J. Math. Kyoto Univ. (JMKYAZ)
37-1 (1997) 169-176

Adjoint actions on the modulo 5 homology groups of E_8 and ΩE_8

By

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1. Introduction

Borel proved in [2] that the integral homology group of the exceptional Lie group E_8 is not 5-torsion free and

$$H(E_8;Z/5) \cong \Lambda(x_3, x_{11}, x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47}) \otimes Z/5[x_{12}]/(x_{12}^5), \text{ with } |x_i| = i,$$

as algebra.

Araki showed the non-commutativity of the Pontrjagin ring $H_*(E_8;Z/5)$ in [1]. The whole Hopf algebra structure and the cohomology operations were determined by Kono in [6]. But it was due to the partial computation of $\mathrm{Cotor}^{H^*(E_8;Z/5)}(Z/5,Z/5)$, which was rather complicated. In [5], using secondary cohomology operations, Kane gave a general theorem to determine the Pontrjagin ring which is non-commutative and determined $H_*(E_8;Z/5)$ as a Hopf algebra over $\mathscr{A}_5$.

Also, for a compact, connected Lie group G, the free loop group of G denoted by $LG(G)$ is the space of free loops on G equiped with multiplication as

$$\phi \cdot \psi(t) = \phi(t) \cdot \psi(t),$$

and has ΩG as its normal subgroup. Thus

$$LG(G)/\Omega G \cong G,$$

and identifying elements of G with constant maps from S^1 to G, $LG(G)$ is equal to the semi-direct product of G and ΩG. This means that the homology of $LG(G)$ is determined by the homology of G and ΩG as module and the algebra structure of $H_*(LG(G);Z/p)$ depends on $H_*(\mathrm{Ad};Z/p)$ where

$$\mathrm{Ad}: G \times \Omega G \to \Omega G$$

is the adjoint map. Since the next diagram commutes where λ, λ' and μ are the multiplication maps of ΩG, $LG(G)$ and G respectively and ω is the composition

Received July 31, 1996

Partially supported by JSPS Research Fellowships for Young Scientists.

$$(1_{\Omega G}\times T\times 1_G)\circ(1_{\Omega G\times G}\times \mathrm{Ad}\times 1_G)\circ(1_{\Omega G}\times \Delta_G\times 1_{\Omega G\times G}),$$

$$\begin{array}{ccccc}\Omega G\times G\times\Omega G\times G & \xrightarrow{\omega} & \Omega G\times\Omega G\times G\times G & \xrightarrow{\lambda\times\mu} & \Omega G\times G\\ \downarrow\cong\times\cong & & \lambda' & & \downarrow\cong\\ LG(G)\times LG(G) & & \longrightarrow & & LG(G)\end{array}$$

we can determine directly the algebra structure of $H_*(LG(G);Z/p)$ by the knowledge of the Hopf algebra structure of $H_*(G;Z/p)$, $H_*(\Omega G;Z/p)$ and induced homology map $H_*(\mathrm{Ad};Z/p)$. See Theorem 6.12 of [4] for detail. Moreover, in [8], it is showed that provided G is simply connected, $H^*(\mathrm{Ad};Z/p)$ is equal to the induced homology map of second projection if and only if $H_*(G;Z)$ is p-torsion free. Thus the case of $(G,p)=(E_8,5)$ is non-trivial.

In this paper we determine $H_*(\mathrm{Ad};Z/5)$ for $G=E_8$ and at the same time, we offer a more simple method for the determination of the coproduct and the cohomology operations on $H^*(E_8;Z/5)$ using the adjoint actions of E_8 on ΩE_8. We also determine $H_*(\Omega E_8;Z/5)$ as a Hopf algebra over $\mathcal{A}_5$.

This paper is organized as follows. In the next section we breifly see the algebra structures of $H^*(E;Z/5)$ and $H_*(\Omega E;Z/5)$ using the Serre spectral sequences. In the third section we determine the adjoint action of $H_*(E_8;Z/5)$ on $H_*(\Omega E_8;Z/5)$ which was introduced in [8]. It gives an easy computation of the Hopf algebra structures and the cohomology operations on them.

