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A note on pluricanonical maps for varieties of dimension 4 and 5

By

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1. Introduction

Let X be a nonsingular projective variety of general type with dimension n defined over $\mathbf{C}$. The behavior of its pluricanonical map $\Phi_{|mK_X|}$ is of special interest to the classification theory. For $n \geq 3$, it remains open whether there is an absolute function $m(n)$ such that $\Phi_{|mK_X|}$ is birational for $m \geq m(n)$. The simplest case to this problem is when X be a nonsingular minimal model. For $n \geq 4$, T. Matsusaka first proved the existence of $m(n)$; K. Maehara presented a function $m(n)$; T. Ando ([1]) got $m(4)=16$ and $m(5)=29$.

With I. Reider's results ([6]) and by improving T. Ando's method, we get the following effective result.

Theorem. *Let X be a nonsingular projective variety of dimension $n \geq 4$ with nef and big canonical divisor K_X. Then there is a function $m(n)$ such that $\Phi_{|mK_X|}$ is birational for $m \geq m(n)$, where $m(4) \leq 12$ and $m(5) \leq 18$.*

Throughout this note, most of our notations and terminologies are standard except the following which we are in favour of:

$:=$ —— definition;

$\sim_{lin}$ —— linear equivalence;

$\sim_{num}$ —— numerical equivalence.

2. The main theorem

We begin by introducing I. Reider's result at first.

Lemma 2.1 *(Corollary 2 of [6]). Let S be an algebraic surface, L a nef and big divisor on S. Suppose $L^2 \geq 10$ and the rational map ϕ defined by $|L+K_S|$ is not birational, then S contains a base point free pencil E' with $L \cdot E'=1$ or $L \cdot E'=2$.*

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We obviously obtain the following corollary.

Corollary 2.1 *Let S be an algebraic surface, R a nef and big divisor on S. Then $\Phi_{|K_S+mR|}$ is birational for $m \geq 4$.*

Kawamata-Viehweg's vanishing theorem will be used in our proof with the following form.

Lemma 2.2 *Let X be a nonsingular complete variety, a divisor D on X is nef and big, then $H^i(X, K_X+D)=0$ for all $i>0$.*

Lemma 2.3 *(Lemma 3 of [1]). Let $|M|$ be a complete linear system free from base points, and let D be a divisor with $|D| \neq \emptyset$. Assume that $|M|$ is not composed of a pencil, i.e., $\dim\Phi_{|M|}(X) \geq 2$. If $\Phi := \Phi_{|M+D|}$ is not a birational map, then, for a general member Y of $|M|$, Φ is not birational on Y.*

We have the following theorem.

Theorem 2.1 *Let X be a nonsingular projective variety of dimension n $(n \geq 2)$. Suppose we have a sequence of nef and big divisors $L_0, L_1, \cdots, L_{n-2}$ such that $\dim \Phi_{|L_i|}(X) \geq i$ for $i>0$ and $|K_X+mL_0| \neq \emptyset$, then $\Phi_{|K_X+mL_0+L_1+\cdots+L_{n-2}|}$ is a birational map onto its image, where $m \geq 4$ is a positive integer.*

Proof. We prove the statement by induction on n, the dimension of X.

For $n=2$, it is just corollary 2.1. So the theorem is true in this case.

Suppose the theorem be true for $n=d$, we want to give a proof for $n=d+1$. Let $f: X' \to X$ be blow-ups according to Hironaka such that $\Phi_{|f^*(L_1)|}$ is a morphism. Considering the system

$$|K_{X'}+mf^*(L_0)+f^*(L_1)+\cdots+f^*(L_{d-1})|,$$

set $f^*(L_1) \sim_{lin} M+Z$, M is the moving part and Z the fixed part. Because $\dim \Phi_{|M|}(X') \geq 1$ by assumption, we have two cases.

