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Nonexistence of isolated singularities for nonlinear systems of partial differential equations and some applications

By

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This paper deals with the singularities of solutions of several classes of nonlinear systems of partial differential equations appearing in gas dynamics and differential geometry. Our techniques depend principally on the method of microlocal analysis, for example microlocalized Sobolev spaces H^s_{mcl}, classical theory of the pseudodifferential operators and certainly the theory of paradifferential operators.

1. Motivation of the considerations

1 The following quasilinear system

$$\begin{pmatrix} s \\ \theta \end{pmatrix}_{\Psi} = \begin{pmatrix} 0 & 1 \\ s & 0 \end{pmatrix} \begin{pmatrix} s \\ \Theta \end{pmatrix}_{\Phi} \tag{1.1}$$

was studied by L. Bers [1] and it was said that (1.1) expresses some properties of a Meyer flow of a Tricomi gas for nozzle problem. The symbols s, θ, Ψ and Φ mean the speed, the inclination of velocity, the stream function and the velocity potential respectively. We shall rewrite (1.1) in the form

$$\begin{pmatrix} u \\ v \end{pmatrix}_{x} = \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_{y}, \ u=u(x, y), \ v=v(x, y) \tag{1.2}$$

When investigating the existence of shock waves, Oleinik considered the system

$$\begin{pmatrix} u \\ v \end{pmatrix}_{x} = \begin{pmatrix} 0 & 1 \\ 1-u & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_{y} \tag{1.3}$$

which is hyperbolic for $u<1$ and elliptic for $u>1$. In their paper [7] Tay Ping Liu and Xin studied the system

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t} + \begin{pmatrix} au & b \\ 0 & v \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_{x} = 0, \tag{1.4}$$
$$a=const>0, \quad b=const>0$$

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arising in theory of overcompressive shock waves.

The study of the following system was proposed by B. Kheifitz

$$(f(\omega))_x + (g(\omega))_y = 0, \tag{1.5}$$

where $f=(f_1, f_2)$, $g=(g_1, g_2)$, $\omega=(u, v)$, i. e.

$$A(\omega)\partial_x\omega + B(\omega)\partial_y\omega = 0,\ A(\omega) = df,\ B(\omega) = dg.$$

Monge Ampere equation with Gaussian curvature K

$$u_{xx}u_{yy} - u_{xy}^2 = K(x, y)(1+u_x^2+u_y^2)^2 \tag{1.6}$$

belongs to the same class of hyperbolo-elliptic systems if for each point (x_0, y_0) for which $K(x_0, y_0)=0$ we have that $dK(x_0, y_0)\neq 0$.

Our main goal is to study all the systems mentioned above from the point of view of the microlocal analysis and to find sufficient conditions for non-existence of isolated singularities.

To begin with, we shall consider the linearization (the first variation) of (1.2), namely

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}_x = \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}_y + \begin{pmatrix} 0 & 0 \\ u_y & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + f, \tag{1.7}$$

i. e. we shall obtain a system with the symbol

$$P_1 = I_2(i\xi) - \begin{pmatrix} 0 & 1 \\ u(x, y) & 0 \end{pmatrix}(i\eta) - \begin{pmatrix} 0 & 1 \\ u_y(x, y) & 0 \end{pmatrix}.$$

For simplicity, we assume $u\in C^2$. Thus we have for the principal symbol p_1^0:

$$-ip_1^0 = \begin{pmatrix} \xi & -\eta \\ -u\eta & \xi \end{pmatrix} \text{ and } -ip_0 = i\begin{pmatrix} 0 & 0 \\ u_y & 0 \end{pmatrix},\ I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

From technical reasons we suppose that our operator has the symbol $-iP_1$, i. e.

$$p_1^0 + p_0 = \begin{pmatrix} \xi & -\eta \\ -u\eta & \xi \end{pmatrix} + i\begin{pmatrix} 0 & 0 \\ u_y & 0 \end{pmatrix}, \tag{1.8}$$

$$\mathrm{Char}\,p_1 = \{\det p_1^0 = \xi^2 - u(x, y)\eta^2 = 0\}.$$

The Hamiltonian vactor field of $\det p_1^0$ is given by:

$$H_{\det p_1^0} = 2\xi\partial_x - 2\eta u(x, y)\partial_y + u_x(x, y)\eta^2\partial_\xi + u_y(x, y)\eta^2\partial_\eta.$$

According to Bony's nonlinear microlocal theory if

a) $u(x_0, y_0)<0$, then $\rho^0=(x_0, y_0, \xi_0, \eta_0)$ is an elliptic point for each $|\xi^0|+|\eta^0|>0$ and the microlocal smoothness of $\begin{pmatrix} u \\ v \end{pmatrix}$ at ρ^0 is well known from [3].

b) $u(x_0, y_0)>0$, $\rho^0\in \mathrm{Char} p_1$, then ρ^0 is a hyperbolic point and we have propagation of singularities along the integral curve $\mathscr{L}$ of $H_{\det p_1^0}$ passing through ρ^0 (see [3]).

c) $u(x_0, y_0)=0$, then $\xi^0=0$, $\eta^0\neq 0$ if $\rho\in \mathrm{Char} p_1$.

Assume $u_x(x_0, y_0)\neq 0$. Then $H_{\det p_1^0(\rho_0)}=(\eta^0)^2(u_x(x_0, y_0)\partial_\xi+u_y(x_0, y_0)\partial_\eta)$ is not parallel to $\Xi(\rho^0)$, where $\Xi(\rho^0)=\xi_0\partial_\xi+\eta_0\partial_\eta=\eta_0\partial_\eta$ is the radial vector field at ρ^0. In that case the singularities again propagate along $\mathscr{L}$.

Suppose now

d) $u_x(x_0, y_0)=0$ but $u_y(x_0, y_0)\neq 0$. Then $H_{\det p_1^0(\rho_0)}\|\Xi(\rho^0)$. This is the case to be studied here.

The case where $u(x_0, y_0)=\nabla_{x,y}u(x_0, y_0)=0$ is rather complicated because of the appearance of double characteristics and it remains out of our investigations.

Let us now multiply

$$co\, p_1^0\cdot p_1^0=\begin{pmatrix} \xi & \eta \\ \eta u & \xi \end{pmatrix}\begin{pmatrix} \xi & -\eta \\ -u\eta & \xi \end{pmatrix}=(\xi^2-u\eta^2)I_2. \tag{1.9}$$

So we have a reduction to the scalar case but our operators (their symbols) are not C^∞ smooth with respect to (x, y). They are only C^1 smooth.

2. As we shall see, some of the properties of the linearized operator will remain true for the nonlinear operator under consideration.