2. Algebra structures

Let $n(j)$, $(1\leq j\leq 8)$, be the exponent of E_8, i.e.

$$\{n(j)\}_{1\leq j\leq 8}=\{1,7,11,13,17,19,23,29\}.$$

First we see $H^*(\Omega E_8;Z/5)$ for low dimensions. Let R be the algebra $Z/5[a_{2n(j)}|1\leq j\leq 8]$ with $|a_i|=i$. By Bott ([3]), the Hopf algebra $H^*(\Omega E_8;Z/5)$ is isomorphic to R as a vector space. There is a map q: $SU(9)\to E_8$ which induces an isomorphism of π_3. Then, Ωq: $\Omega SU(9)\to\Omega E_8$ induces an isomorhpism of π_2 and, as showed in [7], $(\Omega q)^*a_2\in H^2(\Omega SU(9);Z/5)$ is nontrivial and $((\Omega q)^*a_2)^5\neq 0$ for the generator $a_2\in H^2(\Omega E_8;Z/5)$. Thus we have $a_2^5\neq 0$. It follows that $H^*(\Omega E_8;Z/5)$ is isomorphic to R for $*<50$ as algebra. Next there is two possibilities (I): $a_2^{25}\neq 0$ and (II): $a_2^{25}=0$. That is, we can assume it is isomorphic to (I): R or (II): $R/(a_2^{25})\otimes Z/5[a_{50}]$, for $*<10\cdot n(2)=70$, where $|a_{50}|=50$.

Consider the following Serre fibre sequences:

$$\tilde{E}_8\xrightarrow{k}E_8\xrightarrow{\iota}K(Z,3),\tag{1}$$

$$K(Z,1)\longrightarrow\Omega\tilde{E}_8\xrightarrow{\Omega k}\Omega E_8,\tag{2}$$

$$\Omega\tilde{E}_8\longrightarrow *\longrightarrow\tilde{E}_8,\tag{3}$$

where ι induces an isomorphism of π_3.

Let $\tilde{R}\equiv Z/5\,[\tilde{a}_{2n(i)}|2\leq i\leq 8]$ with $|\tilde{a}_i|=i$. Computing the Serre spectral sequence associated to (2), we can see that, for $*<70$, $H^*(\Omega\tilde{E}_8;Z/5)$ is isomorphic to (I): $\tilde{R}$ or (II): $\tilde{R}\otimes\Lambda(\tilde{a}_{49})\otimes Z/5\,[a_{50}]$ according to the case: $a_2^{25}\neq 0$ or $a_2^{25}=0$. Let $\tilde{S}\equiv\Lambda(x_{2n(j)+1}|2\leq j\leq 8)$ with $|\tilde{x}_i|=i$. Again computing the spectral sequence associated to (3), we have, for $*<71$, $H^*(\tilde{E}_8;Z/5)$ is isomorphic to (I): $\tilde{S}$ or (II): $\tilde{S}\otimes Z/5\,[\tilde{x}_{50}]\otimes\Lambda(\tilde{x}_{51})$ where $|\tilde{x}_{50}|=50$, $|\tilde{x}_{51}|=51$.

Recall the fact:

$$H^*(K(Z,3);Z/5)\cong\Lambda(u_3,u_{11},u_{51},\cdots)\otimes Z/5\,[u_{12},u_{52},\cdots],\ |u_i|=i, \tag{4}$$

where $u_{11}=\mathcal{P}^1u_3$, $u_{12}=\beta u_{11}$, $u_{51}=\mathcal{P}^5u_{11}$ and $u_{52}=\beta u_{51}$.

Let $x_i=\iota^*(u_i)$, for $i=11$, 12, 51 and 52, in $H^*(E_8;Z/5)$. By the spectral sequence associated to (1), we obtain, for $*<58$, $H^*(E_8;Z/5)\cong$ (I): $S\otimes\Lambda(x_{11},x_{51})\otimes Z/5\,[x_{12},x_{52}]$ or (II): $S\otimes\Lambda(x_{11})\otimes Z/5\,[x_{12}]$, where $S\equiv\Lambda(x_{2n(j)+1}|1\leq j\leq 7)$ with $|x_i|=i$.

As $\dim E_8=248$, we can conclude that the possible case is (II) and $x_{12}{}^5=0$. Moreover, the generators $\{x_i\}$ are enough to generate $H^*(E_8;Z/5)$. We have determined the algebra structure.

