

An explicit integral representation of Whittaker functions for the representations of the discrete series — the case of $SU(2,2)$ —

By

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1. Introduction

In a sense, this paper is a supplement to the paper [12] of Yamashita. Also it is an analogue of a result in [6].

We consider the Lie group $G = SU(2,2)$. The large discrete series representation of G has a Whittaker model with respect to a nondegenerate character of the maximal unipotent subgroup N of G . Using the Schmid operator, Yamashita [12] explicitly computed the differential equations satisfied by the minimal K -type vectors in the Whittaker model of the discrete series representations.

The purpose of this paper is to push this computation one step further to obtain an explicit integral representation of the Whittaker functions representing these vectors belonging to the minimal K -type (Theorems 4.4 and 4.5). There is a general integral representation due to Jacquet for Whittaker functions. But this representation is sometimes intractable for higher rank groups. We hope our formula is useful for the investigation of L -factors of automorphic representations of the discrete series at the real places.

The content of this paper is as follows: In §2, we briefly review the structure of $SU(2,2)$, the discrete series and the representations of the maximal compact subgroup K . Basic notations and definitions are found in [1,2], which we follow. In §3, we review the results of Yamashita and of Kostant on the dimension of the space of Whittaker vectors. We also calculate the radial A -part of the Schmid operator explicitly. In §4, we describe the holonomic system of differential equations of Whittaker functions which has appeared in [12]. Furthermore, we show that the integral representation of the analytic Whittaker function can be obtained from the differential equations under the parity condition of a nondegenerate character of N .

Because we belong to the culture of automorphic forms, the maximal

compact subgroup K appears in the right hand of G and the unipotent group N in the left side. This is different from [12].

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2. The group $SU(2, 2)$ and its discrete series

2.1. Structure of Lie group and Lie algebra. Let G be the special unitary group $SU(2, 2)$ realized as,

$$G = \{g \in SL_4(\mathbf{C}) \mid g^* I_{2,2} g = I_{2,2}\}, \quad I_{2,2} = \text{diag}(1, 1, -1, -1),$$

where $g^* = {}^t \bar{g}$ denotes the adjoint of a matrix g . We fix some notation for this group and its discrete series representations, used throughout this paper.

Let $U(4)$ be the unitary group of degree 4 in $SL_4(\mathbf{C})$. Take a maximal compact subgroup $K = G \cap U(4) = S(U(2) \times U(2))$. We denote by \mathfrak{g} , \mathfrak{k} the Lie algebra of G , K , respectively. Let $\theta(X) = -{}^t \bar{X}$ be a Cartan involution and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition of \mathfrak{g} .

We set, $\mathfrak{a} = \mathbf{R}H_1 + \mathbf{R}H_2$ with, $H_1 = X_{23} + X_{32}$, $H_2 = X_{14} + X_{41}$, where the X_{ij} 's are elementary matrices given by,

$$X_{ij} = (\delta_{ip} \delta_{jq})_{1 \leq p, q \leq 4} \text{ with Kronecker's delta } \delta_{ip}.$$

Then \mathfrak{a} is a maximally \mathbf{R} -split abelian subalgebra of \mathfrak{g} contained in \mathfrak{p} . Then the restricted root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{a})$ is expressed as,

$$\Delta = \Delta(\mathfrak{g}, \mathfrak{a}) = \{\pm \lambda_1 \pm \lambda_2, \pm 2\lambda_1, \pm 2\lambda_2\}.$$

where λ_j is the dual of H_j . We choose a positive system Δ_+ and a fundamental system Δ_{fund} of Δ as follows:

$$\begin{aligned} \Delta^+ &= \{\lambda_1 \pm \lambda_2, 2\lambda_1, 2\lambda_2\}, \\ \Delta_{\text{fund}} &= \{\lambda_1 - \lambda_2, 2\lambda_2\}. \end{aligned}$$

We also denote the corresponding nilpotent subalgebra by $\mathfrak{n} = \sum_{\beta \in \Delta^+} \mathfrak{g}_\beta$. Here \mathfrak{g}_β is the root subspace of \mathfrak{g} corresponding to $\beta \in \Delta^+$. Then one obtains an Iwasawa decomposition of \mathfrak{g} and G :

$$\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}, \quad G = NAK,$$

with $A = \exp \mathfrak{a}$, $N = \exp \mathfrak{n}$. Now let

$$\begin{aligned} E_1 &= H_{13} - \sqrt{-1}X_{13} + \sqrt{-1}X_{31}, \quad E_2 = H_{24} - \sqrt{-1}X_{24} + \sqrt{-1}X_{42}, \\ E_3 &= 1/2(X_{12} - X_{21} - X_{14} + X_{23} + X_{32} - X_{41} - X_{34} + X_{43}), \\ E_4 &= \sqrt{-1}/2(X_{12} + X_{21} - X_{14} - X_{23} + X_{32} + X_{41} - X_{34} - X_{43}), \\ E_5 &= 1/2(X_{12} - X_{21} + X_{14} + X_{23} + X_{32} + X_{41} + X_{34} - X_{43}), \\ E_6 &= \sqrt{-1}/2(X_{12} + X_{21} + X_{14} - X_{23} + X_{32} - X_{41} + X_{34} + X_{43}), \end{aligned}$$