Let

$$L=-\lambda x\xi-y\eta+d,\ 0<\lambda<1,\ d=id_1,\ d_1\in R^1, \tag{1.10}$$

i. e. $L(x, D)=-\lambda xD_x-yD_y+d=i(\lambda x\partial_x+y\partial_y+d_1)$, as ξ is the symbol of the operator $D_x=-i\partial_x$.

As it is known from [2], we can construct a distribution v and such that $Lv\in C^\infty$ while it has a fixed singularity along the conic ray $(0, 1)$, i. e. $WF(v)=(0, 0, 0, 1)=\rho^0$. In the case $\lambda=1$ there exists a function v such that $Lv\in C^\infty$, $WF(v)=(0, 0, \Gamma_\varepsilon)$ where $\Gamma_\varepsilon=\{|\varphi-\frac{\pi}{2}|\leq\frac{\varepsilon}{2},\ \varepsilon>0\}$ and φ is the polar angle in the plane $O\xi\eta$.

Definition. 1. We shall say that $u\in H^{s-0}_{loc}$ iff $u\in H^t_{loc}$, $\forall t<s$ and $u\notin H^s_{loc}$.

There are no difficulties to verify that there exists a solution v of the equation $Lv=f\in C^\infty$ with the properties $v\in H^{s_0-0}_{mcl}(\rho^0)$, $WF(v)=(0, 0, 0, 1)$, $\rho_0=(0, 0, 0, 1)$, $s_0=\frac{\lambda+1}{2}-d_1$, $0<\lambda<1$. In the case $\lambda=1$ we have $s_0=1-d_1$

and we have an isolated singularity but in an angle: $(0, 0, \Gamma_\varepsilon)$.

Let us remark that $s_0 = -\operatorname{Im} L_0'$ and L_0' is the subprincipal symbol of L. The definition of L_0' is given below in (2.3).

In all the cases $0 < \lambda \leq 1$ we have that the Hamiltonian vector field $H_{L(\rho_0)} = \Xi(\rho_0)$.
More precisely, if $0 < \lambda < 1$ then

$$H_{L(0,0,\xi,\eta)} \parallel \Xi(0, 0, \xi, \eta) \text{ } iff \text{ } either \text{ } \xi=0, \eta=\pm 1 \text{ } or \text{ } \xi=\pm 1, \eta=0;$$

$$H_{L(0,0,\xi,\eta)} \parallel \Xi(0, 0, \xi, \eta) \ \forall \ (\xi, \eta) \ if \ \lambda=1.$$

Remark also that the larger $-d_1 > 0$ is the more regular in C^k spaces the solution v is.

It is easy to deduce from the example in [4], p.80 that there exists a solution w of the semilinear equation with analytic coefficients

$$x\partial_x w + y\partial_y w + 2\mu w = 2f\sqrt{w}, \ \sqrt{1}=1, \tag{1.11}$$

and such that

$$w \in H^{s_0-0}_{loc}, \ s_0 > 2, \ w \in H^{2s_0-1-0}_{mcl} \ (0, 0, 0, 1),$$

where $w(0, 0) > 0$, the analytic function $f(0, 0) \neq 0$, $\mu \in R^1$.

Thus starting with $s_0 - 0$ smooth solution we have an optimal microlocal regularity $2s_0 - \frac{n}{2}$, $s_0 > \frac{n}{2} + 1$, $n=2$.

So in the nonlinear case a natural restriction on the microlocal smoothness appears and it is of the type $t < 2s - \frac{n}{2}$.

It is worth studying the properties of the operator (1.10) from the point of view of nonexistence of singularities. Thus consider the operator $P = \sum_{j=1}^n \lambda_j x_j \partial_{x_j}$ in R^n and $L = P + a$, $a = const. \in R^1$, $\lambda_j > 0$.

By the method of characteristics one can prove that

a). we have a unique classical solution $u \in C^1(\Omega)$, $\Omega \ni O$ if $a > 0$ and O is the origin in R^n. Moreover,

$$O \cap \text{singsupp } Lu = O \cap \text{singsupp } u, \ \forall u \in C^1(\Omega);$$

b). we have uniqueness in $C^1(\Omega)$ modulo a constant if $a=0$ and $f(O)=0$; $u \in C^1(\Omega) \Rightarrow O \cap \text{singsupp } Lu = O \cap \text{singsupp } u$;

c). we have nonuniqueness in $C^1(\Omega)$ if $a < 0$. Moreover, $u \in C^1(\Omega)$ implies that $u(O) = 0$.

We have mentioned above that a solution of $Lv = f \in C^\infty$ with an isolated singularity at a ray exists if $s_0 = \frac{1+\lambda}{2} - a$, $0 < \lambda < 1$, $a > 0$. Thus we have $s_0 < 1$. The function $v \notin C^1(\Omega)$, $\Omega \ni O$, as if $v \in C^1(\Omega)$, $f \in C^\infty(\Omega)$ then $v \in C^\infty(\Omega)$ near O.

2. Statement of the problem and formulation of the main results

Let Ω be a domain in R^n and assume that the real-valued function $u \in H^s_{loc}(\Omega)$, $s > \frac{n}{2} + m + 1$, is a solution of the nonlinear system of partial differential equation (PDE):

$$P_j(x, u(x), \dots, \partial^\beta u(x), \dots)_{|\beta| \leq m} = 0, \ 1 \leq j \leq N, \tag{2.1}$$

where $P_j(x, u, \dots, u_\beta, \dots)_{|\beta| \leq m}$ are real-valued C^∞ functions of their arguments $(x, u, \dots, u_\beta, \dots)_{|\beta| \leq m}$, $\partial^\beta = \partial^{\beta_1}_{x_1} \dots \partial^{\beta_n}_{xn}$ if $\beta = (\beta_1, \dots, \beta_n)$, $u = (v_1, \dots, u_N)$, $\partial^\beta u = (\partial^\beta u_1, \dots, \partial^\beta u_N)$, $u_\beta = (u_{1,\beta}, \dots, u_{N,\beta})$.

The linearization (the first variation) of (2.1) contains the next two symbols:

$$i^m \Big[\sum_{|\alpha| = m} \frac{\partial P_j}{\partial u_{k,\alpha}} (x, \partial^\beta u(x))_{|\beta| \leq m} \xi^\alpha - \tag{2.2}$$
$$i \sum_{|\alpha| = m-1} \frac{\partial P_j}{\partial u_{k,\alpha}} (x, \partial^\beta u(x))_{|\beta| \leq m} \xi^\alpha \Big]^{\downarrow j=1,\dots,N}_{\to k=1,\dots,N}$$

Its principal symbol is real-valued,

$$P_m(x, \xi) = \sum_{|\alpha| = m} \frac{\partial P_j}{\partial u_{k,\alpha}} (x, \partial^\beta u(x))_{|\beta| \leq m} \xi^\alpha \in C^{1+\varepsilon}$$

for some $\varepsilon > 0$, while

$$P_{m-1}(x, \xi) = -i \sum_{|\alpha| = m-1} \frac{\partial P_j}{\partial u_{k,\alpha}} (x, \partial^\beta u(x))_{|\beta| \leq m} \xi^\alpha \in C^{1+\varepsilon}$$

is purely imaginary.

