The mod 3 homology of the space of loops on the exceptional Lie groups and the adjoint action

By

Hiroaki Hamanaka* and Shin-ichiro Hara

1. Introduction

Let \( p \) be a prime number and \( G \) be a compact, connected, simply connected and simple Lie group. Let \( \Omega G \) be the loop space of \( G \). Bott showed \( H_* (\Omega G; \mathbb{Z}/p) \) is a finitely generated bicommutative Hopf algebra concentrated in even degrees, and determined it for classical groups \( G \) ([1]).

Here, let \( G \) be an exceptional Lie group, that is, \( G = G_2, F_4, E_6, E_7, E_8 \). In [2], K. Kozima and A. Kono determined \( H_* (\Omega G; \mathbb{Z}/2) \) as a Hopf algebra over \( \mathcal{A}_2 \), where \( \mathcal{A}_p \) is the mod \( p \) Steenrod Algebra and acts on it dually.

Let \( \text{Ad} : G \times G \to G \) and \( \text{ad} : G \times \Omega G \to \Omega G \) be the adjoint actions of \( G \) on \( G \) and \( \Omega G \) respectively. In [3], the cohomology maps of these adjoint actions are studied and it is shown that \( H^* (\text{ad} : Z/p) = H^* (p_2 : Z/p) \) where \( p_2 \) is the second projection if and only if \( H_* (G; Z) \) is \( p \)-torsion free. For \( p = 2, 3 \) and \( 5 \), some exceptional Lie groups have \( p \)-torsions on its homology. Moreover in [8, 9] mod \( p \) homology map of \( \text{ad} \) is determined for \( (G, p) = (G_2, 2), (F_4, 2), (E_6, 2), (E_7, 2) \) and \( (E_8, 5) \). This result is applied to compute the \( \mathcal{A}_5 \) module structure of \( H_* (\Omega E_6; Z/5) \) and \( H_* (\Omega E_8; Z/5) \) in [9].

For a compact and connected Lie group \( G \), the free loop group of \( G \) is denoted by \( LG (G) \), i.e., the space of free loops on \( G \) equipped with multiplication as

\[
\phi \cdot \phi (t) = \phi (t) \cdot \phi (t),
\]

and has \( \Omega G \) as its normal subgroup. Then

\[
LG (G) / \Omega G \cong G,
\]

and identifying elements of \( G \) with constant maps from \( S^1 \) to \( G \), \( LG (G) \) is equal to the semi-direct product of \( G \) and \( \Omega G \). This means that the homology of \( LG (G) \) is determined by the homology of \( G \) and \( \Omega G \) as module and the algebra structure of \( H_* (LG (G); Z/p) \) depends on \( H_* (\text{ad} : Z/p) \) where

\[
\text{ad} : G \times \Omega G \to \Omega G
\]

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is the adjoint map. Since the next diagram commutes where \( \lambda, \lambda', \) and \( \mu \) are the multiplication maps of \( \Omega G, LG(G) \) and \( G \) respectively and \( \omega \) is the composition

\[
(1_{\Omega G} \times T \times 1_G) \circ (1_{\Omega G \times G} \times ad \times 1_G) \circ (1_{\Omega G} \times \Delta_G \times 1_{\Omega G \times G}),
\]

\[
\Omega G \times G \times \Omega G \times G \xrightarrow{\omega} \Omega G \times \Omega G \times G \xrightarrow{\lambda \times \mu} \Omega G \times G
\]

\[
\downarrow \equiv \times \equiv \downarrow \equiv
\]

\[
LG(G) \times LG(G) \xrightarrow{\lambda'} LG(G)
\]

we can determine directly the algebra structure of \( H_*(LG(G); \mathbb{Z}/p) \) by the knowledge of the Hopf algebra structure of \( H_*(G; \mathbb{Z}/p), H_*(\Omega G; \mathbb{Z}/p) \) and induced homology map \( H_*(ad; \mathbb{Z}/p) \). See Theorem 6.12 of [8] for detail.

In this paper we determined the Hopf algebra structure over \( \mathbb{F}_3 \) of the homology group \( H_* (\Omega G; \mathbb{Z}/3) \) for \( G = F_4, E_6, E_7 \) and \( E_8 \) by using adjoint action and determine the mod 3 homology map of \( \text{ad} \) for them. The result is shown in \( \S 2 \).

This paper is organized as follows. We refer to the results of [4, 5, 6] for the structure of \( H^*(G) \) and compute \( H^*(\Omega G) \) for the lower dimensions and their cohomology operations are partially determined. This is done in \( \S 3 \). In \( \S 4 \) we turn to their homology rings. We determine the algebra structure of \( H_* (\Omega G; \mathbb{Z}/3) \) and we partly determine the Hopf algebra structure and cohomology operations on \( H_* (\Omega G; \mathbb{Z}/3) \). Finally in \( \S 5 \) the homology map of the adjoint action and the rest of the Hopf algebra structure and cohomology operations are determined. The computations are completely algebraic.

