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On maps from BS^1 to classifying spaces of certain gauge groups II

By

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1. Introduction

The purpose of this paper is to generalize Theorem 1.2 of [5]. Let π: $P \to X$ be a principal SU(2) bundle over a simply connected closed 4 manifold X and $\mathcal{G}$ its gauge group. $\mathcal{G}$ is identified with $\Gamma(AdP)$, all continuous sections of the adjoint bundle of P, and we give the compact open topology on it.
We show the following result.

Theorem 1.1 *The following three conditions are equivalent.*
1. *There exists a homotopically non trivial map from BS^1 to $B\mathcal{G}$.*
2. *There exists a non trivial homomorphism from S^1 to $\mathcal{G}$.*
3. *The structure group of P reduces to S^1.*

Remark 1.2 In [5], we showed this result under the assumption that X is a smooth simply connected spin 4 manifold or $\mathbf{C}P^2$.

It is clear that *3* implies *2* and by the Appendix of [5], *2* implies *1*. The structure group of P reduces to S^1 if and only if there exists an element $u \in H^2(X)$ such that $c_2(P) = -u^2[X]$. We will show that *1* implies $c_2(P) = -u^2[X]$. In this paper $H_*(\ )$ $((H^*))$ mean the integral (co) homology.

2. Proof of Theorem 1.1

Note that principal SU(2) bundles over X are classified by their 2nd Chern classes. If $c_2(P) = k$, by [1], we have a homotopy equivalence

$$B\mathcal{G} \simeq \mathrm{Map}_k(X, B\mathrm{SU}(2)),$$

where $\mathrm{Map}_k(X, B\mathrm{SU}(2))$ denotes the connected component of $\mathrm{Map}(X, B\mathrm{SU}(2))$ containing the map inducing P and a fibration

$$\mathrm{Map}_k^*(X, B\mathrm{SU}(2)) \to \mathrm{Map}_k(X, B\mathrm{SU}(2)) \xrightarrow{ev} B\mathrm{SU}(2),$$

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where $\mathrm{Map}_k^*(X, B\mathrm{SU}(2))$ is the space of based maps.

Lemma 2.1 ([5]). *Consider a map* $f: BS^1 \to \mathrm{Map}_k(X, B\mathrm{SU}(2))$. *If* $ev \circ f$ *is homotopically trivial, then so is* f.

Let $\rho: S^1 \to \mathrm{SU}(2)$ be a non trivial homomorphism. Denote by $Z(\rho)$ the centralizer of this homomorphism and by $\mathrm{Map}_\rho(BS^1, B\mathrm{SU}(2))$ the component which contains the map $B\rho$. Note that $Z(\rho) = S^1$. The obvious homomorphism

$$Z(\rho) \times S^1 \to \mathrm{SU}(2)$$

induces a map

$$BZ(\rho) \times BS^1 \to B\mathrm{SU}(2),$$

which has as adjoint

$$ad_\rho : BZ(\rho) \to \mathrm{Map}_\rho(BS^1, B\mathrm{SU}(2)).$$

Let X_ρ be the homotopy fiber of ad_ρ. We can compute the homotopy groups of X_ρ (see [3], [5] for details).

$$\pi_i(X_\rho) = \begin{cases} \widehat{\mathbf{Z}}/\mathbf{Z}, & i=0, 1, 2 \\ 0, & \textit{otherwise}, \end{cases}$$

where $\widehat{\mathbf{Z}} = \prod_p \mathbf{Z}_p$ is the product over all p-adic integers. Note that $\widehat{\mathbf{Z}}/\mathbf{Z}$ is a rational vector space. Let $M \to \mathrm{Map}_\rho(BS^1, B\mathrm{SU}(2))$ be the universal covering. Since BS^1 is simply connected, ad_ρ lifts to M. Let F be the homotopy fiber of the lifting $\widetilde{ad}_\rho : BS^1 \to M$ then we have

$$\pi_i(F) = \begin{cases} \widehat{\mathbf{Z}}/\mathbf{Z}, & i=1, 2 \\ 0, & \textit{otherwise}. \end{cases}$$

Denote by $Rep(S^1, \mathrm{SU}(2))$ the set of conjugation classes of homomorphisms.