This is the standard definition of the subprincipal symbol:

$$p'_{m-1}(x, \xi) = p_{m-1}(x, \xi) + \frac{i}{2} \sum_{j=1}^{n} \frac{\partial^2 p_m}{\partial x_j \partial \xi_j}(x, \xi). \tag{2.3}$$

Consider the point $\rho^0 = (x_0, \xi^0) \in T^*(\Omega) \backslash 0$. It is proved in [3] that $\det p_m(\rho^0) \neq 0$ implies that $u \in H^{2s-m-\frac{n}{2}}_{mcl}(\rho^0)$.

So we suppose

(i) $\det \rho_m(\rho^0) = 0$.

Assume that there exists a $N \times N$ square symbol $\tilde{p}_{1-m}(x, \xi)$, positively homogeneous of order $(1-m)$ with respect to ξ and C^∞ smooth for $\xi \neq 0$, $C^{\tilde{\sigma}}$ smooth with respect to x, $\tilde{\sigma} > 1$ and such that in some conical neighbourhood of ρ^0 the following identity is valid:

(ii) $\tilde{p}_{1-m} \cdot p_m = q_1 I_N$ (I_N is the identity matrix in C_N, q_1 is a first order real-valued scalar symbol). The notation $\mathrm{ord}_\xi q_1 = 1$ means that q_1 is a positively homogeneous symbol of order 1 with respect to ξ.

The matrix $\tilde{p}_{1-m}$ is assumed to be real-valued, $q_1 \in C^{\min(1+\varepsilon,\tilde{\sigma})}$ with respect to x. Further on we denote by $\sum_\rho^m$ (Ω) the set of symbols positively homogeneous of order m with respect to ξ, $\xi \neq 0$, C^∞ with respect to $\xi \neq 0$ and belonging to C^ρ with respect to $x \in \Omega$, $\rho > 0$ is not an integer. $Op(\sum_\rho^m)$ stands for the set of corresponding properly supported paradifferential operators (see [3]).

Denote by $\rho^0 = (x_0, \xi^0), |\xi^0| = 1$ such a characteristic point (i. e. $\det p_m(\rho^0) = 0$) for which:

(iii) $H_{q_1(\rho^0)} + c\Xi(\rho^0) = 0$ for some constant $c < 0$.

By $H_{q_1(\rho^0)} = \sum_{j=1}^n \left(\frac{\partial q_1(\rho^0)}{\partial \xi_j} \partial_{x_j} - \frac{\partial q_1(\rho^0)}{\partial_{x_j}} \partial_{\xi_j} \right)$ we denote the Hamiltonian vector field of the scalar function q_1 at ρ^0 and $\Xi(\rho^0) = \sum_{j=1}^n \xi_j^0 \partial_{\xi_j}$ is the radial vector field at ρ^0. Note that $\det p_m(\rho^0) = 0$ implies that $q_1(\rho^0) = 0$.

In [5] Dencker supposed q_1 to be a symbol of real principal type, i. e. $H_{q_1}(\rho^0)$ is not parallel to $\Xi(\rho^0)$.

Unlike him we assume in (iii) that H_{q_1} and Ξ are colinear at $\rho^0 \in \operatorname{Char} P = \{\rho: \det p_m(\rho) = 0\}$. The subprincipal symbol of $\tilde{p}_{1-m}$ is given by:

$$\tilde{p}'_{-m} = \frac{i}{2} \sum_{j=1}^n \frac{\partial^2 \tilde{p}_{1-m}}{\partial x_j \partial \xi_j}.$$

Introduce now the matrix valued symbol

$$R' = -\frac{i}{2} \{\tilde{p}_{1-m}, p_m\} + \tilde{p}_{1-m} p'_{m-1} + \tilde{p}'_{-m} p_m$$

with Hölder coefficients in x and $\operatorname{ord}_\xi R = 0$. As usual, $\{.,.\}$ is the Poisson bracket.

If A is a square $N \times N$ matrix in $\mathbf{C}_N$ and A^* is its adjoint then $\operatorname{Im} A = \frac{A - A^*}{2i}$.

(iv) Suppose that there exists a conical neighbourhood $\Gamma \ni \rho^0$ with the next property: $u \in H^t_{mcl}(\Gamma \backslash \rho^0)$, $t < 2s - 1 - \frac{n}{2} - m$, $s - 1 - m - \frac{n}{2} \notin \mathbf{Z}$ and H^t_{mcl} is the microlocalized Sobolev space at $\Gamma \backslash \rho^0$.

This is our main result.

Theorem 1. *Assume that the solution* $u \in H^s_{loc}(\Omega)$, $s > \frac{n}{2} + m + 1$ *of the system (2.1) satisfies (i)-(iv) and*

(v) $-cs + \min_{\|z\|=1} (\operatorname{Im} R'(\rho^0) z, z) > 0$.

Then $u \in H^t_{mcl}(\Gamma)$.

Thus the singularity at ρ^0 under the condition (v) is not isolated, i. e. it can be removed.

The restriction $t<2s-\frac{n}{2}-1-m$ can be omitted if P is a classical scalar linear differential operator with C^∞ coefficients. So if $Pu\in C^\infty$, $P=p_m+p_{m-1}+\cdots$, $u\in H^s_{loc}$, $-cs+\mathrm{Im}\,p'_{m-1}(\rho^0)>0$, then $u\in H^s_{mcl}(\Gamma\backslash\rho^0)$ implies that $u\in H^t_{mcl}(\Gamma)$ for each $t>s$. Therefore if $u\in H^s_{loc}$, $-cs+\mathrm{Im}\,p'_{m-1}(\rho^0)>0$, $Pu\in C^\infty$, $WF(u)\cap(\Gamma\backslash\rho^0)=\emptyset$ then $\rho^0\notin WF(u)$.

The proof of Th1 is reduced to a theorem from the theory of the paradifferential operators. Our assertion is non trivial if $s<t<2s-m-1-\frac{n}{2}$.