Theorem 1. *There is an algebra isomorphism*:

$$H^*(E_8;Z/5)\cong\Lambda(x_{2n(j)+1}|1\leq j\leq 7)\otimes\Lambda(x_{11})\otimes Z/5\,[x_{12}]/(x_{12}{}^5).$$

In $H^*(\tilde{E}_8;Z/5)$, we can chose $\tilde{x}_{50}$ and $\tilde{x}_{51}$ such that $\tau'\tilde{x}_{50}=u_{51}$ and $\tau'\tilde{x}_{51}=u_{52}$, where τ' is the transgression. Then $\tau'\mathcal{P}^1\tilde{x}_{51}=\mathcal{P}^1u_{52}=\mathcal{P}^1\beta\mathcal{P}^5u_{11}=\mathcal{P}^6\beta u_{11}=\mathcal{P}^6u_{12}=u_{12}{}^5$. So we can chose $\tilde{x}_{59}$ as $\mathcal{P}^1\tilde{x}_{51}$. Thus we have

Proposition 2. *There is an isomorphism for* $*<71$:

$$H^*(\tilde{E}_8;Z/5)\cong\Lambda(\tilde{x}_{2n(j)+1}|2\leq j\leq 8)\otimes Z/5\,[\tilde{x}_{50}]\otimes\Lambda(\tilde{x}_{51}),$$

and

$$\mathcal{P}^1(\tilde{x}_{51})=\tilde{x}_{59}.$$

Because that $\tilde{a}_i$ is transgressed to $\tilde{x}_{i+1}$ and $(\Omega k)^*a_i=\tilde{a}_i$ for $i=50,58$, the next proposition is obtained.

Proposition 3. *There are isomorphisms for* $*<70$:

$$H^*(\Omega\tilde{E};Z/5)\cong Z/5\,[\tilde{a}_{2n(j)}|2\leq j\leq 8]\otimes\Lambda(\tilde{a}_{49})\otimes Z/5\,[\tilde{a}_{50}],$$
$$H^*(\Omega E_8;Z/5)\cong Z/5\,[a_{2n(j)}|2\leq j\leq 8]/(a_2{}^{25})\otimes Z/5\,[a_{50}],$$

with $\mathcal{P}^1(\tilde{a}_{50})\equiv\tilde{a}_{58}$ *and* $\mathcal{P}^1(a_{50})\equiv a_{58}$ *(modulo decomposable)*.

By the use of a Rothenberg-Steenrod spectral sequence ([10]):

$$E_2\cong H^{**}(H_*(\Omega E_8;Z/5))\equiv\mathrm{Ext}_{H_*(\Omega E_8;Z/5)}(Z/5,Z/5)\Rightarrow E_\infty=\mathcal{G}r(H^*(E_8;Z/5)),$$

it is easily seen that

Theorem 4. *There is an algebra isomorphism*:

$$H_*(\Omega E_8;Z/5)\cong Z/5[t_{2n(j)}|1\leq j\leq 8]/(t_2^5)\otimes Z/5[t_{10}].$$

Remark. The algebra was determined first in [9].

Let σ denote the homology suspension. Examining the spectral sequence, we have the following proposition.

Proposition 5. $\sigma(t_{2n(j)})$, $(1\leq j\leq 7)$, *and* $\sigma(t_{10})$ *are nontirivial primitive elements in* $H_*(E_8;Z/5)$.

3. Coproducts, cohomology operations and adjoint actions

Let $(\ )^*$ denote the dual as to the monomial basis of $\{x_i\}$ and put $y_i=(x_i)^*$.

We recall the adjoint action which was mentioned in [8]. Let ad:$G\times G\to G$ and Ad:$G\times\Omega G\to\Omega G$ be the adjoint actions for the Lie group G. Consider the induced maps of homlogy groups:

$$\mathrm{ad}_*: H_*(G)\otimes H_*(G)\to H_*(G),$$
$$\mathrm{Ad}_*: H_*(G)\otimes H_*(\Omega G)\to H_*(\Omega G).$$

Put $y*y'=\mathrm{ad}_*(y\otimes y')$ and $y\cdot t=yt=\mathrm{Ad}_*(y\otimes t)$.

