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On the zeros of the Epstein zeta functions

By

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§ 1. Introduction

The purpose of the present article is to extend our recent results [8] concerning the distribution of the zeros of the Epstein zeta functions. We have shown there that the "k-analogue" of "the GUE law" fails for the Epstein zeta functions $\zeta(s, Q)$. A more precise definition of "the GUE law" and also the meaning of the "k-analogue" will be given below. The Epstein zeta function $\zeta(s, Q)$ is defined by

$$\zeta(s, Q) = \frac{1}{2} \sum_{x,y}{}' Q(x, y)^{-s} \text{ for } \Re(s) > 1,$$

where x, y runs over all integers excluding $(x, y) = (0, 0)$, $s = \sigma + it$ with real numbers σ and t, $Q(x, y) = ax^2 + bxy + cy^2$ is a positive definite quadratic form with discriminant $\Delta = b^2 - 4ac$, a, b and c are real numbers and $a > 0$ and we put

$$k = \frac{\sqrt{|\Delta|}}{2a}.$$

Some of the well known results concerning $\zeta(s, Q)$ will be recalled below. In the present article, we are concerned with the distribution of the zeros of the Epstein zeta functions associated with the positive definite quadratic forms of more variables. However, we shall treat only the simpler cases among them, for simplicity. We shall also give some new results concerning the simplest $\zeta(s, Q)$. A further extension is possible and will appear elsewhere.

Let d be a positive number. Here we are mainly concerned with the Epstein zeta functions of the form

$$G_d(s) = \sum{}' \frac{1}{(m_1^2 + m_2^2 + d(m_3^2 + m_4^2))^s},$$

where $\Re(s) > 2$, the dash indicates that m_j's run over the integers excluding the case $(m_1, m_2, m_3, m_4) = (0, 0, 0, 0)$. We are particularly intersted in the distribution of the zeros of $G_d(s)$. We put for a convenience

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$$\kappa = \sqrt{d}.$$

We shall show below that for $G_d(s)$, the "κ-analogue" of the "Riemann Hypothesis" holds but the "κ-analogue" of "the GUE law" fails. These results are the extensions of both Stark's result [23] on the "κ-analogue" of the "Riemann Hypothesis" and the author's result [8] on the failure of the "κ-analogue" of "the GUE law" for $\zeta(s, Q)$. We shall describe these in a more precise form below.

We start with recalling some of the known results on $\zeta(s, Q)$. For a convenience, we put

$$Z(s, Q) = 2\zeta(s, Q).$$

Then $Z(s, Q)$ is known to have an analytic continuation to the whole complex plane with a simple pole at $s = 1$. The residue is known to be

$$\frac{2\pi}{\sqrt{|\Delta|}}.$$

The following expansion (cf. Theorem 1 in p.14 of Siegel [21]) at $s = 1$ is fundamental.

Kronecker's Limit Formula. *At $s = 1$, we have the following expansion.*

$$Z(s, Q) = \frac{\frac{2\pi}{\sqrt{|\Delta|}}}{s-1} + \frac{2\pi}{\sqrt{|\Delta|}}\left(2C_0 + \log\frac{a}{|\Delta|} - 2\log\left|\eta\left(\frac{b}{2a} + i\frac{\sqrt{|\Delta|}}{2a}\right)\right|^2\right) + A_1(s-1) + \cdots,$$

where C_0 is the Euler constant, we put

$$\eta(z) = e^{\frac{\pi i z}{12}} \prod_{m=1}^{\infty}(1 - e^{2\pi i m z})$$

for complex $z = x + iy$ with $y > 0$ and A_1 is some constant.

It is also well-known that $Z(s, Q)$ has the following functional equation.

Functional Equation.

$$\left(\frac{|\Delta|}{4}\right)^{\frac{s}{2}}\pi^{-s}\Gamma(s)Z(s, Q) = \left(\frac{|\Delta|}{4}\right)^{\frac{1-s}{2}}\pi^{-(1-s)}\Gamma(1-s)Z(1-s, Q),$$

$\Gamma(s)$ being the Γ-function.

Another fundamental formula is the following.

Chowla-Selberg's Formula.

$$Z(s, Q) = 2a^{-s}\zeta(2s) + 2a^{-s}k^{1-2s}\zeta(2s-1)\frac{\Gamma\left(s-\frac{1}{2}\right)\sqrt{\pi}}{\Gamma(s)}$$

$$+\frac{4a^{-s}\pi^s k^{\frac{1}{2}-s}}{\Gamma(s)}\sum_{n=1}^{\infty} n^{s-\frac{1}{2}}\sigma_{1-2s}(n)\cos\left(\frac{n\pi b}{a}\right)\int_0^{\infty} y^{s-\frac{3}{2}}e^{-\pi nk(y+y^{-1})}dy,$$

where $\zeta(s)$ *is the Riemann zeta function and we put*

$$\sigma_{1-2s}(n) = \sum_{d|n} d^{1-2s}.$$

This gives first the Kronecker's limit formula in the following form (cf. (39) of p.532 in vol. 1 of Selberg[20]).

$$Z(s, Q) = \frac{\frac{2\pi}{\sqrt{|\Delta|}}}{s-1} + \frac{2\pi}{\sqrt{|\Delta|}}\left(2C_0 + \log\frac{a}{|\Delta|}\right) + \frac{\pi^2}{3a}$$
$$+ \frac{8\pi}{\sqrt{|\Delta|}}\sum_{n=1}^{\infty}\sigma_{-1}(n)\cos\left(\frac{n\pi b}{a}\right)e^{-\frac{\pi n\sqrt{|\Delta|}}{a}} + A_1(s-1)+\cdots.$$

This implies immediately Kronecker's limit formula as stated above.
For the simplest case, $Q(x, y) = x^2 + y^2$, we have

$$Z(s) \equiv \sum{}'(m_1^2 + m_2^2)^{-s} = \sum_{n=1}^{\infty}\frac{r(n)}{n^s} = 4\zeta(s)L(s, \chi),$$

where χ is the non-principal Dirichlet character mod 4, $L(s, \chi)$ is the corresponding Dirichlet L-function and $r(n)$ for $n \geq 0$ is defined by

$$r(n) = \sum_{\substack{n=m_1^2+m_2^2 \\ -\infty<m_1, m_2<\infty}} 1.$$

Using the properties of $\zeta(s)$ and $L(s,\chi)$, we get the following identities.

$$4(C_0L(1, \chi) + L'(1, \chi)) = \pi\left(2C_0 + \log\frac{1}{4}\right) + \frac{\pi^2}{3} + 4\pi\sum_{n=1}^{\infty}\sigma_{-1}(n)e^{-2\pi n}$$
$$= 2\pi(C_0 - \log 2 - \log|\eta(i)|^2).$$

where $L'(1, \chi)$ is the value of the derivative of $L(s, \chi)$ at $s = 1$. Using the following well known result

$$L(1, \chi) = \frac{\pi}{4},$$

we get

$$L'(1, \chi) = -\sum_{n=1}^{\infty}\frac{\chi(n)\log n}{n} = \sum_{m=1}^{\infty}\left(\frac{\log(4m-1)}{4m-1} - \frac{\log(4m+1)}{4m+1}\right)$$

$$= \frac{1}{8}\log^2 5 - \frac{1}{8}\log^2 3 + \frac{1}{2}\left(\frac{\log 3}{3} - \frac{\log 5}{5}\right)$$
$$+ 4\int_1^{\infty}\left(\{y\} - \frac{1}{2}\right)\left(\frac{\log(4y+1)}{(4y+1)^2} - \frac{\log(4y-1)}{(4y-1)^2}\right)dy$$
$$= \frac{\pi}{4}(C_0 - 2\log 2 - 4\log|\eta(i)|).$$

The last expression is very effective when one wants to have a numerical value of $L'(1, \chi)$ as is already mentioned in p.75 of Siegel [21] and also will be used below.

Chowla-Selberg's formula gives next a very good approximation not only at $s=1$, but also on the real line. In particular, it implies that

$$Z\left(\frac{1}{2}, Q\right) = \frac{1}{\sqrt{a}}\left(2C_0 + \log\left(\frac{|\Delta|}{64\pi^2 a^2}\right)\right)$$
$$+ \frac{4}{\sqrt{a}}\sum_{n=1}^{\infty}\sigma_0(n)\cos\left(\frac{n\pi b}{a}\right)\int_0^{\infty} y^{-1}e^{-\frac{\pi n\sqrt{|\Delta|}}{2a}(y+y^{-1})}dy$$

(cf. (29) of p. 529 in vol. 1. of Selberg [20]). Thus for $\Delta = -43$ or -163, Chowla and Selberg have obtained

$$Z\left(\frac{1}{2}, Q\right) < 0.$$

Using this kind of expression, Bateman-Grosswald [2] has also shown that

$$Z\left(\frac{1}{2}, Q\right) > 0 \text{ if } k \geq 7.0556$$

and

$$Z\left(\frac{1}{2}, Q\right) < 0 \text{ if } \frac{\sqrt{3}}{2} \leq k \leq 7.0554.$$

Thus they have shown that $Z(s, Q)$ has a real zero between $\frac{1}{2}$ and 1 if $k >$ 7.0556. In fact, the above expression on $Z\left(\frac{1}{2}, Q\right)$ gives further the following result which supplements the above results of Bateman-Grosswald [2] when $b = 0$. We shall prove the following slighty more general result.

Corollary 1. *Suppose that $Q(x, y) = ax^2 + bxy + cy^2$ is a positive definite quadratic form, $a > 0$ and $\frac{b}{a}$ is an even integer. Let $k = \frac{\sqrt{|b^2 - 4ac|}}{2a}$ be as above. Then there exist two positive numbers k_1 and k_2 which satisfy $0 < k_1 < k_2$ and the following three properties.*

(i) $Z\left(\frac{1}{2}, Q\right) = 0$ *when $k = k_1$ or $k = k_2$.*

Moreover $s = \frac{1}{2}$ *is a double zero of* $Z(s, Q)$ *for both cases.*

(ii) $Z\left(\frac{1}{2}, Q\right) > 0$ *when* $0 < k < k_1$ *or* $k > k_2$.

Hence in this case, $Z(s, Q)$ *has one real zero in* $0 < \Re(s) < \frac{1}{2}$ *and one real zero in* $\frac{1}{2} < \Re(s) < 1$ *and* $Z(s, Q) \neq 0$ *at* $s = 0, \frac{1}{2}$ *and* 1.

(iii) $Z\left(\frac{1}{2}, Q\right) < 0$ *when* $k_1 < k < k_2$.

This result is certainly more precise and more comprehensive than Bateman and Grosswald's result mentioned above, although there is some restriction to b. Our proof suggests that the last restriction might be relaxed further.

A rough numerical calculation shows that for $b = 0$, we can take the above k_1 and k_2 as follows.

$$k_1 = 0.1417332\cdots$$

and

$$k_2 = 7.055507955448\cdots.$$

To prove that $\frac{1}{2}$ is a double zero in (i) of Corollary 1, we shall use the following result which holds for a general Q.

Corollary 2.

(i)
$$Z'\left(\frac{1}{2}, Q\right) = 2a^{-\frac{1}{2}}\{C_0^2 - 4\log^2 2 - 4\log 2 \cdot \log\frac{\pi}{k} - \left(\log\frac{\pi}{k}\right)^2 - \log a \cdot \left(C_0 - \log\frac{\pi}{k} - 2\log 2\right) + 2\left(\log\frac{\pi}{ak} + C_0 + 2\log 2\right) \cdot \sum_{n=1}^{\infty} \sigma_0(n) \cos\left(\frac{n\pi b}{a}\right) \int_0^{\infty} y^{-1} e^{-\pi n k (y+y^{-1})} dy\}.$$

(ii)
$$Z''\left(\frac{1}{2}, Q\right) = \frac{2}{\sqrt{a}}\Big\{-\frac{1}{2}\log^2 a \cdot \sqrt{a}Z\left(\frac{1}{2}, Q\right) - \log a \cdot \sqrt{a}Z'\left(\frac{1}{2}, Q\right) - \frac{3}{4}\left(\log\frac{\pi}{k}\right)^3 - 8\log 2 \cdot \left(\log\frac{\pi}{k}\right)^2 + (4C_0^2 - 16\log^2 2 - 8C_1)\log\frac{\pi}{k} - \frac{14}{3}\zeta(3) + \frac{3}{8}C_0^3 + 4C_0^2\log 4 - \frac{4}{3}\log^3 4 + 16C_2 - 8C_1C_0 - 8C_1\log 4 + D\left(\frac{1}{2}, Q\right)\Big\{-\pi^2 + 2(C_0 + 2\log 2)^2 + 4(C_0 + 2\log 2)\log\frac{\pi}{k} + 2\left(\log\frac{\pi}{k}\right)^2\Big\} + 2D''\left(\frac{1}{2}, Q\right)\Big\},$$

where we write the Laurent expansion of $\zeta(s)$ at $s=1$ in the following form

$$\zeta(s)=\frac{1}{s-1}+C_0+C_1(s-1)+C_2(s-1)^2+\cdots$$

and we put

$$D(s,Q)=\sum_{n=1}^{\infty} n^{s-\frac{1}{2}}\sigma_{1-2s}(n)\cos\left(\frac{n\pi b}{a}\right)\int_0^{\infty} y^{s-\frac{3}{2}}e^{-\pi nk(y+y^{-1})}dy.$$

Moreover, when k is sufficiently large, then we can locate the real zeros found in (ii) of Corollary 1 more precisely as follows.

Corollary 3. *Suppose that $Q(x,y)=ax^2+bxy+cy^2$ is a positive definite quadratic form with $a>0$. Let $k=\frac{\sqrt{|b^2-4ac|}}{2a}$ be sufficiently large. Then $Z(s,Q)$ has two real zeros ρ_k and $1-\rho_k$, where*

$$\rho_k=\frac{3}{\pi k}\left(1+\frac{\log k}{k}\frac{6}{k}+O\left(\frac{1}{k}\right)\right) \textit{ as } k\to\infty.$$

Chowla-Selberg's formula plays also an important role when one investigates the distribution of the complex zeros of $Z(s,Q)$.

In fact, Stark [23] has shown that

for $k>K$, all the zeros of $\zeta(s,Q)$ in the region $-1<\sigma<2$, $-2k\le t\le 2k$ are simple zeros; with the exception of two real zeros between 0 and 1, all are on the line $\sigma=\frac{1}{2}$ and that for $0<T\le 2k$,

$$N(T,Q)=\frac{T}{\pi}\log\left(\frac{kT}{\pi e}\right)+O(\log\log T),$$

where $N(T,Q)$ denotes the number of the zeros of $\zeta(s,Q)$ in the region $-1<\sigma<2$, $0\le t\le T$.

This is the "k-analogue" of "the Riemann Hypothesis" noticed above. (We have replaced Stark's remainder term $O(\log^{\frac{1}{3}}(T+3)(\log\log(T+3))^{\frac{1}{6}})$ in $N(T,Q)$ by $O(\log\log T)$. (cf. Remark in p.145 of Fujii [8].))

This seems to be a surprising result, because it seems to provide us a key to understand the following two opposite types of results. On one hand, in certain cases the Epstein zeta functions $Z(s,Q)$ have even infinitely many zeros in $\Re(s)>1$ (cf. Davenport and Heilbronn [4]). This is because they do not have Euler product in general. On the other hand, they have infinitely many zeros on the critical line $\Re(s)=\frac{1}{2}$ (cf. Potter-Titchmarsh [18] and Kober [14]) and even strongly under certain hypothesis, almost all the zeros of $Z(s,Q)$ lie on the the critical line $\Re(s)=\frac{1}{2}$, although they have not Euler product in general (cf. Bombieri and Hejhal [3] and Hejhal [11]).

To understand the situation more clearly, we have shown recently that "k-analogue" of "the GUE law" fails for $Z(s, Q)$. This should be distinguished completely from the zeta functions like $\zeta(s)$, as we have seen in the previous Fujii [6] [7] (cf. also Ozluk [17]).

To be more precise, here we shall define the notion "the GUE law". For $\zeta(s)$, we know (cf. p.212 of Titchmarsh [25]) that the number $N(T)$ of the zeros $\beta+i\gamma$ of $\zeta(s)$ in $0 < \gamma < T$, $0 < \beta < 1$ is given by

$$N(T) = \frac{1}{\pi}\vartheta(T) + 1 + S(T),$$

where we suppose that $T \neq \gamma$ for any γ, $\vartheta(T)$ is the continuos function defined by

$$\vartheta(T) = \Im\left(\log\Gamma\left(\frac{1}{4} + \frac{iT}{2}\right)\right) - \frac{1}{2}T\log\pi$$

with

$$\vartheta(0) = 0$$

and

$$S(T) = \frac{1}{\pi}\arg\zeta\left(\frac{1}{2} + iT\right).$$

It is well known that

$$\vartheta(T) = \frac{T}{2}\log\frac{T}{2\pi} - \frac{T}{2} - \frac{\pi}{8} + \frac{1}{48T} + \frac{7}{5760T^3} + \cdots$$

When $T = \gamma$, then we put

$$N(T) = \frac{1}{2}(N(T+0) + N(T-0)).$$

We have shown in [6] under the Riemann Hypothesis and the Montgomery's Conjecture that for $T > T_0$ and for $0 < \alpha = o(\log T)$,

$$\frac{1}{T}\int_0^T\left(S\left(t + \frac{2\pi\alpha}{\log\frac{T}{2\pi}}\right) - S(t)\right)^2 dt = \frac{1}{\pi^2}\{\log(2\pi\alpha) - \mathrm{Ci}(2\pi\alpha)$$
$$- 2\pi\alpha \cdot \mathrm{Si}(2\pi\alpha) + \pi^2\alpha - \cos(2\pi\alpha) + 1 + C_0 + o(1)\}.$$

We call the right hand side the GUE part and the set of α for which the above asymptotic law holds, namely, $\{\alpha; 0 < \alpha = o(\log T)\}$, in this case, the universal range of α. The left hand side is, essentially, the number variance

$$\frac{1}{T}\int_0^T\left(N\left(t + \frac{2\pi\alpha}{\log\frac{T}{2\pi}}\right) - N(t) - \alpha\right)^2 dt.$$

We say that $\zeta(s)$ obeys "the GUE law" if the number variance

$$\frac{1}{T}\int_0^T\left(S\left(G\left(x+\frac{\alpha}{2}\right)\right)-S\left(G\left(x-\frac{\alpha}{2}\right)\right)\right)^2dx$$

or

$$\frac{1}{T}\int_0^T\left(S\left(t+\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\right)-S(t)\right)^2dt$$

is

$$\sim\frac{1}{\pi^2}\{\log(2\pi\alpha)-\mathrm{Ci}(2\pi\alpha)-2\pi\alpha\cdot\mathrm{Si}(2\pi\alpha)+\pi^2\alpha-\cos(2\pi\alpha)+1+C_0\}$$

as $T\to\infty$ in the universal range of α, which includes at least the range where $\alpha\to\infty$ as $T\to\infty$, where $G(x)$ denotes the inverse function of $\frac{1}{\pi}\vartheta(t)$ for $t>t_0$. This notion can be defined generally, in particular, for the Epstein zeta functions, although $G(x)$ or $\frac{2\pi}{\log\frac{T}{2\pi}}$ must be modified (cf. Fujii [7] [8]), as will be seen below.

Now we turn to the Epstein zeta functions and recall our results on the zeros of $Z(s,Q)$. We notice first that for $0<T\leq 2k$,

$$N(T,Q)=L_Q(T)+\Delta_Q(T),$$

where

$$L_Q(T)=\frac{1}{\pi}\arg\left(\frac{k}{\pi}\right)^{\frac{1}{2}+iT}+\frac{1}{\pi}\arg\Gamma\left(\frac{1}{2}+iT\right)+\frac{1}{\pi}\arg\zeta(1+i2T)$$

and

$$|\Delta_Q(T)|\leq C,$$

C belng always some positive constant in this article. Since

$$\frac{1}{\pi}\arg\left(\frac{k}{\pi}\right)^{\frac{1}{2}+iT}+\frac{1}{\pi}\arg\Gamma\left(\frac{1}{2}+iT\right)=\frac{T}{\pi}\log\left(\frac{kT}{e\pi}\right)+O(1),$$

the number variance with which we are concerned is

$$\frac{1}{T}\int_T^{2T-2}\left(S_Q\left(t+\frac{\alpha\pi}{\log\frac{kT}{\pi}}\right)-S_Q(t)\right)^2dt,$$

where we put

$$S_Q(t)=\frac{1}{\pi}\arg\zeta(1+i2t)+\Delta_Q(t).$$

If it obeys"the GUE law", then it must be that

$$\frac{1}{T}\int_T^{2T-2}\left(S_Q\left(t+\frac{\alpha\pi}{\log\frac{kT}{\pi}}\right)-S_Q(t)\right)^2 dt \sim C\log\alpha \text{ as } \alpha\to\infty$$

with some positive constant C. Contrary to this, we have shown in [8] that *for $k>K$ and $0<T\le k$, there exists some positive constant C such that*

$$\frac{1}{T}\int_T^{2T-2}\left(S_Q\left(t+\frac{\alpha\pi}{\log\frac{kT}{\pi}}\right)-S_Q(t)\right)^2 dt \le C$$

uniformly for positive $\alpha \ll T\log\frac{kT}{\pi}$.

We notice that in p.141 of [8], we have stated that this holds uniformly for positive $\alpha\le\frac{1}{\pi}\log\frac{kT}{\pi}$. However, as the proof shows that the last condition can be relaxed as above.

Consequently, we see that as $k\to\infty$

$$\frac{1}{k}\int_k^{2k-2}\left(S_Q\left(t+\frac{\alpha\pi}{\log\frac{k^2}{\pi}}\right)-S_Q(t)\right)^2 dt \le C$$

uniformly for positive $\alpha \ll k\log\frac{k^2}{\pi}$. This is the failure of the "k-analogue" of "the GUE law" for the Epstein zeta functions $\zeta(s,Q)$.

As an intermediate between $\zeta(s,Q)$ and $G_d(s)$, we should mention the study of Chowla and Selberg (cf. pp. 532-534 in vol. 1 of [20]) on the Epstein zeta function of the following form

$$H_d(s)=\sum{}'(m_1^2+m_2^2+dm_3^2)^{-s},$$

where $\Re(s)>\frac{3}{2}$ and the dash indicates that m_j' s run over the integers excluding the case $(m_1,m_2,m_3)=(0,0,0)$. They have proved first that

$$H_d(s)=Z(s)+\frac{2\pi}{(s-1)}\frac{\zeta(2s-2)}{d^{s-1}}$$
$$+\frac{2\pi^s}{\Gamma(s)}d^{\frac{1-s}{2}}\sum_{n=1}^{\infty}n^{\frac{s-1}{2}}\sum_{u^2|n}\frac{r\left(\frac{n}{u^2}\right)}{u^{2s-2}}\int_0^\infty e^{-\pi\sqrt{nd}(y+y^{-1})}y^{s-2}dy,$$

where $Z(s)$ is introduced above. This expression gives an analytic continuation of $H_d(s)$ to the whole complex plane with a simple pole at $s=\frac{3}{2}$ with the residue $\frac{2\pi}{\sqrt{d}}$. Further they have given a Kronecker's limit formula as follows.

$$\lim_{s\to\frac{3}{2}}\left(H_d(s) - \frac{\frac{2\pi}{\sqrt{d}}}{s-\frac{3}{2}}\right)$$

$$= \frac{2\pi}{\sqrt{d}}(2C_0 - 2 - \log d) + Z\left(\frac{3}{2}\right) + \frac{4\pi}{\sqrt{d}}\sum_{n=1}^{\infty}\left(\sum_{u^2|n}\frac{1}{u}r\left(\frac{n}{u^2}\right)\right)e^{-2\pi\sqrt{nd}}.$$

From the above expression they have proved also, using the fact that $\zeta(-2) = 0$ and the functional equation of $\zeta(s)$, that there is a real number ρ_d such that for $d > d_0$

$$H_d(\rho_d) = 0,$$

where $\rho_d \to 0$ as $d \to \infty$ but $\rho_d \neq 0$.

We now proceed to our Epstein zeta function $G_d(s)$. Sometimes it is better understood if one generalize the framework. Here we shall generalize $G_d(s)$ slightly as follows. Let $Q_1(x, y) = a_1x^2 + b_1xy + c_1y^2$ and $Q_2(x, y) = a_2x^2 + b_2xy + c_2y^2$ be positive definite quadratic forms with the discriminants Δ_1 and Δ_2, respectively. For any $d > 0$, let $G_d(s, Q_1, Q_2)$ be defined by

$$G_d(s, Q_1, Q_2) = \sum{}' \frac{1}{(Q_1(m_1, m_2) + dQ_2(m_3, m_4))^s},$$

where $\Re(s) > 2$ and the dash indicates that m_j's run over the integers excluding the case $(m_1, m_2, m_3, m_4) = (0, 0, 0, 0)$. Whenever we shall treat $G_d(s, Q_1, Q_2)$, we always suppose further, to avoid complications, that a_i, b_i and c_i in the definition of $Q_i(x, y)$ are integers for $i = 1$ and 2. We denote $G_d(s, Q, Q)$ by $G_d(s, Q)$. We shall show first that the analogue of Chowla-Selberg's formula described above holds in a symmetric form as follows.

Theorem 1. *When $\Re(s) > 2$, we have*

$$G_d(s, Q_1, Q_2) = Z(s, Q_1) + \frac{2}{\sqrt{|\Delta_1|}}\frac{\pi}{d^{s-1}}\frac{\Gamma(s-1)}{\Gamma(s)}Z(s-1, Q_2)$$
$$+ \left(\frac{2\pi}{\sqrt{d|\Delta_1|}}\right)^s\frac{2\sqrt{d}}{\Gamma(s)}E(s, d, Q_1, Q_2, \Delta_1),$$

where

$$E(s) \equiv E(s, d, Q_1, Q_2, \Delta_1) = \sum_{n=1}^{\infty}n^{\frac{s-1}{2}}\left(\sum_{m|n}\frac{r_{Q_2}(m)\,r_{Q_1}\left(\frac{n}{m}\right)}{m^{s-1}}\right)K_{s-1}\left(\frac{4\pi}{\sqrt{|\Delta_1|}}\sqrt{dn}\right)$$

with

$$K_\nu(z) = \frac{1}{2}\int_0^\infty e^{-z\frac{u+u^{-1}}{2}} u^{\nu-1} du$$

for arbitrary ν *and* $|\arg z| < \frac{\pi}{2}$ *and* $r_Q(m) = \sum_{\substack{m=Q(x,y)\\ -\infty<x,y<\infty}} 1$. *Moreover* $E(s, d, Q_1, Q_2, \Delta_1)$ *is an entire function of s and satisfies the functional equation*

$$E(s, d, Q_1, Q_2, \Delta_1) = E(2-s, d, Q_2, Q_1, \Delta_1).$$

This gives an analytic continuation of $G_d(s, Q_1, Q_2)$ to the whole complex plane with a simple pole at $s=2$ with the residue

$$\frac{4\pi^2}{d\sqrt{|\Delta_1||\Delta_2|}}.$$

We denote $E(s, d, Q, Q, \Delta)$ by $E(s, d, Q)$ below.