Case 1. If $\dim \Phi_{|M|}(X')=1$, let $g := \Phi_{|L_1|} \circ f$, $W_1 := \overline{\Phi_{|L_1|}(X')}$ and

$$X' \xrightarrow{g_1} C \xrightarrow{s_1} W_1$$

be the Stein factorization of g, we have $M \sim_{num} aY$, where Y is a general fiber of the fibration g_1 and Y is a nonsingular projective variety of dimension d. We have the following exact sequence at least over a nonempty Zariski open subset of C:

$$\begin{aligned}0 &\to \mathcal{O}_{X'}(K_{X'}+mf^*(L_0)+f^*(L_2)+\cdots+f^*(L_{d-1}))\\ &\to \mathcal{O}_{X'}(K_{X'}+mf^*(L_0)+M+f^*(L_2)+\cdots+f^*(L_{d-1}))\\ &\to \oplus_{1=1}^{a}\mathcal{O}_{Y_i}(K_{Y_i}+mL_0'+L_1'+\cdots+L_{d-2}')\to 0,\end{aligned}$$

where $L_i' := f^*(L_{i+1})|_{Y_i}$ for $i=1, \cdots, d-2$, $L_0' = f^*(L_0)|_{Y_i}$ and each Y_i is a general fiber of g_1. We obviouly see that L_i' is nef and big on Y_i for $i \geq 0$. By Kawamata-Viehweg's vanishing theorem, we have

$$H^1(X', K_{X'}+mf^*(L_0)+f^*(L_2)+\cdots+f^*(L_{d-1}))=0,$$

and therefore we get the surjective map

$$\begin{aligned}&H^0(X', K_{X'}+mf^*(L_0)+M+f^*(L_2)+\cdots+f^*(L_{d-1}))\\&\quad\to\oplus_{i=1}^{q} H^0(Y_i, K_{Y_i}+mL_0'+L_1'+\cdots+L_{d-2}')\to 0.\end{aligned}$$

This means that the system $|K_{X'}+mf^*(L_0)+M+f^*(L_2)+\cdots+f^*(L_{d-1})|$ can separate fibers of g and disjoint components of a general fiber of g at least over a nonempty Zariski subset of C. Furthermore,

$$\Phi_{|K_{X'}+mf^*(L_0)+M+f^*(L_2)+\cdots+f^*(L_{d-1})|}|_{Y_i}=\Phi_{|K_{Y_i}+mL_0'+L_1'+\cdots+L_{d-2}'|}.$$

Because $\dim \Phi_{|L_i|}(X) \geq i$ for $i>0$, i.e., $\dim \Phi_{|f^*(L_i)|}(X') > i$, therefore

$$\dim \Phi_{|L_i'|}(Y_i) \geq i+1-1=i$$

for $i=1, \cdots, d-2$. Because $|K_{X'}+mf^*(L_0)| \neq \emptyset$ and $f^*(L_0)$ is big $K_{Y_i}+mL_0' = [K_{X'}+M+mf^*(L_0)]|_{Y_i}$ must be effective. Thus, by induction, we see that

$$\Phi_{|K_{Y_i}+mL_0'+L_1'+\cdots+L_{d-2}'|}$$

is birational. Therefore

$$\Phi_{|K_{X'}+mf^*(L_0)+M+mf^*(L_2)+\cdots+f^*(L_{d-2}')|}$$

is birational and finally

$$\Phi_{|K_{X'}+mf^*(L_0)+f^*(L_1)+f^*(L_2)+\cdots+f^*(L_{d-1})|}$$

is birational.