where $H_{ij} = \sqrt{-1}(X_{ii} - X_{jj})$ for $1 \leq i < j \leq 4$. Then it is easily seen that

$$\mathfrak{g}_{2\lambda_j} = \mathbf{R}E_j \ (j = 1, 2), \ \mathfrak{g}_{\lambda_1 + \lambda_2} = \mathbf{R}E_3 + \mathbf{R}E_4, \ \mathfrak{g}_{\lambda_1 - \lambda_2} = \mathbf{R}E_5 + \mathbf{R}E_6.$$

2.2. Parametrization of discrete series. Let us now parametrize the discrete series of $SU(2, 2)$. Take a compact Cartan subalgebra \mathfrak{t} defined by

$$\mathfrak{t} = \mathbf{R}\sqrt{-1}h^1 + \mathbf{R}\sqrt{-1}h^2 + \mathbf{R}\sqrt{-1}I_{2,2}, \quad \text{with } h^1 = X_{11} - X_{22}, \ h^2 = X_{33} - X_{44}$$

and let $\mathfrak{t}_\mathbb{C}$ be its complexification. Then the absolute root system, of type A_3 , is expressed as,

$$\tilde{\Delta} = \tilde{\Delta}(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C}) = \{[\pm 2, 0; 0], [0, \pm 2; 0], [\pm 1, \pm 1; \pm 2]\}.$$

where $\beta = [r, s; u]$ means $r = \beta(h^1)$, $s = \beta(h^2)$ and $u = \beta(I_{2,2})$. Let

$$\tilde{\Delta}^+ = \{[2, 0; 0], [0, 2; 0], [\pm 1, \pm 1; 2]\}.$$

We write the set of compact positive roots by $\tilde{\Delta}_c^+ = \{[2, 0; 0], [0, 2; 0]\}$ and we fix it hereafter. The Weyl group $\tilde{W} = \tilde{W}(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ is generated by s_1, s_2, s_3 where,

$$\begin{aligned} s_1[r, s; u] &= [-r, s; u], \\ s_2[r, s; u] &= [(r-s+u)/2, (-r+s+u)/2; r+s], \\ s_3[r, s; u] &= [r, -s; u]. \end{aligned}$$

We identify \tilde{W} and the symmetric group \mathfrak{S}_4 of degree 4 by the map: $s_j \mapsto (j, j+1)$. The compact Weyl group is given by $\tilde{W}_c = \langle s_1, s_3 \rangle$, also identified canonically with the subgroup $\mathfrak{S}_2 \times \mathfrak{S}_2$.

There are exactly six positive systems $\tilde{\Delta}_1^+, \tilde{\Delta}_{II}^+, \dots, \tilde{\Delta}_{VI}^+$ containing $\tilde{\Delta}_c^+$, defined by $\tilde{\Delta}_j^+ = w_j \tilde{\Delta}^+$, where the elements $w_j \in \tilde{W}$ are given as,

$$w_I = 1, \ w_{II} = s_2, \ w_{III} = s_2 s_3, \ w_{IV} = s_2 s_1, \ w_V = s_2 s_3 s_1, \ w_{VI} = s_2 s_1 s_3 s_2.$$

We denote by $\tilde{\Delta}_{n,j}^+$ the noncompact positive roots in $\tilde{\Delta}_j^+$.

By definition, the space of the Harish-Chandra parameters \mathcal{E}_c is given by,

$$\mathcal{E}_c = \{\Lambda \in \mathfrak{t}_\mathbb{C}^* \mid \Lambda \text{ is } \tilde{\Delta}\text{-regular, } K\text{-analytically integral and } \tilde{\Delta}_c^+\text{-dominant}\}.$$

Put

$$\mathcal{E}_J = \{\Lambda \in \mathcal{E}_c \mid \tilde{\Delta}_J^+\text{-dominant}\}.$$

