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Spectral decompositions of Berezin transformations on C^n related to the natural $U(n)$-action

Dedicated to Professor Takeshi Hirai on his 60th birthday

By

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Introduction

The Berezin transformation, which links the covariant symbol (the Berezin symbol) and the contravariant symbol (the symbol for a Toeplitz operator) of an operator A, plays an important role in Berezin's theory of quantization, see [4]. Let us begin the present paper with the definition of Berezin transformation. Consider a domain D in $\boldsymbol{C}^n$ and a Borel measure μ on D. Let $\mathfrak{H}$ be a closed subspace of $L^2(D, d\mu)$ consisting of continuous functions and we denote by P the orthogonal projection $L^2(D, d\mu) \to \mathfrak{H}$. For each $\varphi \in L^\infty(D)$ we define the Toeplitz operator $T(\varphi)$ with symbol φ by $T(\varphi)h := P(\varphi h)$ $(h \in \mathfrak{H})$. We assume that $\mathfrak{H}$ has a reproducing kernel $\kappa(z, w)$. The Berezin symbol of a bounded operator A on $\mathfrak{H}$ is the function $\sigma(A)$ on D given by

$$\sigma(A)(z) := \frac{(A\kappa(\cdot, z) \mid \kappa(\cdot, z))_{\mathfrak{H}}}{\kappa(z, z)}.$$

Then by [15, 1.19], the maps T and σ are adjoint to each other in a suitable sense. We will accordingly write σ^* for T. The *Berezin transformation* B associated to $\mathfrak{H}$ is, by definition, the positive selfadjoint operator $\sigma\sigma^*$, which turns out to be a bounded operator on $L^2(D, d\mu_0)$, where $d\mu_0 := \kappa(z, z)\, d\mu$. Moreover B is an integral operator with integral kernel given by $\dfrac{|\kappa(z, w)|^2}{\kappa(z, z)\kappa(w, w)}$, see [4] and [15].

When $\mathfrak{H}$ carries an irreducible unitary representation of a Lie group G acting on D, the operator B is G-invariant, so that it is a very interesting problem to find its spectrum. In the case where $D = \boldsymbol{C}^n$, $\mathfrak{H}$ the Fock space and G the Heisenberg group, one knows that B is expressed as the exponential of the euclidean Laplacian Δ on $\boldsymbol{C}^n$: $B = \exp(\Delta/4)$, see [4, §4], [15, 1.27] and [11, §1] etc. If D is the open unit disk $\boldsymbol{D}$ in $\boldsymbol{C}$ and if $\mathfrak{H} = \mathfrak{H}_\alpha$ $(\alpha > -1)$ is the Hilbert space of holomorphic functions on $\boldsymbol{D}$ which are square integrable rela-

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tive to the measure $\frac{\alpha+1}{\pi}(1-|z|^2)^\alpha dxdy\ (z=x+iy)$ (note that $\mathfrak{H}_\alpha$ carries a holomorphic discrete series representation of the universal covering group of $SU(1,1)$, see [2, §9] for example), then $B=\frac{\left|\Gamma\left(\alpha+\frac{3}{2}+i\Lambda\right)\right|^2}{\Gamma(\alpha+1)\Gamma(\alpha+2)}$ with $\Lambda:=(-\Delta_D-1/4)^{1/2}$, where $\Delta_D:=(1-|z|^2)^2\frac{\partial^2}{\partial z\partial\bar{z}}$ is the Möbius-invariant Laplacian on $\boldsymbol{D}$ and the substitution of the operator Λ into the gamma function Γ is done through the spectral analysis using the spherical Fourier transformation as developed in [8], see [1, §10], [6, §4] and [12, Example 2] for details. This example was generalized to the open unit ball in $\boldsymbol{C}^n$ by [5], see also [12, Example 2′], and has been further generalized recently to the case of bounded symmetric domains by [15].