Remark 1. *As we shall see in the proof of Theorem* 1, *we can express* $G_d(s, Q_1, Q_2)$ *in another way as follows.*

$$G_d(s, Q_1, Q_2) = d^{-s}Z(s, Q_2) + \frac{2}{\sqrt{|\Delta_2|}}\frac{\pi}{d}\frac{\Gamma(s-1)}{\Gamma(s)}Z(s-1, Q_1) + \left(\frac{2\pi}{\sqrt{|\Delta_2|}}\right)^s \frac{2d^{-\frac{s}{2}-\frac{1}{2}}}{\Gamma(s)} E\left(s, \frac{1}{d}, Q_2, Q_1, \Delta_2\right).$$

We can also derive from Theorem 1 a Kronecker's limit formula for $G_d(s, Q_1, Q_2)$ in the following form.

Corollary 4.

$$\lim_{s\to 2}\left(G_d(s, Q_1, Q_2) - \frac{\frac{4\pi^2}{d\sqrt{|\Delta_1||\Delta_2|}}}{s-2}\right) = Z(2, Q_1)$$
$$+ \frac{4\pi^2}{d\sqrt{|\Delta_1||\Delta_2|}}\left\{2C_0 + \log\frac{a_2}{|\Delta_2|} - 2\log\left|\eta\left(\frac{b_2}{2a_2} + i\frac{\sqrt{|\Delta_2|}}{2a_2}\right)\right|^2 - \log d - 1\right\}$$
$$+ \frac{8\pi^2}{\sqrt{d}|\Delta_1|}\sum_{n=1}^\infty \sqrt{n}\left(\sum_{m|n}\frac{r_{Q_2}(m)\, r_{Q_1}\left(\frac{n}{m}\right)}{m}\right)K_1\left(\frac{4\pi}{\sqrt{|\Delta_1|}}\sqrt{dn}\right).$$

Remark 2. *Remark* 1 *gives also a Kronecker*'s *limit formula for* $G_d(s, Q_1, Q_2)$ *in the following form.*

$$\lim_{s\to 2}\left(G_d(s, Q_1, Q_2) - \frac{\frac{4\pi^2}{d\sqrt{|\Delta_1||\Delta_2|}}}{s-2}\right) = d^{-2}Z(2, Q_2)$$
$$+ \frac{4\pi^2}{d\sqrt{|\Delta_1||\Delta_2|}}\left\{2C_0 + \log\frac{a_1}{|\Delta_1|} - 2\log\left|\eta\left(\frac{b_1}{2a_1} + i\frac{\sqrt{|\Delta_1|}}{2a_1}\right)\right|^2 - 1\right\}$$

$$+\frac{8\pi^2 d^{-\frac{3}{2}}}{|\Delta_2|}\sum_{n=1}^{\infty}\sqrt{n}\left(\sum_{m|n}\frac{r_{Q_1}(m)r_{Q_2}\left(\frac{n}{m}\right)}{m}\right)K_1\left(\frac{4\pi}{\sqrt{|\Delta_2|}}\sqrt{\frac{n}{d}}\right).$$

We should compare Corollary 4 with Remark 2. We shall state one consequence in (i) of Corollary 11 below.

Hereafter, we are mainly concerned with $G_d(s)$, for simplicity. However we shall mention briefly how one can extend to a more general case.

Theroem 1 implies also the following result on the trivial zero free region of $G_d(s)$, the trivial zeros of $G_d(s, Q)$ and the value of $G_d(s, Q)$ at $s = 0$.

Corollary 5.

(i) *If $d > 8.6$, then for any $\sigma \geq \frac{5}{2}$, and for any t,*

$$G_d(\sigma + it) \neq 0.$$

(ii) *For any $d > 0$ and for any positive definite integral quadratic form $Q(x, y)$,*

$$G_d(s, Q) = 0 \text{ at } s = -1, -2, -3, \cdots.$$

(iii) *For any $d > 0$ and for any positive definite integral quadratic form $Q(x, y)$*

$$G_d(0, Q) = Z(0, Q) = -1.$$

It is noteworthy that the value of $G_d(s, Q)$ or $Z(s, Q)$ at $s = 0$ does not depend on d or Q.

Naturally, we do not claim that the constant 8.6 in the above corollary is best possible. In fact, it comes from a rough computation of $d_0(\sigma)$ which will be discussed in the section 3. When the class number $h(-\Delta)$ of the positive definite quadratic forms with the discriminant $-\Delta$ is 1, then the proof of (i) of the above Corollary 5 can be modified to get similar results for $G_d(s, Q)$.

As another consequence of Theorem 1, we have the following corollary concerning the less trivial real zeros of $G_d(s, Q)$.

Corollary 6. *For a sufficiently large d and for any positive definite integral quadratic form $Q(x, y)$, there is a real number ρ_d such that*

$$G_d(\rho_d, Q) = 0,$$

where $\rho_d \to 0$ as $d \to \infty$ but $\rho_d \neq 0$.

This is an extension of Chowla-Selberg's result on $H_d(s)$ mentioned above. This will be refined in a more precise form in Theorem 3 below.

It is more convenient to write the formula in Theorem 1 for the case $Q_1 = Q_2 = Q$ in a more symmetric form. Multiplying both sides of that formula by $b(s) \equiv \left(\frac{2\pi}{\sqrt{d|\Delta|}}\right)^{-s}\Gamma(s)$, we get

$$G_d(s,Q)\left(\frac{2\pi}{\sqrt{d|\Delta|}}\right)^{-s}\Gamma(s) = Z(s,Q)\left(\frac{2\pi}{\sqrt{d|\Delta|}}\right)^{-s}\Gamma(s)$$
$$+\frac{2}{\sqrt{|\Delta|}}\frac{\pi}{d^{s-1}}\left(\frac{2\pi}{\sqrt{d|\Delta|}}\right)^{-s}\Gamma(s-1)Z(s-1,Q) + 2\sqrt{d}\,E(s,d,Q)$$
$$= Z(s,Q)\left(\frac{2\pi}{\sqrt{d|\Delta|}}\right)^{-s}\Gamma(s) + Z(2-s,Q)\left(\frac{2\pi}{\sqrt{d|\Delta|}}\right)^{s-2}\Gamma(2-s)$$
$$+ 2\sqrt{d}E(s,d,Q)$$
$$= f(s) + f(2-s) + g(s), \text{ say},$$

where we have used the functional equation of $Z(s,Q)$ mentioned above. Thus we have the following functional equation on $G_d(s,Q)$.

Corollary 7. *When we put*

$$\alpha(s) = G_d(s,Q)\left(\frac{2\pi}{\sqrt{d|\Delta|}}\right)^{-s}\Gamma(s),$$

then we have

$$\alpha(s) = \alpha(2-s).$$

The critical line of $G_d(s,Q)$ is $\Re s = 1$. At the critical point $s = 1$, we have the following consequence.

Corollary 8. *For any $d > 0$ and for any positive definite integral quadratic form $Q(x,y)$, we have*

$$G_d(1,Q) = \frac{2\pi}{\sqrt{|\Delta|}}\left(2C_0 + \log\frac{da^2}{4\pi^2|\Delta|} - 4\log\left|\eta\left(\frac{b}{2a} + i\frac{\sqrt{|\Delta|}}{2a}\right)\right|^2\right)$$
$$+\frac{4\pi}{\sqrt{|\Delta|}}E(1,d,Q),$$

where

$$E(1,d,Q) = \sum_{n=1}^{\infty}\left(\sum_{m|n} r_Q(m)\, r_Q\left(\frac{n}{m}\right)\right)K_0\left(\frac{4\pi}{\sqrt{|\Delta|}}\sqrt{dn}\right).$$

This corresponds to an explicit evaluation of $Z\left(\frac{1}{2},Q\right)$ described above. If one uses the remark after Chowla-Selberg's formula, then one gets also

$$G_d(1,Q) = \frac{4\pi}{\sqrt{|\Delta|}}\left(2C_0 + \log\frac{a}{|\Delta|}\right) + \frac{2\pi^2}{3a}$$
$$+\frac{16\pi}{\sqrt{|\Delta|}}\sum_{n=1}^{\infty}\sigma_{-1}(n)\cos\left(\frac{n\pi b}{a}\right)e^{-\frac{n\pi\sqrt{|\Delta|}}{a}} + \frac{4\pi}{\sqrt{|\Delta|}}E(1,d,Q).$$

We may mention a special case for $G_d(s)$ as follows.

Corollary 8'. *For any $d > 0$, we have*

$$G_d(1) = -\pi\log\frac{\pi^2}{d} + 8L'(1,\chi) + 2\pi E(1,d)$$
$$= -\pi\log\frac{\pi^2}{d} + 2\pi(C_0 - 2\log 2 - 4\log|\eta(i)|) + 2\pi E(1,d)$$

with

$$E(1,d) = \sum_{n=1}^{\infty}\Big(\sum_{m|n} r(n)r\Big(\frac{n}{m}\Big)\Big)K_0(2\pi\sqrt{dn}).$$

The expression in Corollary 8' provides us various information concerning a real zero of $G_d(s)$. We may summarize them in the following theorem.

Theorem 2. *There exist two numbers D_1 and D_2 such that $\frac{1}{16\pi^2} < D_1 < D_2$ and they satisfy the following three properties.*

(i) $G_d(1) = 0$ *when* $d = D_1$ *or* $d = D_2$.

Moreover $s = 1$ is a double zero of $G_d(s)$ for both cases.

(ii) $G_d(1) > 0$ *when* $0 < d < D_1$ *or* $d > D_2$.

Hence in this case, $G_d(s)$ has one real zero in $0 < \Re(s) < 1$, one real zero in $1 < \Re(s) < 2$, and $G_d(s) \neq 0$ for $s = 0, 1,$ and 2.

(iii) $G_d(1) < 0$ *when* $D_1 < d < D_2$.

We shall see below, by our rough computations, that

$$D_1 = 0.156\cdots$$

and

$$D_2 = 6.039\cdots.$$

To prove (i) of Theorem 2, we shall use the following expression of $G_d'(1)$ and $G_d''(1)$.

Corollary 9.

(i) $$G_d'(1) = 8C_0L'(1,\chi) + 4\log\frac{\pi^2}{d}\Big(-C_0\frac{\pi}{4} + L'(1,\chi)\Big) - \Big(\log\frac{\pi^2}{d}\Big)^2\frac{\pi}{2} + 2\pi E(1,d)\Big(\frac{1}{2}\log\frac{\pi^2}{d} + C_0\Big).$$

(ii) $$G_d''(1) = 4C_2\pi + 16C_1L'(1,\chi) + \frac{8L'''(1,\chi)}{3} - 4C_0C_1\pi + \frac{2\pi\Gamma'''(1)}{3} + 2C_0\pi\Gamma''(1) + 8\log\frac{\pi^2}{d}\Big(-\frac{C_1\pi}{4} + C_0L'(1,\chi) - \frac{L''(1,\chi)}{2}\Big)$$

$$+ 8\left(\log\frac{\pi^2}{d}\right)^2\left(-\frac{C_0\pi}{8}+\frac{L'(1,\chi)}{2}\right)-\left(\log\frac{\pi^2}{d}\right)^3\frac{\pi}{3}+2\pi E^{(2)}(1,d)$$

$$+ 2\pi E(1,d)\left\{\frac{1}{4}\left(\log\frac{\pi^2}{d}\right)^2+C_0\log\frac{\pi^2}{d}+2C_0^2-\Gamma''(1)\right\},$$

where $E^{(2)}(s, d)$ is the second derivative of $E(s, d)$ with respect to s.

Part (ii) of the above Theorem 2 gives an analogue of Theorem 3 in Bateman and Grosswald [2].

When we use Corollary 9, we need the estimate of $L''(1,\chi)$ and $L'''(1,\chi)$. We notice the following corollary which is a result of the definition and the Chowla-Selberg's formula.

Corollary 10.

(i) $$L''(1,\chi) = \frac{2}{3}C_1\pi + 2\zeta'(2) - \frac{\pi}{2}C_0^2 - \pi\log 2 + \frac{\pi^3}{12} + \pi(\log 2)^2$$

$$- \frac{\pi^2}{6}\log\pi - \frac{\pi^2}{6}C_0 - 2\pi\log\pi\cdot\log|\eta(i)| + 2\pi D'(1),$$

where we put

$$D(s) = \sum_{n=1}^{\infty} n^{s-\frac{1}{2}}\sigma_{1-2s}(n)\int_0^{\infty} y^{s-\frac{3}{2}}e^{-\pi n(y+y^{-1})}dy$$

$$= D(1) + D'(1)(s-1) + \frac{D''(1)}{2}(s-1)^2 + \cdots.$$

(ii) $$L'''(1,\chi) = \frac{21}{2}C_2\pi - 9\pi C_1\log 2 + 3\pi C_0(\log 2)^2 + \frac{1}{4}C_0\pi^3 - 6\pi C_0C_1$$

$$- 2\pi(\log 2)^3 - \frac{1}{2}\pi^3\log 2 - 3\pi\zeta(3) + 6\zeta''(2) - 6C_0\zeta'(2)$$

$$+ \frac{3}{2}\pi C_0^3 + 3\pi C_0^2\log 2 - \frac{\pi^2}{4}(\log\pi)^2 + \frac{\pi^2C_0^2}{4} + \frac{\pi^4}{24}$$

$$+ \log|\eta(i)|\left(6C_1\pi - 3\pi C_0^2 + \frac{\pi^3}{2} - 3\pi(\log\pi)^2\right)$$

$$+ 6\pi D'(1)\log\pi + 3\pi D''(1).$$

Remark 3. *For our purpose, the following rough estimates of $L''(1,\chi)$ and $L'''(1,\chi)$ are enough. Since*

$$\frac{d}{dx}\left(\frac{\log^2 x}{x}\right) = \frac{\log x}{x^2}(2-\log x),$$

$\dfrac{\log^2 x}{x}$ *is monotone increasing for $1 \le x \le e^2$ and monotone decreasing for $x \ge e^2$. Hence, we get*

$$-0.162741<-\sum_{k=1}^{999}\left(\frac{(\log(4k-1))^2}{4k-1}-\frac{(\log(4k+1))^2}{4k+1}\right)-\frac{(\log(3999))^2}{3999}<L''(1,\chi)$$

$$<-\sum_{k=1}^{1000}\left(\frac{(\log(4k-1))^2}{4k-1}-\frac{(\log(4k+1))^2}{4k+1}\right)<-0.14554.$$

Similarly, $\frac{\log^3 x}{x}$ *is monotone increasing for* $1\leq x\leq e^3$ *and monotone decreasing for* $x\geq e^3$, *we get*

$$0.02354<\sum_{k=1}^{999}\left(\frac{(\log(4k-1))^3}{4k-1}-\frac{(\log(4k+1))^3}{4k+1}\right)<L'''(1,\chi)$$

$$<\sum_{k=1}^{999}\left(\frac{(\log(4k-1))^3}{4k-1}-\frac{(\log(4k+1))^3}{4k+1}\right)+\frac{(\log(3999))^3}{3999}<0.16621.$$

In fact, the Euler-Maclaurin summation formula gives us

$$L''(1,\chi)=-0.1541417\cdots$$

and

$$L'''(1,\chi)=-0.0948828\cdots.$$

We notice that when d is sufficiently large, then the two real zeros found in Corollary 6 and (ii) of Theorem 2 can be located more precisely as follows.

Theorem 3. *For a sufficiently large* d *and for any positive definite quadratic form* $Q(x,y)$, $G_d(s,Q)$ *has two real zeros* ρ_d *and* $2-\rho_d$, *where*

$$\rho_d=\frac{1}{d}\frac{4\pi^2}{|\Delta|}\frac{1}{Z(2,Q)}\left(1+\frac{\log d}{d}\frac{4\pi^2}{|\Delta|}\frac{1}{Z(2,Q)}+O\left(\frac{1}{d}\right)\right)$$

as $d\to\infty$.

This gives a refinemennt of Corollary 6.

We turn our attentions to the complex zeros of $G_d(s)$. Concerning this problem, we shall restrict ourselves only to $G_d(s)$. However, as one sees that we can extend our results, namely, Theorems 4, 5 and 6 to $G_d(s,Q)$ with $h(-\Delta)=1$, although we shall not state the results.

We shall first proceed to show the analogue of Stark's results [23] on the "κ-analogue" of the "Riemann Hypothesis". We remark first, by (i) of Corollary 5, that if

$$\kappa=\sqrt{d}>2.94,$$

then $G_d(s)$ has no zeros in $\Re(s)=\sigma\geq\frac{5}{2}$. Hence we have only to treat the zeros in the region

$$-\frac{1}{2} < \Re(s) < \frac{5}{2}.$$

First of all we shall show the following theorem.

Theorem 4. *There exists a number K such that if $\kappa > K$, then all the zeros of $G_d(s)$ in the region $-\frac{1}{2} < \sigma < \frac{5}{2}$, $-\kappa \leq t \leq \kappa$ are simple zeros; with the exception of two real zeros between 0 and 2, all are on the line $\sigma = 1$.*

We shall next show the following theorem.

Theorem 5. *Let $N(T, G)$ denote the number of the zeros of $G_d(s)$ in the region $-\frac{1}{2} < \sigma < \frac{5}{2}$, $0 \leq t \leq T$. If $\kappa > K$ and $0 < T \leq \kappa$, then*

$$N(T, G) = \frac{T}{\pi} \log \frac{\kappa T}{\pi e} + O(\log\log T),$$

uniformly for κ.

This corresponds to Stark's Riemann-von Mangoldt formula for $N(T, Q)$.

Finally, we shall proceed to the analogue of the failure of the "κ-analogue" of "the GUE law" or the corresponding Berry Conjecture [1] for $G_d(s)$. Before stating our results, we shall first clarify the present situation. In the proof of Theorems 4 and 5, we shall see below that

$$N(T, G) = \frac{1}{\pi} \arg\left(\left(\frac{\kappa}{\pi}\right)^{1+iT} \Gamma(1 + iT)\right) + \frac{1}{\pi} \arg Z(1 + iT) + \Delta_G(T),$$

where $|\Delta_G(T)| \leq C$. Hence the number variance with which we are concerned is

$$\frac{1}{\frac{T}{2}} \int_{\frac{T}{2}}^{T-1} \left(S_G\left(t + \frac{\alpha\pi}{\log \frac{\kappa T}{\pi}}\right) - S_G(t)\right)^2 dt,$$

where we put

$$S_G(t) = \frac{1}{\pi} \arg Z(1 + it) + \Delta_G(t).$$

If it obeys "the GUE law", then it must be, as above, that

$$\frac{1}{\frac{T}{2}} \int_{\frac{T}{2}}^{T-1} \left(S_G\left(t + \frac{\alpha\pi}{\log \frac{\kappa T}{\pi}}\right) - S_G(t)\right)^2 dt \sim C \log \alpha \text{ as } \alpha \to \infty$$

with some positive constant C. However, we shall prove below the following theorem.

Theorem 6. *For $\kappa > K$ and $0 < T \leq \kappa$, we have*

$$\frac{1}{\frac{T}{2}}\int_{\frac{T}{2}}^{T-1}\left(S_G\left(t+\frac{\alpha\pi}{\log\frac{\kappa T}{\pi}}\right)-S_G(t)\right)^2 dt \leq C$$

uniformly for $0 < \alpha \ll T\log\frac{\kappa T}{\pi}$.

Hence we see that as $\kappa \to \infty$,

$$\frac{1}{\frac{\kappa}{2}}\int_{\frac{\kappa}{2}}^{\kappa-1}\left(S_G\left(t+\frac{\alpha\pi}{\log\frac{\kappa^2}{\pi}}\right)-S_G(t)\right)^2 dt \leq C$$

uniformly for $0 < \alpha \ll \kappa\log\frac{\kappa^2}{\pi}$.

Thus we see the failure of the "κ-analogue" of "the GUE law" for $G_d(s)$.

Here we should pay an attention to a special case of $G_d(s)$. Namely, when $d = 1$, then we know, as in Theorem 4 of p.30 in Grosswald [9], that for any integer $n \geq 1$,

$$\sum_{\substack{n=m_1^2+m_2^2+m_3^2+m_4^2 \\ -\infty<m_i<\infty}} 1 = \sum_{\substack{d\mid n \\ d\not\equiv 0 (\mathrm{mod}\ 4)}} d$$

Hence, we have

$$G_1(s) = 8(1-2^{2-2s})\zeta(s)\zeta(s-1).$$

This fact is, in fact, strongly noticed by Siegel in pp.145-146 of [21]. Thus $G_1(s)$ has the functional equation and the Euler product. However on the critical line $\Re(s) = 1$, we have only such zeros of $G_1(s)$ as

$$s = 1 + \frac{n\pi i}{\log 2}, \quad n = \pm 1, \pm 2, \pm 3, \cdots.$$

This means that the number of the zeros of $G_1(s)$ on the critical line in the region $0 < \Im(s) \leq T$ is

$$\left[\frac{T\log 2}{\pi}\right].$$

On the other hand, by Selberg [20], we know that on the line $\Re(s) = \frac{1}{2}$ or $\Re(s) = \frac{3}{2}$, the number of the zeros of $G_1(s)$ is at least

$$AT\log T,$$

with some positive constant A. Moreover, the Riemann-von Mangoldt formula for $G_1(s)$ is, using the notations and the results which is mentioned in the previous section,

$$N(T, G_1) = 2N(T) + \left[\frac{T\log 2}{\pi}\right] = \frac{T}{\pi}\log T - \frac{T}{\pi}(1 + \log\pi) + \Delta_{G_1}(T) + 2S(T),$$

where

$$|\Delta_{G_1}(T)| \leq 1.$$

Consequently, we see that

$G_1(s)$ has almost no zeros on the critical line, namely, the Riemann Hypothesis fails strongly for $G_1(s)$.

And that the results mentioned above implies, under certain hypothesis, that *"the GUE law" holds for* $G_1(s)$.

Furthermore, using only the properties of $\zeta(s)$, we see that at $s = 2$,

$$G_1(s) = \frac{6\zeta(2)}{s-2} + 6\zeta(2)C_0 + 4\zeta(2)\log 2 + 6\zeta'(2) + A(s-2) + \cdots,$$

where A is some constant. Comparing this with Corollary 4, we get (ii) of the following corollary.

Corollary 11.

(i) $$L(2, \chi) = 3\lim_{d\to\infty}\sum_{n=1}^{\infty}\sqrt{n}\sum_{m|n}\frac{r(m)r\left(\frac{n}{m}\right)}{m}\,\frac{K_1\left(2\pi\sqrt{\frac{n}{d}}\right)}{d^{\frac{3}{2}}}.$$

(ii) $$L(2, \chi) = -\frac{3}{2}C_0 + \frac{9}{\pi^2}\zeta'(2) + 4\log 2 + \frac{3}{2} + 3\log|\eta(i)|^2 - 3\sum_{n=1}^{\infty}\sqrt{n}\sum_{m|n}\frac{r(m)r\left(\frac{n}{m}\right)}{m}K_1(2\pi\sqrt{n}).$$

We see also that at $s = 1$,

$$G_1(1) = 8\zeta(0)\lim_{s\to 1}((1 - 2^{2-2s})\zeta(s)) = 8\zeta(0)\cdot 2\log 2 = -8\log 2 < 0.$$

In fact, we see that

$$G_1(\sigma) < 0 \text{ in } 0 < \sigma < 2.$$

Combining the present evaluation of $G_1(1)$ with Corollary 8′, we get further an expsression of $L'(1, \chi)$.

Corollary 12.

$$L'(1, \chi) = -\log 2 + \frac{\pi}{4}\log\pi - \frac{\pi}{4}\sum_{n=1}^{\infty}\left(\sum_{m|n}r(m)r\left(\frac{n}{m}\right)\right)K_0(2\pi\sqrt{n}).$$

At $s = 0$, we have

$$G_1(0) = 8(1-2^2)\zeta(0)\zeta(-1) = 8\cdot(-3)\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{1}{12}\right) = -1.$$

This is included in (iii) of Corollary 5.

Finally, we notice another complicated expression of $L'''(1,\chi)$, which comes from a comparison of the coefficients of $(s-1)^2$ in the Taylor expansions of $G_1(s)$ in two ways, namely, Corollary 9 and what comes from the present expression of $G_1(s)$.

Corollary 13.

$$\begin{aligned}L'''(1,\chi) = \frac{3}{8}\Big\{&\frac{16}{3}(\log 2)^3 + 16(\log 2)^2\log\pi - 4C_2\pi + 4C_0C_1\pi \\ &+ 12C_1\pi\log\pi - 4\pi C_0^2\log\pi - 8\pi\log 2\cdot\log\pi - \frac{4}{3}C_0^3\pi \\ &+ \frac{2}{3}\pi^3\log\pi + 8\pi\log\pi\cdot(\log 2)^2 - \frac{4}{3}\pi(\log\pi)^3 + 16\log 2\cdot(\log\pi)^2 \\ &+ 16\zeta''(0)\log 2 + \frac{4\pi}{3}\zeta(3) + 16\zeta'(2)\log\pi - \frac{4}{3}\pi^2(\log\pi)^2 \\ &- \frac{4}{3}C_0\pi^2\log\pi - 16\pi(\log\pi)^2\cdot\log|\eta(i)| + 16\pi D'(1)\log\pi \\ &- 2\pi E(1,1)\left(C_0^2 - \frac{\pi^2}{6} - 2C_1 - (\log\pi)^2\right) - 2\pi E^{(2)}(1,1)\Big\}.\end{aligned}$$

We shall prove Theorem 1 and Corollaries 4 and 8 in the section 2, (i) and (ii) of Corollary 5 in the section 3, Corollaries 8′ and 9 and Theorem 2 in the section 4, (iii) of Corollary 5, Theorem 3 and Corollary 3 in the section 5, Corollaries 1 and 2 in the section 6, Theorems 4 and 5 in the sections 7 and 8 and Theorem 6 in the section 9. Some of the numerical calculations in this article have been done using Mathematica.