We also put $\rho_{G,J} = 2^{-1} \sum_{\beta \in \tilde{\Delta}_J^+} \beta$ (resp. $\rho_K = 2^{-1} \sum_{\beta \in \tilde{\Delta}_c^+} \beta$), the half sum of positive roots (resp. the half sum of compact positive roots.) Then the space $\mathcal{E}_c \subset \mathfrak{t}_\mathbb{C}^*$ are divided into six parts: $\mathcal{E}_c = \cup_{1 \leq j \leq 6} \mathcal{E}_J$. We note that \mathcal{E}_I (resp. \mathcal{E}_{VI})

corresponds to the holomorphic (resp. anti-holomorphic) discrete series. For $\Lambda \in \cup_{I \leq J \leq VI} \mathcal{E}_J$, we denote the corresponding discrete series by π_Λ and call this Λ the Harish-Chandra parameter of π_Λ . As determined in [11, §10.4], the Gelfand-Kirillov dimensions of the discrete series representations π_Λ are given as follows:

$$\text{GK-dim}(\pi_\Lambda) = \begin{cases} 4 & (\Lambda \in \mathcal{E}_I \cup \mathcal{E}_{VI}), \\ 6 & (\Lambda \in \mathcal{E}_{II} \cup \mathcal{E}_V), \\ 5 & (\Lambda \in \mathcal{E}_{III} \cup \mathcal{E}_{IV}). \end{cases}$$

Therefore the representations belonging to $\mathcal{E}_{II} \cup \mathcal{E}_V$ is a large representation in the sense of Vogan [8, Th. 6.2, f)], hence has an algebraic Whittaker model. (Harish-Chandra parameters of discrete series of $SU(2, 2)$ are described as in Figure 1.)

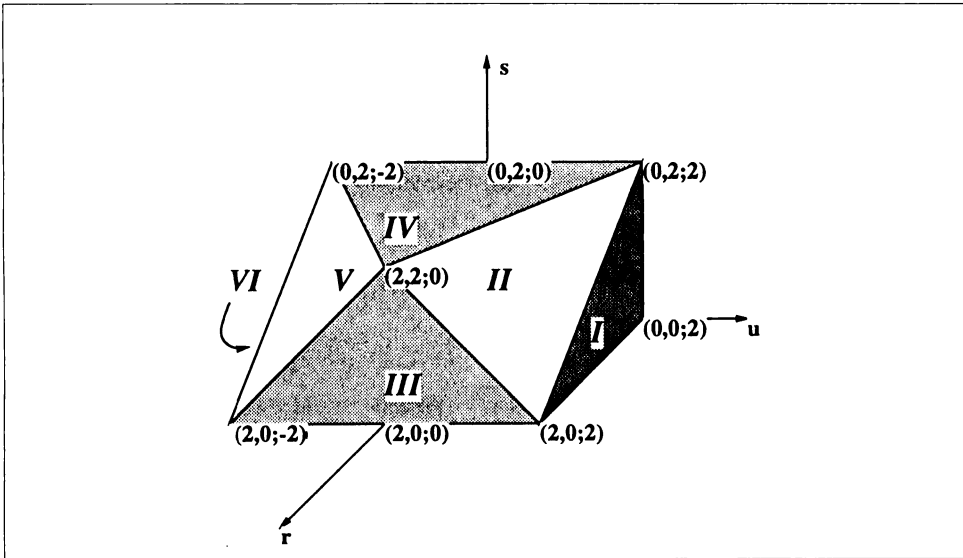


FIGURE 1. Harish-Chandra parameters in t_C^*

2.3. Representations of the maximal compact subgroup. Let

$d_1, d_2 \in \mathbf{Z}_{\geq 0}$ and $d_3 \in \mathbf{Z}$.

For $d = [d_1, d_2; d_3] \in t_C^*$, define $\tau_d \in \widehat{K}$ by the following rule:

$$\begin{aligned} \tau_d(h^j)f_{k_1k_2}^{(d)} &= (2k_j - d_j)f_{k_1k_2}^{(d)}, & (j=1, 2), \\ \tau_d(e_+^i)f_{k_1k_2}^{(d)} &= (d_j - k_j)f_{k_1+\delta_{1j}, k_2+\delta_{2j}}^{(d)}, \\ \tau_d(e_-^i)f_{k_1k_2}^{(d)} &= k_jf_{k_1-\delta_{1j}, k_2-\delta_{2j}}^{(d)}, \\ \tau_d(I_{2,2})f_{k_1k_2}^{(d)} &= d_3f_{k_1k_2}^{(d)}. \end{aligned} \tag{1}$$

Here, $V_d = \{f_{k_1 k_2}^{(d)} | 0 \leq k_j \leq d_j\}_C$ is the standard basis (see [1, §3]) and

$$h^1, h^2, e_+^1 = X_{12}, e_+^2 = X_{34}, e_-^j = {}^t e_+^j$$

are the generators of \mathfrak{k}_C . Then according to [1, Prop. 3.1], \widehat{K} is exhausted by

$$\{(\tau_d, V_d) | d = [d_1, d_2; d_3], d_1 + d_2 + d_3 \text{ is even}\}.$$

The adjoint representation $\text{Ad} = \text{Ad}_{\mathfrak{p}_C}$ of K on \mathfrak{p}_C is decomposed into a direct sum of two irreducible subrepresentations: $\mathfrak{p}_C = \mathfrak{p}_+ + \mathfrak{p}_-$, where,

$$\mathfrak{p}_+ = \text{C}X_{13} + \text{C}X_{14} + \text{C}X_{23} + \text{C}X_{24}, \quad \mathfrak{p}_- = {}^t \mathfrak{p}_+.$$

