss now smoothing properties of the operator Γ. To this end we need to define the scale of Sobolev spaces $W_C^{n,p}(H)$ for arbitrary integer n and $p > 1$. Let $h_1, h_2, \ldots$ be a sequence of vectors in H_0 and let $\mathscr{P}$ denote the space of polynomials on H of the form

$$\phi(x) = P(\phi_{h_1}(x), \ldots, \phi_{h_m}(x)),$$

where $m \geq 0$ is arbitrary and P is any polynomial of m variables. Then we define the first Sobolev norm of $\phi \in \mathscr{P}$ by

$$\|\phi\|_{1,p}^p = \|\phi\|^p + \|C^{1/2}D\phi\|^p$$

and for $n \geq 2$

$$\|\phi\|_{n,p}^p = \|\phi\|_{n-1}^p + \|(C^{1/2}D)^n\phi\|^p,$$

where the norm of the operator $(C^{1/2}D)^n$ is the Hilbert-Schmidt norm in the space $H^{\otimes n}$. Let L_M denote an Ornstein-Uhlenbeck operator (1) with $A = -\frac{1}{2}I$ and $Q = C$ considered in $L^p(H, \mu)$. Then for every $\alpha > 0$ the space $\operatorname{dom}(I - L_M)^{\alpha/2}$ can be endowed with the norm

$$|\phi|_{\alpha,p} = \|(I - L_M)^{\alpha/2}\phi\|_p$$

and by the Meyer inequalities (see p. 28 of [W]) the norms $\|\cdot\|_{n,p}$ and $|\cdot|_{n,p}$ are equivalent for all $n \geq 1$ and $p > 1$. Therefore, we can define the space $W_C^{n,p}(H)$ as the closure of $\mathscr{P}$ in $L^p(H, \mu)$ with respect to the norm $|\cdot|_{n,p}$. Moreover, for any $\alpha > 0$ we denote by $W_C^{\alpha,p}(H)$ the space $\operatorname{dom}(I - L_M)^{\alpha/2}$ with the norm $|\cdot|_{\alpha,p}$. For $\alpha < 0$ we define $W_C^{\alpha,p}(H)$ as the completion of $L^p(H, \mu)$ with respect to the norm $|\cdot|_{\alpha,p}$ and then $W_C^{-\alpha,q}(H)$ can be identified with the dual of $W_C^{\alpha,p}(H)$, where $q = \dfrac{p}{p-1}$. For $p = 2$

$$|\phi|_{\alpha,2}^2 = \sum_{k=0}^{\infty} (k+1)^\alpha \|I_k(\phi)\|^2 \tag{8}$$

and in particular for $\alpha = 1$ we have $\|\phi\|_{1,2} = |\phi|_{1,2}$. In the sequel we consider the space $W_C^{\alpha,p}(H)$ with the norm $|\cdot|_{\alpha,p}$. In the next lemma by $H^{\otimes n}$ we denote the n-fold tensor product of H equipped with the Hilbert-Schmidt norm. By $\Gamma(T) \otimes T$ we denote the tensor of $\Gamma(T)$ and T acting in the space $L^p(H, \mu; H)$. This operator is well defined because $\Gamma(T)$ is positive.

Lemma 3. *For $\phi \in W_C^{n,p}(H)$, $n \geq 1$, $p > 1$, we have*

$$(C^{1/2}D)^n\Gamma(T)\phi = \Gamma(T) \otimes T^{\otimes n}(C^{1/2}D)^n\phi. \tag{9}$$

For $\Phi \in \operatorname{dom}((C^{1/2}D)^n)^* \subset L^q(H, \mu; H^{\otimes n})$

$$((C^{1/2}D)^n)^*\Gamma(T) \otimes T^n\Phi = \Gamma(T)((C^{1/2}D)^n)^*\Phi \tag{10}$$

and for $\phi \in \operatorname{dom}(L_M)$

$$L_M\Gamma(T)\phi = \Gamma(T)L_M\phi. \tag{11}$$

Proof. Note first that by limiting argument

$$C^{1/2}DI_m(\phi_h^m) = mI_{m-1}(\phi_h^{m-1})h$$

for every $h \in H$. Using polarization and the density of polynomials in $W_C^{n,p}(H)$ we can reduce the proof of (9) to the case when $\phi = I_m(\phi_h^m)$ with $m \geq n$. In that case