Finally, we should notice, among many works, the existence of the works 37 (pp.708-734) of Hecke [10], Terras [24] and Hoffstein [12]. 37 of Hecke [10] has shown the existence of infinitely many zeros on the critical line of the general Epstein zeta functions of many variables. Terras [24] has tried to extend Chowla-Selberg [20] and Bateman-Grosswald [2]. Hoffstein [12] has tried to extend Bateman-Grosswald [2].

§ 2. Proof of Theorem 1 and Corollaries 4 and 8 and a remark on (i) of Corollary 11

Let $Q_1(x,y)$, $Q_2(x,y)$, Δ_1 and Δ_2 be the same as in the previous section. We suppose first that $\Re(s) > 2$. Then we have

$$G_d(s, Q_1, Q_2) = \sum{}'(Q_1(m_1, m_2))^{-s}$$

$$+\sum_{m=1}^{\infty} r_{Q_2}(m) \sum_{-\infty<m_1,m_2<\infty} (Q_1(m_1, m_2) + dm)^{-s}$$

$$= Z(s, Q_1) + E_0(s, d, Q_1, Q_2), \text{ say.}$$

We simplify $E_0(s, d, Q_1, Q_2)$ further as follows.

$$E_0(s, d, Q_1, Q_2) = \sum_{m=1}^{\infty} r_{Q_2}(m) \sum_{v=0}^{\infty} r_{Q_1}(v)\,(v + dm)^{-s}.$$

We notice first that

$$\pi^{-s}\Gamma(s) E_0(s, d, Q_1, Q_2) = \int_0^{\infty} x^{s-1} \sum_{m=1}^{\infty} r_{Q_2}(m) \sum_{v=0}^{\infty} r_{Q_1}(v) e^{-\pi(v+dm)x} dx.$$

Using the following transformation formula (cf. p.48 of Siegel [21]),

$$\sum_{n=0}^{\infty} r_{Q_1}(n) e^{-n\pi x} = \frac{2}{\sqrt{|\Delta_1|}} \frac{1}{x} \sum_{n=0}^{\infty} r_{Q_1}(n) e^{-\frac{4n\pi}{|\Delta_1| x}},$$

we get next

$$\pi^{-s}\Gamma(s) E_0(s, d, Q_1, Q_2)$$

$$= \frac{2}{\sqrt{|\Delta_1|}} \int_0^{\infty} x^{s-2} \sum_{m=1}^{\infty} r_{Q_2}(m) \sum_{v=0}^{\infty} r_{Q_1}(v) e^{-\pi(dmx + \frac{4v}{|\Delta_1| x})} dx$$

$$= \frac{2}{\sqrt{|\Delta_1|}} \int_0^{\infty} x^{s-2} \sum_{m=1}^{\infty} r_{Q_2}(m)\, e^{-\pi dmx} dx$$

$$+ \frac{2}{\sqrt{|\Delta_1|}} \int_0^{\infty} x^{s-2} \sum_{m=1}^{\infty} r_{Q_2}(m) \sum_{v=1}^{\infty} r_{Q_1}(v)\, e^{-\pi(dmx + \frac{4v}{|\Delta_1| x})} dx$$

$$= \Psi_1(s) + \Psi_2(s), \text{ say.}$$

$$\Psi_1(s) = \frac{2}{\sqrt{|\Delta_1|}} \sum_{m=1}^{\infty} r_{Q_2}(m) \int_0^{\infty} x^{s-2} e^{-\pi dmx} dx$$

$$= \frac{2}{\sqrt{|\Delta_1|}} \frac{\Gamma(s-1)}{(\pi d)^{s-1}} \sum_{m=1}^{\infty} \frac{r_{Q_2}(m)}{m^{s-1}} = \frac{2}{\sqrt{|\Delta_1|}} \frac{\Gamma(s-1)}{(\pi d)^{s-1}} Z(s-1, Q_2).$$

By a change of variable $x = \sqrt{\frac{4v}{|\Delta_1| dm}}\, y$, we get

$$\Psi_2(s) = \frac{2}{\sqrt{|\Delta_1|}} \sum_{m=1}^{\infty} r_{Q_2}(m) \sum_{v=1}^{\infty} r_{Q_1}(v) \left(\frac{4v}{|\Delta_1| dm}\right)^{\frac{s-1}{2}}$$

$$\cdot \int_0^{\infty} y^{s-2} e^{-\pi \frac{2}{\sqrt{|\Delta_1|}} \sqrt{dvm}\,(y+y^{-1})} dy$$

$$= \frac{2}{\sqrt{|\Delta_1|}} \frac{1}{d^{\frac{s-1}{2}}} \left(\frac{4}{|\Delta_1|}\right)^{\frac{s-1}{2}} \sum_{n=1}^{\infty} \sum_{mv=n} \left(\frac{v^2}{n}\right)^{\frac{s-1}{2}} r_{Q_2}(m) r_{Q_1}(v)$$

$$\cdot \int_0^{\infty} y^{s-2} e^{-\pi \frac{2}{\sqrt{|\Delta_1|}} \sqrt{dn}\,(y+y^{-1})} dy$$

$$= \frac{2}{\sqrt{|\Delta_1|}} \frac{1}{d^{\frac{s-1}{2}}} \left(\frac{4}{|\Delta_1|}\right)^{\frac{s-1}{2}} \sum_{n=1}^{\infty} n^{\frac{s-1}{2}} \sum_{m|n} \frac{r_{Q_2}(m) r_{Q_1}\left(\frac{n}{m}\right)}{m^{s-1}}$$

$$\cdot \int_0^{\infty} y^{s-2} e^{-\pi \frac{2}{|\Delta_1| x} \sqrt{dn}\,(y+y^{-1})} dy$$

$$= \frac{2}{\sqrt{|\Delta_1|}} \frac{2}{d^{\frac{s-1}{2}}} \left(\frac{4}{|\Delta_1|}\right)^{\frac{s-1}{2}} \sum_{n=1}^{\infty} n^{\frac{s-1}{2}} \sum_{m|n} \frac{r_{Q_2}(m) r_{Q_1}\left(\frac{n}{m}\right)}{m^{s-1}} \cdot K_{s-1}\left(\frac{4\pi}{\sqrt{|\Delta_1|}} \sqrt{dn}\right).$$

Thus we have proved that

$$G_d(s, Q_1, Q_2) = Z(s, Q_1) + \frac{2}{\sqrt{|\Delta_1|}} \frac{\pi}{d^{s-1}} \frac{\Gamma(s-1)}{\Gamma(s)} Z(s-1, Q_2)$$

$$+ \left(\frac{2\pi}{\sqrt{|\Delta_1|}\sqrt{d}}\right)^s \frac{2\sqrt{d}}{\Gamma(s)} \sum_{n=1}^{\infty} n^{\frac{s-1}{2}} \left(\sum_{m|n} \frac{r_{Q_2}(m) r_{Q_1}\left(\frac{n}{m}\right)}{m^{s-1}}\right) K_{s-1}\left(\frac{4\pi}{\sqrt{|\Delta_1|}} \sqrt{dn}\right).$$

Since

$$\sum_{n=1}^{\infty} n^{\frac{s-1}{2}} \left(\sum_{m|n} \frac{r_{Q_2}(m) r_{Q_1}\left(\frac{n}{m}\right)}{m^{s-1}}\right) K_{s-1}\left(\frac{4\pi}{\sqrt{|\Delta_1|}} \sqrt{dn}\right)$$

is an entire function of s, this gives an analytic continuation of $G_d(s, Q_1, Q_2)$ to the whole complex plane with a simple pole at $s = 2$.

Using Kronecker's limit formula for $Z(s, Q_2)$ as described in the previous section, we get the following expansion at $s = 2$.

$$\frac{2}{\sqrt{|\Delta_1|}} \frac{\pi}{d^{s-1}} \frac{\Gamma(s-1)}{\Gamma(s)} Z(s-1, Q_2)$$

$$= \frac{2\pi}{\sqrt{|\Delta_1|}} \left(\frac{1}{d} - \frac{\log d}{d}(s-2) + \cdots\right) \cdot (1 - (s-2) + \cdots) \cdot \left(\frac{\frac{2\pi}{\sqrt{|\Delta_2|}}}{s-2}\right.$$

$$\left. + \frac{2\pi}{\sqrt{|\Delta_2|}} \left(2C_0 + \log \frac{a_2}{|\Delta_2|} - 2\log\left|\eta\left(\frac{b_2}{2a_2} + i\frac{\sqrt{|\Delta_2|}}{2a_2}\right)\right|^2\right) + \cdots\right)$$

$$= \frac{\frac{4\pi^2}{d\sqrt{|\Delta_1||\Delta_2|}}}{s-2} + \frac{4\pi^2}{d\sqrt{|\Delta_1||\Delta_2|}} \cdot \Big(2C_0 + \log\frac{a^2}{|\Delta_2|}$$
$$- 2\log\Big|\eta\Big(\frac{b_2}{2a_2} + i\frac{\sqrt{|\Delta_2|}}{2a_2}\Big)\Big|^2 - \log d - 1\Big) + A'_1(s-2) + \cdots.$$

Since the other terms in $G_d(s, Q_1, Q_2)$ in the above expression are regular at $s = 2$, $s = 2$ is a simple pole of $G_d(s, Q_1, Q_2)$ and the residue is

$$\frac{4\pi^2}{d\sqrt{|\Delta_1||\Delta_2|}}.$$

We may remark here that although $s = 1$ is a simple pole of $Z(s, Q_1)$ in the above expression of $G_d(s, Q_1, Q_2)$, $s = 1$ is a removable sigularity for $G_d(s, Q_1, Q_2)$. Because using the functional equation of $Z(s, Q_2)$ and Kronecker's limit formula for $Z(s, Q_2)$, we get first that

$$\frac{2}{\sqrt{|\Delta_1|}} \frac{\pi}{d^{s-1}} \frac{\Gamma(s-1)}{\Gamma(s)} Z(s-1, Q_2)$$
$$= \frac{2}{\sqrt{|\Delta_1|}} \frac{\pi}{d^{s-1}} \frac{1}{\Gamma(s)} \Big(\frac{|\Delta_2|}{4}\Big)^{\frac{3-2s}{2}} \pi^{2s-3}\Gamma(2-s) Z(2-s, Q_2)$$
$$= \frac{\sqrt{|\Delta_2|}}{\sqrt{|\Delta_1|}}\Big(1 + (s-1)\log\frac{4\pi^2}{d|\Delta_2|} + \cdots\Big)$$
$$\cdot (1 - \Gamma'(1)(s-1) + \cdots)(1 - \Gamma'(1)(s-1) + \cdots) \cdot \Bigg(\frac{\frac{2\pi}{\sqrt{|\Delta_2|}}}{1-s}$$
$$+ \frac{2\pi}{\sqrt{|\Delta_2|}}\Big(2C_0 + \log\frac{a_2}{|\Delta_2|} - 2\log\Big|\eta\Big(\frac{b_2}{2a_2} + i\frac{\sqrt{|\Delta_2|}}{2a_2}\Big)\Big|^2\Big) + \cdots\Bigg)$$
$$= \frac{\frac{2\pi}{\sqrt{|\Delta_1|}}}{1-s} + \frac{2\pi}{\sqrt{|\Delta_1|}}\Big(2C_0 + \log\frac{a_2}{|\Delta_2|} - 2\log\Big|\eta\Big(\frac{b_2}{2a_2} + i\frac{\sqrt{|\Delta_2|}}{2a_2}\Big)\Big|^2\Big)$$
$$+ \frac{2\pi}{\sqrt{|\Delta_1|}}\Big(-\log\frac{4\pi^2}{d|\Delta_2|} + 2\Gamma'(1)\Big) + A''_1(s-1) + \cdots.$$

Hence, applying again Kronecker's limit formula to $Z(s, Q_1)$, we get at $s = 1$,

$$Z(s, Q_1) + \frac{2}{\sqrt{|\Delta_1|}} \frac{\pi}{d^{s-1}} \frac{\Gamma(s-1)}{\Gamma(s)} Z(s-1, Q_2)$$
$$= \frac{2\pi}{\sqrt{|\Delta_1|}}\Big(4C_0 + \log\frac{a_1a_2}{|\Delta_1||\Delta_2|} - 2\log\Big|\eta\Big(\frac{b_1}{2a_1} + i\frac{\sqrt{|\Delta_1|}}{2a_1}\Big)\Big|^2$$
$$- 2\log\Big|\eta\Big(\frac{b_2}{2a_2} + i\frac{\sqrt{|\Delta_2|}}{2a_2}\Big)\Big|^2\Big)$$
$$+ \frac{2\pi}{\sqrt{|\Delta_1|}}\Big(-\log\frac{4\pi^2}{d|\Delta_2|} + 2\Gamma'(1)\Big) + A'''_1(s-1) + \cdots.$$

Thus $s=1$ is a removable singularity of $G_d(s, Q_1, Q_2)$.

Under the translation $s \to 2-s$,

$$n^{\frac{s-1}{2}} \sum_{m|n} \frac{r_{Q_2}(m) r_{Q_1}\left(\frac{n}{m}\right)}{m^{s-1}}$$

becomes

$$n^{\frac{s-1}{2}} \sum_{m|n} \frac{r_{Q_1}(m) r_{Q_2}\left(\frac{n}{m}\right)}{m^{s-1}}$$

and the integral

$$\int_0^\infty y^{s-2} e^{-\pi \frac{2}{\sqrt{|\Delta_1|}} \sqrt{dn}\,(y+y^{-1})} dy$$

is invariant. Hence, we see that

$$E(s, d, Q_1, Q_2, \Delta_1) = E(2-s, d, Q_2, Q_1, \Delta_1),$$

where $E(s, d, Q_1, Q_2, \Delta_1)$ is introduced in the statement of Theorem 1. This proves our Theorem 1.

As we have written down explicitly the Laurent expansion of

$$\frac{2}{\sqrt{|\Delta_1|}} \frac{\pi}{d^{s-1}} \frac{\Gamma(s-1)}{\Gamma(s)} Z(s-1, Q_2)$$

at $s=2$, we get immediately

$$\lim_{s \to 2}\left(G_d(s, Q_1, Q_2) - \frac{\frac{4\pi^2}{d\sqrt{|\Delta_1||\Delta_2|}}}{s-2}\right)$$

$$= Z(2, Q_1) + \lim_{s \to 2}\left(\frac{2}{\sqrt{|\Delta_1|}} \frac{\pi}{d^{s-1}} \frac{\Gamma(s-1)}{\Gamma(s)} Z(s-1, Q_2) - \frac{\frac{4\pi^2}{d\sqrt{|\Delta_1||\Delta_2|}}}{s-2}\right)$$

$$+ \frac{8\pi^2}{\sqrt{d}\,|\Delta_1|} \sum_{n=1}^\infty \sqrt{n}\left(\sum_{m|n} \frac{r_{Q_2}(m) r_{Q_1}\left(\frac{n}{m}\right)}{m}\right) K_1\left(\frac{4\pi}{\sqrt{|\Delta_1|}}\sqrt{dn}\right)$$

$$= Z(2, Q_1) + \frac{4\pi^2}{d\sqrt{|\Delta_1||\Delta_2|}}\left(2C_0 + \log\frac{a_2}{|\Delta_2|}\right.$$

$$\left. - 2\log\left|\eta\left(\frac{b_2}{2a_2} + i\frac{\sqrt{|\Delta_2|}}{2a_2}\right)\right|^2 - \log d - 1\right)$$

$$+ \frac{8\pi^2}{\sqrt{d}\,|\Delta_1|} \sum_{n=1}^\infty \sqrt{n}\left(\sum_{m|n} \frac{r_{Q_2}(m) r_{Q_1}\left(\frac{n}{m}\right)}{m}\right) K_1\left(\frac{4\pi}{\sqrt{|\Delta_1|}}\sqrt{dn}\right).$$

This proves our Corollary 4 as described in the previous section.
The proof of Corollary 8 is also included in the above argument.
Finally, we shall give a short notice on (i) of Corollary 11.
Corollary 4 and Remark 2 imply first that for $Q = Q_1 = Q_2$ and $\Delta = \Delta_1 = \Delta_2$,

$$Z(2, Q)\left(1 - \frac{1}{d^2}\right) = \frac{4\pi^2 \log d}{d|\Delta|} + \frac{8\pi^2}{\sqrt{d}|\Delta|}\sum_{n=1}^{\infty}\sqrt{n}\left(\sum_{m|n}\frac{r_Q(m)r_Q\left(\frac{n}{m}\right)}{m}\right)$$

$$\cdot\left(\frac{K_1\left(\frac{4\pi}{\sqrt{|\Delta|}}\sqrt{\frac{n}{d}}\right)}{d} - K_1\left(\frac{4\pi}{\sqrt{|\Delta|}}\sqrt{dn}\right)\right).$$

Taking $Q = x^2 + y^2$ and letting d tend to ∞, we get (i) of Corollary 11.

In fact, it can be easily seen that

$$Z(2, Q) \ll \lim_{d\to\infty}\frac{1}{d^{\frac{3}{2}}}\sum_{n=1}^{\infty}\sqrt{n}\left(\sum_{m|n}\frac{r(m)r\left(\frac{n}{m}\right)}{m}\right)K_1\left(2\pi\sqrt{\frac{n}{d}}\right) \ll Z(2, Q).$$

Since for $a>0$,

$$K_1(a) = \frac{1}{2}\left(\int_1^{\infty} e^{-a\frac{y+y^{-1}}{2}}\frac{1}{y^2}dy + \int_1^{\infty} e^{-a\frac{y+y^{-1}}{2}}dy\right) \leq \frac{1}{2}\left(e^{-a} + \frac{2e^{-\frac{a}{2}}}{a}\right) \ll \frac{e^{-\frac{a}{2}}}{a},$$

we get

$$\frac{1}{d^{\frac{3}{2}}}\sum_{n=1}^{\infty}\sqrt{n}\left(\sum_{m|n}\frac{r(m)r\left(\frac{n}{m}\right)}{m}\right)K_1\left(2\pi\sqrt{\frac{n}{d}}\right)$$

$$\ll \frac{1}{d}\sum_{kl<d}\frac{r(k)}{k}r(l) + \frac{1}{d}\sum_{j=0}^{\infty}e^{-\pi\sqrt{2^j}}\sum_{2^j d<kl\leq 2^{j+1}d}\frac{r(k)}{k}r(l)$$

$$\ll \sum_{k\leq d}\frac{r(k)}{k^2} + \sum_{j=0}^{\infty}e^{-\pi\sqrt{2^j}}2^j\sum_{k\leq 2^{j+1}d}\frac{r(k)}{k^2} \ll \sum_{k=1}^{\infty}\frac{r(k)}{k^2} \ll Z(2, Q).$$

On the other hand, for $a>0$,

$$K_1(a) \geq e^{-\frac{a}{2}}\int_1^{\infty} e^{-\frac{ay}{2}}dy \geq \frac{2e^{-2a}}{a}.$$

Hence, we get

$$\frac{1}{d^{\frac{3}{2}}}\sum_{n=1}^{\infty}\sqrt{n}\left(\sum_{m|n}\frac{r(m)r\left(\frac{n}{m}\right)}{m}\right)K_1\left(2\pi\sqrt{\frac{n}{d}}\right) \geq \frac{1}{d}\sum_{d<kl\leq 2d}\frac{r(k)}{k}r(l)e^{-4\pi\sqrt{\frac{kl}{d}}}$$

$$\geq e^{-4\pi\sqrt{2}}\frac{1}{d}\sum_{1<k\leq 2d}\frac{r(k)}{k}\sum_{\frac{d}{k}<k\leq\frac{2d}{k}} r(l) \ll \sum_{1<k\leq 2d}\frac{r(k)}{k^2} \gg Z(2, Q).$$

§ 3. Proof of (i) and (ii) of Corollary 5

We start from the following decomposition done at the beginning of the previous section.

$$G_d(s) = Z(s) + E_0(s, d),$$

where $E_0(s, d) = E_0(s, d, Q, Q)$ for $Q(x, y) = x^2 + y^2$.

Suppose that $\Re(s) = \sigma > 2$. We notice first that

$$|Z(s)| = 4\left|\prod_p\left(1-\frac{1}{p^s}\right)^{-1}\prod_p\left(1-\frac{\chi(p)}{p^s}\right)^{-1}\right| \geq 4\prod_p\left(1+\frac{1}{p^\sigma}\right)^{-1}\prod_{p\neq 2}\left(1+\frac{1}{p^\sigma}\right)^{-1}$$

$$= 4\left(1+\frac{1}{2^\sigma}\right)\left(\prod_p\left(1+\frac{1}{p^\sigma}\right)^{-1}\right)^2 = 4\left(1+\frac{1}{2^\sigma}\right)\left(\prod_p\frac{\left(1-\frac{1}{p^{2\sigma}}\right)^{-1}}{\left(1-\frac{1}{p^\sigma}\right)^{-1}}\right)^2$$

$$= 4\left(1+\frac{1}{2^\sigma}\right)\frac{\zeta^2(2\sigma)}{\zeta^2(\sigma)},$$

where p runs over the prime numbers and χ is the nonprincipal Dirichlet character mod 4 as is introduced in the section 1. On the other hand,

$$|E_0(s, d)| \leq \sum_{m=1}^{\infty} r(n)\sum_{v=0}^{\infty} r(v)(v+dm)^{-\sigma}$$

$$\leq \frac{1}{d^\sigma}\sum_{m=1}^{\infty}\frac{r(m)}{m^\sigma} + \sum_{m=1}^{\infty} r(m)\sum_{v=1}^{\infty} r(v)(v+dm)^{-\sigma}$$

$$\leq \frac{1}{d^\sigma}\sum_{m=1}^{\infty}\frac{r(m)}{m^\sigma} + \sum_{m=1}^{\infty} r(m)\sum_{m=1}^{\infty}\frac{r(v)}{(4vdm)^{\frac{\sigma}{2}}}$$

$$\leq \frac{4}{d^\sigma}\zeta(\sigma)L(\sigma, \chi) + \frac{1}{2^\sigma d^{\frac{\sigma}{2}}}\left(\sum_{m=1}^{\infty}\frac{r(m)}{m^{\frac{\sigma}{2}}}\right)^2$$

$$\leq \frac{4}{d^\sigma}\zeta(\sigma)L(\sigma, \chi) + \frac{16}{(4d)^{\frac{\sigma}{2}}}\zeta\left(\frac{\sigma}{2}\right)^2 L\left(\frac{\sigma}{2}, \chi\right)^2.$$

Thus $|Z(s)| > |E_0(s, d)|$,
provided that

$$4\left(1+\frac{1}{2^\sigma}\right)\frac{\zeta^2(2\sigma)}{\zeta^2(\sigma)} > \frac{4}{d^\sigma}\zeta(\sigma)L(\sigma, \chi) + \frac{16}{(4d)^{\frac{\sigma}{2}}}\zeta\left(\frac{\sigma}{2}\right)^2 L\left(\frac{\sigma}{2}, \chi\right)^2.$$

The last condition is satisfied if

$$d >$$
$$\left(\frac{\frac{4}{2^\sigma}\zeta\left(\frac{\sigma}{2}\right)^2 L\left(\frac{\sigma}{2},\chi\right)^2+\sqrt{\left(\frac{4}{2^\sigma}\zeta\left(\frac{\sigma}{2}\right)^2 L\left(\frac{\sigma}{2},\chi\right)^2\right)^2+4\left(1+\frac{1}{2^\sigma}\right)\frac{\zeta^2(2\sigma)}{\zeta^2(\sigma)}\zeta(\sigma)L(\sigma,\chi)}}{2\left(1+\frac{1}{2^\sigma}\right)\frac{\zeta^2(2\sigma)}{\zeta^2(\sigma)}}\right)^{\frac{2}{\sigma}}.$$

We put

$$d_0(\sigma)$$
$$=\left(\frac{2^{1-\sigma}\zeta\left(\frac{\sigma}{2}\right)^2 L\left(\frac{\sigma}{2},\chi\right)^2+\sqrt{2^{2-2\sigma}\zeta\left(\frac{\sigma}{2}\right)^4 L\left(\frac{\sigma}{2},\chi\right)^4+\left(1+\frac{1}{2^\sigma}\right)\frac{\zeta^2(2\sigma)}{\zeta(\sigma)}L(\sigma,\chi)}}{\left(1+\frac{1}{2^\sigma}\right)\frac{\zeta^2(2\sigma)}{\zeta^2(\sigma)}}\right)^{\frac{2}{\sigma}}.$$

Thus we see that for any $\sigma > 2$, if $d > d_0(\sigma)$, then

$$G_d(\sigma+it) \neq 0 \text{ for any } t.$$

Now to prove (i) of Corollary 5, suppose that

$$d_1 = d_0(\sigma_1)$$

for any $\sigma_1 > 2$. Then for $d > d_1$ and for $\sigma \geq \sigma_1$, we have

$$\begin{aligned}|Z(\sigma+it)| &\geq 4\left(1+\frac{1}{2^\sigma}\right)\frac{\zeta^2(2\sigma)}{\zeta^2(\sigma)} \geq 4\left(1+\frac{1}{2^{\sigma_1}}\right)\frac{\zeta^2(2\sigma_1)}{\zeta^2(\sigma_1)}\\ &= \frac{4}{d_1^{\sigma_1}}\zeta(\sigma_1)L(\sigma_1,\chi)+\frac{16}{(4d_1)^{\frac{\sigma_1}{2}}}\zeta\left(\frac{\sigma_1}{2}\right)^2 L\left(\frac{\sigma_1}{2},\chi\right)^2\\ &> \frac{4}{d^{\sigma_1}}\zeta(\sigma_1)L(\sigma_1,\chi)+\frac{16}{(4d)^{\frac{\sigma_1}{2}}}\zeta\left(\frac{\sigma_1}{2}\right)^2 L\left(\frac{\sigma_1}{2},\chi\right)^2\\ &\geq E_0(\sigma_1,d) \geq E_0(\sigma,d) \geq |E_0(\sigma+it,d)|.\end{aligned}$$

Thus we see that for any $d > d_1 = d_0(\sigma_1)$ and $\sigma \geq \sigma_1$,

$$G_d(\sigma+it) \neq 0.$$

In particular, since

$$d_0\left(\frac{5}{2}\right) = 8.595913749\cdots,$$

$$G_d(s) \neq 0$$

for $d > 8.595913749\cdots$ and for any $\Re s = \sigma \geq \frac{5}{2}$.