2. Results

Let \( G (l) \) be the compact, connected, simply connected and simple exceptional Lie group of rank \( l \) where \( l = 4, 6, 7 \) or 8. The exponents of \( G (l) \) are the integers \( n(1) < n(2) < \cdots n(l) \) which are given by the following table:

<table>
<thead>
<tr>
<th>( l )</th>
<th>( n(1) )</th>
<th>( n(2) )</th>
<th>( n(3) )</th>
<th>( n(4) )</th>
<th>( n(5) )</th>
<th>( n(6) )</th>
<th>( n(7) )</th>
<th>( n(8) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1</td>
<td>5</td>
<td>7</td>
<td>11</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>4</td>
<td>5</td>
<td>7</td>
<td>8</td>
<td>11</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>11</td>
<td>13</td>
<td>17</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>7</td>
<td>11</td>
<td>13</td>
<td>17</td>
<td>19</td>
<td>23</td>
<td>29</td>
</tr>
</tbody>
</table>

Put \( E(l) = \{ n(1), \cdots, n(l) \} \) and \( \phi(t) = \Delta_*(t) - (t \otimes 1 + 1 \otimes t) \) where \( \Delta \) is the diagonal map. \( \mathcal{P}^*_k \) is the dual of the Steenrod operation \( \mathcal{P}^*_k \). Then the results are following:
Theorem 1. As a Hopf Algebra over $\mathbb{A}_3$, 

\[ H_*(\Omega G(1); \mathbb{Z}/3) \cong \begin{cases} 
\mathbb{Z}/3 \left[ t_{2j} \in E(1) \cup \{3\} \right] / (t_2^3), & \text{if } 1 = 4, 6, 7 \\
\mathbb{Z}/3 \left[ t_{2j} \in E(8) \cup \{3, 9\} \right] / (t_2^3, t_6^3), & \text{if } 1 = 8 
\end{cases} \]

where $|t_{2j}| = 2j$.

\[ -\phi(t_{2j}) = \begin{cases} 
0, & \text{if } j \neq 3, 9, \\
-t_2^2 \otimes t_2 - t_2 \otimes t_2^2, & \text{if } j = 3, \\
t_2 t_6^2 \otimes t_2 + t_2 t_6^2 \otimes t_2^2 - t_6^2 \otimes t_6 - t_2^2 t_6 \otimes t_2 t_6, \\
-t_2 t_6 \otimes t_2^2 t_6 - t_6 \otimes t_6^2 + t_2^2 \otimes t_2 t_6^2 + t_2 \otimes t_2^2 t_6^2, & \text{if } j = 9.
\end{cases} \]

\[ \mathcal{P}_{\mathfrak{a}t_{2j}} = 0, \quad \text{if } r \geq 3, \]

\[ \mathcal{P}_{\mathfrak{a}t_{2j}} = \begin{cases} 
t_{22}, & \text{if } j = 29, \\
0, & \text{otherwise}.
\end{cases} \]

$\mathcal{P}_{\mathfrak{a}t_{2j}}$ and $\mathcal{P}_{\mathfrak{a}t_{2j}}$ are given by the following table:

<table>
<thead>
<tr>
<th>$t_{2j}$</th>
<th>$t_2$</th>
<th>$t_6$</th>
<th>$t_{10}$</th>
<th>$t_{14}$</th>
<th>$t_{16}$</th>
<th>$t_{18}$</th>
<th>$t_{22}$</th>
<th>$t_{26}$</th>
<th>$t_{34}$</th>
<th>$t_{38}$</th>
<th>$t_{45}$</th>
<th>$t_{58}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{P}<em>{\mathfrak{a}t</em>{2j}}$</td>
<td>0</td>
<td>$t_2$</td>
<td>0</td>
<td>0</td>
<td>$t_{10}$</td>
<td>$t_{14}$</td>
<td>$t_{16}$</td>
<td>$t_{18}$</td>
<td>$t_{22}$</td>
<td>$t_{26}$</td>
<td>$t_{34}$</td>
<td>$t_{38}$</td>
</tr>
<tr>
<td>$\mathcal{P}<em>{\mathfrak{a}t</em>{2j}}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$t_6$</td>
<td>0</td>
<td>$t_{14}$</td>
<td>$t_{22}$</td>
<td>$t_{26}$</td>
<td>$t_{34}$</td>
<td>0</td>
</tr>
</tbody>
</table>

where $\varepsilon$ and $\kappa$ are 1 or $-1$.

Remark. In Theorem 1, if $t_{2j}$ does not exist in $H_*(\Omega G(1); \mathbb{Z}/3)$, we regard $t_{2j}$ as 0 for such $j$.

Let Ad: $G \times G \rightarrow G$ and ad: $G \times \Omega G \rightarrow \Omega G$ be the adjoint actions of a Lie group $G$ defined by $Ad(g, h) = ghg^{-1}$ and $ad(g, l)(t) = gl(t) g^{-1}$ where $g, h \in G$, $l \in \Omega G$ and $t \in [0, 1]$. These induce the homology maps

$Ad_*: H_*(G; \mathbb{Z}/3) \otimes H_*(G; \mathbb{Z}/3) \rightarrow H_*(G; \mathbb{Z}/3)$

$ad_*: H_*(G; \mathbb{Z}/3) \otimes H_*(\Omega G; \mathbb{Z}/3) \rightarrow H_*(\Omega G; \mathbb{Z}/3)$.

Theorem 2. There are generators $y_8$ in $H_*(G(1); \mathbb{Z}/3)$ for $l = 4, 6, 7$ and $y_8$ and $y_{20}$ in $H_*(E_6; \mathbb{Z}/3)$. We can choose these generators so that $ad_* (y_i \otimes t_{2j})$ $(i = 8, 20)$ is given by the following table.
where $\delta, \varepsilon \in \mathbb{Z}/3\mathbb{Z}$ and $\varepsilon \neq 0$. For other generators $y_i \in H_*(G(l); \mathbb{Z}/3)$, $ad(y_i \otimes t_{2j}) = 0$ for all $j$.