Now from the above it is possible to define the Berezin transformation provided one has a subspace of L^2 which possesses a reproducing kernel. A situation for this occurs when a compact Lie group U acts linearly on a finite-dimensional complex vector space V in a multiplicity-free way, see [9], [3]. This means that the space $\mathcal{P}(V)$ of holomorphic polynomial functions on V decomposes into a direct sum of mutually inequivalent U-irreducible subspaces $\mathcal{P}_\alpha(V)$ $(\alpha\in A)$. The spaces $\mathcal{P}_\alpha(V)$, though finite-dimensional, provide plenty of reproducing kernel subspaces of $L^2(V, d\mu)$, $d\mu$ being the normalized Gaussian measure on V. In §1 of this paper, we treat the Berezin transformation B_α associated to $\mathcal{P}_\alpha(V)$. Let κ_α be the reproducing kernel of $\mathcal{P}_\alpha(V)$. To exhibit various B_α $(\alpha\in A)$ within a single fixed space, we transfer B_α from $L^2(V, \kappa_\alpha(z, z)d\mu)$ to the ordinary Lebesgue L^2-space $L^2(V)$. Then we show in Theorem 1.2 that the (transferred) Berezin transformation acts on the U-invariant functions as the one-dimensional orthogonal projection onto $\boldsymbol{C}\phi_\alpha$, where $\phi_\alpha(z):=\kappa_\alpha(z, z)^{1/2}e^{-\|z\|^2/2}$. In §2 we treat the case $V=\boldsymbol{C}^n$, $U=U(n)$ in detail and describe the spectral decomposition of B_k $(k=0, 1, \ldots)$ explicitly: note in this case that the parameter set A for $\mathcal{P}_\alpha(V)$ is the set of non-negative integers $\boldsymbol{Z}_+$ reflecting the degree of homogeneity. To describe our result we need some notational preparations. Let $\mathcal{Y}_{jj}$ be the space of spherical harmonics of type (j, j) on $S^{2n-1}\subset\boldsymbol{C}^n$. In other words, $\mathcal{Y}_{jj}$ is the space of the restrictions to S^{2n-1} of harmonic polynomials $h(z, \bar{z})$ which are homogeneous of degree j both in z and $\bar{z}$. Then, denoting by E_{kj} the orthogonal projection $L^2(V)\to\boldsymbol{C}\varphi_k\otimes\mathcal{Y}_{jj}$, where $\varphi_k(r)=r^ke^{-r^2/2}\ (r>0)$, we show in Theorem 2.8 that

$$B_k=\sum_{j=0}^k\binom{n+j+k-1}{j}^{-1}\binom{k}{j}\cdot E_{kj}.$$

The contents of this paper form a part of the first author's master thesis submitted to Kyoto University.

§1. Generalities

Let V be a finite-dimensional complex vector space and U a compact Lie group acting linearly on V. We will denote by π the corresponding action on functions on V: $\pi(u)f(x):=f(u^{-1}x)$ $(u\in U)$. We fix a U-invariant hermitian inner product $(\cdot|\cdot)$ on V. Suppose that the U-action on V is multiplicity-free. This means that the space $\mathscr{P}(V)$ of holomorphic polynomial functions on V has a decomposition $\mathscr{P}(V)=\sum_{\alpha\in A}\mathscr{P}_\alpha(V)$ into mutually inequivalent U-irreducible subspaces, where A is an index set. Note that $\mathscr{P}_\alpha(V)$ is finite-dimensional. Let $\mathfrak{F}$ denote the Fock space, that is, $\mathfrak{F}$ is the Hilbert space of holomorphic functions f on V such that

$$\|f\|_{\mathfrak{F}}^2:=\frac{1}{\pi^n}\int_V |f(z)|^2 e^{-\|z\|^2}\,dm(z)<\infty,$$

where $n:=\dim V$, $\|z\|^2:=(z\mid z)$ and dm is the Lebesgue measure on V defined by the euclidean structure $\mathrm{Re}(\cdot|\cdot)$. The space $\mathfrak{F}$ has an orthogonal decomposition $\mathfrak{F}=\bigoplus_{\alpha\in A}\mathscr{P}_\alpha(V)$. The Hilbert space $\mathfrak{F}$ has the reproducing kernel $\kappa(z,w)$ given by $\kappa(z,w):=e^{(z\mid w)}$ $(z,w\in V)$. This means that $f(w)=(f\mid\kappa(\cdot,w))_{\mathfrak{F}}$ for any $f\in\mathfrak{F}$. Moreover, the function $\kappa_\alpha(z,w)$ defined through the orthogonal decomposition $\kappa(\cdot,w)=\sum_{\alpha\in A}\kappa_\alpha(\cdot,w)$ is easily seen to be the reproducing kernel for the space $\mathscr{P}_\alpha(V)$. Since $\mathscr{P}_\alpha(V)$ is U-invariant, κ_α has the property

$$\kappa_\alpha(uz,uw)=\kappa_\alpha(z,w)\qquad\text{for all } u\in U. \tag{1.1}$$

Proposition 1.1. *There is an open dense subset $\mathcal{O}$ in V such that $\kappa_\alpha(w,w)\neq 0$ for any $w\in\mathcal{O}$.*