To justify the above argument we shall show that

$$4\left(1+\frac{1}{2^{\sigma}}\right)\frac{\zeta^{2}(2\sigma)}{\zeta^{2}(\sigma)}=4a(\sigma),\ \text{say,}$$

is monotone increasing for $\sigma>\dfrac{\log(1+\sqrt{5})}{\log 2}$ and that $E_0(\sigma,d)$ is monotone decreasing for $\sigma>2$.

The latter is clear from the definition of $E_0(\sigma,d)$.

To prove the former, we see first that

$$a'(\sigma)=\frac{\zeta^{2}(2\sigma)}{\zeta^{2}(\sigma)}\left\{-\frac{\log 2}{2^{\sigma}}+4\left(1+\frac{1}{2^{\sigma}}\right)\frac{\zeta'}{\zeta}(2\sigma)-2\left(1+\frac{1}{2^{\sigma}}\right)\frac{\zeta'}{\zeta}(\sigma)\right\}$$

$$=\frac{\zeta^{2}(2\sigma)}{\zeta^{2}(\sigma)}\left\{-\frac{\log 2}{2^{\sigma}}-4\left(1+\frac{1}{2^{\sigma}}\right)\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{2\sigma}}+2\left(1+\frac{1}{2^{\sigma}}\right)\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\sigma}}\right\},$$

where $\Lambda(n)=\log p$ if $n=p^{k}$ with a prime number p and an integer $k\geq 1$, and $=0$ otherwise.

For $\sigma>\dfrac{\log(1+\sqrt{5})}{\log 2}$, we have

$$\frac{\log 2}{2^{\sigma}}\left(1-\frac{2}{2^{\sigma}}-\frac{4}{2^{2\sigma}}\right)>0 \text{ and } 1-\frac{2}{n^{\sigma}}>0 \text{ for } n\geq 3.$$

Hence

$$a'(\sigma)>0$$

for $\sigma>\dfrac{\log(1+\sqrt{5})}{\log 2}$. Consequently, $a(\sigma)$ is monotone increasing for $\sigma>\dfrac{\log(1+\sqrt{5})}{\log 2}$.

This proves (i) of Corollary 5.

By the functional equation of $G_d(s,Q)$, namely, Corollary 7 stated in the section 1, we get trivial zeros as described in the statement (ii) of Corollary 5.

§ 4. Proof of Corollaries 8', 9 and 10 and Theorem 2

Corollary 8 implies the second equality of Corollary 8'. We shall notice the alternative expression of $G_d(1)$ mentioned in Corollary 8'. At $s=1$, we have

$$Z(s)=4\left(\frac{1}{s-1}+C_0+C_1(s-1)+\cdots\right)$$
$$\cdot\left(L(1,\chi)+L'(1,\chi)(s-1)+\frac{L''(1,\chi)}{2}(s-1)^{2}+\cdots\right),$$

By the functional equation of $Z(s)$ mentioned in the section 3, we have at $s=1$,

$$\frac{\pi}{d^{s-1}}\frac{\Gamma(s-1)}{\Gamma(s)}Z(s-1) = \left(\frac{\pi}{\sqrt{d}}\right)^{2s-2}\frac{\Gamma(2-s)Z(2-s)}{\Gamma(s)}$$

$$= \left(1 + 2\log\frac{\pi}{\sqrt{d}}(s-1) + \frac{1}{2}(2\log\frac{\pi}{\sqrt{d}})^2(s-1)^2 + \cdots\right)$$

$$\cdot\ (1 - 2\Gamma'(1)(s-1) + 2C_0^2(s-1)^2 + \cdots)$$

$$\cdot\ 4\left(\frac{1}{1-s} + C_0 + C_1(1-s) + \cdots\right)$$

$$\cdot\ (L(1,\chi) + L'(1,\chi)(1-s) + \frac{L''(1,\chi)}{2}(1-s)^2 + \cdots).$$

Thus we see that the value of

$$Z(s) + \frac{\pi}{d^{s-1}}\frac{\Gamma(s-1)}{\Gamma(s)}Z(s-1)$$

at $s = 1$ is

$$8\left(L(1,\chi)\left(\frac{1}{2}\log d - \log\pi\right) + L'(1,\chi)\right).$$

Hence, we get

$$G_d(1) = 8\left(L(1,\chi)\left(\frac{1}{2}\log d - \log\pi\right) + L'(1,\chi)\right) + 2\pi E(1,d).$$

This shows the first equality in Corollary 8'.

The above expansion also gives the explicit evaluation of the coefficiant of $(s-1)$ in the Taylor expansion of

$$Z(s) + \frac{\pi}{d^{s-1}}\frac{\Gamma(s-1)}{\Gamma(s)}Z(s-1)$$

at $s = 1$ as

$$4\log\frac{\pi^2}{d}(-C_0L(1,\chi) + L'(1,\chi)) + 8C_0L'(1,\chi) - 2\left(\log\frac{\pi^2}{d}\right)^2 L(1,\chi).$$

To get an explicit evaluation of $G'_d(1)$, we need the value of the derivative with respect to s of the function $\left(\frac{\pi}{\sqrt{d}}\right)^s \frac{2\sqrt{d}}{\Gamma(s)}E(s,d)$ at $s = 1$. It is

$$= 2\pi E(1,d)(\log\frac{\pi}{\sqrt{d}} + C_0) + 2\pi\left\{\sum_{n=1}^{\infty}\frac{1}{2}\log n\left(\sum_{m|n} r(m)r\left(\frac{n}{m}\right)\right)K_0(2\pi\sqrt{dn})\right.$$

$$+ \sum_{n=1}^{\infty}\left(\sum_{m|n}(-\log m)r(m)r\left(\frac{n}{m}\right)\right)K_0(2\pi\sqrt{dn})$$

$$+\sum_{n=1}^{\infty}\Big(\sum_{m|n} r(m) r\Big(\frac{n}{m}\Big)\Big)\frac{1}{2}\int_0^{\infty} e^{-2\pi\sqrt{dn}\frac{y+y^{-1}}{2}}\frac{\log y}{y}dy\Big\}.$$

Since

$$\sum_{m|n}\Big(\frac{1}{2}\log n-\log m\Big) r(m) r\Big(\frac{n}{m}\Big)=\sum_{m|n}\Big(\frac{1}{2}\log n-\log\frac{n}{m}\Big) r\Big(\frac{n}{m}\Big) r(m)$$

$$=\sum_{m|n}\Big(\log m-\frac{1}{2}\log n\Big) r(m) r\Big(\frac{n}{m}\Big)=-\sum_{m|n}\Big(\frac{1}{2}\log n-\log m\Big) r(m) r\Big(\frac{n}{m}\Big),$$

we get

$$\sum_{m|n}\Big(\frac{1}{2}\log n-\log m\Big) r(m) r\Big(\frac{n}{m}\Big)=0.$$

Since

$$\int_0^1 e^{-2\pi\sqrt{dn}\frac{y+y^{-1}}{2}}\frac{\log y}{y}dy=-\int_1^{\infty} e^{-2\pi\sqrt{dn}\frac{y+y^{-1}}{2}}\frac{\log y}{y}dy,$$

we get

$$\int_0^{\infty} e^{-2\pi\sqrt{dn}\frac{y+y^{-1}}{2}}\frac{\log y}{y}dy=0.$$

Consequently, we get

$$G_d'(1)=4\log\frac{\pi^2}{d}\Big(-C_0\frac{\pi}{4}+L'(1,\chi)\Big)+8C_0L'(1,\chi)-2\Big(\log\frac{\pi^2}{d}\Big)^2\frac{\pi}{4}$$
$$+2\pi E(1,d)\Big(\frac{1}{2}\log\frac{\pi^2}{d}+C_0\Big).$$

This is (i) of our Corollary 9.

Let $E'(1,d)$ and $E''(1,d)$ be the first and the second derivative of $E(1,d)$ with respect to d, respectively. Then we see easily that

$$E'(1,d)=-\frac{\pi}{\sqrt{d}}\sum_{n=1}^{\infty}\sqrt{n}\Big(\sum_{m|n} r(m) r\Big(\frac{n}{m}\Big)\Big)K_1(2\pi\sqrt{dn})$$

and

$$E''(1,d)=\frac{\pi}{2d^{\frac{3}{2}}}\sum_{n=1}^{\infty}\sqrt{n}\Big(\sum_{m|n} r(m) r\Big(\frac{n}{m}\Big)\Big)K_1(2\pi\sqrt{dn})$$
$$+\frac{\pi^2}{2d}\sum_{n=1}^{\infty} n\Big(\sum_{m|n} r(m) r\Big(\frac{n}{m}\Big)\Big)\big(K_2(2\pi\sqrt{dn})+K_0(2\pi\sqrt{dn})\big).$$

Thus we see that

$$E'(1, d) < 0 \quad \text{for} \quad d > 0$$

and

$$E''(1, d) > 0 \quad \text{for} \quad d > 0.$$

Hence both

$$-8\left(L(1, \chi)\left(\frac{1}{2}\log d - \log \pi\right) + L'(1, \chi)\right)$$

and

$$2\pi E(1, d)$$

are monotone decreasing convex and continuous function of d.

Since

$$-8\left(L(1, \chi)\left(\frac{1}{2}\log 4 - \log \pi\right) + L'(1, \chi)\right) > 2\pi E(1, 4),$$

$$-8\left(L(1, \chi)\left(\frac{1}{2}\log \frac{1}{16\pi^2} - \log \pi\right) + L'(1, \chi)\right) < 2\pi E\left(1, \frac{1}{16\pi^2}\right),$$

$$E(1, d) > 0 \text{ for any } d > 0,$$

and

$$-8\left(L(1, \chi)\left(\frac{1}{2}\log d - \log \pi\right) + L'(1, \chi)\right) \to -\infty \text{ as } d \to \infty,$$

the equation

$$G_d(1) = 0$$

has at least two real solutions D_1 and D_2, $D_1 < D_2$, in $d > \dfrac{1}{16\pi^2}$.

We notice next that

$$\begin{aligned}\frac{dG_z(1)}{dz} &= \frac{\pi}{\sqrt{z}}\left\{\frac{1}{\sqrt{z}} - 2\pi\sum_{n=1}^{\infty}\sqrt{n}\left(\sum_{m|n} r(m) r\left(\frac{n}{m}\right)\right)K_1(2\pi\sqrt{zn})\right\} \\ &\leq \frac{\pi}{\sqrt{z}}\left\{\frac{1}{\sqrt{z}} - 2\pi \cdot 16 \cdot K_1(2\pi\sqrt{z})\right\} \\ &\leq \frac{\pi}{\sqrt{z}}\left\{\frac{1}{\sqrt{z}} - 16\pi\int_1^{\infty} e^{-2\pi\sqrt{z}y}\left(1 + \frac{1}{y^2}\right)dy\right\} \leq \frac{\pi}{\sqrt{z}}\left\{\frac{1}{\sqrt{z}} - \frac{8}{\sqrt{z}}e^{-2\pi\sqrt{z}}\right\}.\end{aligned}$$

Hence $\dfrac{dG_z(1)}{dz} < 0$, provided that $1 < 8e^{-2\pi\sqrt{z}}$, namely that $0 < z < \left(\dfrac{\log 8}{2\pi}\right)^2$. Hence, $G_z(1)$ is monotone decreasing for $0 < z < \left(\dfrac{\log 8}{2\pi}\right)^2$. On the other hand,

$$\frac{dG_z(1)}{dz} = 0$$

has at most one real solution in $z > \frac{1}{4\pi^2}$. Because

$$\sqrt{z}\sum_{n=1}^{\infty}\sqrt{n}\left(\sum_{m|n} r(m)\, r\left(\frac{n}{m}\right)\right)K_1(2\pi\sqrt{zn})$$

is monotone decreasing for $z > \frac{1}{4\pi^2}$. This can be seen as follows.

$$\frac{d}{dz}\left(\sqrt{z}\sum_{n=1}^{\infty}\sqrt{n}\left(\sum_{m|n} r(m)\, r\left(\frac{n}{m}\right)\right)K_1(2\pi\sqrt{zn})\right)$$

$$= \frac{1}{2\sqrt{z}}\sum_{n=1}^{\infty}\sqrt{n}\left(\sum_{m|n} r(m)\, r\left(\frac{n}{m}\right)\right)\frac{1}{2}\int_0^{\infty} e^{-\pi\sqrt{zn}\,(y+y^{-1})}dy$$

$$-\frac{\pi}{2}\sum_{n=1}^{\infty} n\left(\sum_{m|n} r(m)\, r\left(\frac{n}{m}\right)\right)\frac{1}{2}\int_0^{\infty} e^{-\pi\sqrt{zn}\,(y+y^{-1})}(y+y^{-1})\,dy$$

and

$$\frac{\pi}{2}\sum_{n=1}^{\infty} n\left(\sum_{m|n} r(m)\, r\left(\frac{n}{m}\right)\right)\frac{1}{2}\int_0^{\infty} e^{-\pi\sqrt{zn}\,(y+y^{-1})}(y+y^{-1})\,dy$$

$$\geq \frac{2\pi}{2}\sum_{n=1}^{\infty} n\left(\sum_{m|n} r(m)\, r\left(\frac{n}{m}\right)\right)\frac{1}{2}\int_0^{\infty} e^{-\pi\sqrt{zn}\,(y+y^{-1})}y$$

$$> \frac{1}{2\sqrt{z}}\sum_{n=1}^{\infty}\sqrt{n}\left(\sum_{m|n} r(m)\, r\left(\frac{n}{m}\right)\right)\frac{1}{2}\int_0^{\infty} e^{-\pi\sqrt{zn}\,(y+y^{-1})}dy,$$

provided that $\frac{1}{\sqrt{z}} < 2\pi$, namely that $\frac{1}{4\pi^2} < z$.

Consequently, we see that the equation

$$G_d(1) = 0$$

has exactly two real solutions D_1 and D_2, $D_1 < D_2$, in $d > 0$.

If $d > D_2$, then

$$G_d(1) > 0.$$

Since

$$\lim_{\sigma\to 2-0} G_d(\sigma) = -\infty,$$

$G_d(\sigma)$ must have a zero in the interval $(1, 2)$.

If $0 < d < D_1$, we have also

$$G_d(1) > 0$$

and $G_d(\sigma)$ must have a zero in the interval $(1, 2)$.

This proves (ii) of Theorem 2.

We get at the same time (iii) of Theorem 2.

We shall give proofs to several inequalities used above. It is clear that

$$E(1, d) > 0 \text{ for any } d > 0,$$

and

$$-8\left(L(1, \chi)\left(\frac{1}{2}\log d - \log\pi\right) + L'(1, \chi)\right) \to -\infty \text{ as } d \to \infty.$$

Using lower bound for $K_\nu(z)$ described below, we have

$$\begin{aligned}
2\pi E\left(1, \frac{1}{16\pi^2}\right) &\geq 2\pi \cdot 16 \cdot K_0\left(2\pi\sqrt{\frac{1}{16\pi^2}}\right) \\
&\geq 2\pi \cdot 16 \cdot e^{-\frac{2\pi}{4\pi}}\sqrt{\pi}\left(1 - \frac{1}{16\pi\sqrt{\frac{1}{16\pi^2}}}\right) = 32\pi\sqrt{\frac{\pi}{e}}\frac{3}{4} \\
&> 2\pi(2\log\pi + \log 4) = -8\frac{\pi}{4}\left(\frac{1}{2}\log\frac{1}{16\pi^2} - \log\pi\right) \\
&> -8\frac{\pi}{4}\left(\frac{1}{2}\log\frac{1}{16\pi^2} - \log\pi\right) - 8L'(1, \chi),
\end{aligned}$$

since $L'(1, \chi) > 0$. Finally, using an upper bound for $K_\nu(z)$ described below, we get

$$\begin{aligned}
E(1, 4) &= \sum_{n=1}^{\infty}\left(\sum_{m|n} r(k)r(l)\right)K_0(4\pi\sqrt{n}) \leq \sum_{n=1}^{\infty}\left(\sum_{n=kl} r(k)r(l)\right)e^{-4\pi\sqrt{n}}\sqrt{\frac{\pi}{2\cdot 4\pi\sqrt{n}}} \\
&= \frac{1}{2\sqrt{2}}\sum_{n=1}^{\infty}\left(\sum_{n=kl} r(k)r(l)\right)\frac{e^{-4\pi\sqrt{n}}}{n^{\frac{1}{4}}} \leq \frac{e^{-4\pi}}{2\sqrt{2}}\sum_{n=1}^{\infty}\frac{\left(\sum_{n=kl} r(k)r(l)\right)}{n^2} \\
&= \frac{e^{-4\pi}}{2\sqrt{2}}\left(\sum_{n=1}^{\infty}\frac{r(n)}{n^2}\right)^2 \leq \frac{e^{-4\pi}}{2\sqrt{2}}16\zeta^4(2) = \frac{e^{-4\pi}\pi^8}{162\sqrt{2}}.
\end{aligned}$$

Here we notice that

$$\begin{aligned}
-4\log|\eta(i)| &= \frac{\pi}{3} + 4\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{e^{-2\pi mk}}{k} = \frac{\pi}{3} + 4\sum_{k=1}^{\infty}\frac{1}{k(e^{2\pi k}-1)} \\
&= \frac{\pi}{3} + 4\sum_{k=1}^{50}\frac{1}{k(e^{2\pi k}-1)} + 4\sum_{k=51}^{\infty}\frac{1}{k(e^{2\pi k}-1)}.
\end{aligned}$$

Since

$$4\sum_{k=1}^{50}\frac{1}{k\,(e^{2\pi k}-1)}=0.0074907297990741844625543 17\cdots$$

and

$$4\sum_{k=51}^{\infty}\frac{1}{k\,(e^{2\pi k}-1)}\leq\frac{2}{25}\frac{e^{-100\pi}}{e^{2\pi}-1}\leq 1.091\times 10^{-140},$$

we get

$$-4\log|\eta(i)|=1.054688280995671930616768779\cdots.$$

Since

$$L'(1,\chi)=\frac{\pi}{4}(C_0-2\log 2-4\log|\eta(i)|),$$

we get

$$L'(1,\chi)=0.192901316796912429363 1\cdots.$$

Using this numerical value of $L'(1,\chi)$, we get

$$-8\left(L(1,\chi)\left(\frac{1}{2}\log 4-\log\pi\right)+L'(1,\chi)\right)$$

$$\geq -8\frac{\pi}{4}\left(\frac{1}{2}\log 4-\log\pi\right)-8\cdot 0.192901316797$$

$$>1.29416728>0.0009075>2\pi\frac{e^{-4\pi}\pi^8}{162\sqrt{2}}\geq 2\pi E(1,4).$$

These justify above argument.

We shall next prove that $s=1$ is a double zero for both $G_{D_1}(s)$ and $G_{D_2}(s)$.

We shall notice first that $s=1$ is not a simple zero for both $G_{D_1}(s)$ and $G_{D_2}(s)$. Because if d satisfies

$$G_d(1)=0,$$

then

$$G_d'(1)=0.$$

In other words, if d satisfies

$$-\pi\log\frac{\pi^2}{d}+8L'(1,\chi)+2\pi E(1,d)=0,$$

then

$$8C_0L'(1,\chi)+4\log\frac{\pi^2}{d}\Big(-C_0\frac{\pi}{4}+L'(1,\chi)\Big)-\Big(\log\frac{\pi^2}{d}\Big)^2\frac{\pi}{2}$$
$$+2\pi E(1,d)\Big(\frac{1}{2}\log\frac{\pi^2}{d}+C_0\Big)=0.$$

To proceed further we need to locate D_1 and D_2 more precisely. We shall use several numerical values which come from a rough use of a machine.

We start with recalling Lemma 3 of Bateman-Grosswald [2] which states that for $0\le\nu\le\frac{1}{2}$ and for $z>0$, we have

$$0<\Big(\frac{2z}{\pi}\Big)^{\frac{1}{2}}e^zK_\nu(z)\le 1$$

and

$$1-\frac{1-4\nu^2}{8z}\le\Big(\frac{2z}{\pi}\Big)^{\frac{1}{2}}e^zK_\nu(z)\le 1-\frac{1-4\nu^2}{8z}+\frac{(1-4\nu^2)(9-4\nu^2)}{2!\,(8z)^2}.$$

In particular, we have for any $z>0$,

$$e^{-z}\sqrt{\frac{\pi}{2z}}\Big(1-\frac{1}{8z}\Big)\le K_0(z)\le e^{-z}\sqrt{\frac{\pi}{2z}}\Big(1-\frac{1}{8z}+\frac{9}{128z^2}\Big).$$

In a similar manner we have a more precise approximation as follows.

$$e^{-z}\sqrt{\frac{\pi}{2z}}\Big(1-\frac{1}{8z}+\frac{9}{128z^2}-\frac{75}{1024z^3}\Big)\le K_0(z)$$
$$\le e^{-z}\sqrt{\frac{\pi}{2z}}\Big(1-\frac{1}{8z}+\frac{9}{128z^2}-\frac{75}{1024z^3}+\frac{3675}{32768z^4}\Big).$$

We shall give an upper bound for $E(1,d)$ first. For any

$$d>\frac{1}{4\pi^2}\Big(\frac{75\cdot 128}{9\cdot 1024}\Big)^2,$$

we have

$$E(1,d)\le\sum_{n=1}^{50}\Big(\sum_{m|n}r(m)r\Big(\frac{n}{m}\Big)\Big)e^{-2\pi\sqrt{dn}}\sqrt{\frac{\pi}{4\pi\sqrt{dn}}}\Big(1-\frac{1}{16\pi\sqrt{dn}}$$
$$+\frac{9}{512\pi^2dn}-\frac{75}{8192\pi^3(dn)^{\frac{3}{2}}}+\frac{3675}{524288\pi^4(dn)^2}\Big)$$
$$+\sum_{n=51}^{\infty}\Big(\sum_{m|n}r(m)r\Big(\frac{n}{m}\Big)\Big)e^{-2\pi\sqrt{dn}}\sqrt{\frac{\pi}{4\pi\sqrt{dn}}}\Big(1-\frac{1}{16\pi\sqrt{dn}}$$
$$+\frac{9}{512\pi^2dn}-\frac{75}{8192\pi^3(dn)^{\frac{3}{2}}}+\frac{3675}{524288\pi^4(dn)^2}\Big)=E_1+E_2,\text{ say.}$$

If $d \geq \left(\frac{7\log 51}{4\pi}\right)^2 \frac{1}{51}$ and $n \geq 51$, then $e^{-2\pi\sqrt{dn}} \leq \frac{e^{-\pi\sqrt{51d}}}{n^{\frac{7}{4}}}$. Hence, if $d \geq \left(\frac{7\log 51}{4\pi}\right)^2 \frac{1}{51}$, then we have

$$E_2 \leq \frac{e^{-\pi\sqrt{51d}}}{2d^{\frac{1}{4}}} \sum_{n=51}^{\infty} \frac{\sum_{m|n} r(m) r\left(\frac{n}{m}\right)}{n^2}\left(1 - \frac{1}{16\pi\sqrt{dn}} + \frac{9}{512\pi^2 dn} - \frac{75}{8192\pi^3 (dn)^{\frac{3}{2}}} + \frac{3675}{524288\pi^4 (dn)^2}\right)$$

$$= \frac{8e^{-\pi\sqrt{51d}}}{d^{\frac{1}{4}}}\{\zeta(2)^2 L(2,\chi)^2 - \frac{1}{16\pi\sqrt{d}}\zeta\left(\frac{5}{2}\right)^2 L\left(\frac{5}{2},\chi\right)^2 + \frac{9}{512\pi^2 d}\zeta(3)^2 L(3,\chi)^2$$

$$- \frac{75}{8192\pi^3 d^{\frac{3}{2}}}\zeta\left(\frac{7}{2}\right)^2 L\left(\frac{7}{2},\chi\right)^2 + \frac{3675}{524288\pi^4 d^2}\zeta(4)^2 L(4,\chi)^2\} - \frac{e^{-\pi\sqrt{51d}}}{2d^{\frac{1}{4}}}\{A(2)$$

$$- \frac{1}{16\pi\sqrt{d}} A\left(\frac{5}{2}\right) + \frac{9}{512\pi^2 d} A(3) - \frac{75}{8192\pi^3 d^{\frac{3}{2}}} A\left(\frac{7}{2}\right) + \frac{3675}{524288\pi^4 d^2} A(4)\}$$

$= q_2^*(d)$, say,

where we put

$$A(x) = \sum_{n=1}^{50} \frac{\sum_{m|n} r(m) r\left(\frac{n}{m}\right)}{n^x}.$$

The value of $\sum_{m|n} r(m) r\left(\frac{n}{m}\right)$ at $n = 1, 2, 3, \cdots, 50$ can be evaluated easily. In fact, the corresponding values are

16, 32, 0, 48, 64, 0, 0, 64, 32, 128, 0, 0, 64, 0, 0, 80, 64, 64, 0,
192, 0, 0, 0, 0, 160, 128, 0, 0, 64, 0, 0, 96, 0, 128, 0, 96, 64, 0,
0, 256, 64, 0, 0, 0, 128, 0, 0, 0, 32, 320,

respectively. We notice also thaw by a rough calculation, we get

$$A(2) = 35.0746035004719623386 7\cdots$$
$$A\left(\frac{5}{2}\right) = 25.8012242491000305926\cdots$$
$$A(3) = 21.69437774734406166682\cdots$$
$$A\left(\frac{7}{2}\right) = 19.56275687289406969882\cdots$$
$$A(4) = 18.33050920944640605528\cdots.$$

Since

$$\zeta(2)^2 L(2,\chi)^2 = 2.270153960110982728\cdots,$$
$$\zeta\left(\frac{5}{2}\right)^2 L\left(\frac{5}{2},\chi\right)^2 = 1.619420555337452648\cdots,$$
$$\zeta(3)^2 L(3,\chi)^2 = 1.356592254853679817\cdots,$$
$$\zeta\left(\frac{7}{2}\right)^2 L\left(\frac{7}{2},\chi\right)^2 = 1.222748185229687098\cdots,$$

and

$$\zeta(4)^2 L(4,\chi)^2 = 1.145665531455488938\cdots,$$

we have

$$\begin{aligned} q_2^*(d) \le{} & \frac{8e^{-\pi\sqrt{51d}}}{d^{\frac{1}{4}}}\{2.270153960111 - 1.619420555337\cdot\frac{1}{16\pi\sqrt{d}} \\ & + 1.35659225485368\cdot\frac{9}{512\pi^2 d} - 1.22274818522968\cdot\frac{75}{8192\pi^3 d^{\frac{3}{2}}} \\ & + 1.14566553145549\cdot\frac{3675}{524288\pi^4 d^2}\} - \frac{e^{-\pi\sqrt{51d}}}{2d^{\frac{1}{4}}}\{35.07460350047 \\ & - 25.801222424911\cdot\frac{1}{16\pi\sqrt{d}} + 21.694377747344061\cdot\frac{9}{512\pi^2 d} \\ & - 19.5627568728941\cdot\frac{75}{8192\pi^3 d^{\frac{3}{2}}} + 18.3305092094464\cdot\frac{3675}{524288\pi^4 d^2}\} \\ ={} & q_2^{**}(d),\ \text{say}. \end{aligned}$$

We shall give next a lower bound for $E(1,d)$ as follows. If $d > \frac{(0.402467)^2}{8\pi^2} = 0.002051497\cdots$, then we have

$$\begin{aligned} E(1,d) \ge{} & \left(\sum_{m|1} r(m)r\left(\frac{n}{m}\right)\right)e^{-2\pi\sqrt{d}}\sqrt{\frac{\pi}{4\pi\sqrt{d}}}\left(1-\frac{1}{16\pi\sqrt{d}}\right) \\ & + \sum_{n=2}^{50}\left(\sum_{m|n} r(m)r\left(\frac{n}{m}\right)\right)e^{-2\pi\sqrt{dn}}\sqrt{\frac{\pi}{4\pi\sqrt{dn}}}\left(1-\frac{1}{16\pi\sqrt{dn}}\right. \\ & \left.+\frac{9}{512\pi^2 dn}-\frac{75}{8192\pi^3(dn)^{\frac{3}{2}}}\right) = q_3^{**}(d),\ \text{say}. \end{aligned}$$

We denote E_1 by $q_1^*(d)$.