3. The mod 3 cohomology groups

We recall the results of [4, 5, 6] for the structure of $H^*(G(l); \mathbb{Z}/3)$ as the Hopf algebra over $A_3$.

**Theorem 3.** There is an isomorphism:

$$H^*(G(l); \mathbb{Z}/3) \cong \begin{cases} \Lambda(x_{2l+1}) \otimes \mathbb{Z}/3[x_8] / (x_8^2), & \text{if } l = 4, 6, 7, \\ \Lambda(x_{2l+1}) \otimes \mathbb{Z}/3[x_8, x_{20}] / (x_8^2, x_{20}^3), & \text{if } l = 8, \end{cases}$$

the coproduct is given by:

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>$\varphi x_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{11}$</td>
<td>$x_8 \otimes x_3$</td>
</tr>
<tr>
<td>$x_{15}$</td>
<td>$x_8 \otimes x_7$</td>
</tr>
<tr>
<td>$x_{17}$</td>
<td>$x_8 \otimes x_9$</td>
</tr>
<tr>
<td>$x_{27}$</td>
<td>$x_{20} \otimes x_{19} + x_8 \otimes x_7$</td>
</tr>
<tr>
<td>$x_{35}$</td>
<td>$x_{20} \otimes x_{19}$</td>
</tr>
<tr>
<td>$x_{39}$</td>
<td>$x_{20} \otimes x_{19}$</td>
</tr>
<tr>
<td>$x_{47}$</td>
<td>$-x_8 \otimes x_{39} - x_{20} \otimes x_{27} - x_{20} x_8 \otimes x_{19} + x_8 x_{20} \otimes x_7$</td>
</tr>
<tr>
<td>others</td>
<td>0</td>
</tr>
</tbody>
</table>

and the cohomology operations are determined by the following table:

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>$x_3$</th>
<th>$x_7$</th>
<th>$x_9$</th>
<th>$x_{11}$</th>
<th>$x_{15}$</th>
<th>$x_{17}$</th>
<th>$x_{19}$</th>
<th>$x_{20}$</th>
<th>$x_{27}$</th>
<th>$x_{35}$</th>
<th>$x_{39}$</th>
<th>$x_{47}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta x_i$</td>
<td>0</td>
<td>$x_8$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-x_8$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-x_8 x_{20}$</td>
<td>$-x_8 x_{20}$</td>
<td>0</td>
</tr>
<tr>
<td>$\mathcal{P}^1 x_i$</td>
<td>$x_7$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$x_{15}$</td>
<td>$x_{19}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$x_{39}$</td>
<td>0</td>
</tr>
<tr>
<td>$\mathcal{P}^3 x_i$</td>
<td>0</td>
<td>$x_{19}$</td>
<td>$x_{20}$</td>
<td>0</td>
<td>0</td>
<td>$x_{27}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-x_{39}$</td>
<td>$x_{47}$</td>
<td>0</td>
</tr>
</tbody>
</table>

where $\varepsilon$ is 1 or $-1$.

If $r > 1$ then $\mathcal{P}^r x_i = 0$.

**Remark.** We consider $x_i$ in these tables as 0 when $x_i \in H^*$. 
Recall a Serre fibration:

\[ \Omega G(l) \to \ast \to G(l). \]  

First, we compute \( H^*(\Omega G(l); \mathbb{Z}/3) \) by the Serre spectral sequence associated with the fibration (A). This spectral sequence has a Hopf algebra structure. We can proceed to compute it using degree-reason and Kudo’s transgression theorem ([7]) from the previous theorem. For \( j \in E(l) - \{9, 11, 29\} \), there are universally transgressive elements \( a_{2j} \in H^*(\Omega G(l); \mathbb{Z}/3) \), such that \( \tau a_{2j} = x_{2j+1} \). Thus we can show that for \( j = 9, 11, 15, 21, 27 \), there are \( a_{2j} \) such that satisfy

\[
\begin{align*}
    d_7(1 \otimes a_{18}) &= x_7 \otimes a_6^2, & \text{for } l = 4, 6, 7, \\
    d_{11}(1 \otimes a_{36}) &= x_{11} \otimes a_{10}^2, & \text{for } l = 4, 6, 7, \\
    d_{15}(1 \otimes a_{54}) &= x_{15} \otimes a_{14}^2, & \text{for } l = 8, \\
    d_{19}(1 \otimes a_{72}) &= x_{19} \otimes a_{18}^2, & \text{for } l = 4, 6, 7, 8, \\
    d_{19}(1 \otimes a_{90}) &= x_{19} \otimes a_{22}^2, & \text{for } l = 8.
\end{align*}
\]

\( a_{2j} \)'s are generators of the cohomology group in the low dimensions. The results are the following:

**Proposition 4.** For the dimensions less than \( 2n(l) + 2 \), the next isomorphism holds:

\[
H^*(\Omega G(l); \mathbb{Z}/3) \cong \begin{cases} 
\mathbb{Z}/3 [a_{2j}] & \text{if } l = 4, 6, \\
\mathbb{Z}/3 [a_{2j}] / (a_2^9), & \text{if } l = 7, \\
\mathbb{Z}/3 [a_{2j}] / (a_2^{27}, a_{14}^3), & \text{if } l = 8.
\end{cases}
\]

Now we start to determine the cohomology operations and the coproducts on \( a_{2j} \).