Theorem 2.2 ([3]). *The map*

$$Rep(S^1, \mathrm{SU}(2)) \to [BS^1, B\mathrm{SU}(2)]$$

is a bijection.

Suppose there exists a non trivial map $f: BS^1 \to \mathrm{Map}_k(X, B\mathrm{SU}(2))$. By Lemma 2.1, $ev \circ f$ is homotopically nontrivial and by Lemma 2.2 there exists a non trivial homomorphism $\rho: S^1 \to \mathrm{SU}(2)$ such that $ev \circ f \simeq B\rho$. Taking abjoint of f we obtain a map $g: X \to \mathrm{Map}_\rho(BS^1, B\mathrm{SU}(2))$. Since X is simply connected, g lifts to M.

$$\begin{array}{ccc} & & BS^1 \\ & & \downarrow \widetilde{ad}_\rho \\ X & \xrightarrow{\tilde{g}} & M \\ & & \downarrow ev \circ p \\ & & BSU(2). \end{array}$$

Note that $ev \circ p \circ \tilde{g}$ induces P and $ev \circ p \circ \widetilde{ad}_\rho \simeq Bi : BS^1 \to BSU(2)$, where $i : S^1 \hookrightarrow SU(2)$ is an inclusion. $Bi^* c_2 = -c_1^2$, where $c_2 \in H^4(BSU(2))$ is the universal 2nd Chern class and $c_1 \in H^2(BS^1)$ is the universal 1st Chern class and $(ev \circ p \circ \tilde{g})^* c_2 = c_2(P)$.

Proposition 2.3 *$\widetilde{ad}_\rho^* : H^2(M) \to H^2(BS^1)$ is an isomorphism. The kernel of $\widetilde{ad}_\rho^* : H^4(M) \to H^4(BS^1)$ is a rational vector space.*

By this proposition, we can prove Theorem 1.1 as follows. There exists an element $c \in H^2(M)$ such that $\widetilde{ad}_\rho^*(c) = c_1$. Then $\widetilde{ad}_\rho^*(c^2 + (ev \circ p)^* c_2) = 0$. Since $H^4(X) = \mathbf{Z}$, $\tilde{g}^*(c^2 + (ev \circ p)^* c_2) = 0$. Therefore

$$c_2(P) = (ev \circ p \circ \tilde{g})^* c_2 = -(\tilde{g}^* c)^2$$

hence the structure group of P reduces to S^1 and Theorem 1.1 is proved.

In the rest of the paper we prove Proposition 2.3. By the homotopy exact sequence for the fibration

$$F \to BS^1 \xrightarrow{\widetilde{ad}_\rho} M,$$

we have the following short exact sequence

$$0 \to \mathbf{Z} \xrightarrow{\widetilde{ad}_{\rho *}} \pi_2 \to \hat{\mathbf{Z}}/\mathbf{Z} \to 0, \tag{1}$$

where $\pi_2 = \pi_2(M)$.

Lemma 2.4. If A is a rational vector space, so is $\mathrm{Ext}^1(A, \mathbf{Z})$.

Proof. Note that an abelian group A is a rational vector space if and only if the homomorphism $\varphi_l = l \times : A \to A$ is an isomorphism for any non zero integer l. If φ_l is an isomorphism, so is $\mathrm{Ext}^1(\varphi_l) : \mathrm{Ext}^1(A, \mathbf{Z}) \to \mathrm{Ext}^1(A, \mathbf{Z})$.

