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Recurrence and conservativeness of symmetric diffusion processes by Girsanov transformations

Dedicated to Professor Hiroshi Kunita on the occasion of his sixtieth birthday

By

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1. Introduction

Recurrence and conservativeness of symmetric Markov processes have been studied by several people. Firstly K. Ichihara [13], M. Fukushima [8] gave a recurrence criterion for symmetric diffusion processes on $\boldsymbol{R}^d$. K. Ichihara [16] also gave a conservativeness test for such processes. Y. Oshima [24] extended their criteria and gave the general criteria of recurrence and conservativeness for symmetric Markov processes. M. Takeda [32] also gave the conservativeness test for diffusion processes on $\boldsymbol{R}^d$, which is sharper than [16], [24], by using the Lyons-Zheng decomposition. On the other hand, K. Ichihara [14], [15], M. P. Gaffney [10], S. Y. Cheng and S. T. Yau [2], A. A. Grigor'yan [11], [12], and M. Takeda [33] gave the criteria for Brownian motions on Riemannian manifolds. K. Th. Sturm [30] extended their works for strong local regular Dirichlet forms by using the Carathéodory (intrinsic) metric. He assumed the relative compactness of balls by this metric (see also [31]). H. Ôkura [23] also gave the recurrence criteria for regular Dirichlet forms by using the capacitary inequality. He assumed the local integrability of metric by jumping measure with the relative compactness of balls. H. Kaneko [17], [18] extended the results of [23], [30]. He used a class of exhaustion functions instead of the Carathéodory metric. In the case of regular Dirichlet forms on locally compact state space, M. Takeda [35] gave the conservativeness test for symmetric diffusion processes transformed by supermartingale multiplicative functionals by using the Carathéodory metric with the relative compactness of balls. Also Y. Oshima [25], Y. Oshima and K. Th. Sturm [26] gave a criterion of conservativeness for the time-dependent Dirichlet forms. In this paper, we will give the recurrence and conservativeness criteria for symmetric diffusion processes on a separable

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metric space transformed by supermartingale multiplicative functionals. We also use the Carathéodory metric. Our metric is slightly different from what was used in [30], [31], but is very related to the metric used in [4], [5], [27]. To show the results, it is essential that the cut-off function of this metric belongs to the domain of Dirichlet forms. In the case of C_0-regular Dirichlet forms on locally compact state space, C_0-regularity assures the above argument. In the case of quasi-regular Dirichlet forms on separable metric spaces, we can not carry out this procedure, since any balls are not necessarily relatively compact and the domain of forms do not necessarily contain continuous functions with compact support. Instead of the relative compactness of balls, we assume the finiteness of 1-capacity of balls. Then we have that the cut-off function is in the domain of forms by using the characterization of the domain of forms associated with the transformed processes due to P. J. Fitzsimmons [6] and the ideal property of Dirichlet space in the reflected Dirichlet space (see Theorem 3.1 and Theorem 3.2). So we can get the criteria for recurrence and conservativeness. Our criteria are slightly sharper than M. Takeda [35], but are indicated by K. Th. Sturm [30] in the framework of C_0-regular Dirichlet forms.

The organization of this paper is as follows. In Section 2, we review the basic property of quasi-regular Dirichlet forms. In Section 3, we present the transformations by supermartingale multiplicative functionals and the characterization of the associated Dirichlet space. In Section 4, we investigate the Carathéodory metric of quasi-regular strong local Dirichlet forms and show that the cut-off function of metric belongs to the domain of forms under the finiteness of 1-capacity of balls (see condition (**A**), (**B**)). In Section 5, we give the criteria for recurrence and conservativeness and prove them under (**A**), (**B**) and the finiteness of local energy of φ (see condition (**E**)). In the last section, we collect some examples.