**Theorem 5.** For \( j \in E(l) - \{9, 11, 29\} \), \( a_{2j} \in H^*(\Omega G(l); \mathbb{Z}/3) \) is primitive and cohomology operations are determined by

\[
\begin{array}{c|cccccccc}
   a_{2j} & a_2 & a_8 & a_{10} & a_{14} & a_{16} & a_{26} & a_{34} & a_{38} \\
\hline
   \mathcal{P}^1a_{2j} & a_2^3 & 0 & a_{14} & \varepsilon a_2^9 & 0 & 0 & \varepsilon a_{38} & 0 & 0 \\
   \mathcal{P}^2a_{2j} & 0 & 0 & 0 & a_{26} & 0 & -a_{38} & a_{46} & 0 & 0
\end{array}
\]

If \( r > 1 \) then \( \mathcal{P}^{3r}a_{2j} = 0 \).

**Proof.** For \( j \in E(l) - \{9, 11, 29\} \), \( a_{2j} \) is transgressive, therefore \( \mathcal{P}^1a_{2j} = \mathcal{P}^{1}x_{2j+1} = \sigma \mathcal{P}^{1}x_{2j+1} \). Thus this can be determined by Theorem 3.

For the investigation of \( a_{2j} \) which is not transgressive we start from the
following theorem. In the next theorem, $\phi$ means the coproduct of $H^*(\Omega G; Z/3)$ and we set $\phi(a) = \phi(a) - (a \otimes 1 + 1 \otimes a)$.

**Theorem 6.** For $j = 9, 15, 21, 27$, $\tilde{\varphi}_{a_{2j}}$ is given by the following formula:

$$
\tilde{\varphi}_{a_{2j}} =
\begin{cases}
    a_2^3 \otimes a_2^6 + a_2^6 \otimes a_2^3, & \text{if } j = 9, \\
    a_{10} \otimes a_{10}^2 + a_{10}^2 \otimes a_{10}, & \text{if } j = 15, \\
    a_{14} \otimes a_{14}^2 + a_{14}^2 \otimes a_{14}, & \text{if } j = 21, \\
    a_2^9 \otimes a_{2}^{18} + a_2^{18} \otimes a_2^9, & \text{if } j = 27.
\end{cases}
$$

**Proof.** To begin with, we investigate the element $a_{18}$. Let $a_2'$ be the generator of $H^2(\Omega F_4; Z)$. $H^*(\Omega F_4; Z)$ has no torsion and is a commutative Hopf algebra over $Z$. Since $a_2^3 = 0$, there is $a_{18}'$ such that $a_2^9 = 3a_{18}'$ and $\rho a_{18}' \neq 0$, where $\rho$ is modulo 3 reduction. Then we can choose $a_{18}$ as $\rho a_{18}'$. The coproduct of $a_{18}'$ is computed as follows:

$$
\phi a_{18}' = 1/3 \phi a_2^9 = 1/3 (1 \otimes a_2 + a_2 \otimes 1)^9 = a_{18}' \otimes 1 + a_2^3 \otimes a_2^6 + a_2^6 \otimes a_2^3 + 1 \otimes a_{18}' \pmod{3}.
$$

Thus $\tilde{\varphi}_{a_{18}} = a_2^3 \otimes a_2^6 + a_2^6 \otimes a_2^3$ is shown.

Consider the inclusion $c: F_4 \rightarrow E_7$, we chose $a_{18} \in H^*(\Omega E_7; Z/3)$ so as to satisfy $(\Omega c)^* a_{18} = a_{18}$. Because $(\Omega c)^*$ is injective for degrees less than 18, $\tilde{\varphi}_{a_{18}} = a_2^3 \otimes a_2^6 + a_2^6 \otimes a_2^3$ is shown again for this $a_{18}$. And in the similar way we put $a_{30} = 1/3a_{18}^3$, $a_{42} = 1/3a_{14}^3$ and $a_{54} = 1/3a_2^{27}$ and obtain the coproduct formulas of the statement.

We remark that we can assume that $a_{22}$ and $a_{58}$ are primitive.

**Theorem 7.** In Proposition 4 we have that $P_1 a_{18} = \pm a_{22}$.

Let $G(l)$ be the 3-connected cover of $G(l)$ and

$$
\Omega \tilde{G}(l) \rightarrow \rightarrow \tilde{G}(l)
$$

be Serre fibrations. To prove Theorem 7 we have to compute $H^*(\Omega \tilde{G}; Z/3)$ and $H^*(\tilde{G}; Z/3)$.

Let $\tilde{a}_{2j}$ be $\Omega(p)^* a_{2j}$, for $j \neq 1$. Using the Serre spectral sequence associated with the fibration (D), one can easily show that there are generators $\tilde{a}_{17} \in H^{17}$ for $l = 4, 6$, and $\tilde{a}_{53} \in H^{53}$ for $l = 8$. We have the
following proposition. Let denote $E(t) - (1)$ as $\tilde{E}(t)$.