Now we have the following inequality

$$q_*(d) \le G_d(1) \le q^*(d),$$

where we put

$$q_*(d) = -\pi\log\frac{\pi^2}{d} + 8\times 0.19290131679691242 9363 + 2\pi q_3^{**}(d)$$

and

$$q^*(d) = -\pi\log\frac{\pi^2}{d} + 8\times 0.1929013167969124293632 + 2\pi(q_1^*(d) + q_2^{**}(d)).$$

Furthermore, a rough calculation shows that

$$\begin{aligned} q^*(6.039009) &= -5.704270722439\cdots\times 10^{-8}\\ q^*(6.039001) &= -5.0218085245745\cdots\times 10^{-9}\\ q^*(0.16563) &= -0.0001300811639577\cdots\\ q^*(0.1656) &= 0.0070947906833076\cdots \end{aligned}$$

and that

$$\begin{aligned} q_*(6.039002) &= 5.134382159\cdots\times 10^{-7}\\ q_*(6.039) &= -5.269797505\cdots\times 10^{-7}\\ q_*(0.165155) &= -0.00028315072\cdots\\ q_*(0.165152) &= 0.0000610434307333\cdots. \end{aligned}$$

These imply that $G_d(1) = 0$ at $d = D_1$ and $d = D_2$, where

$$0.165152 < D_1 < 0.165663$$

and

$$6.039001 < D_2 < 6.039002.$$

Hence to complete the proof of (i) of Theorem 2, we have to prove that

$$G_d''(1) \neq 0 \text{ for } d = D_1 \text{ and } D_2.$$

For this purpose, we shall first evaluate $G_d''(1)$. We see first after a simple evaluation that the coefficient of $(s-1)^2$ in the Taylor series expansion of

$$Z(s) + \frac{\pi}{d^{s-1}}\frac{\Gamma(s-1)}{\Gamma(s)}Z(s-1)$$

at $s = 1$ is

$$\begin{aligned} = 4\Big\{&C_2\frac{\pi}{2} + 2C_1L'(1,\chi) + \frac{L'''(1,\chi)}{3} - C_0C_1\frac{\pi}{2} + \frac{\pi\Gamma'''(1)}{12} + \frac{C_0\pi\Gamma''(1)}{4}\\ &+ \log\frac{\pi^2}{d}\Big(-\frac{C_1\pi}{4} + C_0L'(1,\chi) - \frac{L''(1,\chi)}{2}\Big) + \Big(\log\frac{\pi^2}{d}\Big)^2\Big(-\frac{C_0\pi}{8} + \frac{L'(1,\chi)}{2}\Big)\\ &-\Big(\log\frac{\pi^2}{d}\Big)^3\frac{\pi}{24}\Big\}. \end{aligned}$$

Similarly, we see that the coefficient of $(s-1)^2$ in the Taylor series expansion of

$$\left(\frac{\pi}{\sqrt{d}}\right)^s \frac{2\sqrt{d}}{\Gamma(s)} E(s, d)$$

at $s = 1$ is

$$\pi E(1, d)\left\{\frac{1}{4}\left(\log\frac{\pi^2}{d}\right)^2 + C_0\log\frac{\pi^2}{d} + 2C_0^2 - \Gamma''(1)\right\} + \pi E^{(2)}(1, d),$$

where $E^{(2)}(s, d)$ is the second derivative of $E(s,d)$ with respect to s.

Consequently, we get

$$\begin{aligned} G_d''(1) = {} & 4C_2\pi + 16C_1L'(1, \chi) + \frac{8L'''(1, \chi)}{3} - 4C_0C_1\pi + \frac{2\pi\Gamma'''(1)}{3} \\ & + 2C_0\pi\Gamma''(1) + 8\log\frac{\pi^2}{d}\left(-\frac{C_1\pi}{4} + C_0L'(1, \chi) - \frac{L''(1, \chi)}{2}\right) \\ & + 8\left(\log\frac{\pi^2}{d}\right)^2\left(-\frac{C_0\pi}{8} + \frac{L'(1, \chi)}{2}\right) - \left(\log\frac{\pi^2}{d}\right)^3\frac{\pi}{3} \\ & + 2\pi E(1, d)\left\{\frac{1}{4}\left(\log\frac{\pi^2}{d}\right)^2 + C_0\log\frac{\pi^2}{d} + 2C_0^2 - \Gamma''(1)\right\} + 2\pi E^{(2)}(1, d). \end{aligned}$$

This proves (ii) of Corollary 9.

We now complete the proof of Theorem 2.

Suppose that $G_d(1) = 0$. Then

$$2\pi E(1, d) = \pi\log\frac{\pi^2}{d} - 8L'(1, \chi).$$

Substituting this into the explicit formula of $G_d''(1)$ given above, we get

$$\begin{aligned} G_d''(1) = {} & -\frac{\pi}{12}\left(\log\frac{\pi^2}{d}\right)^3 + 2L'(1, \chi)\left(\log\frac{\pi^2}{d}\right)^2 + \log\frac{\pi^2}{d}\left(\pi C_0^2 - \frac{\pi^3}{6}\right. \\ & \left. - 2\pi C_1 - 4L''(1, \chi)\right) + 4C_2\pi + 8L'(1, \chi)\left(2C_1 - C_0^2 + \frac{\pi^2}{6}\right) \\ & + \frac{8L'''(1, \chi)}{3} - 4C_0C_1\pi + \frac{4\pi C_0^3}{3} - \frac{4\pi\zeta(3)}{3} + 2\pi E^{(2)}(1, d), \end{aligned}$$

where we have used the following formulas

$$\Gamma''(1) = C_0^2 + \frac{\pi^2}{6} \text{ and } \Gamma'''(1) = -C_0^3 - \frac{C_0\pi^2}{2} - 2\zeta(3).$$

We put

$$\begin{aligned} \mathrm{f}_1(x) = {} & -\frac{\pi}{12}x^3 + 2L'(1, \chi)x^2 + x\left(\pi C_0^2 - \frac{\pi^3}{6} - 2\pi C_1 - 4L''(1, \chi)\right) \\ & + 4C_2\pi + 8L'(1, \chi)\left(2C_1 - C_0^2 + \frac{\pi^2}{6}\right) \\ & + \frac{8L'''(1, \chi)}{3} - 4C_0C_1\pi + \frac{4\pi C_0^3}{3} - \frac{4\pi\zeta(3)}{3}. \end{aligned}$$

We see that

$$\begin{aligned} f_1'(x) &= -\frac{\pi}{4}\left(x - \frac{8}{\pi}L'(1,\chi)\right)^2 + \frac{16}{\pi}L'(1,\chi)^2 + \pi C_0^2 \\ &\quad -\frac{\pi^3}{6} - 2\pi C_1 - 4L''(1,\chi) \\ &< 0 \qquad \text{for any } x, \end{aligned}$$

because

$$\begin{aligned} &\frac{16}{\pi}L'(1,\chi)^2 + \pi C_0^2 - \frac{\pi^3}{6} - 2\pi C_1 - 4L''(1,\chi) \\ &< \frac{16}{\pi}\cdot(0.192902)^2 + \pi C_0^2 - \frac{\pi^3}{6} - 2\pi\cdot(0.0728) + 4\cdot 0.162741 \\ &< -3.737 < 0. \end{aligned}$$

Hence, $f_1(x)$ is monotone decreasing function of x. We notice that

$$3.5 < 4.08 < \log\frac{\pi^2}{0.165663} < \log\frac{\pi^2}{D_1} < \log\frac{\pi^2}{0.165152} < 4.09\cdots < 4.5$$

and that

$$-36.584 < f_1\left(\log\frac{\pi^2}{D_1}\right) < -22.367.$$

We notice next that

$$0.3 < 0.4912\cdots < \log\frac{\pi^2}{6.039002} < \log\frac{\pi^2}{D_2} < \log\frac{\pi^2}{6.039001} < 0.4912\cdots\cdots < 0.6$$

and that

$$-4.8252 < f_1\left(\log\frac{\pi^2}{D_2}\right) < -3.276.$$

On the other hand,

$$\begin{aligned} E^{(2)}(1,d) &= \sum_{n=1}^{\infty}\left(\frac{1}{2}\log n\right)^2\left(\sum_{m|n} r(m)\, r\left(\frac{n}{m}\right)\right)K_0(2\pi\sqrt{dn}) \\ &\quad + \sum_{n=1}^{\infty}\frac{1}{2}\log n\left(\sum_{m|n}(-\log m)\, r(m)\, r\left(\frac{n}{m}\right)\right)K_0(2\pi\sqrt{dn}) \\ &\quad + \sum_{n=1}^{\infty}\frac{1}{2}\log n\left(\sum_{m|n} r(m)\, r\left(\frac{n}{m}\right)\right)\frac{1}{2}\int_0^{\infty} e^{-2\pi\sqrt{dn}\frac{y+y^{-1}}{2}}\frac{\log y}{y}\,dy \\ &\quad + \sum_{n=1}^{\infty}\frac{1}{2}\log n\left(\sum_{m|n}(-\log m)\, r(m)\, r\left(\frac{n}{m}\right)\right)K_0(2\pi\sqrt{dn}) \end{aligned}$$

$$+\sum_{n=1}^{\infty}\Big(\sum_{m|n}(-\log m)^2 r(m)\, r\Big(\frac{n}{m}\Big)\Big)K_0(2\pi\sqrt{dn})$$

$$+\sum_{n=1}^{\infty}\Big(\sum_{m|n}(-\log m)\, r(m)\, r\Big(\frac{n}{m}\Big)\Big)\frac{1}{2}\int_0^{\infty} e^{-2\pi\sqrt{dn}\frac{y+y^{-1}}{2}}\frac{\log y}{y}dy$$

$$+\sum_{n=1}^{\infty}\frac{1}{2}\log n\Big(\sum_{m|n} r(m)\, r\Big(\frac{n}{m}\Big)\Big)\frac{1}{2}\int_0^{\infty} e^{-2\pi\sqrt{dn}\frac{y+y^{-1}}{2}}\frac{\log y}{y}dy$$

$$+\sum_{n=1}^{\infty}\Big(\sum_{m|n}(-\log m)\, r(m)\, r\Big(\frac{n}{m}\Big)\Big)\frac{1}{2}\int_0^{\infty} e^{-2\pi\sqrt{dn}\frac{y+y^{-1}}{2}}\frac{\log y}{y}dy$$

$$+\sum_{n=1}^{\infty}\Big(\sum_{m|n} r(m)\, r\Big(\frac{n}{m}\Big)\Big)\frac{1}{2}\int_0^{\infty} e^{-2\pi\sqrt{dn}\frac{y+y^{-1}}{2}}\frac{\log^2 y}{y}dy$$

$$=\sum_{n=1}^{\infty}\Big(\sum_{m|n}\log m\,(\log m-\frac{1}{2}\log n)\, r(m)\, r\Big(\frac{n}{m}\Big)\Big)K_0(2\pi\sqrt{dn})$$

$$+\sum_{n=1}^{\infty}\Big(\sum_{m|n} r(m)\, r\Big(\frac{n}{m}\Big)\Big)\int_1^{\infty} e^{-2\pi\sqrt{dn}\frac{y+y^{-1}}{2}}\frac{\log^2 y}{y}dy$$

$= f_2(d)+f_3(d)$ say.

We notice first that since for $n \geq 1$ and for $d > 0$,

$$K_0(2\pi\sqrt{dn}) \leq \frac{1}{2}e^{-2\pi\sqrt{dn}}\frac{1}{d^{\frac{1}{4}}n^{\frac{1}{4}}},$$

we have

$$K_0(2\pi\sqrt{D_1 n}) \leq \frac{e^{-6\pi\sqrt{D_1}}}{2D_1^{\frac{1}{4}}n^2} \text{ for } n \geq 28$$

and

$$K_0(2\pi\sqrt{D_2 n}) \leq \frac{e^{-2\pi\sqrt{D_2}}}{2D_2^{\frac{1}{4}}n^2} \text{ for } n \geq 1.$$

Hence, we get

$$0 < f_2(D_1) \leq \frac{1}{2D_1^{\frac{1}{4}}}\sum_{n=1}^{28}\frac{\sum_{m|n}\log m\cdot\Big(\log m-\frac{1}{2}\log n\Big)r(m)\,r\Big(\frac{n}{m}\Big)}{n^{\frac{1}{4}}}e^{-2\pi\sqrt{D_1 n}}$$

$$+\frac{e^{-6\pi\sqrt{D_1}}}{2D_1^{\frac{1}{4}}}\Big\{\sum_{n=1}^{\infty}\frac{\sum_{m|n}\log m\cdot(\log m-\frac{1}{2}\log n)\,r(m)\,r\Big(\frac{n}{m}\Big)}{n^2}$$

$$-\sum_{n=1}^{28}\frac{\sum_{m|n}\log m\cdot\left(\log m-\frac{1}{2}\log n\right)r(m)r\left(\frac{n}{m}\right)}{n^2}\Bigg\}.$$

We notice next that

$$\sum_{n=1}^{\infty}\frac{\sum_{m|n}\log m\cdot(\log m-\frac{1}{2}\log n)r(m)r\left(\frac{n}{m}\right)}{n^2}$$

$$=\frac{1}{2}\Big(\sum_{n=1}^{\infty}\frac{r(m)}{m^2}\log^2 m\Big)\cdot\Big(\sum_{n=1}^{\infty}\frac{r(m)}{m^2}\Big)-\frac{1}{2}\Big(\sum_{n=1}^{\infty}\frac{r(m)}{m^2}\log m\Big)^2$$

$$=\frac{1}{2}(Z''(2)Z(2)-Z'(2)^2)=\frac{1}{2}Z^2(2)\left(\frac{Z'}{Z}\right)'(2)$$

$$\leq\frac{1}{2}(4\zeta(2)L(2,\chi))^2\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^2}\log n\cdot(1+\chi(n))$$

$$\leq 16\zeta^2(2)|\zeta(2)\zeta''(2)-\zeta'(2)^2|.$$

Since for $L>L_0$,

$$|\zeta''(2)-\sum_{n=2}^{L}\frac{\log^2 n}{n^2}|\leq\frac{\log L}{L}\left(\log L+4+\frac{2\log L}{L}\right)=R_1(L),\text{ say,}$$

and

$$|\zeta'(2)+\sum_{n=2}^{L}\frac{\log n}{n^2}|\leq\frac{1}{L}\left(\log L+1+\frac{2\log L}{L}\right)=R_2(L),\text{ say,}$$

we get

$$|\zeta(2)\zeta''(2)-\zeta'(2)^2|\leq\left|\frac{\pi^2}{6}\sum_{n=2}^{L}\frac{\log^2 n}{n^2}-\Big(\sum_{n=2}^{L}\frac{\log n}{n^2}\Big)^2\right|+\frac{\pi^2}{6}R_1(L)$$
$$+2R_2(L)\sum_{n=2}^{L}\frac{\log n}{n^2}+R^2_2(L).$$

Since

$$\sum_{n=2}^{1000}\frac{\log^2 n}{n^2}=1.925771492486574817\cdots,$$

$$\sum_{n=2}^{1000}\frac{\log n}{n^2}=0.929643951846542088\cdots,$$

$$R_1(1000) \leq 0.07544353827622274147$$

and

$$R_2(1000) \leq 0.00792157078954010 1327,$$

we get

$$|\zeta(2)\zeta''(2) - \zeta'(2)^2| \leq 2.442420134016605379.$$

Thus we get

$$\sum_{n=1}^{\infty} \frac{\sum_{m|n} \log m \cdot \left(\log m - \frac{1}{2}\log n\right) r(m) r\left(\frac{n}{m}\right)}{n^2} \leq 105.7395223012017539.$$

The values of $\sum_{m|n} \log m \cdot \left(\log m - \frac{1}{2}\log n\right) r(m) r\left(\frac{n}{m}\right)$ for $n=1,2,3,\cdots,28$ are

$0, 8 \cdot \log^2 2, 0, 8 \cdot \log^2 4, 16 \cdot \log^2 5, 0, 0,$
$80 \cdot \log^2 2, 8 \cdot \log^2 9, 32 \cdot (\log^2 2 + \log^2 5)$
$0, 0, 16 \cdot \log^2 13, 0, 0, 16 \cdot (\log^2 2 + \log^2 8), 16\log^2 17,$
$16 \cdot (\log^2 2 + \log^2 9), 0, 16 \cdot (3\log^2 5 + 2\log^2 4),$
$0, 0, 0, 0, 24 \cdot \log^2 25, 32 \cdot (\log^2 2 + \log^2 13), 0, 0,$

respectively. Hence we get

$$\sum_{n=1}^{28} \frac{\sum_{m|n} \log m \cdot \left(\log m - \frac{1}{2}\log n\right) r(m) r\left(\frac{n}{m}\right)}{n^2} = 8.465929760628084512\cdots.$$

Similarly, we get

$$\sum_{n=1}^{28} \frac{\sum_{m|n} \log m \cdot \left(\log m - \frac{1}{2}\log n\right) r(m) r\left(\frac{n}{m}\right)}{n^{\frac{1}{4}}} e^{-2\pi\sqrt{D_1 n}}$$

$$\leq 0.000471188781950964.$$

Combining all of these estimates, we get

$$0 < f_2(D_1) \leq 0.271564050120363.$$

In the same manner, we get

$$0 < f_2(D_2) \leq \frac{e^{-2\pi\sqrt{D_2}}}{2D_2^{\frac{1}{4}}} \sum_{n=1}^{\infty} \frac{\sum_{m|n} \log m \cdot (\log m - \frac{1}{2}\log n) r(m) r\left(\frac{n}{m}\right)}{n^2}$$

$$\leq \frac{e^{-2\pi\sqrt{D_2}}}{2D_2^{\frac{1}{4}}} 16\zeta^2(2)|\zeta(2)\zeta''(2) - \zeta'(2)^2|$$

$$\leq \frac{e^{-2\pi\sqrt{D_2}}}{2D_2^{\frac{1}{4}}} \times 105.74 \leq 6.640912062 \times 10^{-6}.$$

We shall next estimate $f_3(d)$. For this purpose we put, for simplicity, $W = \pi\sqrt{d}$ and $V = e^{\frac{3+\sqrt{5}}{2}}$. Then we get for $d > 0.16$,

$$f_3(d) \leq \sum_{n=1}^{\infty}\Big(\sum_{m|n} r(m) r\Big(\frac{n}{m}\Big)\Big)\int_1^{\infty} e^{-W\sqrt{n}y} \frac{\log^2 y}{y}\, dy$$

$$\leq \frac{2}{W^2}\sum_{n=1}^{\infty}\frac{1}{n}\Big(\sum_{m|n} r(m) r\Big(\frac{n}{m}\Big)\Big) e^{-WV\sqrt{n}} \int_V^{\infty} \frac{\log^2 y}{y^3}\, dy$$

$$\leq \frac{\log^2 V}{(VW)^2} e^{-WV}\Big(\sum_{m=1}^{\infty}\frac{r(m)}{m^2}\Big)^2 \leq \frac{\log^2 V}{(VW)^2} e^{-WV} 16\zeta(2)^4.$$

Hence, we get

$$f_3(D_1) \leq 6.56797543 \times 10^{-8}$$

and

$$f_3(D_2) \leq 7.795 \times 10^{-48}.$$

Consequently, we get

$$E^{(2)}(1, D_1) \leq 0.271564116.$$

and

$$E^{(2)}(1, D_2) \leq 0.000006640912063.$$

Hence, we get

$$G''_{D_1}(1) < -22.36 + 1.71 < -20 < 0$$

and

$$G''_{D_2}(1) < -3.27 + 2\pi \times 0.000006641 < -3 < 0.$$

Thus $s = 1$ is a double zero for $d = D_1$ or D_2.

This completes the proof of Theorem 2.

Finally, we shall give a proof of Corollary 10 briefly. We shall describe the Taylor expansion of $Z(s)$ as $s=1$ in two ways.

$$Z(s) = 4\zeta(s)L(s, \chi) = \frac{\pi}{s-1} + 4\Big(C_0\frac{\pi}{4} + L'(1, \chi)\Big)$$

$$+4\left(C_1\frac{\pi}{4}+C_0L'(1,\chi)+\frac{L''(1,\chi)}{2}\right)(s-1)$$
$$+4\left(C_2\frac{\pi}{4}+C_1L'(1,\chi)+\frac{C_0L''(1,\chi)}{2}+\frac{L'''(1,\chi)}{6}\right)(s-1)^2+\cdots.$$

On the other hand, Chowla-Selberg's formula gives us

$$Z(s)=2\zeta(2s)+2\zeta(2s-1)\frac{\Gamma\left(s-\frac{1}{2}\right)\sqrt{\pi}}{\Gamma(s)}+\frac{4\pi^s}{\Gamma(s)}D(s).$$

It is easily seen that

$$2\zeta(2s)+2\zeta(2s-1)\frac{\Gamma\left(s-\frac{1}{2}\right)\sqrt{\pi}}{\Gamma(s)}=\frac{\pi}{s-1}+(2\zeta(2)+B_0)$$
$$+(s-1)(4\zeta'(2)+B_1)+(s-1)^2(4\zeta''(2)+B_2)+\cdots,$$

where we put

$$B_0=2\pi(C_0-\log 2),$$
$$B_1=4\pi C_1-4\pi C_0\log 2+\frac{\pi^3}{6}+2\pi(\log 2)^2$$

and

$$B_2=-2\pi\zeta(3)-\frac{1}{3}\pi^3\log 2-\frac{4}{3}\pi(\log 2)^3+8\pi C_2-8\pi C_1\log 2$$
$$+\frac{1}{3}C_0\pi^3+4\pi C_0(\log 2)^2.$$

Finally,

$$\frac{4\pi^2}{\Gamma(s)}D(s)=4\pi D(1)+(s-1)(4\pi D'(1)+4\pi D(1)(\log\pi+C_0))$$
$$+(s-1)^2(2\pi D''(1)+4\pi D'(1)(\log\pi+C_0)+2\pi D(1)((\log\pi)^2$$
$$+2C_0\log\pi-\Gamma''(1)+2C^2_0))+\cdots.$$

Comparing the coefficients, we get first

$$L''(1,\chi)=\frac{3}{2}C_1\pi+2\zeta'(2)-\frac{\pi}{2}C^2_0-\pi\log 2+\frac{\pi^3}{12}+\pi(\log 2)^2$$
$$+2C_0\pi\log|\eta(i)|+2\pi D(1)(\log\pi+C_0)+2\pi D'(1)$$

and

$$L'''(1,\chi)=\frac{21}{2}C_2\pi-9\pi C_1\log 2+3\pi C_0(\log 2)^2+\frac{1}{4}C_0\pi^3-6\pi C_0C_1$$
$$-2\pi(\log 2)^3-\frac{1}{2}\pi^3\log 2-3\pi\zeta(3)+6\zeta''(2)-6C_0\zeta'(2)$$

$$+\frac{3}{2}\pi C_0^3+3\pi C_0^2\log 2+(6C_1\pi-6C_0^2\pi)\log|\eta(i)|$$

$$+3\pi D(1)\left((\log\pi)^2-C_0^2-\frac{\pi^2}{6}\right)+6\pi D'(1)\log\pi+3\pi D''(1).$$

Since

$$D(1)=\sum_{n=1}^{\infty}\sum_{d|n}\frac{1}{d}e^{-2\pi n}=-\frac{\pi}{12}-\log|\eta(i)|,$$

we get the expressions as stated in Corollary 10.