Lemma 2.5 *The sequence* (1) *is a split short exact sequence.*

Proof. By killing homotopy groups, we have an inclusion $j : BSU(2) \to K(\mathbf{Z}, 4)$. Note that the composite map

$$BS^1 \xrightarrow{B\rho} BSU(2) \xrightarrow{j} K(\mathbf{Z}, 4)$$

is represented by $-l^2c_1^2$, where l is a non zero integer and j induces a map $M \to \mathrm{Map}_{-l^2}(BS^1, K(\mathbf{Z}, 4))$. We show that the homomorphism

$$\pi_2(BS^1) \xrightarrow{\widetilde{ad}_{\rho*}} \pi_2(M) \xrightarrow{j_*} \pi_2(\mathrm{Map}_{-l^2}(BS^1, K(\mathbf{Z}, 4)))$$

is not a zero map. Consider the following isomorphism.

$$\begin{aligned}\pi_2(\mathrm{Map}_{-l^2}(BS^1, K(\mathbf{Z}, 4))) &\cong \pi_2(\mathrm{Map}_0(BS^1, K(\mathbf{Z}, 4)))\\ &\cong \pi_2(\mathrm{Map}_0^*(BS^1, K(\mathbf{Z}, 4)))\\ &= [S^2 \wedge BS^1, K(\mathbf{Z}, 4)]\\ &\cong \mathbf{Z}.\end{aligned}$$

The element $ac_1 \in H^4(S^2 \wedge BS^1)$ represents a generator ε of $\pi_2(\mathrm{Map}_0^*(BS^1, K(\mathbf{Z}, 4)))$ where $a \in H^2(S^2; \mathbf{Z})$ is a generator. The generator of $\pi_2(\mathrm{Map}_{-l^2}(BS^1, K(\mathbf{Z}, 4)))$ corresponds to ac_1 under the isomorphism above is represented by

$$ac_1 - l^2c_1^2 : S^2 \times BS^1 \to K(\mathbf{Z}, 4).$$

The map

$$S^2 \times BS^1 \hookrightarrow BS^1 \times BS^1 \to BSU(2) \xrightarrow{j} K(\mathbf{Z}, 4)$$

is represented by $-2lac_1 - l^2c_1^2$, which is $-2l \times \varepsilon$, therefore $j_*\widetilde{ad}_{\rho*} \neq 0$. The short exact sequence (1) induces a long exact sequence

$$0 \to \mathrm{Hom}(\pi_2, \mathbf{Z}) \xrightarrow{(\widetilde{ad}_{\rho*})^*} \mathrm{Hom}(\mathbf{Z}, \mathbf{Z}) \to \mathrm{Ext}^1(\widehat{\mathbf{Z}}/\mathbf{Z}, \mathbf{Z}) \to \cdots.$$

Since $\widehat{\mathbf{Z}}/\mathbf{Z}$ is a rational vector space, so is $\mathrm{Ext}^1(\widehat{\mathbf{Z}}/\mathbf{Z}, \mathbf{Z})$ by Lemma 2.4, hence $(\widetilde{ad}_{\rho*})^*$ must be epic or zero. As we saw, $(\widetilde{ad}_{\rho*})^*(j_*) = j_*\widetilde{ad}_{\rho*} \neq 0$. Therefore $(\widetilde{ad}_{\rho*})^*$ is epic. An element $\alpha \in \mathrm{Hom}(\pi_2, \mathbf{Z})$ such that $i^*(\alpha) = 1$ gives a splitting.

Corollary 2.6 *The homotopy groups of M is given by*

$$\pi_j(M) = \begin{cases} \mathbf{Z} \oplus \widehat{\mathbf{Z}}/\mathbf{Z}, & j=2 \\ \widehat{\mathbf{Z}}/\mathbf{Z}, & j=3 \\ 0, & \textit{otherwise.} \end{cases}$$

Thus we have a fibration

$$K(\widehat{\mathbf{Z}}/\mathbf{Z}, 3) \to M \xrightarrow{\pi} K(\mathbf{Z}, 2) \times K(\widehat{\mathbf{Z}}/\mathbf{Z}, 2) \qquad (2)$$

and we may assume

$$BS^1 \xrightarrow{\widetilde{ad}_\rho} M \xrightarrow{\pi} K(\mathbf{Z}, 2) \times K(\widehat{\mathbf{Z}}/\mathbf{Z}, 2) \xrightarrow{p_1} K(\mathbf{Z}, 2) = BS^1 \qquad (3)$$

is identity, where p_1 is the first projection. Put $B = BS^1 \times K(\widehat{\mathbf{Z}}/\mathbf{Z}, 2)$. Note that $H_i(K(\widehat{\mathbf{Z}}/\mathbf{Z}, j))$ are rational vector spaces and

$$H^j(K(\widehat{\mathbf{Z}}/\mathbf{Z}, 3)) = 0, \ j \leq 3$$
$$H^4(K(\widehat{\mathbf{Z}}/\mathbf{Z}, 3)) \cong \mathrm{Ext}^1(\widehat{\mathbf{Z}}/\mathbf{Z}, \mathbf{Z}).$$