We shall verify Th1 applying Th3 from [3] with $d=m$, $\rho=s-\varepsilon-\frac{n}{2}$, $0<\varepsilon\ll 1$, $\sigma=\rho-m$. Thus, there exists a paradifferential operator $\mathscr{P}\in Op(\Sigma^m_\sigma)$, $\sigma>1$, $\sigma\notin\mathbf{Z}$ with the symbol (2.2) and such that

$$\mathscr{P}u\in H^{s-m+\sigma}_{loc}=H^{s+\sigma-1+(1-m)}_{loc},\ u\in H^s_{loc}(\Omega)\subset C^\rho,\ \rho>m+1,$$

$\rho\notin\mathbf{Z}$, $u\in H^t_{mcl}(\Gamma\backslash\rho^0)$. As $t<s+\sigma-1$ we have that $\mathscr{P}u\in H^{t-m+1}_{loc}$. Without loss of generality we assume $\tilde{p}_{1-m}\in\Sigma^{1-m}_\sigma$. So $\mathscr{P}u=f\in H^{t-m+1}_{loc}$, $u\in H^s_{loc}$, $u\in H^t_{mcl}(\Gamma\backslash\rho^0)$ and therefore $\tilde{p}\mathscr{P}u=f_1\in H^t_{loc}$, $\tilde{p}\mathscr{P}\in Op\,(\Sigma^1_\sigma)$.

The paradifferential calculus gives us:

$$\tilde{p}\mathscr{P}=\tilde{p}_{1-m}p_m+\tilde{p}_{1-m}\,p_{m-1}+\frac{1}{i}\sum_{j=1}^{n}\frac{\partial\tilde{p}_{1-m}}{\partial\xi_j}\frac{\partial p_m}{\partial x_j}+\widetilde{R},$$

where $\widetilde{R}$ is $(-1+\sigma)$ smoothing operator in each Sobolev space (i. e. $\widetilde{R}: H^s\to H^{s-1+\sigma}$, $\widetilde{R}$ is continuous for each s).

According to (ii)

$$\frac{\partial^2}{\partial x_j\partial\xi_j}(\tilde{p}_{1-m}\,p_m)=\frac{\partial^2\tilde{p}_{1-m}}{\partial x_j\partial\xi_j}p_m+\tilde{p}_{1-m}\frac{\partial^2 p_m}{\partial x_j\partial\xi_j}+$$

$$+\frac{\partial\tilde{p}_{1-m}}{\partial\xi_j}\cdot\frac{\partial p_m}{\partial x_j}+\frac{\partial\tilde{p}_{1-m}}{\partial x_j}\cdot\frac{\partial p_m}{\partial\xi_j}=\frac{\partial^2 q_1}{\partial x_j\partial\xi_j}I_N.$$

So

$$\tilde{p}\mathscr{P}=\left(q_1-\frac{i}{2}\sum_{j=1}^{n}\frac{\partial^2 q_1}{\partial x_j\partial\xi_j}\right)I_N-\frac{i}{2}\,\{\tilde{p}_{1-m},p_m\}+$$

$$+\tilde{p}_{1-m}p'_{m-1}+\tilde{p}'_{-m}\,p_m+\widetilde{R}.$$

Let us put

$$R=\tilde{p}\mathscr{P}=q_1I_N-\frac{i}{2}\sum_{j=1}^{n}\frac{\partial^2 q_1}{\partial x_j\partial\xi_j}I_N+R'+\widetilde{R},\tag{2.4}$$

$R\in Op(\Sigma^1_\sigma)$.

Obviously, the subprincipal symbol $R'_0=R'$.

Using standard bootstrap arguments we can see that Th1 is a corollary from the following

Theorem 2. *Let the paradifferential operator* $R \in Op(\Sigma_\sigma^1)$, $\sigma > 1$, $\sigma = 1 + \frac{1}{l}$, $l \in \mathbf{N}$, $l > 1$ *be given by* (2.4). *Suppose that* $u \in H_{comp}^{t-\frac{1}{2l}}(\Omega)$, $Ru \in H_{mcl}^t(\Gamma)$, $u \in H_{mcl}^t(\Gamma \backslash \rho^0)$
and

$$-ct + \min_{\|z\|=1} (\operatorname{Im} R_0'(\rho^0) z, z) > 0. \tag{2.5}$$

Then $u \in H_{mcl}^t(\Gamma)$.

/$\min_{\|z\|=1}(\operatorname{Im} R_0'(\rho^0)z, z)$ is the smallest eigenvalue of the Hermitian matrix $\operatorname{Im} R_0'(\rho^0)$ and $\max_{\|z\|=1}(\operatorname{Im} R_0'(\rho^0)z, z)$ is its largest one/.

(A) In many cases we multiply p_m by $\tilde{p}_{m_1} \in \Sigma_\sigma^{m_1}$, $\sigma \notin \mathbf{Z}$ and then $\tilde{p}_{m_1} p_m = q_{m+m_1} I_N$, $\det p_m(\rho^0) = 0$. So $q_{m+m_1}(\rho^0) = 0$; $H_{q_{m+m_1}(\rho^0)} + c\Xi(\rho^0) = 0$, $c < 0$.

Replacing (v) by $(v)'$,

$$(v)' - c\left(s - \frac{m + m_1 - 1}{2}\right) + \min_{\|z\|=1} (\operatorname{Im} R_0'(\rho^0)\, z, z) > 0,$$

where

$$R_0' = -\frac{i}{2}\{\tilde{p}_{m_1}, p_m\} + \tilde{p}_{m_1} p_{m-1}' + \tilde{p}_{m_1-1}' p_m,$$

we get the same conclusion that $u \in H_{mcl}^t(\Gamma)$ if $u \in H_{loc}^s$, $s > \frac{n}{2} + m + 1$, $t < 2s - \frac{n}{2} - m - 1$.

More specially, if $\tilde{p}_{m_1} = {}^{co}\, p_m \in \Sigma_\sigma^{m(N-1)}$, i. e. $m_1 = m(N-1)$, $\det p_m \in \Sigma_\sigma^{mN}$ we can reformulate our main result for the systems of the type (1.2), (1.3).

(B) If $c > 0$ Th1 remains true with $(v)''$ instead of $(v)'$, namely

$$(v)'' \quad c\left(s - \frac{m + m_1 - 1}{2}\right) - \max_{\|z\|=1} (\operatorname{Im} R_0'(\rho^0)\, z, z) > 0.$$

In the scalar case $\tilde{p}_{m_1} = 1$ and then (v) is replaced by

$$(v)''' \quad -c\left(s - \frac{m-1}{2}\right) + \operatorname{Im} p_{m-1}'(\rho^0) > 0.$$

Let us consider now the operator (1.10) with $Lu \in C^\infty$, i. e. $m = 1$, $c = -1$, $u \in H_{loc}^s$, $s + \operatorname{Im} L_0' > 0$. The last inequality is equivalent to $s > s_0$.

Then $u \in H_{mcl}^t(\Gamma \backslash \rho^0)$, $\rho^0 = (0, 0, 0, 1)$, $\rho^0 \in \Gamma$ implies $u \in H_{mcl}^t(\Gamma)$, $t > s$.