In fact, $\text{Ad}_{\pm} = \text{Ad}|_{\mathfrak{p}_{\pm}}$ is isomorphic to $\tau_{[1,1; \pm 2]}$, respectively. For later use, we fix the K -isomorphisms $\iota_{\pm}: \mathfrak{p}_{\pm} \rightarrow V_{[1,1; \pm 2]}$ as follows:

$$\begin{aligned} \iota_+ : (X_{23}, X_{13}, X_{24}, X_{14}) &\mapsto (f_{00}^{(11)}, f_{10}^{(11)}, -f_{01}^{(11)}, -f_{11}^{(11)}), \\ \iota_- : (X_{41}, X_{31}, X_{42}, X_{32}) &\mapsto (f_{00}^{(11)}, f_{10}^{(11)}, -f_{01}^{(11)}, -f_{11}^{(11)}) \quad ([1, \text{Prop. 3.10}]). \end{aligned}$$

The irreducible decomposition of \mathfrak{k}_C -module $V_d \otimes \mathfrak{p}_C$ is given as follows:

$$V_d \otimes \mathfrak{p}_C = V_d \otimes \mathfrak{p}_+ \oplus V_d \otimes \mathfrak{p}_-, \quad V_d \otimes \mathfrak{p}_{\pm} \cong \bigotimes_{\epsilon_1, \epsilon_2 \in \{\pm 1\}} V_{[r+\epsilon_1, s+\epsilon_2; u \pm 2]}.$$

The projectors

$$\begin{aligned} P_{rs}^{(\epsilon_1, \epsilon_2)} : V_d \otimes \mathfrak{p}_+ &\rightarrow V_{[r+\epsilon_1, s+\epsilon_2; u+2]}, \\ \overline{P}_{rs}^{(\epsilon_1, \epsilon_2)} : V_d \otimes \mathfrak{p}_- &\rightarrow V_{[r+\epsilon_1, s+\epsilon_2; u-2]}, \end{aligned}$$

are explicitly given by [1, Lemma 3.12].

3. Radial A -part of the differential operator $\mathcal{D}_{\eta, d}$

Let η be a unitary character of $N = \exp \mathfrak{n}$. For $F(g) \in C_{\eta, \tau_d}^{\infty}(N \backslash G/K)$, let

$$\nabla_{\eta, d} F(g) = \sum_k R_{X_k} F(g) \otimes X_k, \quad \{X_k\}: \text{ orthonormal basis of } \mathfrak{p}$$

be the Schmid operator. Put $\mathcal{D}_{\eta, d}^{(J)} = P_d^{(J)} \circ \nabla_{\eta, d}$ with the projectors

$$P_d^{(J)} : V_d \otimes \mathfrak{p}_C \rightarrow V_d^- = \bigoplus_{\beta \in \overline{J}_d} V_{d-\beta}.$$

Let π_{Λ} be the discrete series representation of G where $\Lambda \in \overline{E}_J$ is a Harish-Chandra parameter. Then $d = [r, s; u] = \Lambda + \rho_{G, J} - 2\rho_K$, called the Blattner parameter of π_{Λ} , is the highest weight of the minimal K -type of π_{Λ} . For a Whittaker vector $\Phi \in \text{Hom}_{(g, K)}(\pi_{\Lambda}^*, C_{\eta}^{\infty}(N \backslash G))$, define $\Phi_{\pi_{\Lambda}, \tau_d} \in C_{\eta, \tau_d}^{\infty}(N \backslash G/K)$ by the following:

$$\langle v^*, \Phi_{\pi_{\Lambda}, \tau_d}(g) \rangle = \Phi(v^*)(g) \quad (v^* \in V_{\tau_d}^*, g \in G).$$

The algebraic Whittaker function $\Phi_{\pi_{\Lambda}, \tau_d}$ for the representation of the discrete

series is uniquely determined at the value $a \in A = \exp(\mathfrak{a})$ by virtue of the Iwasawa decomposition.

Theorem 3.1 ([11]). *Let $\Lambda \in \mathcal{E}_J$. Then,*

$$\text{Hom}_{(\mathfrak{g}, K)}(\pi_\Lambda^*, C_\eta^\infty(N \backslash G)) \cong \ker(\mathcal{D}_{\eta, d}^{(J)}),$$

where d is the Blattner parameter of π_Λ .