Proof. Let $H=U_{\mathbf{C}}\subset GL(V)$, the complexification of the compact Lie group U. We have

$$\kappa_\alpha(h^{-1}z,h^*w)=\kappa_\alpha(z,w)\qquad\text{for all } h\in H,$$

where h^* stands for the adjoint of h relative to the inner product $(\cdot|\cdot)$ we are fixing. Now it is known by [14, Theorem 6.2] and [17, Theorem 2] that the H-action on V possesses an open dense orbit $\mathcal{O}$. We claim that $\kappa_\alpha(w,w)\neq 0$ for any $w\in\mathcal{O}$. In fact suppose $\kappa_\alpha(w_0,w_0)=0$ for some $w_0\in\mathcal{O}$. Then $\|\kappa_\alpha(\cdot,w_0)\|_{\mathfrak{F}}^2=\kappa_\alpha(w_0,w_0)=0$, so that $\kappa_\alpha(z,w_0)=0$ for all $z\in V$. Let $w\in\mathcal{O}$ be arbitrary and take $h\in H$ such that $w=hw_0$. Then we have

$$\kappa_\alpha(z,w)=\kappa_\alpha(z,hw_0)=\kappa_\alpha(h^*z,w_0)=0\qquad\text{for all } z\in V. \tag{1.2}$$

Since κ_α is the reproducing kernel of $\mathscr{P}_\alpha(V)$, (1.2) implies that any $f\in\mathscr{P}_\alpha(V)$ vanishes on the open dense set $\mathcal{O}$, whence the contradiction $\mathscr{P}_\alpha(V)=\{0\}$.

Let P_α be the orthogonal projection $L^2(V,e^{-\|z\|^2}dm)\to\mathscr{P}_\alpha(V)$. Making use of P_α, we define the Toeplitz operators $\sigma_\alpha^*(\varphi)$ $(\varphi\in L^\infty(V))$ on $\mathscr{P}_\alpha(V)$ by

$\sigma_\alpha^*(\varphi)p := P_\alpha(\varphi p)$ $(p \in \mathcal{P}_\alpha(V))$. Noting Proposition 1.1, we set $e_w^\alpha(z) := \frac{\kappa_\alpha(z, w)}{\kappa_\alpha(w, w)^{1/2}}$. For every bounded operator A on $\mathfrak{F}$, the Berezin symbol $\sigma_\alpha(A)$ of A associated to $\mathcal{P}_\alpha(V)$ is defined to be $\sigma_\alpha(A)(z) := (Ae_z^\alpha \mid e_z^\alpha)_{\mathfrak{F}}$. Put

$$d\mu_\alpha(z) := \frac{1}{\pi^n} \kappa_\alpha(z, z) e^{-\|z\|^2} dm(z).$$

Then by [15, 1.19], the Berezin transformation $\sigma_\alpha \sigma_\alpha^*$ associated to $\mathcal{P}_\alpha(V)$ is the integral operator on $\mathcal{L}_\alpha := L^2(V, d\mu_\alpha)$ with kernel $|(e_z^\alpha \mid e_w^\alpha)|_{\mathfrak{F}}^2$. We transfer the Berezin transformation from $\mathcal{L}_\alpha$ to $L^2(V) := L^2(V, dm)$ via the unitary transformation I_α given by

$$I_\alpha h(z) := \frac{1}{\pi^{n/2}} \kappa_\alpha(z, z)^{1/2} e^{-\|z\|^2/2} h(z) \qquad (h \in \mathcal{L}_\alpha).$$

Then by a simple computation, we see that the Berezin transformation B_α on $L^2(V)$ is an integral operator

$$B_\alpha f(z) = \int_V b_\alpha(z, w) f(w)\, dm(w)$$

with kernel given by

$$b_\alpha(z, w) = \frac{1}{\pi^n} e^{-\|z\|^2/2} e^{-\|w\|^2/2} \frac{|\kappa_\alpha(z, w)|^2}{\kappa_\alpha(z, z)^{1/2} \kappa_\alpha(w, w)^{1/2}}.$$

By (1.1) we have $b_\alpha(uz, uw) = b_\alpha(z, w)$ for all $u \in U$, so that B_α is a U-invariant operator on $L^2(V)$, that is, B_α commutes with $\pi(u)$ for all $u \in U$. Moreover B_α is selfadjoint and positive.

Let $L^2(V)^U$ be the closed subspace of $L^2(V)$ consisting of U-invariant functions. By U-invariance of B_α, it is clear that $L^2(V)^U$ is stable under B_α. The action of B_α on $L^2(V)^U$ is given by the following theorem.

Theorem 1.2. *Let $\phi_\alpha \in L^2(V)^U$ be the unit vector defined by*

$$\phi_\alpha(z) := \frac{1}{\pi^{n/2} d_\alpha^{1/2}} \kappa_\alpha(z, z)^{1/2} e^{-\|z\|^2/2},$$

where $d_\alpha := \dim \mathcal{P}_\alpha(V)$. Then B_α acts on $L^2(V)^U$ as the one-dimensional orthogonal projection $\phi_\alpha \otimes \phi_\alpha$.