(C) Here is a slight generalization of Theorem 1 for quasilinear and semilinear systems.

(a) $u \in H^s$, $s > \frac{n}{2} + m + 1$, $t < 2s - 1 - \frac{n}{2} - m$ for fully nonlinear systems

(b) $s > \frac{n}{2} + m$, $t < 2s - \frac{n}{2} - m$ in the quasilinear system case

(c) $s>\frac{n}{2}+m-1$, $t<2s-\frac{n}{2}-m+1$ in the semilinear case.

The conclusion is that $u\in H^t_{mcl}(\Gamma)$.

3. Some applications of the previous result

1. Let us consider at first the case c). for the system (1.2).

Having in mind that the principal symbol $\det p_1^0=\xi^2-u(x, y)\eta^2$ we have that if u $(x_0, y_0)=0$ then $\xi^0=0$, $\eta^0=\pm 1$ if $(x_0, y_0, \xi^0, \eta^0)\in \mathrm{Char}\det p_1$.

Let us consider now the curve γ: $u=0$, $u(x_0, y_0)=0$. According to the implicit function theorem (say $u_x(x_0, y_0)\neq 0$) we find $x=x(y)\in C^3$, $x_0=x(y_0)$.

The corresponding to $\det p_1^0$ Hamiltonian system is

$$\begin{gathered}\dot{x}=2\xi,\ x(0)=x_0\\ \dot{y}=2u\eta,\ y(0)=y_0\\ \dot{\xi}=-u_x\eta^2,\ \xi(0)=0\\ \dot{\eta}=-u_y\eta^2,\ \eta(0)=\pm 1\end{gathered}$$

Obviously, $\dot{x}(0)=0$, $\dot{y}(0)=0$,

$$\begin{gathered}\ddot{x}(0)=-2u_x(x_0, y_0)\neq 0,\\ \ddot{y}=2(u_x\dot{x}+u_y\dot{y})\eta+2u\dot{\eta}\Rightarrow \ddot{y}(0)=0,\\ \dddot{y}=2(u_{xx}\dot{x}^2+2u_{xy}\dot{x}\dot{y}+u_{yy}\dot{y}^2)\eta+\\ 2(u_x\ddot{x}+u_y\ddot{y})\eta+2(u_x\dot{x}+u_y\dot{y})\dot{\eta}+2u\ddot{\eta}+2\dot{u}\dot{\eta}\Rightarrow\\ \dddot{y}(0)=-4u_x^2(x_0, y_0),\end{gathered}$$

i. e. the characteristic curve in the plane

$$\delta\left|\begin{array}{l} x(t)=x_0-u_x(x_0, y_0)t^2+O(t^3)\\ y(t)=y_0-\frac{2}{3}u_x^2(x_0, y_0)t^3+O(t^3)\end{array}\right.$$

has a cusp point at (x_0, y_0) and it is located in the domain where $u>0$.

Conclusion. Let us consider the system (1.2) in the case c). $u_x(x_0, y_0)\neq 0$ and suppose that the solution $u\in H^s(\Omega)$, $s>4$, $\Omega\ni(x_0, y_0)$ is such that $u\in C^\infty(\Omega_+)$. As usual $\Omega_+=\{(x, y): u(x, y)>0\}$. Then $\begin{pmatrix}u\\ v\end{pmatrix}\in C^\infty$ in a full neighbourhood of the point (x_0, y_0).

If the curve γ: $u(x, y)=0$, $u(x_0, y_0)=0$, $\nabla u(x_0, y_0)\neq 0$ has one and only one horizontal tangent passing through the point (x_0, y_0) then $u\in H^s$, $s>4$, $\Omega\ni(x_0, y_0)$, $u\in C^\infty(\Omega_+)$ implies $\begin{pmatrix}u\\ v\end{pmatrix}\in C^\infty(\Omega)$ according to the previous remark and Theorem 1. Bellow we give the details.

We shall apply now Th1 to the system (1.2).

Assume that $\begin{pmatrix} u \\ v \end{pmatrix} \in H^s_{loc}$, $s>2$ (i. e. $u \in C^{1+\varepsilon}$, $\varepsilon>0$) and u

$u_x(x_0, y_0) = 0$, $u_y(x_0, y_0) \neq 0$, $\xi^0 = 0$, $\eta^0 = \pm 1$. So $c = -$

Moreover, $R'(\rho^0) = \eta^0 i u_y(x_0, y_0) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Thus, for 2

solutions of (1.2) with isolated H^t singularities at $(x_0, y_0; 0,$

Similar result is true for (1.3) with $u(x_0, y_0) = 1$ instea

2. Let us consider now a slight generalization of (1.6):

$$\det \begin{vmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{vmatrix} = K(x, y)\, f(x, y, u, \nabla u),$$

where $K(x_0, y_0) = 0$, $\nabla K(x_0, y_0) \neq 0$, $K \in C^\infty$, $f \in C^\infty$, $f(x, y, u, p, q) > 0$ everywhere, $(p = u_x, q = u_y)$. Certainly, u to be real-valued. Let $u \in H^s_{loc}$, $s > 4$, (i. e. $u \in C^{3+\varepsilon}(\Omega)$) suppose that

(3.2) The curve $\gamma = \{(x, y) \in \Omega: K(x, y) = 0\}$ is ch point (x_0, y_0) for the linearized on u of equation (3.1).

It is easy to see that at each point $(x, y) \in \gamma$ at least on and normal vectors is non characteristic for the linearized e

Proposition. Let $u \in C^\infty(\Omega \backslash (x_0, y_0))$ and (3.2) be f Then $u \in C^\infty(\Omega)$.

By a rotation of coordinates leaving the equation (3.1) assume that

$$K(x, y) = y + O(x^2 + y^2),$$

i. e. $(x_0, y_0) = 0$, $\nabla K(0) = (0, 1)$, the tangential vector to γ a

The linearized on u of equation (3.1) has the following

$$p = u_{yy}\xi^2 - 2u_{xy}\xi\eta + u_{xx}\eta^2 + iKf_p\xi + iKf_q\eta - Kf_u,$$

where

$$f_p = f_p(x, y, u, u_x, u_y).$$

According to (3.2): $u_{xx}(0, 0) = 0$, $u_{yy}(0, 0) \neq 0$ and therefore $u_{xy}(0, 0) = 0$. The point $\rho^0 = (0, 0, \xi_0, \eta_0)$ is non-characteri if $\xi_0 \neq 0$. Then $u \in H^{2s-3}_{mcl}(\rho^0)$. So let $\rho^0 = (0, 0, 0, \eta^0)$, $\eta_0 \neq 0$

Thus $p^0_2(\rho^0) = 0$, $H_{p^0_2(\rho^0)} = -\eta^{02}(u_{xxx}(0, 0)\partial_\xi + u_{xxy}(0, 0)\partial_\eta$

Differentiating (3.1) with respect to x, y we find that:

$u_{xxx}(0, 0)=0$, i. e. $0 \neq H_{p_2^0(\rho^0)} \| \Xi(\rho^0)$, $\eta_0^2=1$.