§ 5. Proof of (iii) of Corollary 5, Theorem 3 and Corollary 3

Suppose that $0\leq x<\frac{1}{2}$. We start with the formula

$$G_d(x,Q)=Z(x,Q)+\frac{2}{\sqrt{|\Delta|}}\frac{\pi}{d^{x-1}}\frac{\Gamma(x-1)}{\Gamma(x)}Z(x-1,Q)$$
$$+\left(\frac{2\pi}{\sqrt{d|\Delta|}}\right)^x\frac{2\sqrt{d}}{\Gamma(x)}E(x,d,Q).$$

Since for $0\leq x<\frac{1}{2}$

$$\left|K_{x-1}\left(\frac{4\pi}{\sqrt{|\Delta|}}\sqrt{dn}\right)\right|=\left|K_{1-x}\left(\frac{4\pi}{\sqrt{|\Delta|}}\sqrt{dn}\right)\right|$$
$$\leq e^{-\frac{4\pi}{\sqrt{|\Delta|}}\sqrt{dn}}\left(\frac{|\Delta|}{dn}\right)^{\frac{1}{4}}\frac{1}{2\sqrt{2}}\left(1+\frac{\frac{3}{2}-x}{8\pi}\sqrt{\frac{|\Delta|}{dn}}\right),$$

we have

$$|E(x,d,Q)|\ll\frac{1}{d^{\frac{1}{4}}}\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{4}-\frac{x}{2}}}\left(\sum_{m|n}m^{1-x}r(m)r\left(\frac{n}{m}\right)\right)e^{-\frac{4\pi}{\sqrt{|\Delta|}}\sqrt{dn}}\left(1+\frac{1}{\sqrt{dn}}\right),$$

where we have used Lemma 2 of Bateman and Grosswald [2], in stead of Lemma 3 of Bateman and Grosswald [2] used in the previous section. The right hand side is bounded for any $d>0$ and if d is sufficiently large, then it is

$$\ll\frac{1}{d^A}\sum_{n=1}^{\infty}\frac{1}{n^A}\ll\frac{1}{d^A},$$

where A is a sufficiently large constant and we have used a trivial upper bound

$$r_Q(n)\leq|\{(x,y);ax^2+bxy+cy^2\leq n\}|\ll n.$$

Since $E(x, d, Q)$ is bounded for $0 \le x < \frac{1}{2}$, for any $d > 0$ and for any $|\Delta|$ and $Z(-1, Q) = 0$, we have

$$G_d(0, Q) = Z(0, Q).$$

Using the functional equation and the Kronecker's limit formula, for $Z(s, Q)$, we get at $s = 0$,

$$\begin{aligned}
Z(s, Q) &= \left(\frac{|\Delta|}{4}\right)^{\frac{1}{2}-s} \pi^{2s-1} \frac{\Gamma(1-s)}{\Gamma(s)} Z(1-s, Q) \\
&= \left(\frac{|\Delta|}{4}\right)^{\frac{1}{2}-s} \pi^{2s-1} \frac{s\Gamma(1-s)}{\Gamma(s-1)} \left\{ \frac{\frac{2\pi}{\sqrt{|\Delta|}}}{-s} \right. \\
&\quad \left. + \frac{2\pi}{\sqrt{|\Delta|}} \left(2C_0 + \log\frac{a}{|\Delta|} - 2\log\left|\eta\left(\frac{b}{2a} + i\frac{\sqrt{|\Delta|}}{2a}\right)\right|^2\right) - sA_1 + \cdots \right\} \\
&= -\frac{2\pi}{\sqrt{|\Delta|}} \left(\frac{|\Delta|}{4}\right)^{\frac{1}{2}-s} \pi^{2s-1} \frac{\Gamma(1-s)}{\Gamma(s+1)} + sA_2 + \cdots.
\end{aligned}$$

Hence, we get

$$Z(0, Q) = -1.$$

Consequently, we get

$$G_d(0, Q) = Z(0, Q) = -1 \neq 0.$$

This is (iii) of Corollary 5.

Now by the functional equation of $Z(s, Q)$, we have for any $\frac{1}{2} > x > 0$

$$\begin{aligned}
Z(-x-1, Q) &= \left(\frac{2\pi}{\sqrt{|\Delta|}}\right)^{-3-2x} \frac{\Gamma(2+x)}{\Gamma(-x-1)} Z(2+x, Q) \\
&= \left(\frac{2\pi}{\sqrt{|\Delta|}}\right)^{-3-2x} \frac{-x(-1-x)\Gamma(2+x)}{\Gamma(1-x)} Z(2+x, Q) > 0
\end{aligned}$$

and

$$\begin{aligned}
Z(x-1, Q) &= \left(\frac{2\pi}{\sqrt{|\Delta|}}\right)^{2x-3} \frac{\Gamma(2-x)}{\Gamma(x-1)} Z(2-x, Q) \\
&= \left(\frac{2\pi}{\sqrt{|\Delta|}}\right)^{2x-3} \frac{(x-1)\Gamma(2-x)}{\Gamma(x)} Z(2-x, Q) < 0.
\end{aligned}$$

Hence, using again the fact that $Z(0, Q) = -1$, $G_d(x, Q)$ must have a zero near $x = 0$ when d is sufficiently large.

We shall now locate it more precisely. We suppose that d is sufficiently large.

For this purpose we shall solve the following equation for $\frac{1}{2} > x > 0$.

$$d^{1-x}=\frac{\sqrt{|\Delta|}}{2}\frac{1-x}{Z(x-1,Q)\pi}\Big(Z(x,Q)+2\frac{(2\pi)^x}{\Gamma(x)|\Delta|^{\frac{x}{2}}}d^{\frac{1-x}{2}}E(x,d,Q)\Big).$$

By the functional equation, this is equivalent to

$$d^{1-x}=\frac{\sqrt{|\Delta|}}{2\pi}\frac{1}{Z(2-x,Q)\Gamma(2-x)}\Big(-\Big(\frac{2\pi}{\sqrt{|\Delta|}}\Big)^2\Gamma(1-x)Z(1-x,Q)$$
$$-2d^{\frac{1-x}{2}}\Big(\frac{2\pi}{\sqrt{|\Delta|}}\Big)^{3-x}E(x,d,Q)\Big).$$

Since at $s=1$,

$$Z(s,Q)=\frac{\frac{2\pi}{\sqrt{|\Delta|}}}{s-1}+A_0+(s-1)A_1+\cdots,$$

where A_0 denotes the constant term in Kronecker's limit formula mentioned in the section 1 and A_1 is some constant. Hence, we get

$$Z(1-x,Q)=-\frac{\frac{2\pi}{\sqrt{|\Delta|}}}{x}+A_0-xA_1+\cdots.$$

Hence our equation to be solsed becomes

$$d^{1-x}=\frac{1}{xZ(2-x,Q)\Gamma(2-x)\frac{2\pi}{\sqrt{|\Delta|}}}\Big\{\Big(\frac{2\pi}{\sqrt{|\Delta|}}\Big)^2\Gamma(1-x)\Big(\frac{2\pi}{\sqrt{|\Delta|}}-xA_0$$
$$+x^2A_1+\cdots\Big)-2x\sqrt{d}\Big(\frac{1}{\sqrt{d}}\Big)^x\Big(\frac{2\pi}{\sqrt{|\Delta|}}\Big)^{3-x}E(x,d,Q)\Big\}$$
$$=\frac{1}{xZ(2-x,Q)\Gamma(2-x)\frac{2\pi}{\sqrt{|\Delta|}}}\Big\{\Big(\frac{2\pi}{\sqrt{|\Delta|}}\Big)^3+O(x(1+d^{-A}))\Big\}.$$

Taking a logarithm, this is reduced to

$$(1-x)\log d=\log\Bigg(\frac{1}{Z(2-x,Q)\Gamma(2-x)\frac{2\pi}{\sqrt{|\Delta|}}}\Bigg)$$
$$-\log x+\log\Big(\Big(\frac{2\pi}{\sqrt{|\Delta|}}\Big)^3\Big)+O(x(1+d^{-A})).$$

Dividing by $\log d$, we get

$$x=1+\frac{\log x}{\log d}+\frac{1}{\log d}\log\Bigg(\frac{\frac{2\pi}{\sqrt{|\Delta|}}Z(2,Q)}{\Big(\frac{2\pi}{\sqrt{|\Delta|}}\Big)^3}\Bigg)+O\Big(\frac{x(1+d^{-A})}{\log d}\Big).$$

Here we put

$$x=\frac{1}{d}\frac{4\pi^2}{|\Delta|Z(2,Q)}\Big(1+\frac{\log d}{d}\frac{4\pi^2}{|\Delta|Z(2,Q)}+y\Big).$$

Then we have

$$\frac{1}{d}\frac{4\pi^2}{|\Delta|Z(2,Q)}\Big(1+\frac{\log d}{d}\frac{4\pi^2}{|\Delta|Z(2,Q)}+y\Big)$$
$$=1+\frac{1}{\log d}\log\Big(\frac{1}{d}\frac{4\pi^2}{|\Delta|Z(2,Q)}\Big(1+\frac{\log d}{d}\frac{4\pi^2}{|\Delta|Z(2,Q)}+y\Big)\Big)$$
$$+\frac{1}{\log d}\log\frac{|\Delta|Z(2,Q)}{4\pi^2}+O\Big(\frac{1}{d\log d}(1+y)\Big).$$

This is simplified to

$$\frac{1}{d}\frac{4\pi^2}{|\Delta|Z(2,Q)}\Big(1+\frac{\log d}{d}\frac{4\pi^2}{|\Delta|Z(2,Q)}+y\Big)$$
$$=\frac{1}{\log d}\log\Big(1+\frac{\log d}{d}\frac{4\pi^2}{|\Delta|Z(2,Q)}+y\Big)+O\Big(\frac{1}{d\log d}(1+y)\Big).$$

At this stage we see easily that y satisfies

$$y=O\Big(\frac{1}{d}\Big)$$

and get

$$x=\frac{1}{d}\frac{4\pi^2}{|\Delta|Z(2,Q)}\Big(1+\frac{\log d}{d}\frac{4\pi^2}{|\Delta|Z(2,Q)}+O\Big(\frac{1}{d}\Big)\Big).$$

This proves our Theorem 3.

In the same manner, we can prove Corollary 3. We shall give only a sketch of the proof.

We suppose that $k\gg 1$ and $0\leq x<\frac{1}{2}$. By Chowla-Selberg's formula for $Z(s,Q)$ and the functional equation of $\zeta(s)$, we have for $0<x<\frac{1}{2}$,

$$\frac{1}{2}Z(x,Q)a^x\Gamma(x)\pi^{1-2x}$$
$$=(k^{1-2x}\zeta(2-2x)\Gamma(1-x)+\zeta(1-2x)\Gamma\Big(\frac{1}{2}-x\Big)\sqrt{\pi})$$
$$+4\pi^{1-x}k^{\frac{1}{2}-x}\sum_{n=1}^{\infty}n^{x-\frac{1}{2}}\sigma_{1-2x}(n)\cos\Big(\frac{n\pi b}{a}\Big)K_{x-\frac{1}{2}}(2\pi nk)$$
$$=U_1(x)+U_2(x),\ \text{say}.$$

$U_1(x)$ and $U_2(x)$ are continuous functions.

$$\begin{aligned}|K_{x-\frac{1}{2}}(2\pi nk)| &= |K_{\frac{1}{2}-x}(2\pi nk)|\\ &\ll e^{-2\pi nk}\sqrt{\frac{1}{nk}}\Big(1+\frac{1-x}{4\pi nk}\Big),\end{aligned}$$

we have

$$\begin{aligned}U_2(x) &\ll k^{\frac{1}{2}-x}\Big|\sum_{n=1}^{\infty}n^{x-\frac{1}{2}}\sigma_{1-2x}(n)\cos\Big(\frac{n\pi b}{a}\Big)K_{x-\frac{1}{2}}(2\pi nk)\Big|\\ &\ll k^{\frac{1}{2}-x}\sum_{n=1}^{\infty}n^{x-\frac{1}{2}}\sigma_{1-2x}(n)e^{-2\pi nk}\frac{1}{\sqrt{nk}}\Big(1+\frac{1-x}{4\pi nk}\Big)\\ &\ll \frac{1}{k^A}\end{aligned}$$

for $0 < x < \frac{1}{2}$, where A is a sufficiently large constant.

The Laurent expansion of $U_1(x)$ at $x=0$ is

$$\begin{aligned}U_1(x) = -\frac{\pi}{2x} + k\zeta(2) + C_0\pi + \frac{1}{2}\Gamma'\Big(\frac{1}{2}\Big)\sqrt{\pi} + \Big\{k(-2\zeta(2)\log k - 2\zeta'(2)\\ -\Gamma'(1)\zeta(2)) - 2C_1\pi - C_0\sqrt{\pi}\Gamma'\Big(\frac{1}{2}\Big) - \frac{1}{4}\sqrt{\pi}\Gamma''\Big(\frac{1}{2}\Big)\Big\}x + \cdots.\end{aligned}$$

Hence we see that

$$\lim_{x\to+0} U_1(x) = -\infty$$

and

$$U_1\Big(\frac{1}{A\log k}\Big) \geq A'k,$$

where A is sufficiently large and A' is some positive constant.

Hence, $U_1(x) + U_2(x) = 0$ must have a solution in $0 < x < \frac{1}{2}$. In fact, it has only one solution.

To lacate it more precisely, we start with the following equation.

$$\begin{aligned}k^{1-2x} = \frac{-\Gamma(x)\pi^{1-2x}}{\zeta(2-2x)\Gamma(1-x)}\Big(\frac{\zeta(1-2x)\zeta\big(\frac{1}{2}-x\big)\pi^{-\frac{1}{2}+2x}}{\Gamma(x)}\\ + \frac{4\pi^x k^{\frac{1}{2}-x}}{\Gamma(x)}\sum_{n=1}^{\infty}n^{x-\frac{1}{2}}\sigma_{1-2x}(n)\cos\Big(\frac{n\pi b}{a}\Big)K_{x-\frac{1}{2}}(2\pi nk)\Big).\end{aligned}$$

Using the Laurent expansion of $\zeta(s)$ at $s=1$ and the above estimate on the last sum, the problem is reduced to

$$k^{1-2x}=\frac{1}{x\zeta(2-2x)\Gamma(1-x)}\Big(\frac{\pi}{2}+O(x(1+k^{-A})\Big).$$

Taking a logarithm, it is reduced to

$$x=\frac{1}{2}+\frac{1}{2\log k}\log\Big(\frac{\zeta(2)2}{\pi}\Big)+\frac{\log x}{2\log k}+O\Big(\frac{x}{\log k}(1+k^{-A})\Big).$$

We take

$$x=\frac{1}{k}\frac{\pi}{\zeta(2)2}\Big(1+\frac{2\log k}{k}\frac{\pi}{\zeta(2)2}+y\Big)$$

and get our Corollary 3 as described in the introduction.

§ 6. Proof of Corollaries 1 and 2

We proceed in the same manner as in the section 4. By Chowla-Selberg's formula stated in the section 1, we get the following expansion at $s=\frac{1}{2}$ for a general Q.

$$\begin{aligned}Z(s,Q)=\ &a^{-\frac{1}{2}}\Big\{\log\frac{|\Delta|}{64a^2\pi^2}+2C_0+4\sum_{n=1}^{\infty}\sigma_0(n)\cos\Big(\frac{n\pi b}{a}\Big)\int_0^{\infty}y^{-1}e^{-\pi nk(y+y^{-1})}dy\Big\}\\&+\Big(s-\frac{1}{2}\Big)2\,a^{-\frac{1}{2}}\Big\{C_0^2-4\log^2 2-4\log 2\cdot\log\frac{\pi}{k}-\Big(\log\frac{\pi}{k}\Big)^2\\&-\log a\cdot\Big(C_0-\log\frac{\pi}{k}-2\log 2\Big)+2\Big(\log\frac{\pi}{ak}+C_0\\&+2\log 2\Big)\sum_{n=1}^{\infty}\sigma_0(n)\cos\Big(\frac{n\pi b}{a}\Big)\int_0^{\infty}y^{-1}e^{-\pi nk(y+y^{-1})}dy\Big\}+\cdots.\end{aligned}$$

Thus we get

$$\begin{aligned}Z'\Big(\frac{1}{2},Q\Big)=\ &2a^{-\frac{1}{2}}\Big\{C_0^2-4\log^2 2-4\log 2\cdot\log\frac{\pi}{k}-\Big(\log\frac{\pi}{k}\Big)^2\\&-\log a\cdot(C_0-\log\frac{\pi}{k}-2\log 2)+2\Big(\log\frac{\pi}{ak}+C_0+2\log 2\Big)\\&\cdot\sum_{n=1}^{\infty}\sigma_0(n)\cos\Big(\frac{n\pi b}{a}\Big)\int_0^{\infty}y^{-1}e^{-\pi nk(y+y^{-1})}dy\Big\}.\end{aligned}$$

This proves our Corollary 2.

We now suppose that $\frac{b}{a}$ is an even integer and put

$$H(k)=\sum_{n=1}^{\infty}\sigma_0(n)\int_0^{\infty}y^{-1}e^{-\pi nk(y+y^{-1})}dy.$$

Then

$$\sqrt{a}\,Z\left(\frac{1}{2}, Q\right) = 2\log k - 2\log 4\pi + 2C_0 + 4H(k) = J(k), \text{ say.}$$

We notice first that

$$H'(k) = -2\pi \sum_{n=1}^{\infty} n\sigma_0(n) \int_0^{\infty} e^{-\pi nk(y+y^{-1})} dy < 0$$

for $k > 0$ and

$$H''(k) = 2\pi^2 \sum_{n=1}^{\infty} n^2\sigma_0(n) \left(\int_0^{\infty} y e^{-\pi nk(y+y^{-1})} dy + \int_0^{\infty} y^{-1} e^{-\pi nk(y+y^{-1})} dy \right) > 0$$

for $k > 0$.

Hence

$$-2\log k + 2\log 4\pi - 2C_0$$

and

$$4H(k)$$

are monotone decreasing convex continuous function of $k > 0$.

Since $J(4) < 0$, $J\left(\frac{1}{4\pi}\right) > 0$ and

$$J(k) \to \infty \quad \text{as} \quad k \to \infty,$$

the equation $J(k) = 0$ has at least two real solutions k_1 and k_2 in

$$k_2 > k_1 > \frac{1}{4\pi}.$$

We notice next that

$$J'(k) = \frac{2}{k} + 4H'(k) = \frac{2}{k} - 8\pi \sum_{n=1}^{\infty} \sigma_0(n) \int_0^{\infty} e^{-\pi nk(y+y^{-1})} dy.$$

Since

$$\int_0^{\infty} e^{-\pi nk(y+y^{-1})} dy = \int_0^{\infty} e^{-\pi nk(y+y^{-1})} \left(1 + \frac{1}{y^2}\right) dy$$
$$\geq \int_1^{\infty} e^{-2\pi nky} dy \geq \frac{e^{-2\pi nk}}{2\pi nk},$$

we get

$$J'(k) \leq \frac{2}{k} - \frac{4}{k}\sum_{n=1}^{\infty}\sigma_0(n)\,e^{-2\pi nk} = \frac{2}{k}(1 - 2\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}e^{-2\pi mjk})$$

$$= \frac{2}{k}\Big(1 - 2\sum_{m=1}^{\infty}\frac{1}{e^{2\pi mk}-1}\Big) < \frac{2}{k}\Big(1 - 2\frac{1}{e^{2\pi k}-1}\Big) < 0,$$

provided that $1 < \frac{2}{e^{2\pi k}-1}$, namely, that $0 < k < \frac{\log 3}{2\pi}$. Hence $J(k)$ is monotone decreasing for $0 < k < \frac{\log 3}{2\pi}$. Moreover, $J'(k) = 0$ has at most one real positive solution, because $k\sum_{n=1}^{\infty} n\sigma_0(n)\int_0^{\infty} e^{-\pi nk(y+y^{-1})}dy$ is monotone decreasing for $k > 0$. This can be seen as follows.

$$\frac{d}{dk}\Big(k\sum_{n=1}^{\infty} n\sigma_0(n)\int_0^{\infty} e^{-\pi nk(y+y^{-1})}dy\Big) = \sum_{n=1}^{\infty} n\sigma_0(n)\int_0^{\infty} e^{-\pi nk(y+y^{-1})}dy$$

$$- \pi\sum_{n=1}^{\infty} n^2\sigma_0(n)\int_0^{\infty} e^{-\pi nk(y+y^{-1})}(y + y^{-1})dy$$

and

$$\pi\sum_{n=1}^{\infty} n^2\sigma_0(n)\int_0^{\infty} e^{-\pi nk(y+y^{-1})}(y + y^{-1})dy$$

$$\geq \pi\sum_{n=1}^{\infty} n^2\sigma_0(n)\int_0^{\infty} e^{-\pi nk(y+y^{-1})}dy > \sum_{n=1}^{\infty} n\sigma_0(n)\int_0^{\infty} e^{-\pi nk(y+y^{-1})}dy.$$

Consequently, $J(k) = 0$ has exactly two real solutions k_1 and k_2 in $k_2 > k_1 > 0$.

We shall justify some of the above arguments. First, we shall prove that

$$J\Big(\frac{1}{4\pi}\Big) > 0.$$

$$4H\Big(\frac{1}{4\pi}\Big) = 8\sum_{n=1}^{\infty}\sigma_0(n)K_0\Big(2\pi n\frac{1}{4\pi}\Big) \geq 8K_0\Big(2\pi\frac{1}{4\pi}\Big) \geq 8\sqrt{\frac{\pi}{e}}\,\frac{3}{4} > 6.45$$

$$> -2C_0 = -2\log\frac{1}{4\pi} + 2\log 4\pi - 2C_0.$$

Hence, we get $J\Big(\frac{1}{4\pi}\Big) > 0$.

We shall next show that

$$J(4) < 0.$$

$$4H(4) = 8\sum_{n=1}^{\infty}\sigma_0(n)K_0(8\pi n) \le 8\sum_{n=1}^{\infty}\sigma_0(n)\sqrt{\frac{\pi}{2\cdot 8\pi n}}e^{-8\pi n}$$
$$\le 2e^{-8\pi}\sum_{n=1}^{\infty}\frac{\sigma_0(n)}{n^2} = 2e^{-8\pi}\zeta(2)^2 = \frac{\pi^4}{18}e^{-8\pi}.$$

Hence, we get

$$J(4) = 2\log 4 - 2\log 4\pi + 2C_0 + 4H(4) \le -2\log\pi + 2C_0 + \frac{\pi^4}{18}e^{-8\pi} < 0.$$

From the above argument it is clear that if $k > k_2$ or $0 < k < k_1$, then

$$Z\left(\frac{1}{2}, Q\right) > 0.$$

Since

$$\lim_{\sigma\to 1-0} Z(\sigma, Q) = -\infty,$$

$Z(\sigma, Q)$ must have a real zero in the interval $\left(\frac{1}{2}, 1\right)$.

This proves (ii) of Corollary 1.

We get at the same time (iii) of Corollary 1.

We notice first that for k_1 and k_2, $s = \frac{1}{2}$ is not a simple zero of $Z(s, Q)$. Because if k satisfies

$$2\log k - 2\log 4\pi + 2C_0 + 4H(k) = 0,$$

then

$$C_0^2 - 4\log^2 2 - 4\log 2\cdot\log\frac{\pi}{k} - \left(\log\frac{\pi}{k}\right)^2 - \log a\cdot\left(C_0 - \log\frac{\pi}{k} - 2\log 2\right)$$
$$+ 2\left(\log\frac{\pi}{ak} + C_0 + 2\log 2\right)\dot{H}(k) = 0.$$

To prove that $s = \frac{1}{2}$ is a double zero for these cases, we need to locate k_1 and k_2 more precisely. Here we use the same method as to locate D_1 and D_2 more precisely. Here the situation is much simpler because the functions involved are much simpler and we get rough numerical values

$$k_1 = 0.1417332\cdots$$

and

$$k_2 = 7.055507955448\cdots$$

as noticed in the section 1.

As an example, we shall locatc k_2. We put

$$k=\frac{4\pi}{e^{C_0}}(1\pm\varepsilon) \text{ with } \varepsilon=10^{-14}.$$

Then we have for this k,

$$\frac{1}{2}\sqrt{a}Z\Big(\frac{1}{2}, Q\Big)=\log(1\pm\varepsilon)+2H\Big(\frac{4\pi}{e^{C_0}}(1\pm\varepsilon)\Big).$$

Now

$$\begin{aligned}0<2H\Big(\frac{4\pi}{e^{C_0}}(1\pm\varepsilon)\Big)&\leq 4\sum_{n=1}^{\infty}\sigma_0(n)K_0\Big(\frac{8\pi^2}{e^{C_0}}(1\pm\varepsilon)n\Big)\\ &\leq\frac{4}{2}\sqrt{\frac{e^{C_0}}{4\pi}}(1+10^{-10})\sum_{n=1}^{\infty}\frac{\sigma_0(n)}{\sqrt{n}}e^{-\frac{8\pi^2}{e^{C_0}}\frac{9}{10}n}\\ &\leq\frac{4}{2}\sqrt{\frac{e^{C_0}}{4\pi}}(1+10^{-10})\Big(e^{-\frac{8\pi^2}{e^{C_0}}\frac{9}{10}}-e^{-\frac{8\pi^2}{e^{C_0}}\frac{8}{5}}+e^{-\frac{8\pi^2}{e^{C_0}}\frac{8}{5}}\zeta(2)^2\Big)\leq 4\cdot 10^{-16}.\end{aligned}$$

On the other hand,

$$\log(1+\varepsilon)>\varepsilon-\frac{\varepsilon^2}{2}>10^{-15}.$$

Similarly, we have

$$\log(1-\varepsilon)<-\varepsilon\leq-10^{-14}.$$

Thus at $k=\frac{4\pi}{e^{C_0}}(1+10^{-14})$,

$$Z\Big(\frac{1}{2}, Q\Big)>0$$

and at $k=\frac{4\pi}{e^{C_0}}(1-10^{-14})$,

$$Z\Big(\frac{1}{2}, Q\Big)<0.$$

Since

$$\frac{4\pi}{e^{C_0}}=7.05550795544818276277064853817\cdots,$$

we get

$$k_2=7.055507955448\cdots.$$

We now proceed to complete the proof of Corollary 1.

We start with Chowla-Selberg's formula stated in the section 1.

$$a^s Z(s, Q) = 2\zeta(2s) + 2k^{1-2s}\zeta(2s-1)\frac{\Gamma\left(s-\frac{1}{2}\right)\sqrt{\pi}}{\Gamma(s)} + \frac{4\pi^s k^{\frac{1}{2}-s}}{\Gamma(s)} D(s),$$

where $D(s)$ is introduced in the statement of Corollary 2 in the section 1.