On the other hand

$$\begin{aligned} H_2(B) &\cong H_2(BS^1) \oplus H_2(K(\widehat{\mathbf{Z}}/\mathbf{Z}, 2)) \\ H_3(B) &= 0 \\ H_4(B) &\cong H_4(BS^1) \oplus H_2(BS^1) \otimes H_2(K(\widehat{\mathbf{Z}}/\mathbf{Z}, 2)) \oplus H_4(K(\widehat{\mathbf{Z}}/\mathbf{Z}, 2)) \\ &\cong \mathbf{Z} \oplus \widehat{\mathbf{Z}}/\mathbf{Z} \oplus H_4(K(\widehat{\mathbf{Z}}/\mathbf{Z}, 2)) \\ H_5(B) &\cong H_5(K(\widehat{\mathbf{Z}}/\mathbf{Z}, 2)) \end{aligned}$$

therefore

$$\begin{aligned} H^2(B) &\cong H^2(BS^1) \\ H^4(B) &\cong \mathrm{Hom}(H_4(B), \mathbf{Z}) \oplus \mathrm{Ext}^1(H_3(B), \mathbf{Z}) \\ &\cong \mathbf{Z} \\ H^5(B) &\cong \mathrm{Hom}(H_5(B), \mathbf{Z}) \oplus \mathrm{Ext}^1(H_4(B), \mathbf{Z}) \\ &\cong \mathrm{Ext}^1(\widehat{\mathbf{Z}}/\mathbf{Z} \oplus H_4(K(\widehat{\mathbf{Z}}/\mathbf{Z}, 2)), \mathbf{Z}). \end{aligned}$$

Consider the Serre spectral sequence for the fibration (2). Since

$$\sum_{p+q=2} E_\infty^{p,q} \cong E_2^{2,0} = H^2(B),$$

we have an isomorphism

$$H^2(BS^1) \xrightarrow[p_1^*]{\cong} H^2(B) \xrightarrow[\pi^*]{\cong} H^2(M)$$

and by the fact that the composition of maps in (3) is identity, $\widetilde{ad}_\rho{}^*: H^2(M) \to H^2(BS^1)$ is an isomorphism.

$E_\infty^{0,4} \cong E_5^{0,4}$ and $E_5^{0,4}$ is the kernel of

$$H^4(K(\widehat{\mathbf{Z}}/\mathbf{Z}, 3)) \cong \mathrm{Ext}^1(\widehat{\mathbf{Z}}/\mathbf{Z}, \mathbf{Z}) = E_4^{0,4} \xrightarrow{d_4} E_4^{5,0} = H^5(B).$$

Since $H^5(B)$ is torsion free and $\mathrm{Ext}^1(\widehat{\mathbf{Z}}/\mathbf{Z}, \mathbf{Z})$ is a rational vector sace, $E_\infty^{0,4}$ is a rational vector space. Note that $E_2^{4,0} \cong E_\infty^{4,0}$ and π^* is given by

$$H^4(B) = E_2^{4,0} \cong E_\infty^{4,0} \hookrightarrow H^4(M).$$

Since $p_1^*: H^4(BS^1) \to H^4(B)$ is an isomorphism, we have a short exact sequence

$$0 \to H^4(BS^1) \to H^4(M) \to E_\infty^{0,4} \to 0$$

and $\widetilde{ad}_\rho{}^* : H^4(M) \rightarrow H^4(BS^1)$ gives a splitting of this sequence. Therefore the kernel of $\widetilde{ad}_\rho{}^*$ is isomorphic to $E_\infty^{0,4}$ which is a rational vector space.

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