The subprincipal symbol p_2' $(\rho^0)=0$ and therefore the conditions $(v)'''$, $(v)''$ are satisfied. Let Γ be sufficiently small conic neighbourhood of $\rho^0=(0, 0, 0, \eta^0)$. Then we know that $u \in H_{mcl}^{2s-3}$ $(\Gamma \setminus \rho^0)$[1] and therefore $u \in H_{mcl}^{2s-4}(\rho^0)$ which implies $u \in H^{2s-4}$ (0) (i. e. $\exists\, \varphi \in C_0^\infty$, $\varphi \equiv 1$ near 0, $\varphi u \in H^{2s-4}$). This way we raise the smoothness of u at 0 with $s-4>0$. Repeating the same procedure we arrive at $u \in C^\infty$ near $(0, 0)$.

Corollary. *Consider the equation (3.1) and assume that the curve γ is non characteristic for the linearized on u of equation (3.1) with the exception of the point $(x_0, y_0) \in \gamma$. Then $u \in C^\infty(\Omega_-)$, $\Omega_-=\{(x, y): K(x, y)<0\}$, implis that $u \in C^\infty$ in a full neighbourhood of (x_0, y_0).*

4. Proof of Theorem 2

Let us consider the symbol

$$c_1=\kappa_1(x)\gamma_1(\xi)(1+\mu^2|\xi|^2)^{-\delta},\ \delta=const>0,\ \mu \in (0, 1]$$

($\mu \to 0$ further on, i. e. $0<\mu \ll 1$), $\kappa_1 \in C_0^\infty$, $\kappa_1 \equiv 1$ near x_0, $\mathrm{ord}_\xi \gamma_1=r, \gamma_1|\xi|^{-r} \equiv 1$ in a conic neighbourhood of ξ^0, conesupp $c_1 \in \Gamma_1 \subset\subset \Gamma$.

A simple calculation shows that

$\{q_1, c_1\}=(1+\mu^2|\xi|^2)^{-\delta} A_r$, where $A_r=\{q_1, \kappa_1\gamma_1\}+2\delta\kappa_1\gamma_1 \times \sum_{j=1}^n \frac{\partial q_1}{\partial x_j}\mu^2\xi_j(1+\mu^2|\xi|^2)^{-1}$. But $(1+\mu^2|\xi|^2)^{-1}$ is a bounded family in $\sum_\lambda^0$, $\forall \lambda>0$ and $\frac{\mu^2\xi_j}{1+\mu^2|\xi|^2}$ is a bounded family in $\sum_\lambda^{-1}$, $\forall \lambda>0$, $\mu \in (0, 1]$. Thus $A_r \in \sum_{\sigma-1}^r$ uniformly with respect to μ and conesupp $A_r \in \Gamma_1 \subset\subset \Gamma$.

Let us define the cutoff symbol η, $\eta \equiv 1$ in a conic neighbourhood of ρ^0, $\eta \in S_{1,0}^0$, conesupp $\eta \in \{(x, \xi): \kappa_1(x)\gamma_1(\xi) \neq 0\}$.

Thus

$$\{q_1, c_1\}=\eta^2 c_1 \frac{A_r}{\kappa_1\gamma_1}+(1-\eta^2)(1+\mu^2|\xi|^2)^{-\delta}A_r=$$
$$=\eta^2 c_1 B_0+(1-\eta^2)(1+\mu^2|\xi|^2)^{-\delta}A_r$$

and $B_0 \in \sum_{\sigma-1}^0$ uniformly with respect to μ, conesupp $B_0 \subset\subset \Gamma$.

So

$$B_0=D_0+2\delta(1+\mu^2|\xi|^2)^{-1}\sum_{j=1}^n \frac{\partial q_1}{\partial x_j}\mu^2\xi_j$$

[1] Here we use the fact that $u \in C^\infty$ $(\Omega \setminus (x_0, y_0))$.

and $D_0(\rho)=\frac{\{q_1, \kappa_1\gamma_1\}}{\kappa_1\gamma_1}=\{q_1, \kappa_1\gamma_1\}$ assuming $\rho=(x, \xi)$, $|\xi|=1$ in a tiny conic neighbourhood of ρ^0.

But

$$B_0(\rho^0)=<\nabla_\xi q_1(\rho^0), \nabla_x(\kappa_1\gamma_1)(\rho^0)>-<\nabla_x q_1(\rho^0), \nabla_\xi(\kappa_1\gamma_1)(\rho^0)>$$
$$+2\delta(1+\mu^2)^{-1}\mu^2<\nabla_x q_1(\rho^0), \xi^0>=-c<\xi^0, \nabla_\xi\gamma_1\xi^0)>+$$
$$+2\delta c\frac{\mu^2}{1+\mu^2}=-c\left(r-\frac{2\delta\mu^2}{1+\mu^2}\right)=-c(r+O(\mu^2)), \mu\to 0.$$

(Here we have used Euler's identity for the homogeneous function γ_1, $\mathrm{ord}_\xi\gamma_1=r$).

Let us consider now the symbol

$$C_\mu^2=\kappa^2(x)\gamma^2(\xi)|\xi|^{2t}(1+\mu^2|\xi|^2)^{-2}$$

where κ, γ, $\mathrm{ord}_\xi\,\gamma=0$ have the same properties as κ_1 and γ_1, i. e. $\delta=2$, $r=2t$ ($\Rightarrow C_\mu\in\sum_\sigma^{t-2}$ for each fixed $\mu>0$).

Then we get

$$\{q_1, C_\mu^2\}=\eta^2C_\mu^2B_0+(1-\eta^2)(1+\mu^2|\xi|^2)^{-2}A_{2t}, \tag{4.1}$$

where $B_0\in\sum_{\sigma-1}^0$ uniformly with respect to μ, $A_{2t}\in\sum_{\sigma-1}^{2t}$ uniformly with respect to μ, conesupp B_0 and conesupp $A_{2t}\subset\subset\Gamma$ and

$$B_0(\rho^0)=-2c(t+O(\mu^2)), \mu\to 0. \tag{4.2}$$

We point out that $u\in H_{mcl}^t$(conesupp $(1-\eta^2)A_{2t}$) and that $t>s$ and (v) imply

$$\frac{1}{2}B_0(\rho^0)+\min_{\|z\|=1}(\mathrm{Im}\,R_0'(\rho^0)z, z)>0. \tag{4.3}$$

$(C_\mu u=C_\mu I_N u,\ R=q_1I_N+R_0+\widetilde{R}\ )$.