The coefficient of $\left(s-\frac{1}{2}\right)^2$ of the Taylor expansion of

$$2\zeta(2s) + 2k^{1-2s}\zeta(2s-1)\frac{\Gamma\left(s-\frac{1}{2}\right)\sqrt{\pi}}{\Gamma(s)}$$

at $s=\frac{1}{2}$ is

$$= -\frac{4}{3}\left(\log\frac{\pi}{k}\right)^3 + 4\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\left(\log\frac{\pi}{k}\right)^2 - 4\left(\log\frac{\pi}{k}\right)\left(\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\right)^2$$

$$-\frac{\Gamma'\left(\frac{1}{2}\right)\Gamma''\left(\frac{1}{2}\right)}{\Gamma^2\left(\frac{1}{2}\right)} + 2\left(\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\right)^3 - 8C_0\log\frac{\pi}{k}\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} + 4C_0\left(\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\right)^2$$

$$+ 4C_0\left(\log\frac{\pi}{k}\right)^2 - 8C_1\log\frac{\pi}{k} + 8C_1\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} + 16C_2 + \frac{1}{3}\frac{\Gamma'''\left(\frac{1}{2}\right)}{3\Gamma\left(\frac{1}{2}\right)}.$$

Similarly, the coefficient of $\left(s-\frac{1}{2}\right)^2$ of the Taylor expansion of

$$\frac{4\pi^s k^{\frac{1}{2}-s}}{\Gamma(s)} D(s) \quad \text{at} \quad s = \frac{1}{2}$$

is

$$2D''\left(\frac{1}{2}\right) + D\left(\frac{1}{2}\right)\left\{-\pi^2 + 2(C_0 + 2\log 2)^2 + 4(C_0 + 2\log 2)\log\frac{\pi}{k} + 2\left(\log\frac{\pi}{k}\right)^2\right\}.$$

This implies that

$$\frac{1}{2}\sqrt{a}Z''\left(\frac{1}{2}\right) = -\frac{1}{2}\log^2 a \cdot \sqrt{a}Z\left(\frac{1}{2}\right) - \log a \cdot \sqrt{a}Z'\left(\frac{1}{2}\right)$$

$$-\frac{4}{3}\left(\log\frac{\pi}{k}\right)^3 + 4\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\left(\log\frac{\pi}{k}\right)^2 - 4\left(\log\frac{\pi}{k}\right)\left(\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\right)^2$$

$$-\frac{\Gamma'\left(\frac{1}{2}\right)\Gamma''\left(\frac{1}{2}\right)}{\Gamma^2\left(\frac{1}{2}\right)} + 2\left(\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\right)^3 - 8C_0\log\frac{\pi}{k}\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} + 4C_0\left(\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\right)^2$$

$$+ 4C_0\left(\log\frac{\pi}{k}\right)^2 - 8C_1\log\frac{\pi}{k} + 8C_1\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} + 16C_2 + \frac{1}{3}\frac{\Gamma'''\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$$

$$+ 2D''\left(\frac{1}{2}\right) + D\left(\frac{1}{2}\right)\Big\{- \pi^2 + 2(C_0 + 2\log 2)^2$$

$$+ 4(C_0 + 2\log 2)\log\frac{\pi}{k} + 2\left(\log\frac{\pi}{k}\right)^2\Big\}.$$

Here we notice that

$$\left(\frac{\Gamma'}{\Gamma}\right)'\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty}\frac{1}{\left(\frac{1}{2}+n\right)^2} = 4\left(\zeta(2) - \sum_{n=1}^{\infty}\frac{1}{(2n)^2}\right) = \frac{\pi^2}{2}$$

and

$$\left(\frac{\Gamma'}{\Gamma}\right)''\left(\frac{1}{2}\right) = -2\sum_{n=0}^{\infty}\frac{1}{\left(\frac{1}{2}+n\right)^3} = -16\left(\zeta(3) - \sum_{n=1}^{\infty}\frac{1}{(2n)^3}\right) = -14\zeta(3).$$

Hence, we get

$$\left(\frac{\Gamma''}{\Gamma}\right)\left(\frac{1}{2}\right) = \frac{\pi^2}{2} + \left(\left(\frac{\Gamma'}{\Gamma}\right)\left(\frac{1}{2}\right)\right)^2$$

and

$$\left(\frac{\Gamma'''}{\Gamma}\right)\left(\frac{1}{2}\right) = -14\zeta(3) + \frac{3\pi^2}{2}\left(\frac{\Gamma'}{\Gamma}\right)\left(\frac{1}{2}\right) + \left(\left(\frac{\Gamma'}{\Gamma}\right)\left(\frac{1}{2}\right)\right)^3.$$

Consequently, we get

$$-\frac{\Gamma'\left(\frac{1}{2}\right)\Gamma''\left(\frac{1}{2}\right)}{\Gamma^2\left(\frac{1}{2}\right)} + 2\left(\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\right)^3 + 4C_0\left(\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\right)^2 + 8C_1\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} + 16C_2 + \frac{1}{3}\frac{\Gamma'''\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{4}{3}\left(\left(\frac{\Gamma'}{\Gamma}\right)\left(\frac{1}{2}\right)\right)^3 + 4C_0\left(\left(\frac{\Gamma'}{\Gamma}\right)\left(\frac{1}{2}\right)\right)^2 + 8C_1\left(\frac{\Gamma'}{\Gamma}\right)\left(\frac{1}{2}\right) + 16C_2 - \frac{14}{3}\zeta(3)$$

$$= 16C_2 - \frac{14}{3}\zeta(3) + \frac{8}{3}C_0^3 - 8C_1C_0 - 8C_1\log 4 + 4C_0^2\log 4 - \frac{4}{3}\log^3 4.$$

This implies (ii) of Corollary 2.

We now suppose that

$$Z\left(\frac{1}{2}\right) = 0 \text{ and } Z'\left(\frac{1}{2}\right) = 0.$$

Then we have

$$2D\left(\frac{1}{2}\right)=\log\frac{\pi}{k}+2\log 2-C_0.$$

We substitute this into above formula for $\frac{1}{2}\sqrt{a}Z''\left(\frac{1}{2}\right)$. Then we get

$$\begin{aligned}\frac{1}{2}\sqrt{a}Z''\left(\frac{1}{2}\right)= &-\frac{4}{3}\left(\log\frac{\pi}{k}\right)^3-8\log 2\cdot\left(\log\frac{\pi}{k}\right)^2-4\log\frac{\pi}{k}\left(2C_1-C_0^2+\log^2 4\right)\\ &-\frac{14}{3}\zeta(3)+\frac{8}{3}C_0^3+4C_0^2\log 4-\frac{4}{3}\log^3 4+16C_2-8C_1C_0-8C_1\log 4\\ &+D\left(\frac{1}{2}\right)\left\{-\pi^2+2(C_0+2\log 2)^2+4(C_0+2\log 2)\log\frac{\pi}{k}+2\left(\log\frac{\pi}{k}\right)^2\right\}\\ &+2D''\left(\frac{1}{2}\right)\\ =&-\frac{1}{3}\left(\log\frac{\pi}{k}\right)^3+\left(\log\frac{\pi}{k}\right)^2(C_0-2\log 2)\\ &+\log\frac{\pi}{k}\left(-\frac{\pi^2}{2}-8C_1+3C_0^2+4C_0\log 2-4\log^2 2\right)\\ &-\frac{14}{3}\zeta(3)+\frac{5}{3}C_0^3+6C_0^2\log 2+4C_0\log^2 2+\frac{1}{2}\pi^2C_0\\ &-\log 2\cdot\pi^2-\frac{8}{3}\log^3 2+16C_2-8C_1C_0-8C_1\log 4+2D''\left(\frac{1}{2}\right).\end{aligned}$$

We put

$$\begin{aligned}f_4(x)= &-\frac{1}{3}x^3+x^2(C_0-2\log 2)+x\left(-\frac{\pi^2}{2}-8C_1+3C_0^2+4C_0\log 2-4\log^2 2\right)\\ &-\frac{14}{3}\zeta(3)+\frac{5}{3}C_0^3+6C_0^2\log 2+4C_0\log^2 2\\ &+\frac{1}{2}\pi^2C_0-\log 2\cdot\pi^2-\frac{8}{3}\log^3 2+16C_2-8C_1C_0-8C_1\log 4.\end{aligned}$$

Then we see easily that

$$\begin{aligned}f_4'(x)&=-(x-(C_0-2\log 2))^2+4C_0^2-\frac{\pi^2}{2}-8C_1\\ &<0\quad\text{for any}\quad x.\end{aligned}$$

Hence $f_4(x)$ is a monotone decreasing function of x. We notice that

$$3<\log\frac{\pi}{k_1}=\log\frac{\pi}{0.141\cdots}=3.098538\cdots<4$$

and that

$$f_4(4)<-62\quad\text{and}\quad f_4(3)<-39.$$

Similarly, we have

$$-1<\log\frac{\pi}{k_2}=\log\frac{\pi}{7.05550\cdots}=-0.809\cdots<-0.7$$

and

$$f_4(-1) < -4 \quad \text{and} \quad f_4(-0.7) < -5.$$

We shall estimate next $D''\left(\frac{1}{2}\right)$. First, by the definition, we get

$$\begin{aligned} D''(s) = &\sum_{n=1}^{\infty} \log^2 n \cdot n^{s-\frac{1}{2}} \sum_{d|n} d^{1-2s} \cos\left(\frac{n\pi b}{a}\right) \int_0^{\infty} y^{s-\frac{3}{2}} e^{-\pi n k(y+y^{-1})} dy \\ &+ \sum_{n=1}^{\infty} \log n \cdot n^{s-\frac{1}{2}} \sum_{d|n} (-2\log d) d^{1-2s} \cos\left(\frac{n\pi b}{a}\right) \int_0^{\infty} y^{s-\frac{3}{2}} e^{-\pi n k(y+y^{-1})} dy \\ &+ \sum_{n=1}^{\infty} \log n \cdot n^{s-\frac{1}{2}} \sum_{d|n} d^{1-2s} \cos\left(\frac{n\pi b}{a}\right) \int_0^{\infty} \log y \cdot y^{s-\frac{3}{2}} e^{-\pi n k(y+y^{-1})} dy \\ &+ \sum_{n=1}^{\infty} \log n \cdot n^{s-\frac{1}{2}} \sum_{d|n} (-2\log d) d^{1-2s} \cos\left(\frac{n\pi b}{a}\right) \int_0^{\infty} y^{s-\frac{3}{2}} e^{-\pi n k(y+y^{-1})} dy \\ &+ \sum_{n=1}^{\infty} n^{s-\frac{1}{2}} \sum_{d|n} (-2\log d)^2 d^{1-2s} \cos\left(\frac{n\pi b}{a}\right) \int_0^{\infty} y^{s-\frac{3}{2}} e^{-\pi n k(y+y^{-1})} dy \\ &+ \sum_{n=1}^{\infty} n^{s-\frac{1}{2}} \sum_{d|n} (-2\log d) d^{1-2s} \cos\left(\frac{n\pi b}{a}\right) \int_0^{\infty} \log y \cdot y^{s-\frac{3}{2}} e^{-\pi n k(y+y^{-1})} dy \\ &+ \sum_{n=1}^{\infty} \log n \cdot n^{s-\frac{1}{2}} \sum_{d|n} d^{1-2s} \cos\left(\frac{n\pi b}{a}\right) \int_0^{\infty} \log y \cdot y^{s-\frac{3}{2}} e^{-\pi n k(y+y^{-1})} dy \\ &+ \sum_{n=1}^{\infty} n^{s-\frac{1}{2}} \sum_{d|n} (-2\log d) d^{1-2s} \cos\left(\frac{n\pi b}{a}\right) \int_0^{\infty} \log y \cdot y^{s-\frac{3}{2}} e^{-\pi n k(y+y^{-1})} dy \\ &+ \sum_{n=1}^{\infty} n^{s-\frac{1}{2}} \sum_{d|n} d^{1-2s} \cos\left(\frac{n\pi b}{a}\right) \int_0^{\infty} \log^2 y \cdot y^{s-\frac{3}{2}} e^{-\pi n k(y+y^{-1})} dy. \end{aligned}$$

Hence, we get

$$\begin{aligned} D''\left(\frac{1}{2}\right) &= 4 \sum_{n=1}^{\infty} \sum_{d|n} \left(\log d - \frac{1}{2}\log n\right)^2 \cos\left(\frac{n\pi b}{a}\right) \int_0^{\infty} \frac{1}{y} e^{-\pi n k(y+y^{-1})} dy \\ &\quad + \sum_{n=1}^{\infty} \sigma_0(n) \cos\left(\frac{n\pi b}{a}\right) \int_0^{\infty} \frac{\log^2 y}{y} e^{-\pi n k(y+y^{-1})} dy \\ &= f_5(k) + f_6(k), \text{ say.} \end{aligned}$$

We shall estimate $f_5(k_2)$ first.

$$f_5(k_2)=8\sum_{n=1}^{\infty}\sum_{d|n}\left(\log d-\frac{1}{2}\log n\right)^2 K_0(2\pi nk_2)$$
$$\leq \frac{4}{\sqrt{k_2}}\sum_{n=1}^{\infty}\frac{e^{-2\pi nk_2}}{\sqrt{n}}\sum_{d|n}\left(\log d-\frac{1}{2}\log n\right)^2$$
$$\leq \frac{4}{\sqrt{k_2}}e^{-2\pi k_2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{d|n}\left(\log d-\frac{1}{2}\log n\right)^2$$
$$\leq \frac{2e^{-2\pi k_2}}{\sqrt{k_2}}|\zeta''(2)\zeta(2)-\zeta'(2)^2|\leq 1.027713\times 10^{-19}.$$

Similarly, we get

$$f_5(k_1)\leq \frac{4}{\sqrt{k_1}}\sum_{n=1}^{14}\frac{e^{-2\pi nk_1}}{\sqrt{n}}\sum_{d|n}\left(\log d-\frac{1}{2}\log n\right)^2$$
$$+\frac{4}{\sqrt{k_1}}e^{-20\pi k_1}\left(\sum_{n=1}^{\infty}\frac{\sum_{d|n}\left(\log d-\frac{1}{2}\log n\right)^2}{n^2}-\sum_{n=1}^{14}\frac{\sum_{d|n}\left(\log d-\frac{1}{2}\log n\right)^2}{n^2}\right).$$

Since

$$\sum_{n=1}^{14}\frac{e^{-2\pi nk_1}}{\sqrt{n}}\sum_{d|n}\left(\log d-\frac{1}{2}\log n\right)^2\leq 0.07896949867,$$
$$\sum_{n=1}^{\infty}\frac{\sum_{d|n}\left(\log d-\frac{1}{2}\log n\right)^2}{n^2}=\frac{1}{2}(\zeta''(2)\zeta(2)-\zeta'(2)^2)$$
$$\leq\frac{1}{2}\times 2.44242013401660538$$

and

$$\sum_{n=1}^{14}\frac{\sum_{d|n}\left(\log d-\frac{1}{2}\log n\right)^2}{n^2}=0.533458527928099\cdots,$$

we get

$$f_5(k_1)\leq 0.85.$$

To estimate $f_6(k)$, we put $W=k\pi$ and $V=e^{\frac{3+\sqrt{5}}{2}}$, for simplicity. Then we have

$$|f_6(k)|\leq\sum_{n=1}^{\infty}\sigma_0(n)\int_0^{\infty}\frac{\log^2 y}{y}e^{-Wn(y+y^{-1})}dy\leq 2\sum_{n=1}^{\infty}\sigma_0(n)\int_1^{\infty}\frac{\log^2 y}{y}e^{-Wny}dy$$

$$= \frac{2}{W^2}\sum_{n=1}^{\infty}\frac{\sigma_0(n)}{n^2}\int_1^{\infty}\frac{2-6\log y+2\log^2 y}{y^3}e^{-Wny}dy$$

$$\leq \frac{4}{W^2}\sum_{n=1}^{\infty}\frac{\sigma_0(n)}{n^2}\int_V^{\infty}\frac{1-3\log y+\log^2 y}{y^3}e^{-Wny}dy$$

$$\leq \frac{4}{W^2}\sum_{n=1}^{\infty}\frac{\sigma_0(n)}{n^2}e^{-WVn}\int_V^{\infty}\frac{\log^2 y}{y^3}dy \leq \frac{4\log^2 V}{W^2V^2}\sum_{n=1}^{\infty}\frac{\sigma_0(n)}{n^2}e^{-WVn}$$

$$= \frac{4\log^2 V}{W^2V^2}\left\{e^{-WV}+\frac{1}{2}e^{-2WV}+\frac{2}{9}e^{-3WV}+e^{-4WV}\sum_{n=4}^{\infty}\frac{\sigma_0(n)}{n^2}\right\}$$

$$= \frac{4\log^2 V}{W^2V^2}\left\{e^{-WV}+\frac{1}{2}e^{-2WV}+\frac{2}{9}e^{-3WV}+e^{-4WV}\left(\zeta^2(2)-1-\frac{1}{2}-\frac{2}{9}\right)\right\}.$$

This implies that

$$|f_6(k_1)|\leq|f_3(0.131733)|\leq 0.001646$$

and

$$|f_6(k_2)|\leq|f_3(7.0555)|\leq 3.22\times 10^{-136}.$$

Hence, we get

$$\frac{1}{2}\sqrt{a}Z''\left(\frac{1}{2}\right) < -39+1.8 < -37 < 0 \text{ at } k=k_1.$$

Similarly, we get

$$\frac{1}{2}\sqrt{a}Z''\left(\frac{1}{2}\right) < -4+1.8 < -2 < 0 \text{ at } k=k_2.$$

This proves all of Corollaries 1 and 2.

§ 7. Preliminaries for the proof of Theorems 4 and 5

We need another expression of $E(s)$ which gives a better upper bound of $E(s)$ for a larger $|t|$. This will be used in the next section to prove some lemmas which is necessary for the proof of Theorems 4 and 5.

We start with expressing

$$E_0(s) = \sum_{m=1}^{\infty}r(m)\sum_{-\infty<m_1,m_2<\infty}(m_1^2+m_2^2+dm)^{-s}$$

in another way.

$$E_0(s) = \sum_{m=1}^{\infty}r(m)\sum_{m_1=-\infty}^{\infty}\int_{-\infty}^{\infty}(m_1^2+y^2+dm)^{-s}dy$$

$$+\sum_{m=1}^{\infty} r(m) \sum_{m_1=-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\{y\} - \frac{1}{2}\right) \frac{d}{dy}((m_1^2 + y^2 + dm)^{-s}) dy$$

$$= \sum_{m=1}^{\infty} r(m) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x^2 + y^2 + dm)^{-s} dx dy$$

$$+ \sum_{m=1}^{\infty} r(m) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\{x\} - \frac{1}{2}\right) \frac{d}{dx}((x^2 + y^2 + dm)^{-s}) dx dy$$

$$+ \sum_{m=1}^{\infty} r(m) \sum_{m_1=-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\{y\} - \frac{1}{2}\right) \frac{d}{dy}((m_1^2 + y^2 + dm)^{-s}) dy$$

$$= \Phi_1(s) + \Phi_2(s) + \Phi_3(s), \quad \text{say.}$$

Since

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x^2 + y^2 + dm)^{-s} dx\, dy = \int_0^{2\pi} \int_0^{\infty} (r^2 + dm)^{-s} r\, dr\, d\theta$$

$$= 2\pi (dm)^{-s+1} \int_0^{\infty} (u^2 + 1)^{-s} u du = \frac{\pi}{s-1} (dm)^{-s+1},$$

we get

$$\Phi_1(s) = \frac{\pi}{d^{s-1}} \frac{\Gamma(s-1)}{\Gamma(s)} Z(s-1).$$

By a change of variable $x = \sqrt{y^2 + dm}\, u$ and by a repeated use of partial integral, we get for each integer $j \geq 1$

$$\Phi_2(s) = \sum_{m=1}^{\infty} r(m) \int_{-\infty}^{\infty} (y^2 + dm)^{-s} \int_{-\infty}^{\infty} B_1(\{\sqrt{y^2 + dm}\, u\})$$

$$\cdot \frac{d}{du}((u^2 + 1)^{-s}) du dy$$

$$= -\sum_{m=1}^{\infty} r(m) \int_{-\infty}^{\infty} (y^2 + dm)^{-s-j+\frac{1}{2}} \int_{-\infty}^{\infty} \frac{B_{2j}(\{\sqrt{y^2 + dm}\, u\})}{2j!}$$

$$\cdot \frac{d^{(2j)}}{du^{(2j)}}((u^2 + 1)^{-s}) du dy,$$

where $B_{2j}(x)$ is the $2j$—th Bernoulli polynomial.

Now suppose that $|s| \leq \sqrt{d} + 3$, $\sigma \geq 1$, $j = [C\log d]$ and d is sufficiently large. Then we have for any positive constant A

$$|\Phi_2(s)| \leq \sum_{m=1}^{\infty} r(m) \int_{-\infty}^{\infty} (y^2 + dm)^{-\sigma-j+\frac{1}{2}}$$

$$\cdot \int_{-\infty}^{\infty} \frac{2\zeta(2j)}{(2\pi)^{2j}} 2^j (u^2+1)^{-\sigma-j}(2|s|+2j-1)^{2j} du dy$$
$$\leq \frac{2\zeta(2j)}{(2\pi)^{2j}} 2^j (2|s|+2j-1)^{2j}$$
$$\cdot \sum_{m=1}^{\infty} r(m) \int_{-\infty}^{\infty} (y^2+dm)^{-\sigma-j+\frac{1}{2}} dy \int_{-\infty}^{\infty} (u^2+1)^{-\sigma-j} du$$
$$\leq d^{-\sigma+1-j} \frac{2\zeta(2j)}{(2\pi)^{2j}} 2^j (2|s|+2j-1)^{2j}$$
$$\cdot \sum_{m=1}^{\infty} \frac{r(m)}{m^{\sigma-1+j}} \int_{-\infty}^{\infty} (v^2+1)^{-\sigma-j+\frac{1}{2}} dv) \left(\int_{-\infty}^{\infty} (u^2+1)^{-\sigma-j} du\right)$$
$$\ll \left(\frac{4|s|+4j-2}{2\pi\sqrt{d}}\right)^{2j} \ll \left(\frac{5\sqrt{d}}{2\pi\sqrt{d}}\right)^{C\log d} \ll d^{-A}.$$

In a similar manner we can estimate $\Phi_3(s)$. Since

$$\sum_{m=1}^{\infty} r(m) \sum_{n=-\infty}^{\infty} (n^2+dm)^{-\sigma-j+\frac{1}{2}}$$
$$\ll \sum_{m=1}^{\infty} r(m) \left((dm)^{-\sigma-j+\frac{1}{2}} + \int_{1}^{\infty} (x^2+dm)^{-\sigma-j+\frac{1}{2}} dx \right.$$
$$\left. + (|\sigma|+j) \int_{1}^{\infty} (x^2+dm)^{-\sigma-j-\frac{1}{2}} dx \right)$$
$$\ll \sum_{m=1}^{\infty} r(m) (dm)^{-\sigma-j+\frac{1}{2}} \left(1+\sqrt{dm} \int_{-\infty}^{\infty} (u^2+1)^{-\sigma-j+\frac{1}{2}} du \right.$$
$$\left. + (|\sigma|+j) \frac{1}{\sqrt{dm}} \int_{-\infty}^{\infty} (u^2+1)^{-\sigma-j-\frac{1}{2}} du \right),$$

we get for $\sigma \geq 1$

$$|\Phi_3(s)| \ll d^{-A}.$$

Thus we get for $|s| \leq \sqrt{d}+3$, $\sigma \geq 1$ and $d > d_0$, using the notations introduced in the section 1,

$$\left|\frac{g(s)}{b(s)}\right| = \left|\frac{2\sqrt{d}}{b(s)} E(s,d)\right| \ll |\Phi_2(s)| + |\Phi_3(s)| \ll d^{-A}.$$

This will be used in the next section.

§ 8. Proof of Theorems 4 and 5

We shall extend the argument in Stark [23]. We shall provide several lemmas which will be used in the proof of Theorems 4 and 5. We shall give a full proof only to Lemma 1. The proofs of the other lemmas will be omitted, because it is clear how to modify Stark's proof in the present situation.