$$\begin{aligned}(C^{1/2}D)^n\Gamma(T)I_m(\phi_h^m)(x) &= (C^{1/2}D)^n I_m(\phi_{Th}^m)(x)\\ &= m(m-1)\dots(m-n+1)I_{m-n}(\phi_{Th}^{m-n})(x)(Th)^{\otimes n}\\ &= m(m-1)\dots(m-n+1)\Gamma(T)\otimes T^{\otimes n}(I_{m-n}(\phi_h^{m-n})h^{\otimes n})(x)\\ &= \Gamma(T)\otimes T^{\otimes n}(C^{1/2}D)^m I_m(\phi_h^m)(x).\end{aligned}$$

To prove (10) note first that by (9)

$$C^{1/2}D\Gamma(T^*) = \Gamma(T^*)\otimes T^*C^{1/2}D.$$

Hence

$$(C^{1/2}D)^*\Gamma(T)\otimes T = (\Gamma(T^*)\otimes T^*C^{1/2}D)^* = (C^{1/2}D\Gamma(T^*))^* = \Gamma(T)(C^{1/2}D)^*$$

and (10) follows for $n=1$. For $n>1$ the proof is similar. The last identity (11) follows from (9), (10) and the fact that $L_M = -\frac{1}{2}(C^{1/2}D)^*C^{1/2}D$. Note that (11) follows easily from the fact that $e^{tL_M} = \Gamma(e^{-t/2}I)$.

Proposition 3. *Let $\|T\| \le 1$. Then the following holds.*

a) For all $p, q > 1$ *and* $\alpha \in \mathbf{R}$ *the operator* $\Gamma(T)$ *has a unique extension (restriction) to a norm one contraction* $\Gamma(T)\colon W_C^{\alpha,p}(H) \to W_C^{\alpha,q}(H)$ *provided* $q \le q_0$, *where* q_0 *is given by* (4).

b) *Assume that T is a strict contraction, $\alpha, \beta \in \mathbf{R}$ and $p, q > 1$. If $\alpha < \beta$ and $q < q_0$, where q_0 is given by* (4) *then the operator $\Gamma(T)\colon W_C^{\alpha,p}(H) \to W_C^{\beta,q}(H)$ is bounded. Moreover, if T is compact then $\Gamma(T)\colon W_C^{\alpha,p}(H) \to W_C^{\beta,q}(H)$ is also compact for $q < q_0$.*

c) *Assume $p = q = 2$. Then the operator $\Gamma(T)\colon W_C^{\alpha,2}(H) \to W_C^{\beta,2}(H)$ is bounded for all $\alpha, \beta \in \mathbf{R}$ if and only if T is a strict contraction and in that case for $\alpha < \beta$*

$$\|\Gamma(T)\|_{W^{\alpha,2}\to W^{\beta,2}} \le c(T)^{(\beta-\alpha)/2}\frac{1}{\|T\|^{1-c(T)}}$$

with

$$c(T) = \frac{\beta-\alpha}{2\log\dfrac{1}{\|T\|}}$$

provided $\|T\| \ge e^{-(\beta-\alpha)/2}$. If $\|T\| < e^{-(\beta-\alpha)/2}$ then $\|\Gamma(T)\|_{\alpha\to\beta} = 1$.

Proof. The part a) is an immediate consequence of (11).