The bilinear form $(C_\mu f_1, C_\mu u)$, $f_1=Ru$ is well defined in H^0 as $C_\mu f_1\in H_{comp}^2$, $C_\mu u\in H_{comp}^{2-\frac{1}{2l}}$ (conesupp $C_\mu\subset\subset\Gamma$, $\forall\mu>0$).

Thus,

$$\mathrm{Im}(C_\mu f_1, C_\mu u)=\mathrm{Im}(C_\mu q_1I_N u, C_\mu u)+\mathrm{Im}(C_\mu R_0u, C_\mu u)+K, \tag{4.4}$$

$|K|\le C\|u\|_{t-\frac{\sigma-1}{2}}^2$,

where the constant C does not depend on μ.

In fact, $C_\mu\in\sum_\sigma^t$ uniformly with respect to μ and $\widetilde{R}$ is a smoothing operator of order $-(\sigma-1)$, i. e. $|(C_\mu\tilde{R}u, C_\mu u)|\le\|C_\mu\tilde{R}u\|_{\frac{\sigma-1}{2}}\|C_\mu u\|_{\frac{1-\sigma}{2}}\le$ $C\|u\|_{t-\frac{\sigma-1}{2}}^2$, $\frac{\sigma-1}{2}=\frac{1}{2l}$.

The left-hand side of (4.4) can be estimated very easy as $f_1\in H_{mcl}^t(\Gamma)$, conesupp $C_\mu\in\Gamma$ imply

$$|(C_\mu u, C_\mu f_1)| \leq \varepsilon \| C_\mu u\|^2_{0,mcl(\Gamma)} + C(\varepsilon)\|f_1\|^2_{t,mcl(\Gamma)},$$

$\forall \varepsilon > 0$ and $C(\varepsilon)$ does not depend on μ.

The identity $C_\mu u = \eta C_\mu u + (1-\eta) C_\mu u$, $(\eta - 1) C_\mu \in \Sigma^t_\sigma$ uniformly with respect to μ, the relations $u \in H^t_{mcl}(\Gamma \setminus \rho^0)$ and conesuppη is concentrated near ρ^0, $\eta \equiv 1$ near ρ^0 show that $\| C_\mu u \|_0 \leq \| \eta C_\mu u\|_0 + d_1 \| u \|_{t,mcl(\Gamma'')}$, $\rho^0 \notin \bar{\Gamma}''$.

So

$$|(C_\mu f_1, C_\mu u)| \leq \varepsilon \| \eta C_\mu u\|_0^2 + C(\varepsilon)\|f_1\|^2_{t,mcl(\Gamma)} + C_1(\varepsilon)\|u\|^2_{t,mcl(\Gamma'')}, \ \forall \varepsilon > 0. \tag{4.5}$$

Obviously, $\mathrm{Im}(C_\mu q_1 I_N u, C_\mu u) = \mathrm{Im}(q_1 C_\mu u, C_\mu u) + \mathrm{Im}([C_\mu, q_1]u, C_\mu u)$.
Put $v = C_\mu u$. Then $(q_1 v, v) = (v, q_1^* v)$, $q_1 = q_1 I_N$ and according to [3] the symbol of the L_2 adjoint operator q_1^* of q_1 is given by the formula:

$$q_1^* = q_1 - i \sum_{|\alpha|=1} q_{1(\alpha)}^{(\alpha)} + \tilde{\tilde{\tilde{R}}},$$

$\tilde{\tilde{\tilde{R}}}$ being a smoothing continuous operator of order $-(\sigma-1)$. So

$$(q_1 v, v) = (v, q_1 v) + i \sum_{|\alpha|=1} (v, q_{1(\alpha)}^{(\alpha)}) + O(\|u\|^2_{t-\frac{\sigma-1}{2}}) \Rightarrow$$
$$2\mathrm{Im}(q_1 v, v) = \sum_{|\alpha|=1} (v, q_{1(\alpha)}^{(\alpha)} v) + O(\|u\|^2_{t-\frac{\sigma-1}{2}}).$$

Having in mind the fact that $q_{1(\alpha)}^{(\alpha)} \in \Sigma^0_{\sigma-1}$ is real-valued we get, applying again Theorem 3.3 from [3]:

$$2\mathrm{Im}(q_1 v, v) = \sum_{|\alpha|=1} (q_{1(\alpha)}^{(\alpha)} v, v) + O(\|u\|^2_{t-\frac{\sigma-1}{2}}).$$

Thus,

$$\mathrm{Im}(q_1 C_\mu u, C_\mu u) = \frac{1}{2} \sum_{|\alpha|=1} (q_{1(\alpha)}^{(\alpha)} C_\mu u, C_\mu u) + O(\|u\|^2_{t-\frac{1}{2l}}), \tag{4.6}$$

and the remainder $O(\cdot)$ is independent of μ.

In a similar way

$$\mathrm{Im}(C_\mu R_0 u, C_\mu u) = \mathrm{Im}(R_0 C_\mu u, C_\mu u) + O(\|u\|^2_{t-\frac{1}{2l}}).$$

In fact, $[C_\mu, R_0] \in \Sigma^{t-1}_{\sigma-2}$, for $\sigma > 2$ as $R_0 \in \Sigma^0_{\sigma-1}$ and $[C_\mu, R_0]$ is $-t+\sigma-1$ regularizing operator, uniformly with respect to μ, in the case $1 < \sigma < 2$. Then

$$\mathrm{Im}(R_0 C_\mu u, C_\mu u) = (\mathrm{Im} R_0 C_\mu u, C_\mu u) + O(\|u\|^2_{t-\frac{1}{2l}}) \tag{4.7}$$

and $O(\cdot)$ is independent of μ, $\mathrm{Im}\ R_0 = \dfrac{R_0 - R_0^*}{2i}$ is a Hermitian self-adjoint matrix,

We know that $[C_\mu, q_1] \in \Sigma^t_{\sigma-1}$ uniformly with respect to μ and has the principal symbol $\frac{1}{i}\{C_\mu, q_1\}$. So

$$\operatorname{Im}([C_\mu, q_1]u, C_\mu u) = -\operatorname{Re}(\{C_\mu, q_1\}u, C_\mu u) + O(\|u\|^2_{t-\frac{1}{2l}}),$$

i. e.

$$\operatorname{Im}([C_\mu, q_1]u, C_\mu u) = \frac{1}{2}\operatorname{Re}(\{q_1, C_\mu^2\}u, u) + O(\|u\|^2_{t-\frac{1}{2l}}). \tag{4.8}$$