To prove Theorem 1.2 we need

Lemma 1.3. *For $\alpha, \beta \in A$, one has*

$$\int_U \kappa_\alpha(uz, w) \overline{\kappa_\beta(uz, w)}\, du = \delta_{\alpha\beta} \cdot \frac{1}{d_\alpha} \kappa_\alpha(z, z) \kappa_\alpha(w, w),$$

where the left hand side is the integration over the compact Lie group U with re-

spect to the normalized Haar measure du.

Proof. This is a simple consequence of Schur's orthogonality relations. In fact it suffices to note that the reproducing property together with (1.1) yields

$$\kappa_\alpha(uz, w) = (\kappa_\alpha(\cdot, w) \mid \kappa_\alpha(\cdot, uz))_{\mathfrak{F}} = (\kappa_\alpha(\cdot, w) \mid \pi(u)\kappa_\alpha(\cdot, z))_{\mathfrak{F}}.$$

Then the equality $\|\kappa_\alpha(\cdot, w)\|_{\mathfrak{F}}^2 = \kappa_\alpha(w, w)$ immediately gives Lemma 1.3, because by the assumption $\mathscr{P}_\alpha(V)$ and $\mathscr{P}_\beta(V)$ carry inequivalent irreducible representations of U if $\alpha \neq \beta$.

Proof of Theorem 1.2. Let $f \in L^2(V)^U$. Then we have $B_\alpha f \in L^2(V)^U$, so that

$$\begin{aligned} B_\alpha f(z) &= \int_U B_\alpha f(uz)\, du = \int_U du \int_V b_\alpha(uz, w) f(w)\, dm(w) \\ &= \frac{1}{\pi^n} \frac{e^{-\|z\|^2/2}}{\kappa_\alpha(z, z)^{1/2}} \int_U du \int_V \frac{|\kappa_\alpha(uz, w)|^2}{\kappa_\alpha(w, w)^{1/2}} f(w) e^{-\|w\|^2/2}\, dm(w). \end{aligned}$$

Changing the order of integration and applying Lemma 1.3, we find that $B_\alpha f(z) = (f \mid \phi_\alpha)_2 \phi_\alpha(z)$, where $(\cdot|\cdot)_2$ denotes the inner product of $L^2(V)$. To see that $\|\phi_\alpha\|_2 = 1$, we recall that $\kappa_\alpha(z, w)$ is the reproducing kernel of $\mathscr{P}_\alpha(V)$. Thus $\kappa_\alpha(z, z) = \sum_{j=1}^{d_\alpha} |\varphi_j(z)|^2$ for any orthonormal basis $\{\varphi_j\}_{j=1}^{d_\alpha}$ of $\mathscr{P}_\alpha(V) \subset \mathfrak{F}$. Hence

$$\frac{1}{\pi^n} \int_V \kappa_\alpha(z, z) e^{-\|z\|^2}\, dm(z) = d_\alpha.$$

This clearly implies $\|\phi_\alpha\|_2^2 = 1$.

§2. Spectral decomposition: the case of $U(n)$-action on C^n

Throughout this section we treat the case $V = \boldsymbol{C}^n$ and $U = U(n)$ in detail and describe the spectral decomposition of the Berezin transformation. The canonical hermitian inner product on $\boldsymbol{C}^n$ will be denoted by $z \cdot \bar{w}$ instead of $(\cdot|\cdot)$. The natural action of $U(n)$ on $\boldsymbol{C}^n$ is known to be multiplicity-free. In fact denoting by $\mathscr{P}_k(\boldsymbol{C}^n)$ the space of homogeneous holomorphic polynomial functions on $\boldsymbol{C}^n$ of degree k, we have a decomposition $\mathscr{P}(\boldsymbol{C}^n) = \sum_{k=0}^{\infty} \mathscr{P}_k(\boldsymbol{C}^n)$ into mutually inequivalent $U(n)$-irreducibles and the corresponding orthogonal decomposition $\mathfrak{F} = \bigoplus_{k=0}^{\infty} \mathscr{P}_k(\boldsymbol{C}^n)$ for the Fock space $\mathfrak{F}$. The expansion $e^{z \cdot \bar{w}} = \sum_{k=0}^{\infty} \frac{(z \cdot \bar{w})^k}{k!}$ shows that the reproducing kernel $\kappa_k(z, w)$ of $\mathscr{P}_k(\boldsymbol{C}^n)$ is given by $\kappa_k(z, w) = \frac{(z \cdot \bar{w})^k}{k!}$. Thus the Berezin transformation B_k associated to $\mathscr{P}_k(\boldsymbol{C}^n)$ is the integral operator

$$B_k f(z)=\int_{C^n} b_k(z,w) f(w)\, dm(w)$$

on $L^2(\boldsymbol{C}^n)$ with kernel given by

$$b_k(z,w)=\frac{1}{\pi^n k!} e^{-\|z\|^2/2} e^{-\|w\|^2/2} \frac{|z\cdot\bar{w}|^{2k}}{\|z\|^k \|w\|^k}. \tag{2.1}$$