According to the result of Kostant [4, Th. 6.8.1], the dimension formula can be obtained:

Theorem 3.2.

$$\dim_{\mathbb{C}}(\text{Hom}_{(\mathfrak{g}, K)}(\pi_\Lambda^*, C_\eta^\infty(N \backslash G))) = \begin{cases} 4 & (J = \text{II}, \text{V}), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It is sufficient to consider the case of $\Lambda \in \mathcal{E}_{\text{II}} \cup \mathcal{E}_{\text{V}}$ because of the largeness of π_Λ . In order to specify the dimension, consider

$$F^\# = \exp(\mathfrak{a}_{\mathbb{C}}) = \langle \text{diag}(1, -1, 1, -1), \alpha = \sqrt{-1} \begin{pmatrix} & 1_2 \\ 1_2 & \end{pmatrix} \rangle.$$

Define another representation $\pi^{(\alpha)}$ by $\pi^{(\alpha)}(g) = \pi(\alpha^{-1}g\alpha)$. Then, we see from

$$\tau_{[r, s; u]}(\alpha^{-1}X\alpha) = \tau_{[s, r; -u]}(X) = \tau_{w_0 w_{\text{II}}^{-1}[r, s; u]}(X)$$

that the Harish-Chandra parameter of $\pi_\Lambda^{(\alpha)}$ belongs to \mathcal{E}_{V} . Using the same argument in the proof of [2, Th. 4.2], we obtain the theorem.

It is also known that the rapidly-decreasing solution is unique if exists.

For $a = (a_1, a_2) = \exp((\log a_1)H_1 + (\log a_2)H_2) \in A$, let $\partial_j = a_j \partial / \partial a_j$ and

$$L_1 = 1/2 \partial_1, L_2^\pm = 1/2 (\partial_2 \pm \eta_2 a_2^2), \mathcal{S} = \xi(a_1/a_2)^2, \mathcal{S}' = \xi'(a_1/a_2)^2.$$

We also put, for a character η of N ,

$$\eta_2 = \sqrt{-1} \eta(E_2), \xi = \eta(E_5) + \sqrt{-1} \eta(E_6), \xi' = \eta(E_5) - \sqrt{-1} \eta(E_6), \eta_0 = \xi \xi'.$$

The following lemma is obtained by direct calculation similar to [1, Lemma 6.5].

Lemma 3.3. For $F(g) = \sum_{k,l} c_{k,l}(g) f_{k,l}^{(d)} \in C_{\eta,\tau_d}^\infty(N \backslash G/K)$, $d = [r, s; u]$, we have,

$$P^{(-,-)} \circ \nabla^+ c_{k,l} = \begin{pmatrix} (k-r)(s-l)\mathcal{S} \\ (k-r)(l+1)(L_1 - (2k-2l+r+3s-u+6)/4) \\ (k+1)(s-l)(L_2^- - (2k-2l-r+s-u+2)/4) \\ (k+1)(l+1)\mathcal{S}' \end{pmatrix} \begin{pmatrix} c_{k,l} \\ c_{k,l+1} \\ c_{k+1,l} \\ c_{k+1,l+1} \end{pmatrix},$$

$$\bar{P}^{(-,-)} \circ \nabla^- c_{k,l} = \begin{pmatrix} (k-r)(s-l)\mathcal{S} \\ (r-k)(l+1)(L_2^+ + (2k-2l-r+s-u+2)/4) \\ (k+1)(l-s)(L_1 + (2k-2l-3r-s-u-6)/4) \\ (k+1)(l+1)\mathcal{S}' \end{pmatrix} \begin{pmatrix} c_{k,l} \\ c_{k,l+1} \\ c_{k+1,l} \\ c_{k+1,l+1} \end{pmatrix}.$$

4. Integral representation of discrete series Whittaker function

In the following, η is assumed to be nondegenerate. According to Theorem 3.2, we treat the cases of $J = \text{II}$ and V . Moreover, since we have $\eta^{(\alpha)}(E_2) = -\eta(E_2)$, the Whittaker functions of π_Λ ($\Lambda \in \mathcal{E}_\text{V}$) can be obtained from $\pi_{\Lambda'}$ ($\Lambda' = w_{\text{II}} w_{\text{V}}^{-1} \Lambda \in \mathcal{E}_{\text{II}}$), by taking the parameters $(s, r, -u, -\eta_2)$ in place of (r, s, u, η_2) . Thus we only treat the case of $\Lambda \in \mathcal{E}_{\text{II}}$. Then, the Blattner parameter of π_Λ is $d = \Lambda + [0, 0; 2]$.