Our result is the following.

Theorem 6. *In* $H_*(E_8;Z/5)$, *there are* $y_{2n(j)+1}$, $(1\leq j\leq 7)$, y_{11} and y_{12} *satisfying that*

y_i	y_3	y_{11}	y_{12}	y_{15}	y_{23}	y_{27}	y_{35}	y_{39}	y_{47}
$y_{12}*y_i$	y_{15}	y_{23}	0	y_{27}	y_{35}	y_{39}	y_{47}	0	0
$\mathscr{P}^1_*y_i$	0	y_3	0	0	y_{15}	0	y_{27}	0	y_{39}
β_*y_i	0	0	y_{11}	0	0	0	0	0	0

All y_i *are primitive and* $y_{12}*y_i=[y_{12},y_i]=y_{12}y_i-y_iy_{12}$.

Remark. This result coincides with that of §46-2 of [5].

From now on, we prove this theorem combining the adjoint actions on $H^*(E_8;Z/5)$ and $H^*(\Omega E_8;Z/5)$.

Dualizing the properties of ad* and Ad* stated in [8], we have

Proposition 7. *For* $y,y',y''\in H_*(G)$ *and* $t,t',t''\in H_*(\Omega G)$

(1) $1*y=y$, $1\cdot t=t$.

(2) $y*1=0$ *and* $y\cdot 1=0$, *if* $|y|>0$.

(3) $(yy')t=y(y't)$.

(4) $y(tt')=\sum(-1)^{|y''||t|}(y't)(y''t')$, *where* $\Delta_*y=\sum y'\otimes y''$ *is the coproduct*.

(5) $\phi(y\cdot t)=\Delta_*(y)\cdot\phi(t)$, *where* ϕ *is the coproduct and* $(y'\otimes y'')\cdot(t'\otimes t'')=(-1)^{|y''||t'|}(y't'\otimes y''t'')$.

(6) $\sigma(y\cdot t)=y*\sigma(t)$, *where* σ *is the homology suspension.*

(7) *If* y *is primitive then* $y*y'=[y,y']$, *where* $[y,y']=yy'-(-1)^{|y\cdot y'|}y'y$.

(8) *If* t *is primitive then* $y\cdot t$ *is also primitive.*

(9) $\mathscr{P}^n_*(y*y')=\sum_i\mathscr{P}^{n-i}_*y*\mathscr{P}^i_*y'$ *and* $\mathscr{P}^n_*(y\cdot t)=\sum_i\mathscr{P}^{n-i}_*y\cdot\mathscr{P}^i_*t$.

Remark. In our case, $|t|$ and $|t'|$ are always even. So $y(tt')=\sum(y't)(y''t')$ and $(y'\otimes y'')\cdot(t'\otimes t'')=(y't'\otimes y''t'')$.

To state the non-commutativity of $H_*(E_8;Z/5)$, we need only the fact:

Lemma 8. $[y_{12},y_3]\neq 0$.

Proof. Suppose that $[y_{12},y_3]=0$. Then $H_*(E_8;Z/5)\cong\Lambda(y_3,y_{11},y_{15})\otimes Z/5[y_{12}]$ for $*<23$. Let $\{E_r'\}$ be the Rothenberg-Steenrod spectral sequece conversing to $H^*(BE_8;Z/5)$. Then we have

$$E_2'\cong Z/5[s(y_3),s(y_{11}),s(y_{15})]\otimes\Lambda(s(y_{12}))$$

for total degree <24. Since $E_2'=E_\infty'$ in these degrees, there are indecomposable elements z_4, z_{12}, z_{16} and z_{13} in $H^*(BE_8;Z/5)$ corresponding to $s(y_3)$, $s(y_{11})$, $s(y_{15})$ and $s(y_{12})$, respectively. Especially, $z_4z_{13}\neq 0$. It is a contradiction. (For detail, see Lemma 5.3 and 5.4 of [6].)

Therefore $[y_{12},y_3]$ is the nontrivial primitive element. So we may define y_{15} by that.

Proposition 9. $[y_{12},y_3]=y_{15}$.