Through the polar coordinates $z=ru$ $(r>0,\ u\in S^{2n-1})$, we have $dm(ru)=r^{2n-1}dr\, d\sigma(u)$, where $d\sigma$ is the canonical rotation-invariant measure on the sphere S^{2n-1}. Hence

$$L^2(\boldsymbol{C}^n)=L^2((0,\infty),r^{2n-1}dr)\otimes L^2(S^{2n-1},d\sigma).$$

In order to study the operators B_k we need a decomposition of $L^2(S^{2n-1},d\sigma)$ into $U(n)$-irreducibles, which we now describe. Our reference is the books [16, Chapter 11] and [13, Kapitel V].

Let $\mathcal{P}_{pq}$ be the space of polynomial functions $h(z,\bar{z})$ on $\boldsymbol{C}^n$ which are homogeneous of degree p in z and degree q in $\bar{z}$. We denote by $\mathcal{H}_{pq}$ the harmonic polynomials in $\mathcal{P}_{pq}$. Then

$$\mathcal{P}_{pq}=\sum_{j=0}^{\min(p,q)} \|z\|^{2j}\cdot\mathcal{H}_{p-j,q-j}. \tag{2.2}$$

Moreover putting $\mathcal{Y}_{pq}:=\{h|_{S^{2n-1}};\, h\in\mathcal{H}_{pq}\}$, we have the following orthogonal decomposition into mutually inequivalent irreducible $U(n)$-modules $\mathcal{Y}_{pq}$:

$$L^2(S^{2n-1},d\sigma)=\bigoplus_{p,q=0}^{\infty}\mathcal{Y}_{pq}. \tag{2.3}$$

We have

$$\dim\mathcal{Y}_{pq}=\frac{(n+p+q-1)\,(n+p-2)!\,(n+q-2)!}{(n-1)!\,(n-2)!\,p!\,q!}. \tag{2.4}$$

We put $\mathfrak{H}_{pq}:=L^2((0,\infty),r^{2n-1}dr)\otimes\mathcal{Y}_{pq}$. Then we have $L^2(\boldsymbol{C}^n)=\bigoplus_{p,q=0}^{\infty}\mathfrak{H}_{pq}$ and every $\mathfrak{H}_{pq}$ is invariant under B_k.

Lemma 2.1. *Unless $p=q\leqq k$, the restriction of B_k to $\mathfrak{H}_{pq}$ is zero.*

Proof. Suppose that $f\in\mathfrak{H}_{pq}$ is of the form $f(ru)=f_0(r)Y(u)$ $(r>0,\ u\in S^{2n-1})$ with $Y\in\mathcal{Y}_{pq}$. Then

$$B_k f(sv)=\frac{s^k e^{-s^2/2}}{\pi^n k!}\int_0^\infty r^{2n+k-1} f_0(r) e^{-r^2/2}\,dr\int_{S^{2n-1}} Y(u)\,|v\cdot\bar{u}|^{2k}\,d\sigma(u),$$

where $s>0$ and $v\in S^{2n-1}$. Since the function $S^{2n-1}\ni u\mapsto|v\cdot\bar{u}|^{2k}$ belongs to $\bigoplus_{j=0}^{k}\mathcal{Y}_{jj}$ in view of (2.2), we get the lemma by (2.3).

Therefore we have only to consider the action of B_k on $\mathfrak{H}_{jj}$ for $0 \leqq j \leqq k$. The proof of Lemma 2.1 indicates that it suffices to decompose the function $|v \cdot \bar{u}|^{2k} = |u \cdot \bar{v}|^{2k}$. To do this we consider $\mathbf{e}_n := {}^t(0, \ldots, 0, 1) \in S^{2n-1}$ and denote by L the stabilizer in $U(n)$ at the vector $\mathbf{e}_n$. Then

$$L = \left(\begin{array}{c|c} U(n-1) & 0 \\ \hline 0 & 1 \end{array} \right).$$

Put $\eta(z) := |z \cdot \mathbf{e}_n|^{2k}$ $(z \in \boldsymbol{C}^n)$. Then η belongs to $\mathscr{P}_{kk}$ and is L-invariant: $\eta(lz) = \eta(z)$ for all $l \in L$. Decompose η as $\eta(z) = \sum_{j=0}^{k} \|z\|^{2(k-j)} \eta_j(z)$ according as (2.2). Then η_j belongs to $\mathscr{H}_{jj}$ and is L-invariant. We quote here the following proposition, see [16, 11.3.2] or [13, V.2.10].