**Proposition 8.** For the dimensions less than $2n(t) + 2$, the next isomorphism holds:

$$H^*(\Omega G(t); Z/3) \cong \begin{cases} 
Z/3[\tilde{a}_2] \in \tilde{E}(l) \cup \{9\}] \otimes \Lambda(\tilde{a}_{17}), & \text{if } l = 4, 6, \\
Z/3[\tilde{a}_2] \in \tilde{E}(l) \cup \{15\}] / (\tilde{a}_{10}^3), & \text{if } l = 7, \\
Z/3[\tilde{a}_2] \in \tilde{E}(l) \cup \{21, 27\}] / (\tilde{a}_{14}^3) \otimes \Lambda(\tilde{a}_{53}), & \text{if } l = 8.
\end{cases}$$

By computing the Serre spectral sequence associated with (B), it is easy to see $\tilde{a}_{2j}$, $(j \neq 15, 21)$ is universally transgressive. Let $\tilde{x}_{i+1}$ be $\tau\tilde{a}_i$. Then we have the following:

**Proposition 9.** For the dimensions less than $2n(t) + 2$, the next isomorphism holds:

$$H^*(\tilde{G}(l); Z/3) \cong \begin{cases} 
\Lambda(\tilde{x}_{2j+1}] \in \tilde{E}(l) \cup \{9\}) \otimes Z/3[\tilde{x}_{18}], & \text{if } l = 4, 6, \\
\Lambda(\tilde{x}_{2j+1}] \in \tilde{E}(7)], & \text{if } l = 7, \\
\Lambda(\tilde{x}_{2j+1}] \in \tilde{E}(8) \cup \{27\}) \otimes Z/3[\tilde{x}_{54}], & \text{if } l = 8.
\end{cases}$$

**Proof of Theorem 7.** It is possible to show that $P^1a_{18}$ is not zero as follows. Let $\sigma'$ denotes the cohomology suspension associated to the fibration (C) for $l = 4$. It is easy to see $\tilde{x}_{18} = \sigma' \beta P^1u_3$ and $\tilde{x}_{23} = \sigma'(\beta P^1u_3)^3$, where $u_3$ is the generator of $H^3(K(Z, 3); Z/3)$. So we get $P^1\tilde{x}_{19} = \sigma' P^1\beta P^1u_3 = \sigma' P^1\beta P^1u_3 = \sigma'(\beta P^1u_3)^3 = \tilde{x}_{23}$, and from this, we have $(\Omega p)^*P^1a_{18} = P^1(\Omega p)^*a_{18} = P^1\sigma \tilde{x}_{19} = \sigma P^1\tilde{x}_{19} = \sigma \tilde{x}_{23} = a_{22}$, where $\sigma$ is the cohomology suspension associated to (B). Thus $P^1a_{18} \neq 0$. We fix $a_{22}$ as $P^1a_{18}$.

4. **Homology groups**

**Theorem 10.** The homology ring of $\Omega G(l)$ is

$$H_*(\Omega G(l); Z/3) \cong \begin{cases} 
Z/3[t_2] \in E(l) \cup \{3\}] / (t_2^3), & \text{if } l = 4, 6, 7, \\
Z/3[t_2] \in E(8) \cup \{3, 9\}] / (t_2^3, t_6^3), & \text{if } l = 8.
\end{cases}$$

where $|t_2| = 2j$. The coproduct is given by

$$\bar{\phi}(t_2) = \begin{cases} 
0, & \text{if } j \neq 3, 9, 11, 29, \\
-t_2^2 \otimes t_2 - t_2 \otimes t_2^2, & \text{if } j = 3, \\
t_2 t_6 \otimes t_2 + t_2 t_6^2 \otimes t_2^2 - t_6^2 \otimes t_6 - t_2^2 t_6 \otimes t_6^2 - t_2 t_6 \otimes t_6^2 - t_2 \otimes t_2^2 t_6^2, & \text{if } j = 9.
\end{cases}$$
Proof. Let $t_{2j}$ be the dual element of $a_{2j} \in H_* (\Omega G; \mathbb{Z}/3)$ as to the monomial basis for $j \in E (l) - \{9\}$ and $t_6, t_{18}$ be the dual element of $a_3, a_9$, respectively. It is easy to see $t_3^3 = t_6^3 = 0$ and to show the coproduct formula for $t_6$ and $t_{18}$. Thus we can say that statement (1) is true for $\ast < 2n (l) + 2$.

Now it is possible to show that there is no truncation in $H_* (\Omega G (l); \mathbb{Z}/3)$ other than the parts generated by $t_2$ and $t_6$ and that (1) holds for all dimensions. Since $H_* (\Omega G (l); \mathbb{Z}/3)$ is the even degree concentrated commutative Hopf algebra, we may suppose

$$H_* (\Omega G (l); \mathbb{Z}/3) = \mathbb{Z}/3 [u_i | i \in E] \otimes \mathbb{Z}/3 [v_j | j \in E] / (v_j^{3^{|l|}} | j \in E).$$

Consider an Eilenberg-Moore spectral sequence:

$$E_2 = \text{Ext}_{H_* (\Omega G (l); \mathbb{Z}/3)} (\mathbb{Z}/3, \mathbb{Z}/3) \Rightarrow E_\infty = \mathcal{G} (H^* (G (l); \mathbb{Z}/3)).$$

Since $E_2 = \Lambda (su_i | i \in E') \otimes \Lambda (sv_j | j \in E) \otimes \mathbb{Z}/3 [\theta v_j | j \in E]$, where $\deg su_i = (1, |u_i|)$, $\deg sv_j = (1, |v_j|)$, and $\deg \theta v_j = (2, 3^{|v_j|})$, the essential differentials have the forms:

$$d_{su_i} = (\theta v_j)^{3^{v_j}} (k_j \geq 1) \text{ and } d_{sv_j} = (\theta v_j)^{3^{v_j}} (l_j \geq 1).$$