Lemma 4.1. The projector $P_d^{(\text{II})}$ decomposes into four projectors as follows:

$$P_d^{(\text{II})} = P^{(-,-)} \oplus \bar{P}^{(-,-)} \oplus \bar{P}^{(-,+)} \oplus \bar{P}^{(+,-)}. \tag{2}$$

Proof. We find that

$$\tilde{\Delta}_{\text{II}}^+ = \{[1, -1; 2], [1, 1; \pm 2], [-1, 1; 2], [0, 2; 0], [2, 0; 0]\}$$

and that

$$\tilde{\Delta}_{\text{II}}^+ = \{[1, -1; 2], [1, 1; \pm 2], [-1, 1; 2]\}.$$

Thus the lemma follows.

According to Theorem 3.1, Whittaker functions are characterized by the differential equations derived by the composition of the Schmid operator and projectors which appears in the decomposition of $P_d^{(\text{II})}$.

Let $\Phi_{\pi,\tau_d}(a) = \sum_{k,l} c_{k,l}(a) f_{k,l}$. For notation, we write $(\Phi)_{k,l} = c_{k,l}$. Then, $c_{k,l}$'s satisfy the following system which is equivalent to $\mathcal{D}_{\eta,d}^{(\text{II})} \Phi = 0$:

$$\begin{aligned}
(\widetilde{C}_1^+ : 1)_{kl} & (P^{(-,-)} \circ \nabla \Phi)_{k,l} = 0, \\
(\widetilde{C}_2^- : 1)_{kl} & (\overline{P}^{(-,-)} \circ \nabla \Phi)_{k,l} - (s-l) (\overline{P}^{(-,+)} \circ \nabla \Phi)_{k,l+1} = 0, \\
(\widetilde{C}_2^- : 2)_{kl} & (\overline{P}^{(-,-)} \circ \nabla \Phi)_{k,l} + (l+1) (\overline{P}^{(-,+)} \circ \nabla \Phi)_{k,l+1} = 0, \\
(\widetilde{C}_3^- : 1)_{kl} & (\overline{P}^{(-,-)} \circ \nabla \Phi)_{k,l} - (k+1) (\overline{P}^{(+,-)} \circ \nabla \Phi)_{k+1,l} = 0, \\
(\widetilde{C}_3^- : 2)_{kl} & (\overline{P}^{(-,-)} \circ \nabla \Phi)_{k,l} - (r-k) (\overline{P}^{(+,-)} \circ \nabla \Phi)_{k+1,l} = 0,
\end{aligned}$$

where we write $\nabla = \nabla_{\eta,d}$ for simplicity.

Given a $d = [r, s; u]$, we put,

$$\begin{aligned}
b_0 = b_0(d) &= (r+s+u)/2, & b_1 = b_1(d) &= (-r+s+u)/2, \\
b_2 = b_2(d) &= (r-s+u)/2, & b_3 = b_3(d) &= (-r-s+u)/2.
\end{aligned} \tag{3}$$

By Lemma 3.3 as well as [1, Lemma 6.5], the coefficients $\{c_{k,l}\}$ satisfy the following concrete equations:

$$\begin{aligned}
(C_1^+ : 1)_{kl} & (k-r)(l+1)(L_1 + (b_3 - s - k + l - 3)/2)c_{k,l+1} \\
& + (k+1)(s-l)(L_2 + (b_2 - k + l - 1)/2)c_{k+1,l} \\
& + (k+1)(l+1)\mathcal{S}'c_{k+1,l+1} + (k-r)(s-l)\mathcal{S}c_{k,l} = 0, \\
(C_2^- : 1)_{kl} & (r-k)(L_2^+ - (b_2 - k + l + 1)/2)c_{k,l+1} \\
& + (k+1)\mathcal{S}'c_{k+1,l+1} + (k+1)(s-l)c_{k+1,l} = 0, \\
(C_2^- : 2)_{kl} & (k+1)(L_1 - (b_0 + r - k - l + 1)/2)c_{k+1,l} - (k-r)\mathcal{S}c_{k,l} = 0, \\
(C_3^- : 1)_{kl} & (l+1)(L_2^+ - (b_2 - k + l + 1)/2)c_{k,l+1} - (s-l)\mathcal{S}c_{k,l} + (k+1)(s-l)c_{k+1,l} = 0, \\
(C_3^- : 2)_{kl} & (s-l)(L_1 - (b_1 + k + l + 3)/2)c_{k+1,l} - (l+1)\mathcal{S}'c_{k+1,l+1} = 0.
\end{aligned}$$

Each equation $(C_p^\pm : q)$ is derived from $(\widetilde{C}_p^\pm : q)$.