Proposition 2.2. *Let $\mathscr{Y}_{jj}^L$ be the space of L-invariant functions in $\mathscr{Y}_{jj}$. Then* $\dim \mathscr{Y}_{jj}^L = 1$ *and $\mathscr{Y}_{jj}^L$ consists of the scalar multiples of the function $Y_j(u) := P_j^{(n-2,0)}(2|u \cdot e_n|^2 - 1)$, where $P_j^{(\alpha,\beta)}$ stands for the Jacobi polynomial of degree j defined through the Gauss' hypergeometric function ${}_2F_1$:*

$$P_j^{(\alpha,\beta)}(t) = \binom{\alpha+j}{j} \cdot {}_2F_1\left(-j, j+\alpha+\beta+1, \alpha+1 \,;\, \frac{1-t}{2}\right).$$

Since $\mathscr{Y}_{jj}$ is a finite-dimensional space consisting of continuous functions on S^{2n-1}, it possesses a reproducing kernel $\Phi_j(u, v)$. The $U(n)$-invariance of $\mathscr{Y}_{jj}$ implies

$$\Phi_j(gu, gv) = \Phi_j(u, v) \qquad \text{for all } g \in U(n). \tag{2.5}$$

In particular $\Phi_j(\cdot, \mathbf{e}_n) \in \mathscr{Y}_{jj}^L$, so that $\Phi_j(\cdot, \mathbf{e}_n)$ is a constant multiple of the function Y_j in Proposition 2.2. Now for every $v \in S^{2n-1}$ we take $g \in U(n)$ so that $g\mathbf{e}_n = v$. Then by (2.5)

$$\Phi_j(u, v) = \Phi_j(g^{-1}u, \mathbf{e}_n) = C_j \cdot Y_j(g^{-1}u) = C_j \cdot P_j^{(n-2,0)}(2|u \cdot \bar{v}|^2 - 1)$$

for some $C_j \in \boldsymbol{C}$. Though not necessary in the sequel, we compute the constant C_j for completeness.

Proposition 2.3. *The reproducing kernel $\Phi_j(u, v)$ of $\mathscr{Y}_{jj}$ is given by*

$$\Phi_j(u, v) = C_j \cdot P_j^{(n-2,0)}(2|u \cdot \bar{v}|^2 - 1),$$

where $C_j := \dfrac{(n+2j-1)(n+j-2)!}{2\pi^n j!}$. *Note that Φ_j is real-valued.*

Proof. Put $m = \dim \mathscr{Y}_{jj}$. We know by (2.4) that

$$m = \frac{(n+2j-1)[(n+j-2)!]^2}{(n-1)!(n-2)!(j!)^2}.$$

Since Φ_j is the reproducing kernel of $\mathcal{Y}_{jj}$, we have for any orthonormal basis $\{\psi_l\}_{l=1}^m$ of $\mathcal{Y}_{jj}$

$$\sum_{l=1}^{m} |\psi_l(v)|^2 = \Phi_j(v, v) = \Phi_j(\mathbf{e}_n, \mathbf{e}_n) \qquad \text{for all } v \in S^{2n-1},$$

the second equality being a consequence of (2.5). Hence

$$m = \int_{S^{2n-1}} \Phi_j(v, v)\, d\sigma(v) = \Phi_j(\mathbf{e}_n, \mathbf{e}_n)\, \sigma(S^{2n-1}) = C_j \cdot \binom{n-2+j}{j} \frac{2\pi^n}{(n-1)!},$$

which gives the proposition.

Combining Proposition 2.2 with Proposition 2.3, we see that $\mathcal{Y}_{jj}^L = \mathbf{C}\Phi_j(\cdot, \mathbf{e}_n)$. Therefore $\eta_j|_{S^{2n-1}} = a_j^k \cdot \Phi_j(\cdot, \mathbf{e}_n)$ for some $a_j^k \in \mathbf{C}$. For every $v \in S^{2n-1}$ we choose $g \in U(n)$ so that $v = g\mathbf{e}_n$. Then

$$\begin{aligned} |u \cdot \bar{v}|^{2k} &= |g^{-1}u \cdot \mathbf{e}_n|^{2k} = \eta(g^{-1}u) \\ &= \sum_{j=0}^{k} a_j^k \cdot \Phi_j(g^{-1}u, \mathbf{e}_n) = \sum_{j=0}^{k} a_j^k \cdot \Phi_j(u, v). \end{aligned} \tag{2.6}$$

To compute the constants a_j^k we need the following integral formula.