Because $H^* (G (l); \mathbb{Z}/3)$ is a finite dimensional vector space, one can easily show

$$E_\infty = \Lambda (su_i | i \in E') \otimes \Lambda (sv_j | j \in E) \otimes \mathbb{Z}/3 [\theta v_j | j \in E] / ((\theta v_j)^{3^{|v_j|}} | j \in E), \quad (I' \subseteq I, f' \subseteq f)$$

and $|I'| + |f'| = |I|$. Here the total dimension of $E_\infty$ is $2^{|v_j| + |f| + \sum_m (m_j \geq 1)}$ and the total dimension of $H^* (G (l); \mathbb{Z}/3)$ is $2^{E (l) + |f|}$ where $E (l) = 1$ for $l = 4, 6, 7$ and $f (l) = 2$ for $l = 8$. Thus the indices $f$ of the truncation part satisfy that $|f| \leq f (l)$ and $|I| = |E (l)|$. This means that the truncation parts of $H_* (\Omega G; \mathbb{Z}/3)$ is generated by only $t_2$ and $t_6$.

Therefore $H_* (\Omega G (l); \mathbb{Z}/3)$ has the form

$$\mathbb{Z}/3 [u_i | i \in E] \otimes \mathbb{Z}/3 [t_2] / (t_2^3) \quad \text{for } l = 4, 6, 7 \text{ and }$$

$$\mathbb{Z}/3 [u_i | i \in E] \otimes \mathbb{Z}/3 [t_2, t_6] / (t_2^3, t_6^3) \quad \text{for } l = 8.$$

Also Theorem 5 means that for $j \in E (l) - \{9\}$, $t_{2j}$ is primitive and indecomposable and $t_6, t_{18}$ are indecomposable. Thus

$$\{t_{2j} | j \in E (l)\} \cup \{t_6\} \subset \{u_i | i \in E\} \quad \text{for } l = 4, 6, 7 \text{ and }$$

$$\{t_{2j} | j \in E (l)\} \cup \{t_{18}\} \subset \{u_i | i \in E\} \quad \text{for } l = 8.$$

Since $|E| = |E (l)|$, the theorem is proved.

Dualizing the result of Theorem 5 and Theorem 7, we obtain the statement of Theorem 1 except for $P_{4t_{26}}, P_{4t_{34}}, P_{5t_{34}}, P_{4t_{46}}, P_{4t_{58}}$ and $P_{5t_{58}}$. To determine these operations, we use the adjoint action of $H_* (G (l); \mathbb{Z}/3)$ on $H_* (\Omega G (l); \mathbb{Z}/3)$ which is introduced in the next section.

Remark. The computation of dualizing the result of Theorem 5 and Theorem 7 is not difficult except for $P_{4t_{18}}$, because $P_{4t}$ is primitive if $t$ is
primitive. Moreover, it is easily shown
\[ \tilde{\phi}(p_1^t_{18}) = P^t_1 \implies \phi(-t_2t_6^2) \]
and this shows \( P^t_1 t_{18} = -t_2t_6^2 \) modulo primitive elements. By Theorem 5 we can see \( P^t_{a_{14}} = e_{a_2}^9 \) and this shows that \( P^t_{18 t_{18}} = e_{t_{14}} - t_2t_6^2. \)

5. Adjoint action

Put \( y * y' = \text{Ad}_*(y \otimes y') \) and \( y * t = \text{ad}_*(y \otimes t) \) where \( y, y' \in H_*(G; \mathbb{Z}/3) \) and \( t \in H_*(\Omega G; \mathbb{Z}/3). \) The following theorem is the dual result of [3]. Also see [9].

**Theorem 11.** For, \( y, y', y'' \in H_*(G; \mathbb{Z}/3) \) and \( t, t' \in H_*(\Omega G; \mathbb{Z}/3) \)

(i) \( 1 * y = y, 1 * t = t. \)

(ii) \( y * 1 = 0, \) if \( |y| > 0, \) whether \( 1 \in H_*(G; \mathbb{Z}/3) \) or \( 1 \in H_*(\Omega G; \mathbb{Z}/3). \)

(iii) \( (y y') * t = y * (y' * t). \)

(iv) \( y * (t t') = \sum (-1)^{w' * |t|} (y' * t) (y'' * t') \) where \( \Delta_{\ast y} = \sum y' \otimes y''. \)

(v) \( \sigma(y * t) = y * \sigma(t) \) where \( \sigma \) is the homology suspension.

(vi) \( P_{*}^n (y * t) = \sum (P_{*}^t y) * (P_{*}^n - t). \)

(vii) \( P_{*}^n (y * y') = \sum (P_{*}^t y') * (P_{*}^n - t'). \)

And \( \Delta_{* y} (y * t) = (\Delta_{* y} *) \ast (\Delta_{* t}). \)

(viii) \( \Delta_{* y} (y * t) = (\Delta_{* y} \ast (\Delta_{* t}). \)

If \( t \) is primitive then \( y * t \) is primitive.

Also the result of [3] implies the following theorem. See [8].