Since $\sigma(y_{12}t_2)=y_{12}*\sigma(t_2)=y_{12}*y_3=[y_{12},y_3]=y_{15}$, $y_{12}t_2$ is the indecomposable element. Thus we may assume that

$$t_{14}=y_{12}t_2. \tag{5}$$

Then t_{14} is primitive and $\sigma(t_{14})=y_{15}$.

Let ϕ be the coproduct of $H_*(\Omega E_8;Z/5)$ and $\bar\phi(t)=\phi(t)-t\otimes 1-1\otimes t$. $(\ )^*$ denotes the dual as to the monomial basis of $\{t_{2j}\}$. Multiplying a_i and t_i by nonzero scalars or moving them modulo decomposable if we need, we may assume that $a_{2n(j)}=(t_{2n(j)})^*$, $(1\leq j\leq 8)$, $a_2{}^5=(t_{10})^*$ and $a_{50}=(t_{10}{}^5)^*$. As t_{10} is dual to $a_2{}^5$, it is easily verified that

$$\bar\phi(t_{10})=4t_2{}^4\otimes t_2+3t_2{}^3\otimes t_2{}^2+3t_2{}^2\otimes t_2{}^3+4t_2\otimes t_2{}^4. \tag{6}$$

$\mathscr{P}^1a_2=a_2{}^5$ implies $\mathscr{P}^1_*t_{10}=t_2$. Define t_{22}' by $y_{12}t_{10}-t_2{}^4t_{14}$. Then by (6) and Proposition 7, $\bar\phi(t_{22}')=\Delta^*(y_{12})\phi(t_{10})-\phi(t_2)^4\phi(t_{14})=t_{22}'\otimes 1+1\otimes t_{22}'$. On the other hand, since $\mathscr{P}^1_*y_{12}$ and $\mathscr{P}^1_*t_{14}$ are trivial, $\mathscr{P}^1_*t_{22}'=y_{12}\mathscr{P}^1_*t_{10}=y_{12}t_2=t_{14}$. So t_{22}' is nontrivial and primitive. Put $t_{22}=t_{22}'$. Now we obtain the following equations.

$$y_{12}t_{10}=t_{22}-t_2^4t_{14}, \tag{7}$$
$$\mathcal{P}^1_*t_{22}=t_{14}. \tag{8}$$

Using Proposition 7 and $y_{12}^5=0$, we can compute $y_{12}^4t_{22}$, that is,

$$y_{12}^4t_{22}=y_{12}^4(y_{12}t_{10}+t_2^4t_{14})=y_{12}^5t_{10}+y_{12}^4(t_2^4t_{14})=y_{12}^4(t_2^4t_{14})$$

Here, since $y_{12}t_j$ ($j=14,26,38$) is primitive, there exists $\rho_j\in Z/5$ such that $y_{12}t_j=\rho_jt_{j+12}$, where $t_{50}=t_{10}^5$. Note that $y_{12}(t_{10}^5)=0$. Therefore modulo the ideal (t_{26},t_{38},t_{10}^5), we have

$$y_{12}^4(t_2^4t_{14})\equiv4y_{12}^3(t_2^3t_{14}^2)\equiv12y_{12}^2(t_2^2t_{14}^3)\equiv24y_{12}(t_2t_{14}^4)\equiv-t_{14}^5.$$

But, since $y_{12}^4t_{22}$ is primitive, we obtain $y_{12}^4t_{22}=-t_{14}^5$. This means that $y_{12}^it_{22}$, $(1\leq i\leq4)$, are nontrivial primitive elements. Therefore we can define the generators so that

$$t_{22+12i}=y_{12}^it_{22},\ (1\leq i\leq3). \tag{9}$$

Next we will observe $y_{12}^it_{14}$, $(1\leq i\leq3)$. Since $\mathcal{P}^1_*t_{58}$ is primitive, there is $\epsilon\in Z/5$ such that $\mathcal{P}^1_*t_{58}=\epsilon t_{10}^5$. On the other hand, from Proposition 3, $\mathcal{P}^1a_{50}\equiv a_{58}$ (up to non zero coefficient and modulo decomposable). Dualize it, then we can see $\epsilon\neq0$. Re-define t_{58} by $\epsilon^{-1}y_{12}^3t_{22}$. We have