First we recall the well-known results concerning the function $Z(s) = 4\zeta(s)L(s, \chi)$. First of all it satisfies the following functional equation

$$\pi^{-s}\Gamma(s)Z(s) = \pi^{-(1-s)}\Gamma(1-s)Z(1-s).$$

The trivial zeros of $Z(s)$ are

$$s = -1, -2, -3, \cdots.$$

$$Z(\sigma) \neq 0 \text{ for } \sigma > 0.$$

$$Z(s) \neq 0 \text{ for } \sigma \geq 1 - \frac{C}{(\log(|t|+3))^{\frac{2}{3}}(\log\log(|t|+3))^{\frac{1}{3}}}.$$

For $t \geq 3$ and for $\sigma \geq 1 - \frac{C}{(\log(|t|+3))^{\frac{2}{3}}(\log\log(|t|+3))^{\frac{1}{3}}}$, we have

$$\frac{Z'}{Z}(s) \ll (\log t)^{\frac{2}{3}}(\log\log t)^{\frac{1}{3}}$$

and

$$\frac{1}{Z}(s) \ll (\log t)^{\frac{4}{3}}(\log\log t)^{\frac{2}{3}}.$$

These come from Chapter III and IV of Titchmarsh [25] and their extensions to $L(s, \chi)$ (cf. Satz 6.2 of p.295 in Prachar [19] for a weaker result, which is already enough for our present purpose). We put, as already introduced in the section 1,

$$f(s) = Z(s)\left(\frac{\pi}{\kappa}\right)^{-s}\Gamma(s),\ b(s) = \left(\frac{\pi}{\kappa}\right)^{-s}\Gamma(s) \text{ and } g(s) = 2\sqrt{d}E(s, d, Q).$$

Lemma 1. *If* $1 + \frac{1}{(\log\kappa)^{\frac{7}{8}}} \leq \sigma \leq \frac{5}{2}$, $3 \leq t \leq \kappa + 1$, *or if* $\sigma = \frac{5}{2}$, $0 \leq t \leq 3$, *then*

$$|f(s)| > |f(2-s)|$$

and

$$|f(s) + f(2-s)| > |g(s)|.$$

Proof In the above region we have

$$|f(s)| = 4|b(s)||\zeta(s)L(s, \chi)| > 4|b(s)|\frac{1}{(\log t)^{\frac{4}{3}}(\log\log t)^{\frac{2}{3}}}$$

$$> 4|b(s)|\frac{1}{(\log\kappa)^{2}}.$$

For $1+\frac{1}{(\log\kappa)^{\frac{7}{8}}}\leq\sigma\leq 2+\frac{1}{\log\kappa}$ and $3\leq t\leq\kappa+1$, we have

$$\begin{aligned}\left|\frac{f(2-s)}{b(s)}\right| &\ll \left(\frac{\kappa}{\pi}\right)^{2-2\sigma} t^{2-2\sigma}|Z(2-s)| \\ &\ll \log^2\kappa\cdot\left(\frac{\kappa}{\pi}\right)^{2-2\sigma} t^{2-2\sigma}t^{1+\frac{1}{\log\kappa}-(2-\sigma)} \\ &\ll \log^2\kappa\cdot e^{-A(\log\kappa)^{\frac{1}{8}}}\ll(\log\kappa)^{-2},\end{aligned}$$

since

$$|Z(s)|\ll\log^2\kappa\cdot t^{1+\frac{1}{\log\kappa}-\sigma}\quad\text{for}\quad -\frac{1}{\log\kappa}\leq\sigma\leq 1+\frac{1}{\log\kappa},\quad t\geq 3$$

because

$$|Z(s)|\ll\left(\sum_{n=1}^{\infty}\frac{1}{n^{1+\frac{1}{\log\kappa}}}\right)^2\ll\log^2\kappa\quad\text{for}\quad\sigma\geq 1+\frac{1}{\log\kappa},$$

$$\begin{aligned}\left|Z\left(-\frac{1}{\log\kappa}+it\right)\right| &= \left|\frac{\pi^{-(1+\frac{1}{\log\kappa}-it)}\Gamma\left(1+\frac{1}{\log\kappa}-it\right)Z\left(1+\frac{1}{\log\kappa}-it\right)}{\pi^{-(-\frac{1}{\log\kappa}+it)}\Gamma\left(-\frac{1}{\log\kappa}+it\right)}\right| \\ &\ll t^{\frac{2}{\log\kappa}+1}\left|Z\left(1+\frac{1}{\log\kappa}-it\right)\right|\ll\log^2\kappa\cdot t^{\frac{2}{\log\kappa}+1}\end{aligned}$$

and

$$|Z(\sigma+3i)\ll 1\quad\text{for}\quad -\frac{1}{\log\kappa}\leq\sigma\leq 1+\frac{1}{\log\kappa}.$$

For $2+\frac{1}{\log\kappa}\leq\sigma\leq\frac{5}{2}$ and $3\leq t\leq\kappa+1$, we have

$$\begin{aligned}\left|\frac{f(2-s)}{b(s)}\right| &= \frac{\left|\left(\frac{\kappa}{\pi}\right)^{2-s}\Gamma(2-s)Z(2-s)\right|}{|b(s)|} \\ &= \frac{|\kappa^{2-s}||\pi^{-(s-1)}\Gamma(s-1)Z(s-1)|}{|b(s)|} \\ &\ll \kappa^{2-2\sigma}\frac{1}{|s-1|}|Z(s-1)|\ll\kappa^{2-2\sigma}\left|Z\left(1+\frac{1}{\log\kappa}\right)\right| \\ &\ll \kappa^{2-2\sigma}\log^2\kappa\ll\kappa^{-2-\frac{2}{\log\kappa}}\log^2\kappa\ll\frac{\log^2\kappa}{\kappa^2}.\end{aligned}$$

For $\sigma=\frac{5}{2}$, $0\leq t\leq 3$, it is clear that

$$\left|\frac{f(2-s)}{b(s)}\right|\leq(\log\kappa)^{-2}.$$

Hence we get

$$|f(s)| > |f(2-s)|.$$

Consequently, we get

$$|f(s)+f(2-s)| \geq |f(s)| - |f(2-s)|$$
$$\geq \frac{4|b(s)|}{\log^2\kappa} - \frac{|b(s)|}{\log^2\kappa} \geq \frac{3|b(s)|}{\log^2\kappa} > \frac{|b(s)|}{\kappa^A} \geq |g(s)|.$$

Lemma 2. *If* $1-\frac{1}{(\log\kappa)^{\frac{7}{8}}} \leq \sigma \leq 1+\frac{1}{(\log\kappa)^{\frac{7}{8}}}$ and $\kappa \leq t \leq \kappa+1$, *then*

$$\Re\left(\frac{f'}{f}(s)\right) > \log\kappa \quad \textit{and} \quad \left|\Im\left(\frac{f'}{f}(s)\right)\right| < \log\kappa.$$

Proof. Since

$$\frac{f'}{f}(s) = \log\frac{\kappa}{\pi} + \frac{\Gamma'}{\Gamma}(s) + \frac{Z'}{Z}(s),$$

we get

$$\Re\left(\frac{f'}{f}(s)\right) > \log\frac{\kappa}{\pi} + \log t - (\log t)^{\frac{7}{8}} + O\left(\frac{1}{t}\right) > \log\kappa.$$

Similarly, we get

$$\left|\Im\left(\frac{f'}{f}(s)\right)\right| < \log\kappa.$$

Lemma 3. *There exists a number T_0 such that $\kappa \leq T_0 \leq \kappa+1$ and*

$$\arg f(1+iT_0) \equiv 0 \ (2\pi).$$

Thus $f(1+iT_0) > 0$ and $f(1-iT_0) > 0$.

Lemma 4. *For $1 \leq \sigma \leq 1+\frac{1}{(\log\kappa)^{\frac{7}{8}}}$ and $t = T_0$,*

$$|f(s)| \geq |f(2-s)| \ \textit{and} \ |f(s)+f(2-s)| > |g(s)|$$

with equality in the first part if and only if $\sigma = 1$.

Lemma 5. *Let R be the interior of the rectangle corners at $\frac{5}{2} \pm iT_0$, $-\frac{1}{2} \pm iT_0$. Then the number (counted with multiplicity) of the zeros of $\alpha(s)$ in R is exactly*

$$2 + \frac{2}{\pi}\arg f(1+iT_0).$$

Thus we have for $0 < T \leq T_0$

$$N(T) = \frac{1}{\pi}\arg f(1+iT) + O(1)$$

$$= \frac{1}{\pi}\arg\left(\left(\frac{\kappa}{\pi}\right)^{1+iT}\Gamma(1+iT)\right) + \frac{1}{\pi}\arg Z(1+iT) + O(1)$$

$$= \frac{T}{\pi}\log\frac{\kappa}{\pi} + \frac{1}{\pi}(T\log T - T) + \frac{1}{\pi}\arg Z(1+iT) + O(1).$$

Since we know that

$$\arg\zeta(1+iT),\ \arg L(1+iT,\chi) \ll \log\log T,$$

this proves our Theorem 5.

The proof of Theorem 4 can be obtained if we follow the argument in pp.53-54 of Stark [23].

§ 9. Proof of Theorem 6.

For the proof of Theorem 6, we shall prove the following lemma which is more general than what we need.

Lemma 6. *For any $\frac{1}{2} < \sigma \leq 1$, for $T > T_0$ and for $0 < h \ll T$, we have*

$$\int_T^{2T} (\arg(Z(\sigma + i(t+h))) - \arg(Z(\sigma + it)))^2 dt = T\Bigg(\frac{1-\cos(h\log 2)}{2^{2\sigma}}$$

$$+ \sum_{m=1}^{\infty} \frac{\Lambda^2(4m+1)}{(4m+1)^{2\sigma}\log^2(4m+1)}(1-\cos(h\log(4m+1)))\Bigg) + O(T^{1-\delta(\sigma)}),$$

where $\delta(\sigma)$ is an appropriate positive constant which may depend on σ.

It is clear that this result with $\sigma = 1$ implies our Theorem 6.

Now suppose that $2 \leq X = T^a \leq T^2$, a is a sufficiently small positive constant which may depend on σ. We put

$$\sigma_{X,t} = \frac{1}{2} + 2\max_{\rho}\left(\beta - \frac{1}{2}, \frac{2}{\log X}\right),$$

ρ running here through all zeros $\beta + i\gamma$ of $\zeta(s)$ for which

$$|t-\gamma| \leq \frac{X^{3|\beta - \frac{1}{2}|}}{\log X}.$$

We put further

$$\Lambda_X(n)=\begin{cases}\Lambda(n) & \text{for } 1\le n\le X\\[2mm] \Lambda(n)\dfrac{\left(\log\frac{X^3}{n}\right)^2-2\left(\log\frac{X^2}{n}\right)^2}{2(\log X)^2} & \text{for } X\le n\le X^2\\[2mm] \Lambda(n)\dfrac{\left(\log\frac{X^3}{n}\right)^2}{2(\log X)^2} & \text{for } X^2\le n\le X^3.\end{cases}$$

Under these notations, we shall use the following Selberg's explicit formula (cf. p.239 of Selberg [20]) for $\sigma\ge\sigma_{X,t}$ and $t\ge\sqrt{X}$.

$$\frac{\zeta'}{\zeta}(s)=-\sum_{n<X^3}\frac{\Lambda_X(n)}{n^s}+O\left(X^{\frac14-\frac{\sigma}{2}}\left|\sum_{n<X^3}\frac{\Lambda_X(n)}{n^{\sigma_{X,t}+it}}\right|\right)+O\left(X^{\frac14-\frac{\sigma}{2}}\log t\right).$$

Then we get for $t\ge T$ and for $\sigma\ge\sigma_{X,t}$,

$$\begin{aligned}\arg(\zeta(\sigma+it))&=-\Im\left(\int_\sigma^\infty\frac{\zeta'}{\zeta}(u+it)du\right)\\&=\Im\left(\sum_{n<X^3}\frac{\Lambda_X(n)}{n^{\sigma+it}\log n}\right)+O\left(\frac{X^{\frac14-\frac{\sigma}{2}}}{\log X}\left(\left|\sum_{n<X^3}\frac{\Lambda_X(n)}{n^{\sigma_{X,t}+it}}\right|+\log t\right)\right)\\&=M(t)+O(R(t)),\ \text{say.}\end{aligned}$$

We put

$$f(\sigma,t)=\begin{cases}1 & \text{if } \sigma\ge\sigma_{X,t}\\ 0 & \text{otherwise.}\end{cases}$$

Similarly for the Dirichlet L-function $L(s,\chi)$, we define similar notations as follows. We put first

$$\sigma_{X,t,\chi}=\frac12+2\max_{\rho(\chi)}\left(\beta(\chi)-\frac12,\frac{2}{\log X}\right),$$

$\rho(\chi)$ running here through all zeros $\beta(\chi)+i\gamma(\chi)$ of $L(s,\chi)$ for which

$$|t-\gamma(\chi)|\le\frac{X^{3|\beta(\chi)-\frac12|}}{\log X}.$$

Under these notations, we shall use the following explicit formula (Cf. Fujii [5]) for $\sigma\ge\sigma_{X,t,\chi}$ and $t\ge\sqrt{X}$.

$$\frac{L'}{L}(s,\chi)=-\sum_{n<X^3}\frac{\Lambda_X(n)\chi(n)}{n^s}+O\left(X^{\frac14-\frac{\sigma}{2}}\left|\sum_{n<X^3}\frac{\Lambda_X(n)\chi(n)}{n^{\sigma_{X,t,\chi}+it}}\right|\right)+O\left(X^{\frac14-\frac{\sigma}{2}}\log t\right).$$

Then we get for $t\ge T$ and for $\sigma\ge\sigma_{X,t,\chi}$,

$$\arg(L(\sigma+it,\chi)) = -\int_\sigma^\infty \Im \frac{L'}{L}(u+it,\chi)du$$
$$= \Im\Big(\sum_{n<X^3} \frac{\Lambda_X(n)\chi(n)}{n^{\sigma+it}\log n}\Big) + O\Big(\frac{X^{\frac{1}{4}-\frac{\sigma}{2}}}{\log X}\Big(\Big|\sum_{n<X^3} \frac{\Lambda_X(n)\chi(n)}{n^{\sigma_{X,t,\chi}+it}}\Big| + \log t\Big)\Big)$$
$$= M(t,\chi) + O(R(t,\chi)), \quad \text{say.}$$

We put

$$f(\sigma,t,\chi) = \begin{cases} 1 & \text{if } \sigma \geq \sigma_{X,t,\chi} \\ 0 & \text{otherwise.} \end{cases}$$

Then for any $\sigma \geq \max(\sigma_{X,t}, \sigma_{X,t,\chi})$, we have

$$\arg(Z(\sigma+it)) = M(t) + M(t,\chi) + O(R(t) + R(t,\chi))$$
$$= \widetilde{M}(t) + O(\widetilde{R}(t)), \text{ say.}$$

Now for any $\frac{1}{2} < \sigma \leq 1$,

$$\int_T^{2T} (\arg(Z(\sigma+i(t+h))) - \arg(Z(\sigma+it)))^2 dt$$
$$= \int_T^{2T} f(\sigma,t+h)f(\sigma,t)f(\sigma,t+h,\chi)f(\sigma,t,\chi))$$
$$\cdot (\arg(Z(\sigma+i(t+h))) - \arg(Z(\sigma+it)))^2 dt$$
$$+ \int_T^{2T} (1 - f(\sigma,t+h)f(\sigma,t)f(\sigma,t+h,\chi)f(\sigma,t,\chi))$$
$$\cdot (\arg(Z(\sigma+i(t+h))) - \arg(Z(\sigma+it)))^2 dt = S_1 + S_2, \text{ say.}$$

Since

$\arg\zeta(\sigma+it)$, $\arg\zeta(\sigma+i(t+h))$, $\arg L(\sigma+it,\chi)$ and $\arg L(\sigma+i(t+h),\chi)$ $\ll \log T$ for $\sigma \geq \frac{1}{2}$, we get

$$S_2 \ll \log^2 T\Big(\int_T^{2T} (1-f(\sigma,t))dt + \int_T^{2T} (1-f(\sigma,t,\chi))dt$$
$$+ \int_T^{2T} (1-f(\sigma,t+h))dt + \int_T^{2T} (1-f(\sigma,t+h,\chi))dt\Big)$$
$$= \log^2 T \cdot (S_2' + S_2'' + S_2''' + S_2''''), \text{ say.}$$

$$S_2' \leq |\{T \leq t \leq 2T \,;\, \text{there exists } \beta+i\gamma \text{ in the region } \beta > \frac{1}{2} + \frac{1}{\log X},$$

$$T - \frac{X^{\frac{3}{2}}}{\log X} \leq \gamma \leq 2T + \frac{X^{\frac{3}{2}}}{\log X} \quad \text{such that} \quad \max_{\substack{|t-\gamma| \leq \frac{X^{3(\beta-\frac{1}{2})}}{\log X} \\ \beta > \frac{1}{2} + \frac{1}{\log X}}} \beta > \Big(\frac{1}{2} + \sigma\Big)\frac{1}{2}\Big\}|$$

$$\leq 2 \sum_{\substack{T-\frac{X^{\frac{3}{2}}}{\log X}<\gamma\leq 2T+\frac{X^{\frac{3}{2}}}{\log X} \\ \beta>(\frac{1}{2}+\sigma)\frac{1}{2}}} \frac{X^{3(\beta-\frac{1}{2})}}{\log X} \ll \frac{X^{\frac{3}{2}}}{\log X}\Psi(\sigma, T),$$

where we put

$$\Psi(\sigma, T) = \min(T^{\frac{12}{5}(1-(\frac{1}{2}+\sigma)\frac{1}{2})}, T^{4(\frac{1}{2}+\sigma)\frac{1}{2}(1-(\frac{1}{2}+\sigma)\frac{1}{2})}) \log^C T.$$

We notice that we have used Theorem 1 in p.128 and Theorem 1 in p.131 of Karatsuba and Voronin [13]. (One might get a better estimate if one does not use a trivial estimate $X^{3(\beta-\frac{1}{2})} \ll X^{\frac{3}{2}}$.) In a similar manner, we can estimate S_2'', S_2''' and S_2'''' and get

$$S_2 \ll \frac{X^{\frac{3}{2}}}{\log X}\Psi(\sigma, T)\log^2 T,$$

where Theorem 1 in p.128 and Theorem 1 in p.131 of Karatsuba and Voronin [13] can be easily extended to $L(s, \chi)$.

To evaluate S_1, we use the above formula for $\arg(\zeta(\sigma+it))$, $\arg(\zeta(\sigma+i(t+h)))$, $\arg L(\sigma+it, \chi)$, and $\arg L(\sigma+i(t+h), \chi)$. We get first

$$S_1 = \int_T^{2T} f(\sigma, t+h)f(\sigma, t)f(\sigma, t+h, \chi)f(\sigma, t, \chi))\,(\tilde{M}(t+h)-\tilde{M}(t))^2 dt$$
$$+O\Big(\Big(\int_T^{2T}\big(\tilde{M}(t+h)-\tilde{M}(t)\big)^2 dt\Big)^{\frac{1}{2}}\Big(\int_T^{CT}\tilde{R}(t)^2 dt\Big)^{\frac{1}{2}}\Big)+O\Big(\int_T^{CT}\tilde{R}(t)^2 dt\Big)$$
$$= S_3 + O(\sqrt{S_4}\sqrt{S_5}) + O(S_5), \text{ say.}$$

$$S_3 = \int_T^{2T}(\tilde{M}(t+h)-\tilde{M}(t))^2 dt$$
$$+\int_T^{2T}(f(\sigma, t+h)f(\sigma, t)f(\sigma, t+h, \chi)f(\sigma, t, \chi)-1)$$
$$(\tilde{M}(t+h)-\tilde{M}(t))^2 dt$$
$$= S_4 + S_6, \text{ say.}$$

$$S_6 \ll \Big(\Big(\int_T^{2T}(1-f(\sigma, t))dt\Big)^{\frac{1}{2}}+\Big(\int_T^{2T}(1-f(\sigma, t+h))dt\Big)^{\frac{1}{2}}$$
$$+\Big(\int_T^{2T}(1-f(\sigma, t, \chi))dt\Big)^{\frac{1}{2}}+\Big(\int_T^{2T}(1-f(\sigma, t+h, \chi))dt\Big)^{\frac{1}{2}}\Big)$$
$$\cdot\Big(\int_T^{2T}(\tilde{M}(t+h)-\tilde{M}(t))^4 dt\Big)^{\frac{1}{2}}$$
$$\ll \Big(\sqrt{S_2'}+\sqrt{S_2''}+\sqrt{S_2'''}+\sqrt{S_2''''}\Big)\cdot\sqrt{S_7} \text{ say.}$$

So we are left to evaluate S_4, S_7 and S_5.

$$S_4 = \int_T^{2T}\Big(\frac{\eta(t)-\bar\eta(t)}{2i}\Big)^2 dt$$
$$= -\frac{1}{4}\int_T^{2T}\eta^2(t)\,dt - \frac{1}{4}\int_T^{2T}\bar\eta^2(t)\,dt + \frac{1}{2}\int_T^{2T}|\eta(t)|^2 dt$$
$$= S_8 + \overline{S_8} + S_9, \text{ say,}$$

where we put

$$\eta(t) = \sum_{n<X^3}\frac{\Lambda_X(n)}{n^{\sigma+it}\log n}\Big(\frac{1}{n^{ih}}-1\Big)(\chi(n)+1).$$

We get simply,

$$S_8 \ll \sum_{m,n<X^3}\frac{\Lambda_X(m)\Lambda_X(n)}{(mn)^{\sigma}\log m\log n\log(mn)} \ll \Phi(X,\sigma),$$

where we put

$$\Phi(X,\sigma) = \begin{cases} 1 & \text{if } \sigma = 1 \\ \dfrac{X^{6(1-\sigma)}}{\log^3 X} & \text{if } \dfrac{1}{2} < \sigma < 1. \end{cases}$$

By Montgomery and Vaughan [16], we get

$$S_9 = \frac{1}{2}\sum_{n<X^3}(T+O(n))\frac{\Lambda_X^2(n)}{n^{2\sigma}\log^2 n}\Big|\frac{1}{n^{ih}}-1\Big|^2|\chi(n)+1|^2$$
$$= T\Big(\frac{1-\cos(h\log 2)}{2^{2\sigma}} + \sum_{m=1}^{\infty}\frac{\Lambda^2(4m+1)}{(4m+1)^{2\sigma}\log^2(4m+1)}(1-\cos(h\log(4m+1)))\Big)$$
$$+O\Big(T\sum_{n>X}\frac{\Lambda^2(n)}{n^{2\sigma}\log^2 n}\Big) + O\Big(\sum_{n<X^3}\frac{\Lambda^2(n)}{n^{2\sigma-1}\log^2 n}\Big)$$
$$= T\Big(\frac{1-\cos(h\log 2)}{2^{2\sigma}} + \sum_{m=1}^{\infty}\frac{\Lambda^2(4m+1)}{(4m+1)^{2\sigma}\log^2(4m+1)}(1-\cos(h\log(4m+1)))\Big)$$
$$+ O\Big(\frac{T}{X^{2\sigma-1}\log X}\Big) + O(\Phi_1(X,\sigma)),$$

where we put

$$\Phi_1(X,\sigma) = \begin{cases} \log\log X & \text{if } \sigma = 1 \\ \dfrac{X^{6(1-\sigma)}}{\log X} & \text{if } \dfrac{1}{2} < \sigma < 1. \end{cases}$$

Using Montgomery and Vaughan [16] again, we get

$$S_7 \ll \int_T^{2T}\Big|\sum_{m,n<X^3}\frac{\Lambda_X(m)\Lambda_X(n)}{(mn)^{\sigma+it}\log m\,\log n}\Big(\frac{1}{m^{ih}}-1\Big)\Big(\frac{1}{n^{ih}}-1\Big)\Big|^2 dt$$

$$+\int_T^{2T}\Big|\sum_{m,n<X3}\frac{\Lambda_X(m)\Lambda_X(n)\chi(m)\chi(n)}{(mn)^{\sigma+it}\log m\log n}\Big(\frac{1}{m^{ih}}-1\Big)\Big(\frac{1}{n^{ih}}-1\Big)\Big|^2 dt$$

$$\ll\int_T^{2T}\Big|\sum_{k<X^6}\frac{u(k)}{k^{\sigma+it}}\Big|^2 dt+\int_T^{2T}\Big|\sum_{k<X^6}\frac{u(k,\chi)}{k^{\sigma+it}}\Big|^2 dt$$

$$\ll\sum_{k<X^6}(\mathrm{T}+O(k))\frac{|u(k)|^2+|u(k,\chi)|^2}{k^{2\sigma}}$$

$$\ll T\sum_{k<X^6}\frac{\sigma_0(k)^2}{k^{2\sigma}}+\sum_{k<X^6}\sigma_0(k)^2k^{1-2\sigma}\ll T+X^{12(1-\sigma)}\log^3X,$$

where

$$u(k)=\sum_{\substack{mn=k\\ m,n<X^3}}\frac{\Lambda_X(m)\Lambda_X(n)}{\log m\log n}\Big(\frac{1}{m^{ih}}-1\Big)\Big(\frac{1}{n^{ih}}-1\Big)\ll\sigma_0(k)=\sum_{d|k}\cdot 1,$$

$$u(k,\chi)=\sum_{\substack{mn=k\\ m,n<X^3}}\frac{\Lambda_X(m)\Lambda_X(n)\chi(m)\chi(n)}{\log m\log n}\Big(\frac{1}{m^{ih}}-1\Big)\Big(\frac{1}{n^{ih}}-1\Big)\ll\sigma_0(k)$$

and we have used the estimate

$$\sum_{k\le Y}\sigma_0^2(k)\ll Y\log^3Y.$$

Finally,

$$S_5\ll\int_T^{CT}R^2(t)dt+\int_T^{CT}R^2(t,\chi)dt=S_5'+S_5'' \quad \text{, say.}$$

Using pp.248-251 of Selberg[20], we get

$$S_5'\ll\frac{X^{\frac12-\sigma}}{\log^2X}\Big(\int_T^{CT}\Big|\sum_{n<X^3}\frac{\Lambda_X(n)}{n^{\sigma_{X,t}+it}}\Big|^2dt+T\log^2T\Big)$$

$$\ll\frac{X^{\frac12-\sigma}}{\log^2X}\Big(\sqrt{T\log X}\Big(\int_{\frac12}^{\infty}X^{\frac12-\sigma}\int_T^{CT}\Big|\sum_{p<X^3}\frac{\Lambda_X(p)\log(Xp)}{p^{\sigma+it}\log^2X}\Big|^4dtd\sigma\Big)^{\frac12}$$

$$+T\log^2T\Big)\ll\frac{T}{X^{\sigma-\frac12}}.$$

Similarly, using an extension of Selberg's argument to $L(s,\chi)$ (cf. Fujii [5]), we get

$$S_5''\ll\frac{T}{X^{\sigma-\frac12}}.$$

Consequently, we get

$$\int_T^{2T}(\arg(Z(\sigma+i(t+h)))-\arg(Z(\sigma+it)))^2dt$$

$$=T\Big(\frac{1-\cos(h\log 2)}{2^{2\sigma}}+\sum_{m=1}^{\infty}\frac{\Lambda^2(4m+1)}{(4m+1)^{2\sigma}\log^2(4m+1)}(1-\cos(h\log(4m+1)))\Big)$$

$$+O(T^{1-\frac{a}{2}(\sigma-\frac{1}{2})})+O\Big(T^{\frac{1}{2}+\frac{13a}{4}-\frac{7a\sigma}{2}}\frac{1}{\sqrt{\log T}}\Big)+O\Big(\frac{T^{6a(1-\sigma)}}{\log T}\Big)$$

$$+O\Big(T^{\frac{3a}{2}}\Psi(\sigma,T)\log T\Big)+O\Big(T^{\frac{3a}{4}}(T^{6a(1-\sigma)}\log T+\sqrt{T/\log T})\sqrt{\Psi(\sigma,T)}\Big).$$

Here we can choose an optimal a and get our Lemma 6 as described at the beginning of this section, although we shall not describe it explicitly.

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