CASE 2. If $\dim \Phi_{|f^*(L_1)|}(X') \geq 2$, i.e., $|f^*(L_1)|$ is not composed of a pencil, set $f^*(L_1) \sim_{lin} M+Z$, where M is the moving part. By Bertini's theorem, a general member $Y \in |M|$ is a nonsingular projective variety of dimension d. Again, we consider the system

$$|K_{X'}+mf^*(L_0)+M+f^*(L_2)+\cdots+f^*(L_{d-1})|.$$

We have the following exact sequence

$$\begin{aligned}&0\to\mathcal{O}_{X'}(K_{X'}+mf^*(L_0)+f^*(L_2)+\cdots+f^*(L_{d-1}))\\&\quad\to\mathcal{O}_{X'}(K_{X'}+mf^*(L_0)+M+f^*(L_2)+\cdots+f^*(L_{d-1}))\\&\quad\to\mathcal{O}_{Y}(K_Y+mL_0'+L_1'+\cdots+L_{d-2}')\to 0,\end{aligned}$$

where $L_0' := f^*(L_0)|_Y$ and $L_i' := f^*(L_{i+1})|_Y$ for $i=1, \cdots, d-2$. It is obvious that L_i' is nef and big on Y for $i=0, \cdots, d-2$. We can see that $|K_Y+mL_0'| \neq \emptyset$ and $\dim \Phi_{|L_i'|}(Y) \geq i$. Thus, by induction, $\Phi_{|K_Y+mL_0'+L_1'+\cdots+L_{d-2}'|}$ is birational. From lemma 1.3, we see that

$$\Phi_{|K_{X'}+mf^*(L_0)+M+f^*(L_2)+\cdots+f^*(L_{d-1})|}$$

is birational. And therefore

$$\Phi_{|K_{X'}+mf^*(L_0)+f^*(L_1)+\cdots+f^*(L_{d-1})|}$$

is birational.

Defintion 2.1 Let X be a nonsingular projective variety of dimension n. Define

$$r_0(X) := \min\{p|p\geq 5,\ h^0(X, mK_X)>0 \text{ for } m\geq p\};$$
$$r_i(X) := \min\{q|\dim \Phi_{|qK_X|}(X)\geq i\},\ i=1, \cdots, n-2:$$
$$m(n, X) := \sum_{i=0}^{n-2} r_i(X);$$
$$m(n) := \sup_X \{m(n, X)\}.$$

By Matsusaka's theorem ([5]), $m(n)$ is a finite value. We have the following theorem.

Theorem 2.2 *Let X be a nonsingular projective variety of dimension $n\geq 3$. The canonical divisor K_X is nef and big. Then $\Phi_{|mK_X|}$ is birational for $m\geq m(n)$.*

Proof. This is a direct result from theorem 2.1. We only have to take $L_0=K_X$, $m=r_0(X)-1$ and $L_i=r_i(X)K_X$ for $i=1, \cdots, n-2$.

3. m(4) and m(5)

Lemma 3.1 *(See lemma 7' and lemma 8' of [1]). Let X be a nonsingular projective variety with nef and big canonical divisor K_X. $\dim X=n$. Then*

(1) *If $n=4$, we have $h^0(X, mK_X)\geq 2\ (m\geq 3)$; $\dim \Phi_{|mK_X|}(X)\geq 2\ (m\geq 4)$.*

(2) *If $n=5$, we have $h^0(X, mK_X)\geq 2\ (m\geq 3)$; $\dim \Phi_{|mK_X|}(X)\geq 2\ (m\geq 4)$; $\dim \Phi_{|mK_X|}(X)\geq 3\ (m\geq 6)$.*

From the above lemma, we see that $m(4, X)\leq 12$ and $m(5, X)\leq 18$. Thus $m(4)\leq 12$ and $m(5)\leq 18$. Therefore we get the results on 4 and 5 dimensional cases by theorem 2.2 as follows.

Corollary 3.1 *Let X be a nonsingular projective variety of dimension n. Suppose K_X is nef and big, then*

(1) *When $n=4$, $\Phi_{|mK_X|}$ is birational for $m\geq 12$;*

(2) *When $n=5$, $\Phi_{|mK_X|}$ is birational for $m\geq 18$.*

Remark. A direct result of theorem 2.2 for $n=3$ is $m(3)\leq 7$. Certainly, this is a known result by [4]. We recently proved $m(3)\leq 6$ in [2].

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