**Theorem 12.** We set a submodule \( A \) of \( H_*(G; \mathbb{Z}/3) \) as
\[ A = \mathbb{Z}/3[y_8]/(y_8^3) \]
for \( G = F_4, E_6, E_7 \) and
\[ A = \mathbb{Z}/3[y_8, y_{20}]/(y_8^3, y_{20}^3) \]
for \( G = E_8 \)

where \( y_{2i} \) is the dual of \( x_{2i} \) with respect to the monomial basis. Then there exists a retraction \( p: H_*(G; \mathbb{Z}/3) \to A \) and the following diagram commutes.

\[
\begin{array}{ccc}
H_*(G; \mathbb{Z}/3) \otimes H_*(\Omega G; \mathbb{Z}/3) & \xrightarrow{\text{ad}_*} & H_*(\Omega G; \mathbb{Z}/3) \\
\downarrow p \otimes 1 & & \downarrow \text{ad}_* \\
A \otimes H_*(\Omega G; \mathbb{Z}/3) & \xrightarrow{\text{ad}_*} & A \otimes H_*(\Omega G; \mathbb{Z}/3)
\end{array}
\]

**Remark.** By Theorem 3 we can see \( P_{*}^t y_{20} = y_8. \)
Since $A_{d+}$ is agreed with the composition $\mu_\ast (1 \otimes \mu_\ast) \ast (1 \otimes 1 \otimes \iota_\ast) \ast (1 \otimes \iota)$ where $\mu$ is the multiplication of $G(l)$ and $\iota$ is the inverse map, the next theorem follows. See [9].

**Theorem 13.** Let $y, y' \in H_\ast(G)$. If $y$ is primitive,

$$ y \ast y' = [y, y'] $$

where $[y, y'] = yy' - (-1)^{\nu(y) \nu(y')} y' y$. 

Now we give the proof of Theorem 2 and finish the proof of Theorem 1. Let $y_i$ be the dual element of $x_i \in H^\ast(G(l))$ as to the monomial basis. By Theorem 3 and Theorem 13 we see that for $j \in E(l) \cup \{3, 9\} - \{11, 29\}$

$$ y_i \ast y_{2j+1} = \begin{cases} 
 y_{2j+9} & \text{for } j = 1, 3, 4, 9, 13, \\
 -y_{2j+9} & \text{for } j = 19, \\
 0 & \text{others}
\end{cases} $$

and

$$ y_{20} \ast y_{2j+1} = \begin{cases} 
 y_{2j+21} & \text{for } j = 3, 7, 9, \\
 -y_{2j+21} & \text{for } j = 13, \\
 0 & \text{others.}
\end{cases} $$

Since $\sigma y_i = y_{2j+1}$ for $j \in E(l) \cup \{3, 9\} - \{11, 29\}$, Theorem 11 (v) implies

$$ \sigma (y_i \ast t_{2j}) \neq 0 \quad \text{for } j = 1, 3, 4, 9, 13, 19, $$

$$ \sigma (y_i \ast t_{2j}) \neq 0 \quad \text{for } j = 3, 7, 9, 13. \quad (2) $$

Then the equations

$$ y_i \ast t_{2} = t_{10} \quad (3) $$
$$ y_i \ast t_{8} = t_{16} \quad (4) $$
$$ y_i \ast t_{26} = t_{34} \quad (5) $$
$$ y_i \ast t_{38} = -t_{46} \quad (6) $$
$$ y_{20} \ast t_{14} = t_{34} \quad (7) $$
$$ y_{20} \ast t_{26} = -t_{46} \quad (8) $$

are shown by Theorem 11 (viii). Moreover (2) implies

$$ y_i \ast t_{6} = t_{14} \quad (9) $$
$$ y_i \ast t_{18} = t_{26} \quad (10) $$
$$ y_{20} \ast t_{6} = t_{26} \quad (11) $$
$$ y_{20} \ast t_{18} = t_{38} \quad (12) $$

modulo decomposable elements. Since

$$ \overline{\phi}(y_i \ast t_6) = - (y_i \ast t_2) \otimes t_2 - (y_i \ast t_2) \otimes t_2 - t_2 \otimes (y_i \ast t_2) - t_2 \otimes (y_i \ast t_2) $$

$$ = \overline{\phi}(-t_{10}t_2^2), $$
one can see that \( y_8 \ast t_6 = -t_{10}t_2^2 \) mod primitive elements. By this and (9), we have

\[
y_8 \ast t_6 = t_{14} - t_{10}t_2^2. \quad (13)
\]

The equations

\[
\begin{align*}
y_8 \ast t_{18} &= t_{26} + t_{10}t_2^2t_6^2 - t_{14}t_6^2, \\
y_20 \ast t_6 &= t_{26} - (y_20 \ast t_2)t_2^2, \\
y_20 \ast t_{18} &= t_{38} - (y_20 \ast t_6)t_6^2
\end{align*}
\]

are shown in the similar way.