b) For any contraction T we have

$$\Gamma(T) = \Gamma(\|T\|)\Gamma\left(\frac{T}{\|T\|}\right)$$

and by a) the operator

$$\Gamma\left(\frac{T}{\|T\|}\right): W_C^{\alpha,p}(H) \to W_C^{\alpha,p}(H)$$

is bounded. We show first that the operator $\Gamma(\|T\|): W_C^{\alpha,q}(H) \to W_C^{\beta,q}(H)$ is bounded. If t is the solution to the equation

$$\|T\| = e^{-t/2}$$

then

$$\Gamma(\|T\|) = e^{tL_M}.$$

Since the semigroup e^{tL_M} is analytic in $L^q(H, \mu)$ for $q > 1$ we find that

$$\|\Gamma(\|T\|)\|_{W^{\alpha,q}\to W^{\beta,q}} = \|(I - L_M)^{(\beta-\alpha)/2}\Gamma(\|T\|)\|_{L^q\to L^q}$$

$$= \|(I - L_M)^{(\beta-\alpha)/2}e^{tL_M}\|_{L^q\to L^q} \leq \frac{a}{t^{(\beta-\alpha)/2}}$$

for certain $a > 0$ and the boundedness follows. In the next step we use the representation

$$\Gamma(T) = \Gamma(|T|^\varepsilon)\Gamma(|T|^{1-\varepsilon})\Gamma(U),$$

where U is a unitary operator on H. By the first part of the proof the operator

$$\Gamma(|T|^\varepsilon): W_C^{\alpha,q}(H) \to W_C^{\alpha,q}(H)$$

is bounded and by Lemma 2 and the definition of Sobolev spaces the operator

$$\Gamma(|T|^{1-\varepsilon}): W_C^{\alpha,p}(H) \to W_C^{\alpha,q}(H)$$

is bounded for $q < q_0$ by the same argument as in the proof of Proposition 2. Moreover, if T is compact then once more invoking Proposition 2 we obtain the compactness of $\Gamma(T)$ for $q < q_0$.

c) We shall consider now the case $p = q = 2$. Note that for $\phi \in W_C^{2,\alpha}$

$$\|\Gamma(T)\phi\|_{\alpha\to\beta} = \|(I - L_M)^{(\beta-\alpha)/2}\Gamma(T)(I - L_M)^{\alpha/2}\phi\|.$$

Taking into account the definition of $\Gamma(T)$ we obtain

$$(I - L_M)^{(\beta-\alpha)/2}\Gamma(T) = \sum_{k=0}^{\infty} (I - L_M)^{(\beta-\alpha)/2}\Gamma_k(T)I_k(\psi)$$

and now Lemma 1 and properties of L_M yield

$$\|\Gamma(T)\|_{\alpha\to\beta} = \sup_{k\geq 1} \|(I - L_M)^{(\beta-\alpha)/2}\Gamma_k(T)I_k\| = \sup_{k\geq 1} ((k + 1)^{(\beta-\alpha)/2}\|\Gamma_k(T)\|).$$

Hence

$$\|\Gamma(T)\|_{\alpha\to\beta} = \sup_{k\geq 1} ((k + 1)^{(\beta-\alpha)/2}\|T\|^k).$$

To conclude the proof of boundedness it is enough to notice that the above estimate is finite if and only if $\|T\| < 1$ and in that case

$$\sup_{k\geq 0}(k+1)^{\beta-\alpha}\|T\|^{2k} \leq c(T)^{\beta-\alpha}\frac{1}{\|T\|^{2(1-c(T))}}$$

provided $\|T\| \geq e^{-(\beta-\alpha)/2}$. Otherwise

$$\sup_{k\geq 0}(k+1)^{\beta-\alpha}\|T\|^{2k} = 1.$$

Remark 2. It follows from Proposition 2.2 in [DZ3] that the imbedding of $W_C^{1,2}(H)$ into $L^2(H,\mu)$ is not compact. The following example shows that this property holds for an arbitrary pair of Sobolev spaces discussed in this section. It is enough to consider the imbedding of $W_C^{1,p}(H)$ into $L^q(H,\mu)$. The general argument is the same. Let $\{e_k: k \geq 1\}$ be a CONS in H. Then $C^{1/2}D\phi_{e_k} = e_k$ and $\|\phi_{e_k}\|_{1,p} = c_p$ for every $k \geq 1$. If the family of functions $\{\phi_{e_k}: k \geq 1\}$ is relatively compact in $L^q(H,\mu)$ then there exists a subsequence k_j such that $\phi_{e_{k_j}}$ are convergent μ-*a.s.* and therefore convergent in $L^2(H,\mu)$ but this is impossible.