Next, define

$$h_{k,l} = k!l!(r-k)!(s-l)!e^{i\frac{1}{2}\eta/2}a_1^{-b_1-k+l-2}a_2^{-b_2+k-l}c_{k,l}. \tag{4}$$

Then the system satisfied by $h_{k,l}$'s is given as follows:

$$\begin{aligned}
(H_1^+ : 1)_{kl} & (\partial_1 + 2b_3 - 2)h_{k,l+1} - \xi'(a_1/a_2)^2 h_{k+1,l+1} + \xi(a_1/a_2)^2 h_{k,l} \\
& - (\partial_2 - 2\eta_2 a_2^2 + 2b_2 - 2k + 2l)(a_1/a_2)^2 h_{k+1,l} = 0 \\
& (0 \leq k \leq r-1, 0 \leq l \leq s-1), \\
(H_2^- : 1)_{kl} & \partial_2 h_{k,l+1} + \xi'(a_1/a_2)^2 h_{k+1,l+1} + 2(l+1)(a_1/a_2)^2 h_{k+1,l} = 0 \\
& (0 \leq k \leq r-1, 0 \leq l \leq s-1), \\
(H_2^- : 1)_{k0} & \partial_2 h_{k,0} + \xi'(a_1/a_2)^2 h_{k+1,0} = 0 \quad (0 \leq k \leq r-1), \\
(H_2^- : 2)_{kl} & (\partial_1 - 2r + 2k + 2)h_{k+1,l} + \xi h_{k,l} = 0 \quad (0 \leq k \leq r-1),
\end{aligned}$$

$$\begin{aligned}
(H_3^-: 1)_{kl} \quad & \partial_2 h_{k,l+1} - \xi (a_1/a_2)^2 h_{k,l} + 2(r-k) (a_1/a_2)^2 h_{k+1,l} = 0 \\
& (0 \leq k \leq r-1, 0 \leq l \leq s-1), \\
(H_3^-: 1)_{rl} \quad & \partial_2 h_{r,l+1} - \xi (a_1/a_2)^2 h_{rl} = 0 \quad (0 \leq l \leq s-1), \\
(H_3^-: 2)_{kl} \quad & (\partial_1 - 2l) h_{k+1,l} - \xi' h_{k+1,l+1} = 0 \quad (0 \leq l \leq s-1). \\
(H_p^\pm: q)_{kl} & \text{ is a direct consequence of } (C_p^\pm: q)_{kl}. \text{ Concluding, we obtain the} \\
& \text{differential equations satisfied by the Whittaker functions.}
\end{aligned}$$

Proposition 4.2. Let $\Phi_{\pi_1, \tau_d} = \sum_{k,l} c_{k,l} f_{k,l}^{(d)}$. Define $h_{k,l}$ by (4). Then $h_{k,l}$ satisfies the following:

$$\begin{aligned}
\text{(i)} \quad & (\partial_1 + 2b_3 - 2) h_{k,l+1} - (2\partial_1 + \partial_2 - 2\eta_2 a_2^2 + 2b_3 - 2) (a_1/a_2)^2 h_{k+1,l} = 0 \\
& (0 \leq k \leq r-1, 0 \leq l \leq s-1), \\
\text{(ii)} \quad & \partial_2 h_{k,l+1} + (a_1/a_2)^2 (\partial_1 + 2) h_{k+1,l} = 0 \quad (0 \leq k \leq r-1, 0 \leq l \leq s-1), \\
\text{(iii)} \quad & (\partial_1 - 2r + 2k + 2) h_{k+1,l} + \xi h_{k,l} = 0 \quad (0 \leq k \leq r-1), \\
\text{(iv)} \quad & (\partial_1 - 2l) h_{k+1,l} - \xi' h_{k+1,l+1} = 0 \quad (0 \leq l \leq s-1), \\
\text{(v)} \quad & \partial_2 h_{k,0} + \xi' (a_1/a_2)^2 h_{k+1,0} = 0 \quad (0 \leq l \leq s-1), \\
\text{(vi)} \quad & \partial_2 h_{r,l+1} - \xi (a_1/a_2)^2 h_{r,l} = 0 \quad (0 \leq l \leq s-1).
\end{aligned}$$

Proof. Equation (i) can be obtained from Equations $(H_1^+: 1)_{kl}$, $(H_2^+: 1)_{kl}$ and $(H_3^-: 1)_{kl}$. Equation (ii) is from $(H_2^-: 2)_{kl}$ and $(H_3^-: 1)_{kl}$. The others are clear.

Equations (iii) and (iv) in Proposition 4.2 tell us that $h_{r,0}$ determines the system $\{h_{k,l}\}$. By the proposition above, we see that from (iii) and (v),

$$(\partial_1 \partial_2 - \eta_0 (a_1/a_2)^2) h_{r,0} = 0, \quad (5)$$

and from (i), (iii) and (iv),

$$\begin{aligned}
& (\partial_1 + 2b_3 - 2) \partial_1^2 h_{r,0} \\
& + (2\partial_1 + \partial_2 - 2\eta_2 a_2^2 + 2b_3 - 2) (a_1/a_2)^2 \eta_0 h_{r,0} = 0.
\end{aligned} \quad (6)$$

Operating ∂_2 to (6), we obtain the following:

Corollary 4.3.

$$(\partial_1 \partial_2 - \eta_0 (a_1/a_2)^2) h_{r,0} = 0, \tag{7}$$

$$((\partial_1 + \partial_2)^2 - (2b_3 - 2)(\partial_1 + \partial_2) - 2\eta_2 a_2^2 \partial_2) h_{r,0} = 0. \tag{8}$$

Equation (6) can be recovered from (8) by operating ∂_1 . We can also check that this system becomes holonomic of rank 4.