By the equation (13), we can compute \( y_8 \ast t_{26} \) as

\[
y_8^3 \ast t_6 = y_8^2 \ast (t_{14} - t_{10}t_2^2) = y_8^2 \ast t_{14} + t_{10}^3.
\]

Since \( y_8^3 = 0 \), \( y_8^2 \ast t_{14} = -t_{10}^3 \) and this means \( y_8 \ast t_{14} \) is a non-zero primitive indecomposable element. We redefine \( t_{22} \) as

\[ t_{22} = y_8 \ast t_{14}. \quad (17) \]

Then we have

\[
y_8 \ast t_{22} = -t_{10}^3.
\]

By Theorem 7 we can set \( P^1_{\ast}t_{22} = \kappa t_6^3 \) where \( \kappa = \pm 1 \). Since \( P^1_{\ast}t_{22} = P^1_{\ast}(y_8 \ast t_{14}) = y_8 \ast t_{10} \), we have

\[
y_8 \ast t_{10} = \kappa t_6^3.
\]

By the similar manner, we can compute \( y_8^3 \ast t_{18} \) and obtain \( y_8^2 \ast t_{26} = -t_{14}^3 \). Therefore

\[
y_8 \ast t_{34} = y_8^2 \ast t_{26} = -t_{14}^3. \quad (18)
\]

Because \( t_{16} \) and \( t_{46} \) are primitive, we can set

\[
\begin{align*}
y_8 \ast t_{16} &= \rho_3t_8^3, \\
y_8 \ast t_{46} &= \rho_3t_{18}^3.
\end{align*}
\]

Operate \( P^3_{\ast} \) to (20) to obtain

\[
y_8 \ast t_{34} = P^3_{\ast}(y_8 \ast t_{46}) = \rho_3P^3_{\ast}(t_{18}^3) = \rho_3\varepsilon t_{14}^3.
\]

Thus by (18), we conclude that \( \rho_3 = -\varepsilon \). \( y_8 \ast t_{58} \) will be determined after the determination of \( y_{20} \ast t_{58} \).

Here we apply \( P^1_{\ast} \) on (5), (6) and (14), \( P^3_{\ast} \) on (5) to see

\[
\begin{align*}
P^1_{\ast}t_{26} &= P^1_{\ast}(y_8 \ast t_{18} - t_{10}t_2^2t_2^2 + t_{14}t_6^2) \\
&= \varepsilon y_8 \ast t_{14} = \varepsilon t_{22}, \\
P^1_{\ast}t_{34} &= P^1_{\ast}(y_8 \ast t_{26}) = \varepsilon y_8 \ast t_{22} = -\varepsilon t_{10}^3.
\end{align*}
\]
Next we compute $y_{20} \ast t_{26}$. First we apply $\mathcal{P}_k$ to (15) to obtain

$$y_{20} \ast t_2 = \mathcal{P}_k(y_{20} \ast t_6) = \mathcal{P}_k(t_{26} - (y_{20} \ast t_2)^2) = \varepsilon t_{22}.$$ 

From this, (15) and (16) imply that

$$y_{20} \ast t_6 = t_{26} - \varepsilon t_{22} t_2^2,$$
$$y_{20} \ast t_{18} = t_{38} + \varepsilon t_{22} t_6 t_2^2 - t_{26} t_6^2.$$

$y_{20}^3 \ast t_6$ is computed as

$$0 = y_{20}^3 \ast t_6 = y_{20}^2 \ast (y_{20} \ast t_6) = y_{20}^2 \ast (t_{26} - \varepsilon t_{22} t_2^2) = y_{20}^2 \ast t_{26} + \varepsilon t_{22}^3.$$

Thus $y_{20} \ast t_{46} = -y_{20}^2 \ast t_{26} = \varepsilon t_{22}^3$.

The similar computation of $y_{20}^3 \ast t_{18}$ implies

$$y_{20}^2 \ast t_{38} = -t_{26}^3.$$

Thus $y_{20} \ast t_{38}$ is a non zero primitive indecomposable element and we redefine $t_{58}$ as $y_{20} \ast t_{38}$. Hence

$$y_{20} \ast t_{38} = t_{58},$$
$$y_{20} \ast t_{58} = -t_{26}^3.$$ 

By applying $\mathcal{P}_k^3$ to (22), we have

$$y_{8} \ast t_{58} = \mathcal{P}_k^3(y_{20} \ast t_{58}) = -\mathcal{P}_k^3(t_{26}^3) = -\varepsilon t_{22}^3.$$

We obtain also

$$y_{20} \ast t_{22} = \varepsilon \mathcal{P}_k^1(y_{20} \ast t_{26}) = -\mathcal{P}_k^1 t_{46} = -t_{14}^3$$

by applying $\mathcal{P}_k^1$ to (8).

Since $t_{34}$ is primitive, we can set $y_{20} \ast t_{34} = \rho_{418}^3 (\rho_4 \in \mathbb{Z}/3)$. Operating $\mathcal{P}_k^3$ to the both sides of this equation, $\rho_4 \varepsilon t_{14}^3$ is computed as follows:

$$\rho_4 \varepsilon t_{14}^3 = \rho_{4} \mathcal{P}_k^3(t_{18}^3) = \mathcal{P}_k^3(y_{20} \ast t_{34}) = y_{8} \ast t_{34} + y_{20} \ast t_{22} = t_{14}^3.$$

So $y_{20} \ast t_{34} = \varepsilon t_{18}^3$ is shown. Now $a_d \ast$ is determined except for $y_{8} \ast t_{16}$.

Finally we operate $\mathcal{P}_k^1$ to (21) and $\mathcal{P}_k^2$ to (22) and see

$$\mathcal{P}_k^1 t_{58} = \mathcal{P}_k^1(y_{20} \ast t_{38}) = y_{20} \ast (\mathcal{P}_k^1 t_{38}) = \varepsilon y_{20} \ast t_{34} = t_{18}^3.$$
These equations imply that
\[ y_{20} \star (\overline{P}_* t_{58}) = \overline{P}_* (y_{20} \star t_{58}) = - \overline{P}_* (t_{26}^3) = - t_{14}^3. \]

This completes the proof of Theorem 1.

References