3. The Ornstein-Uhlenbeck semigroup as the second quantized operator

We start with the following lemma.

Lemma 4. *If* (A1) *holds then* $S(t)H_0 \subset H_0$ *for all* $t \geq 0$, *and* $S_0(t) = Q_\infty^{-1/2}S(t)Q_\infty^{1/2}$, $t \geq 0$, *is a strongly continuous semigroup of contractions on* H. *Moreover* $\|S_0(t)\| < 1$ *if and only if* $\operatorname{im} Q_t^{1/2} = \operatorname{im} Q_\infty^{1/2}$.

Proof. We show first that $S(t)H_0 \subset H_0$ for every $t > 0$. Note first that for a fixed $t > 0$ the operator $T = Q_\infty^{1/2}S^*(t)Q_\infty^{-1/2}$ is well defined and bounded on H_0 and for $h \in H_0$

$$R_t\phi_h(x) = \int_H \langle S(t)x + y, Q_\infty^{-1/2}h\rangle \mu_t(dy) = \langle x, S^*(t)Q_\infty^{-1/2}h\rangle = \phi_{Th}(x).$$

Hence for $h \in H_0$ and $k \in H$ we obtain

$$\langle R_t\phi_h, \phi_k\rangle = \langle \phi_{Th}, \phi_k\rangle = \langle Th, k\rangle.$$

Therefore

$$|\langle Th, k\rangle| = |\langle R_t\phi_h, \phi_k\rangle| \leq \|h\|\,\|k\|$$

and it follows from this inequality that the operator T has a unique extension to a contraction on H which is denoted by $S_0^*(t)$. Let $S_0(t)$ denote its adjoint on H. Then

$$\langle h, S_0(t)k\rangle = \langle Th, k\rangle = \langle Q_\infty^{-1/2}h, S(t)Q_\infty^{1/2}k\rangle$$

and as a consequence we find that $S(t)Q_\infty^{1/2}k \in H_0$ for every $k \in H$. Hence

$S(t)H_0 \subset H_0$ and $S_0(t) = Q_\infty^{-1/2} S(t) Q_\infty^{1/2}$. Clearly, S_0 enjoys the semigroup property. Taking into account that

$$Q_\infty = Q_t + S(t) Q_\infty S^*(t)$$

we obtain

$$\langle Q_t x, x \rangle = \langle (I - S_0(t) S_0^*(t)) Q_\infty^{1/2} x, Q_\infty^{1/2} x \rangle. \tag{12}$$

It follows from this equation and (A1b) that the operator $I - S_0(t) S_0^*(t)$ is non-negative. If $x \in H$ and $y \in H_0$ then

$$\lim_{t \to 0} \langle S_0(t) x, y \rangle = \langle x, y \rangle$$

and because $S_0(t)$ is uniformly bounded the standard argument shows that S_0 is weakly continuous at zero and hence a C_0-semigroup (see Theorem 1.4 in [P]). The last part of the lemma follows easily from (12).

The next theorem establishes connection between the second quantization operator discussed in Section 1 and the Ornstein-Uhlenbeck semigroup. In a different framework it has been noticed also in [FP].

Theorem 1. *If* (A1) *holds then* $R_t = \Gamma(S_0^*(t))$ and $R_t^* = \Gamma(S_0(t))$. *Moreover,*

$$R_t \phi(x) = \int_H \phi(S(t)x + Q_\infty^{1/2} \sqrt{I - S_0(t) S_0^*(t)}\, Q_\infty^{-1/2} y) \mu(dy) \tag{13}$$

and

$$R_t^* \phi(x) = \int_H \phi(Q_\infty^{1/2} S_0^*(t) Q_\infty^{-1/2} x + Q_\infty^{1/2} \sqrt{I - S_0^*(t) S_0(t)}\, Q_\infty^{-1/2} y) \mu(dy). \tag{14}$$

Proof. The theorem follows straightforwardly from Proposition 1 and Lemma 4.

Remark 2. If ϕ is bounded then (13) holds for al $x \in H$. If moreover, $S(t)(\operatorname{im}(Q_\infty)) \subset \operatorname{im}(Q_\infty)$ then (14) also holds for all $x \in H$.