To get an integral representation, first we consider

$$\mathcal{W}(a_1, a_2) = \int_0^\infty \phi(u) \exp\left(\frac{\eta_0 a_1^2}{u} - \frac{u}{4a_2^2}\right) \frac{du}{u} \tag{9}$$

for $\phi \in C^\infty(\mathbf{R}_{>0})$. Then \mathcal{W} formally satisfies the differential equation (7). Suppose \mathcal{W} satisfy (8), then ϕ should be the solution of the following differential equation,

$$4u^2 \frac{d^2 \phi}{du^2} + 4b_3 \frac{d\phi}{du} - \eta_2 u \phi = 0. \tag{10}$$

Putting $v = \sqrt{u}$ and $\phi(u) = v^{-b_3+1/2} \psi(v)$, we see that Equation (10) becomes

$$\frac{d^2 \psi}{dv^2} + \left(-\frac{1}{4} \cdot (4\eta_2) + \frac{1/4 - (b_3 - 1)^2}{v^2}\right) \psi = 0.$$

If $\eta_2 > 0$ (i.e. $\text{Im}(\eta(E_2)) < 0$), then we can find the unique rapidly-decreasing solution,

$$\psi(v) = W_{0, b_3-1}(2\sqrt{\eta_2}v)$$

where $W_{k,l}(z)$ is the usual Whittaker function. Returning to Equation (9), we can confirm the absolute convergence of the integral in this case. In conclusion,

Theorem 4.4. *Let $\Lambda = [r, s; u - 2] \in \mathcal{E}_{\text{II}}$. Assume that $\text{Im}(\eta(E_2)) < 0$ for a nondegenerate character η on N . Then there exists a rapidly-decreasing Whittaker function Φ_{π_1, τ_d} characterized by the following $h_{r,0}$:*

$$h_{r,0}(a_1, a_2) = C \int_0^\infty t^{-b_3+1/2} W_{0, b_3-1}(t) \exp\left(\frac{4\eta_2 \eta_0 a_1^2}{t^2} - \frac{t^2}{16\eta_2 a_2^2}\right) \frac{dt}{t}, \tag{11}$$

where $\eta_2 = -\text{Im}(\eta(E_2))$, $b_3 = (-r - s + u)/2$ and C is a constant.

Proof. The situation is completely similar to [6, Theorem (9.1)]. So we omit the proof.

From this theorem, $\pi_\Lambda^*(\Lambda \in \mathcal{E}_V)$ has no non-trivial rapidly-decreasing

solution in its four-dimensional space of Whittaker vectors if $\eta_2 > 0$, which agree with the Shalika's multiplicity-one result.

Now recover the $h_{k,l}$'s from $h_{r,0}$. By using Proposition 4.2, we have,

Theorem 4.5. *Let $\Lambda = [r, s; u - 2] \in \Xi_{II}$. Assume that $\text{Im}(\eta(E_2)) < 0$ for a nondegenerate character η . Consider the rapidly-decreasing Whittaker function $\Phi_{\pi, \tau_i}(g) = \sum_{k,l} c_{k,l}(g) f_{k,l}$ and define $\{h_{k,l}\}$ by means of (4). Then they can be expressed as follows:*

$$h_{k,l} = C \sum_{i=0}^l (-1)^i (k-r, l-i) \binom{l}{i} (\xi')^{r-k-l+i} \xi^i (-8\eta_2 a_1^2)^{r-k+i} \quad (12)$$

$$\times \int_0^\infty t^{-b_3+2(k-i-r)+1/2} W_{0,b_3-1}(t) \exp\left(\frac{4\eta_0 \eta_2 a_1^2}{t^2} - \frac{t^2}{16\eta_2 a_2^2}\right) \frac{dt}{t}.$$

Here C is a constant independent of k and l , the binomial $\binom{0}{0} = 1$, and

$$(p; q) = \begin{cases} p(p+1) \cdots (p+q-1) & (q > 0), \\ 1 & (q = 0), \\ 0 & (1-q \leq p \leq 0, q \neq 0), \end{cases}$$

for a non-positive integer p and non negative integer q .

Proof. One can directly check that this formula satisfies the recursive condition in Proposition 4.2. Its initial condition is exactly (11) in Theorem 4.4.

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