Lemma 2.4. *For $f \in L^1(S^{2n-1}, d\sigma)$ one has*

$$\int_{S^{2n-1}} f(u)\, d\sigma(u) = \int_0^{\pi/2} (\sin\theta)^{2n-3} \cos\theta\, d\theta \int_{-\pi}^{\pi} d\varphi \int_{S^{2n-3}} f((\sin\theta)w + (\cos\theta)e^{i\varphi}\mathbf{e}_n)\, d\sigma(w).$$

Proof. We give here a direct proof for readers' convenience. Consider the function $F(z) := e^{-\|z\|^2} f(z/\|z\|)$. Then

$$\begin{aligned} I &:= \int_{C^n} F(z)\, dm(z) = \int_{C^{n-1}} dm(w) \int_C F(w + t\mathbf{e}_n)\, dm(t) \\ &= \int_{C^{n-1}} dm(w) \int_C f\left(\frac{w + t\mathbf{e}_n}{\sqrt{\|w\|^2 + |t|^2}}\right) e^{-(\|w\|^2 + |t|^2)}\, dm(t). \end{aligned}$$

Putting $t = re^{i\varphi}$ and $w = \rho v$ ($\rho > 0$, $v \in S^{2n-3}$), we get

$$I = \int_0^\infty \rho^{2n-3}\, d\rho \int_{S^{2n-3}} d\sigma(v) \int_0^\infty e^{-(\rho^2 + r^2)}\, r dr \int_{-\pi}^{\pi} f\left(\frac{\rho v + re^{i\varphi}\mathbf{e}_n}{\sqrt{\rho^2 + r^2}}\right) d\varphi.$$

Finally setting $r = s\cos\theta$, $\rho = s\sin\theta$ ($0 \leqq \theta \leqq \pi/2$), we arrive at

$$\begin{aligned} I = &\int_0^\infty e^{-s^2} s^{2n-1}\, ds \int_0^{\pi/2} \cos\theta (\sin\theta)^{2n-3}\, d\theta \\ &\times \int_{S^{2n-3}} d\sigma(v) \int_{-\pi}^{\pi} f((\sin\theta)v + (\cos\theta)e^{i\varphi}\mathbf{e}_n)\, d\varphi. \end{aligned}$$

On the other hand, $I=\int_0^\infty e^{-r^2}r^{2n-1}\,dr\int_{S^{2n-1}} f(u)\,d\sigma(u)$. This together with the above computation yields the lemma.

Proposition 2.5. *Recall that* $\eta(u)=|u\cdot \mathbf{e}_n|^{2k}$. *Then*

$$\int_{S^{2n-1}}\eta(u)P_j^{(n-2,0)}(2|u\cdot \mathbf{e}_n|^2-1)\,d\sigma(u)=\frac{2\pi^n k!}{(n+k-1)!}\binom{n+j-2}{j}\cdot\lambda_j^k,$$

where $\lambda_j^k:=\binom{n+j+k-1}{j}^{-1}\binom{k}{j}$.

Proof. Let J be the integral on the left hand side. Applying Lemma 2.4, we get

$$J=2\pi\sigma(S^{2n-3})\int_0^{\pi/2}(\sin\theta)^{2n-3}(\cos\theta)^{2k+1}P_j^{(n-2,0)}(\cos 2\theta)\,d\theta.$$

The formula $P_j^{(\alpha,\beta)}(t)=2^{-j}\sum_{l=0}^{j}\binom{j+\alpha}{l}\binom{j+\beta}{j-l}(t+1)^l(t-1)^{j-l}$ [10, p. 211] gives

$$J=2\pi\sigma(S^{2n-3})\sum_{l=0}^{j}(-1)^{j-l}\binom{j+n-2}{l}\binom{j}{j-l}$$
$$\times\int_0^{\pi/2}(\cos\theta)^{2(l+k)+1}(\sin\theta)^{2(j-l+n-1)-1}\,d\theta.$$

Since $2\int_0^{\pi/2}(\cos\theta)^{2p-1}(\sin\theta)^{2q-1}\,d\theta=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$, we get

$$J=\pi\sigma(S^{2n-3})\sum_{l=0}^{j}(-1)^{j-l}\binom{j+n-2}{l}\binom{j}{j-l}\frac{\Gamma(l+k+1)\Gamma(j-l+n-1)}{\Gamma(n+j+k)}$$
$$=2\pi^n\binom{n+j-2}{j}\frac{(j!)^2}{(n+j+k-1)!}\sum_{l=0}^{j}(-1)^{j-l}\frac{(l+k)!}{(j-l)!(l!)^2},$$

where we have used $\sigma(S^{2n-3})=\frac{2\pi^{n-1}}{(n-2)!}$. Proposition 2.5 now follows from the next combinatorial identity.