It follows from Theorem 1 and the definition of the operator Γ that the space $\mathscr{H}_n$ is invariant for R_t. Let R_t^n denote the semigroup R_t restricted to $\mathscr{H}_n$ and L_n be its generator. Then the set

$$\mathscr{D}_n = \{ I_n(\phi_{h_1} \cdots \phi_{h_n}) : h_1, \ldots, h_n \in \operatorname{dom}(A_0^*) \} \subset \operatorname{dom}(L_n)$$

is a core for the semigroup R_t^n and on $\mathscr{D}_n$

$$L_n I_n(\phi_{h_1} \cdots \phi_{h_n}) = \sum_{i=1}^n I_n(\phi_{h_1} \cdots \phi_{A_0^* h_i} \cdots \phi_{h_n}). \tag{15}$$

Similarly the noncompleted direct sum

$$\mathscr{D} = \sum_{n=1}^{\infty} \mathscr{D}_n \subset \operatorname{dom}(L)$$

is a core for the semigroup R_t.

The following hypercontraction result is an immediate consequence of Lemmas 1 and 4 and Theorem 1.

Theorem 2. *Let* (A1) *and* (A2) *hold. Then for* $p, q \geq 1$ *and* $t \geq 0$

$$\|R_t\|_{L^p \to L^q} = 1$$

if and only if

$$\sqrt{\frac{p-1}{q-1}} \geq \|S_0(t)\|.$$

If this condition is not satisfied then $\|R_t\|_{L^p \to L^q} = \infty$.

Remark 3. In general Theorem 2 does not imply that the generator L of the semigroup R_t is a Sobolev generator in the terminology of Gross [G]. However, if $\|S_0(t)\| \leq e^{-at}$ for certain $a > 0$ then L is a Sobolev generator.

In Theorem 3a) below we give a complete answer to the question of compactness of R_t on the scale of $L^p(H, \mu)$ spaces and generalize some results from [DZ3] and [CG1]. Note that for the limiting exponent q_0 given in Theorem 2, R_t is still bounded but it is not compact. In particular, R_t: $L^1(H, \mu) \to L^1(H, \mu)$ is not compact, unless $S(t) = 0$.

Theorem 3. *Let* (A1) *hold.*

a) *For* $p, q \geq 1$ *and* $t \geq 0$ *the operator* R_t: $L^p(H, \mu) \to L^q(H, \mu)$ *is compact if and only if* (A2) *holds,* $S_0(t)$ *is compact and*

$$q < 1 + \frac{p-1}{\|S_0(t)\|^2}.$$

b) *The operator* R_t *is Hilbert-Schmidt on* $L^2(H, \mu)$ *if and only if* $S_0(t)$ *is Hilbert-Schmidt on* H *and* (A2) *holds. In that case*

$$\|R_t\|_{HS} = \frac{1}{\sqrt{\det(I - S_0(t)S_0^*(t))}}.$$

Proof. The proof follows immediately from Lemma 4 and Proposition 2.

The next theorem shows that the semigroups R_t and R_t^* have some smoothing properties which are described by the estimates (16) and (17) below. These estimates are more precise than the analogous ones obtained in [CG2] by the use of Cameron-Martin formula.

Theorem 4. a) *If* (A1) *holds then* R_t and R_t^* *define strongly continuous semigroups of contractions in all spaces* $W_{Q_\infty}^{\alpha,p}(H)$ for $p > 1$ and $\alpha \in \mathbf{R}$.

b) *If additionally* (A2) *holds then* R_t *and* R_t^* *are bounded operators from* $W_{Q_\infty}^{\alpha,p}(H)$ *to* $W_{Q_\infty}^{\beta,q}(H)$ *for all* α, $\beta \in \mathbf{R}$ *and* $q < q_0$, *where* q_0 *is given by* (4). *Moreover, if* $S_0(t)$ *is compact then* R_t *and* R_t^* *are compact from* $W_{Q_\infty}^{\alpha,p}$ *to* $W_{Q_\infty}^{\beta,q}(H)$ *for all* α, $\beta \in \mathbf{R}$ *and* $q < q_0$.