Proposition 10.

$$y_{12}^3t_{22}=\epsilon t_{58}, \tag{10}$$
$$\mathcal{P}^1_*t_{58}=t_{10}^5. \tag{11}$$

From this, $y_{12}^3t_{14}=y_{12}^3\mathcal{P}^1_*t_{22}=\mathcal{P}^1_*(y_{12}^3t_{22})=\mathcal{P}^1_*(\epsilon t_{58})=\epsilon t_{10}^5$. So we can fix

$$t_{14+12i}=y_{12}^it_{14},\ (1\leq i\leq2). \tag{12}$$

By $\mathcal{P}^1_*(y_{12}^it_{2k})=y_{12}^i\mathcal{P}^1_*t_{2k}$, $\mathcal{P}^1_*$ is determined on all t_{2k}.
We summarize the results.

Theorem 11. *In Therem 4, we can chose the generators satisfying the following table*:

t_{2j}	t_2	t_{10}	t_{14}	t_{22}	t_{26}	t_{34}	t_{38}	t_{46}	t_{58}
$y_{12}t_{2j}$	t_{14}	$t_{22}-t_2^4t_{14}$	t_{26}	t_{34}	t_{38}	t_{46}	ϵt_{10}^5	ϵt_{58}	$-\epsilon^{-1}t_{14}^5$
$\mathcal{P}^1_*t_{2j}$	0	t_2	0	t_{14}	0	t_{26}	0	t_{38}	t_{10}^5

All t_{2k}, $(k\neq5)$ *are primitive and*

$$\bar{\phi}(t_{10})=4t_2^4\otimes t_2+3t_2^3\otimes t_2^2+3t_2^2\otimes t_2^3+4t_2\otimes t_2^4.$$

Proof of Theorem 6. Put $y_{2n(j)+1}=\sigma(t_{2n(j)})$, $(3\leq j\leq7)$. Theorem 6 is an immediate consequence of Theorem 1, Theorem 4 and Proposition 5 with Proposition 7.

Fix the basis of $H_*(E_8;Z/5)$:

$$\{\prod_{j=1}^{7} y_{2n(j)+1}{}^{\epsilon_{2n(j)+1}} y_{11}{}^{\epsilon_{11}} y_{12}{}^{e} | 0 \leq \epsilon_i \leq 1,\ 0 \leq e < 5\}.$$

Let $(\ )^*$ be the dual with respect to the above basis. We may assume that $x_{2n(j)+1} = (y_{2n(j)+1})^*$, $(2 \leq j \leq 7)$. Let φ be the coproduct of $H^*(E_8;Z/5)$ and $\bar{\varphi}(x) = \varphi(x) - x \otimes 1 - 1 \otimes x$. Then the following theorem is easily obtained by dualizing Theorem 6.

Theorem 12. *In Theorem* 1, *we can chose the generators satisfying following tables*:

x_i	x_3	x_{11}	x_{12}	x_{15}	x_{23}	x_{27}	x_{35}	x_{39}	x_{47}
$\mathcal{P}^1 x_i$	x_{11}	0	0	x_{23}	0	x_{35}	0	x_{47}	0
βx_i	0	x_{12}	0	0	$x_{12}{}^2/2$	0	$x_{12}{}^3/3!$	0	$x_{12}{}^4/4!$

x_i	$\bar{\varphi} x_i$
x_{15}	$x_{12} \otimes x_3$
x_{23}	$x_{12} \otimes x_{11}$
x_{27}	$x_{12} \otimes x_{15} + x_{12}{}^2/2 \otimes x_3$
x_{35}	$x_{12} \otimes x_{23} + x_{12}{}^2/2 \otimes x_{11}$
x_{39}	$x_{12} \otimes x_{27} + x_{12}{}^2/2 \otimes x_{15} + x_{12}{}^3/3! \otimes x_3$
x_{47}	$x_{12} \otimes x_{35} + x_{12}{}^2/2 \otimes x_{23} + x_{12}{}^3/3! \otimes x_{11}$

Remark. In [6], $x_{2n(j)+1}$, $(4 \leq j \leq 7)$, are chosen as our $2x_{27}$, $2x_{35}$, $3!x_{39}$ and $3!x_{47}$ respectively.

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