Combining (4.4) - (4.8) we get

$$\varepsilon\|\eta C_\mu u\|_0^2 + C(\varepsilon)\|f_1\|^2_{t,mcl(\Gamma)} + C_1(\varepsilon)\|u\|^2_{t,mcl(\Gamma'')} \geq \tag{4.9}$$

$$\frac{1}{2}\operatorname{Re}(\{q_1, C_\mu^2\}u, u) + (\operatorname{Im}R'_0 C_\mu u, C_\mu u) + O\|u\|^2_{t-\frac{1}{2l}}).$$

The identity (4.1) shows that

$$\operatorname{Re}(\{q_1, C_\mu^2\}u, u) = \operatorname{Re}(\eta C_\mu B_0 u, \eta C_\mu u) + O(\|u\|^2_{t-\frac{1}{2}}) +$$
$$\operatorname{Re}\underbrace{((1-\eta^2)(1+\mu^2|D|^2)^{-2}A_{2t}}_{J}\, u, u),$$

where conesupp $A_{2t}\subset\Gamma$, $A_{2t}\in\sum_{\sigma-1}^{2t}$ uniformly with respect to μ.

According to Bony ([3], Corollary 3.5 a)., b).)

$$|(Ju, u)| \leq d_2(\|u\|^2_{t-\frac{\sigma-1}{2}} + \|u\|^2_{t,mcl(\Gamma'')}). \tag{4.10}$$

Having in mind that

$$(\operatorname{Im}R'_0 C_\mu u, C_\mu u) = (\eta^2 \operatorname{Im}R'_0 C_\mu u, C_\mu u) + \tag{4.11}$$
$$+ \underbrace{(C_\mu^*(1-\eta^2)\operatorname{Im}R'_0 C_\mu}_{J_1}\, u, u),$$

$J_1\in\sum_{\sigma-1}^{2t}$ uniformly with respect to μ, conesupp $J_1\subset\Gamma$, $J_1\equiv 0$ near ρ^0 we conclude that

$$|(J_1 u, u)| \leq d_3(\|u\|^2_{t-\frac{\sigma-1}{2}} + \|u\|^2_{t,mcl(\Gamma'')}) \tag{4.12}$$

with d_2, d_3 independent of μ. Obviously,

$$(\eta^2\operatorname{Im}R'_0 C_\mu u, C_\mu u) = (\operatorname{Im}R'_0\eta C_\mu u, \eta C_\mu u) + O(\|u\|^2_{t-\frac{\sigma-1}{2}})$$

as $[\eta, \operatorname{Im}R'_0]$ is smoothing operator of order $-(\sigma-1)$ and $O(\cdot)$ does not depend on μ.

So

$$\operatorname{Im}(R'_0 C_\mu u, C_\mu u) = (\operatorname{Im}R'_0\eta C_\mu u, \eta C_\mu u) + \tag{4.13}$$
$$O(\|u\|^2_{t-\frac{\sigma-1}{2}} + \|u\|^2_{t,mcl(\Gamma'')})$$

According to (v): 1). $\frac{1}{2}B_0+\mathrm{Im}R'_0\in\sum^0_{\sigma-1}$, and 2). $\frac{1}{2}B_0(\rho^0)I_N+\mathrm{Im}R'_0(\rho^0)$ is Hermitian and positively definite matrix which implies that $\frac{1}{2}B_0(\rho)I_N+\mathrm{Im}R'_0(\rho)$ is Hermitian and positively definite matrix near ρ^0.

Combining (4.9) - (4.13) and having in mind that

$$(\eta C_\mu B_0 u,\ \eta C_\mu u)=(B_0\eta C_\mu u,\ \eta C_\mu u)+O\ (\|u\|^2_{t-\frac{\sigma-1}{2}})$$

we get that $\forall\,\varepsilon>0$

$$\begin{aligned}\varepsilon\|\eta C_\mu u\|^2_0+C(\varepsilon)\|f_1\|^2_{t,mcl(\Gamma)}+d_4\|u\|^2_{t,mcl(\Gamma'')}\geq\\ \mathrm{Re}\Big(\Big(\frac{1}{2}B_0+\mathrm{Im}R'_0\Big)\eta C_\mu u,\ \eta C_\mu u\Big)+d_5\|u\|^2_{t-\frac{1}{2l}}.\end{aligned}\tag{4.14}$$

The constants $C(\varepsilon)$, d_4, d_5 in (4.14) are independent of μ.

Taking conesupp $\eta\subset\{\rho\colon\frac{1}{2}B(\rho)+\mathrm{Im}R'_0(\rho)>0\}$ we can apply Garding's inequality for positive paradifferential operators [3] and obtain:

$$\begin{aligned}\mathrm{Re}\Big(\Big(\frac{1}{2}B_0+\mathrm{Im}R'_0\Big)(\eta C_\mu u),\ \eta C_\mu u\Big)\geq d_6\|C_\mu u\|^2_0\\ +O(\|\eta C_\mu u\|^2_{-\theta})\end{aligned}\tag{4.15}$$

for some $\theta=const>0$ and $d_6=const>0$, $O(.)$ are independent of μ.

The standard interpolation inequality in Sobolev spaces gives us that

$$\|\eta C_\mu u\|_0\leq const,$$

i. e. for sufficiently small $\varepsilon>0$

$$u\in H^t_{mcl(\rho^0)}\Rightarrow u\in H^t_{mcl}(\Gamma).$$

Thus everything is proved.

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References

[1] L. Bers, Mathematical aspects of subsonic and transonic gas dynamics, Wiley, New York, 1958.

[2] V. Guillemin and D. Schaeffer, On a certain calss of Fuchsian PDE, Duke Math. J., **44** (1977), 157-199.

[3] J. M. Bony, Calcul symbolique et propagation des singularités pour les équations aux derivées partielles non-linéares, Ann. Scient. Ec. Norm. Sup., Paris, **14** (1981), 209-246.

[4] P. Popivanov and I. Iordanov, Paradifferential operators and propagation of singulariries for non-linear PDE, Akad. der Wiss. der DDR, Report, Berlin, 1983.

[5] N. Dencker, On the propagation of polarization sets for systems of real principal type. J. of Funct. Analysis, **46**-3 (1982) 351-372.

[6] P. Popivanov, Removable singularities of a class of fully non-linear PDE, Nonlinear Analysis-TMA, **18** (1992), 851-859.

[7] J. Hong and C. Zuily, L_p and Hölder estimates for a class of degenerate elliptic boundary value problems. Applications to the Monge-Ampere equation, Comm. in PDE, **16** (1991), 997-1032.

[8] Tay Ping Liu and Z. Xin, Overcompressive shock waves in B. L. Kheifitz, M. Shearer-editors of Nonlinear evolution equations that change type, The IMA volumes in Math. and its applications 27, Springer, 139-145,1991.