c) *Let* (A1) *hold. Then the following properties hold if and only if* (A2) *is satisfied. The operators* R_t and R_t^* *are bounded from* $W_{Q_\infty}^{\alpha,2}(H)$ to $W_{Q_\infty}^{\beta,2}(H)$ *for all* α, $\beta \in \mathbf{R}$. *Moreover, for* $\beta > \alpha$

$$\|R_t\|_{W_{Q_\infty}^{\alpha,2} \to W_{Q_\infty}^{\beta,2}} \le c_t^{(\beta-\alpha)/2} \frac{1}{\|S_0(t)\|^{1-c_t}} \tag{16}$$

and

$$\|R_t^*\|_{W_{Q_\infty}^{\alpha,2} \to W_{Q_\infty}^{\beta,2}} \le c_t^{(\beta-\alpha)/2} \frac{1}{\|S_0(t)\|^{1-c_t}}, \tag{17}$$

where

$$c_t = \frac{\beta - \alpha}{2 \log \dfrac{1}{\|S_0(t)\|}}$$

provided $\|S_0(t)\| \ge e^{-(\beta-\alpha)/2}$. *Otherwise* $\|R_t\|_{\alpha,\beta} = \|R_t^*\|_{\alpha,\beta} = 1$. *If* $S_0(t)$ *is compact then the operators* R_t: $W_{Q_\infty}^{\alpha,2}(H) \to W_{Q_\infty}^{\beta,2}(H)$ *are also compact.*

Proof. Follows from Proposition 3 and Theorem 1.

Corollary 1. *Assume* (A1) *and* $\|S_0(t)\| \le e^{-at}$. *Then for any* $T > 0$ *there exists* $C > 0$ *such that for* $t \le T$

$$\|Q_\infty^{1/2} DR_t\|_2 + \|Q_\infty^{1/2} DR_t^*\|_2 \le \frac{C}{\sqrt{t}}.$$

Proof. This corollary follows from the equality $\|\cdot\|_{1,2} = |\cdot|_{1,2}$, Theorem 4 and the estimate $c_t \le (2at)^{-1}$.

Theorem 5. *Let* (A1) *and* (A2) *hold. Then the semigroup* R_t *is differentiable on* $L^2(H, \mu)$ *if and only if* $S_0(t)$ *is differentiable.*

Proof. Assume that $S_0^*(t)$ is differentiable. Then by [P]

$$\frac{d}{dt} S_0^*(t) = A_0^* S_0^*(t)$$

in the uniform operator topology. We need to show that R_t is differentiable for $t > 0$ and the operator

$$\frac{d}{dt} R_t = LR_t$$

is bounded. To this end it is enough to show that the operator LR_t extends to a bounded operator on $L^2(H, \mu)$. Note also that for $\phi \in \mathscr{D}$

$$\|LR_t\phi\|^2 = \sum_{n=1}^{\infty} \|L_n R_t^n I_n(\phi)\|^2 .$$

Hence we need to show that for $t > 0$

$$\sup_{n \geq 1} \|L_n R_t^n\| < \infty.$$

Theorem 1 and (15) yield that

$$L_n R_t^n I_n(\phi_{h_1} \cdots \phi_{h_n}) = \sum_{i=1}^{n} I_n(\phi_{S_0^*(t)h_1} \cdots \phi_{S_0^*(t)h_{i-1}} \phi_{A_0^* S_0^*(t)h_i} \phi_{S_0^*(t)h_{i+1}} \cdots \phi_{S_0^*(t)h_n}).$$

Because $A_0^* S_0^*(t)$ is bounded Lemma 1 yields

$$\|L_n R_t^n\| \leq n \|A_0^* S_0^*(t)\| \, \|S_0^*(t)\|^{n-1}$$

and because

$$\sup_{n \geq 1} n \|S_0^*(t)\|^{n-1} < \infty$$

differentiability of R_t follows. To prove necessity notice that if R_t is differentiable then so is R_t^1 and so is $S_0^*(t)$.

Acknowledgement. The authors are grateful to M. Fuhrman for helpful discussion on time regularity of the Ornstein-Uhlenbeck semigroup.

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