Lemma 2.6. *One has*

$$\frac{(j!)^2}{(n+j+k-1)!}\sum_{l=0}^{j}(-1)^{j-l}\frac{(l+k)!}{(j-l)!(l!)^2}=\frac{k!}{(n+k-1)!}\cdot\lambda_j^k.$$

Proof. The left hand side is equal to

$$\frac{j!\,k!}{(n+j+k-1)!}\sum_{l=0}^{j}(-1)^{j-l}\binom{l+k}{l}\binom{j}{l}=$$

$$=\frac{(-1)^j k!}{(n+k-1)!}\binom{n+j+k-1}{j}^{-1}\sum_{l=0}^{j}\binom{-k-1}{l}\binom{j}{l},$$

where we used $\binom{l+k}{l}=(-1)^l\binom{-k-1}{l}$. The sum $S:=\sum_{l=0}^{j}\binom{-k-1}{l}\binom{j}{l}$ is the constant term of the Laurent expansion at $x=0$ of the function

$$f(x):=(1+x)^{-(k+1)}\left(1+\frac{1}{x}\right)^j.$$

Since $f(x)=x^{-j}(1+x)^{j-(k+1)}$, we see that S is the coefficient of x^j of the function $(1+x)^{j-k-1}$. Hence $S=\binom{j-k-1}{j}=(-1)^j\binom{k}{j}$. This clearly yields the lemma.

Proposition 2.7. *The constants a_j^k in* (2.6) *are given by*

$$a_j^k=\frac{2\pi^n k!}{(n+k-1)!}\cdot\lambda_j^k.$$

Proof. Recall that $\eta(u)=|u\cdot\mathbf{e}_n|^{2k}=\sum_{j=0}^{k}a_j^k\cdot\Phi_j(u,\mathbf{e}_n)$ for $u\in S^{2n-1}$. Therefore taking the inner product of both sides with $\Phi_j(\cdot,\mathbf{e}_n)$, we get

$$(2.7)\qquad \int_{S^{2n-1}}\eta(u)\,\Phi_j(u,\mathbf{e}_n)\,d\sigma(u)=a_j^k\cdot\Phi_j(\mathbf{e}_n,\mathbf{e}_n)=a_j^k\,C_j\cdot\binom{n-2+j}{j},$$

where C_j is the constant appearing in Proposition 2.3. Again by Proposition 2.3, we see that the left hand side of (2.7) equals C_j times of the integral in Proposition 2.5. These observations lead us to Proposition 2.7.

To describe the spectral decomposition of B_k we need some notational preparations. Let φ_k be the unit vector in $L^2((0,\infty),r^{2n-1}dr)$ given by

$$\varphi_k(r):=\sqrt{\frac{2}{(n+k-1)!}}\,r^k e^{-r^2/2},$$

and A_k the one-dimensional orthogonal projection of $L^2((0,\infty),r^{2n-1}dr)$ onto $\boldsymbol{C}\varphi_k$. We remark that

$$\sigma(S^{2n-1})^{-1/2}\varphi_k(\|z\|)=\frac{1}{\pi^{n/2}d_k^{1/2}}\,\kappa_k(z,z)^{1/2}e^{-\|z\|^2/2},$$

where $d_k:=\dim\mathscr{P}_k(\boldsymbol{C}^n)=\binom{n+k-1}{k}$. Compare this with the function ϕ_α in Theorem 1.2. We denote by E_j the orthogonal projection $L^2(S^{2n-1},d\sigma)\to\mathscr{Y}_{jj}$.

The operator E_j is an integral operator on $L^2(S^{2n-1}, d\sigma)$ with reproducing kernel $\Phi_j(u, v)$ of $\mathcal{Y}_{jj}$ as integral kernel.

Theorem 2.8. *One has the spectral decomposition*

$$B_k=\sum_{j=0}^{k}\binom{n+j+k-1}{j}^{-1}\binom{k}{j}\cdot(A_k\otimes E_j).$$

Proof. By (2.1), (2.6) and Proposition 2.7 we have for $r, s>0$ and $u, v\in S^{2n-1}$

$$b_k(sv, ru)=\frac{e^{-s^2/2}e^{-r^2/2}r^k s^k}{\pi^n k!}|v\cdot\bar{u}|^{2k}=\frac{e^{-s^2/2}e^{-r^2/2}r^k s^k}{\pi^n k!}\sum_{j=0}^{k}a_j^k\cdot\Phi_j(u, v)$$

$$=\varphi_k(r)\varphi_k(s)\sum_{j=0}^{k}\lambda_j^k\cdot\Phi_j(u, v).$$

This clearly gives Theorem 2.8 in view of the explicit formula for λ_j^k given in Proposition 2.5.

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