2. Review of quasi regular Dirichlet form

Let X be a separable metric space and m a σ-finite Borel measure on X with full topological support. Consider a symmetric Dirichlet space $(\mathscr{E}, \mathscr{F})$ on $L^2(X; m)$, namely $\mathscr{F}$ is a dense subspace of $L^2(X; m)$ and $u^\natural = 0 \vee u \wedge 1 \in \mathscr{F}$ if $u \in \mathscr{F}$ and $\mathscr{E}$ is a symmetric non-negative definite bilinear form on $\mathscr{F} \times \mathscr{F}$ satisfying $\mathscr{E}(u^\natural, u^\natural) \leq \mathscr{E}(u, u)$ for $u \in \mathscr{F}$, and $\mathscr{F}$ is complete with respect to $\mathscr{E}_1^{1/2}$-norm. Here $\mathscr{E}_1(u, v) = \mathscr{E}(u, v) + \int_X uvdm$ for $u, v \in \mathscr{F}$. For a closed subset F of X, we set $\mathscr{F}_F = \{u \in \mathscr{F}: u = 0\ m\text{-a. e. on } X - F\}$. An increasing sequence $\{F_n\}_{n \in N}$ of closed subset of X is said to be an $\mathscr{E}$-nest or generalized nest if $\bigcup_{n=1}^{\infty} \mathscr{F}_{F_n}$ is $\mathscr{E}_1^{1/2}$-dense in $\mathscr{F}$. A. subset N of X is said to be $\mathscr{E}$-polar or $\mathscr{E}$-exceptional if there exists an $\mathscr{E}$-nest $\{F_n\}_{n \in N}$ such that $N \subset \bigcap_{n=1}^{\infty} (X - F_n)$. A statement $P = P(x)$ depending on $x \in X$ is said to be "P $\mathscr{E}$-q.e." if there exists an $\mathcal{E}$-polar set N such that $P(x)$ holds for $x \in X - N$. A function u is said to be $\mathcal{E}$-quasi-continuous if there exists an $\mathcal{E}$-nest $\{F_n\}_{n\in N}$ such that $u|_{F_n}$ is continuous on F_n for each $n \in \boldsymbol{N}$. A subset E of X is said to be $\mathcal{E}$-quasi-open if there exists an $\mathcal{E}$-nest $\{F_n\}_{n\in N}$ such that $E \cap F_n$ is open with respect to the relative topology on F_n for each $n \in \boldsymbol{N}$. $\mathcal{E}$-quasi-closedness can be defined as the dual notion. For two subsets A, B of X, we write $A = B$ $\mathcal{E}$-q.e. if $I_A = I_B$ $\mathcal{E}$-q.e. If a function u has an $\mathcal{E}$-quasi-continuous m-version, we denote it by $\tilde{u}$. We prepare three conditions which are called the conditions of quasi-regularity of $(\mathcal{E}, \mathcal{F})$ as follows:

- **(QR1)** There exists an $\mathcal{E}$-nest of compact sets.
- **(QR2)** There exists an $\mathcal{E}_1^{1/2}$-dense subset of $\mathcal{F}$ whose elements have $\mathcal{E}$-quasi-continuous m-versions.
- **(QR3)** There exist an $\mathcal{E}$-polar set $N \subset X$ and $u_n \in \mathcal{F}$, $n \in \boldsymbol{N}$ having $\mathcal{E}$-quasi-continuous m-versions $\tilde{u}_n$, $n \in \boldsymbol{N}$ such that $\{\tilde{u}_n\}_{n\in N}$ separates the points of $X - N$.

Assume that $(\mathcal{E}, \mathcal{F})$ is quasi-regular, namely **(QR1)**, **(QR2)**, **(QR3)** hold. Then there exists an m-equivalence class of special standard processes $\mathbf{M}/\sim$ properly associated with $(\mathcal{E}, \mathcal{F})$. Fix a special standard process $\mathbf{M} = (\Omega, X_t, \zeta, P_x)$ properly associated with $(\mathcal{E}, \mathcal{F})$. Properly association means that $x \mapsto \int_\Omega f(X_t(\omega))\ P_x(d\omega)$ is an $\mathcal{E}$-quasi-continuous m-version of $T_t f$ for $f \in \mathcal{B}_+(X) \cap L^2(X; m)$. Under the quasi-regularity of Dirichlet space $(\mathcal{E}, \mathcal{F})$, there exists another Dirichlet space $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ on $L^2(\hat{X}, \hat{m})$ with a locally compact separable metric space $\hat{X}$ and a positive Radon measure $\hat{m}$ on $\hat{X}$, which is C_0-regular and $\mathcal{E}$-quasi-homeomorphic to $(\mathcal{E}, \mathcal{F})$. Precisely to say, there exists an $\mathcal{E}$-nest $\{K_n\}$ of compact sets, and a locally compact separable metric space $\hat{X}$, and a map i: $Y = \bigcup_{n=1}^{\infty} K_n \to \hat{X}$ such that $i|_{K_n}$ is a homeomorphism and the image $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ of $(\mathcal{E}, \mathcal{F})$ for $\hat{m} = m \circ i^{-1}$ is a C_0-regular Dirichlet form on $L^2(\hat{X}; \hat{m})$ satisfying that $\{i(K_n)\}$ is an $\hat{\mathcal{E}}$-nest. The definition of the image $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ of $(\mathcal{E}, \mathcal{F})$ is as follows: Define an isometry i^*: $L^2(\hat{X}; \hat{m}) \to L^2(X; m)$ by setting $i^*(u^\#)$ to be the m-class represented by $\check{u} \circ i$ for any $\mathcal{B}(\hat{X})$ -measurable $\hat{m}$-version $\check{u}$ of $u^\# \in L^2(\hat{X}; \hat{m})$. $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ is defined by $\hat{\mathcal{F}} = \{u^\# \in L^2(\hat{X}; \hat{m}) \,|\, i^*(u^\#) \in \mathcal{F}\}$ and $\hat{\mathcal{E}}(u^\#, v^\#) = \mathcal{E}(i^*(u^\#), i^*(v^\#))$ for $u^\#, v^\# \in \hat{\mathcal{F}}$ (cf. Chapter VI Theorem 1.2 in [22]). For a function u on X, we set $u^\#$ by $u^\#(y) = u(x)$ if $y = i(x)$ and otherwise $u^\#(y) = 0$. Then representations u, v of m-classes in $\mathcal{F}$ satisfy $u^\#, v^\# \in \hat{\mathcal{F}}$, $i^*(u^\#) = u$, $i^*(v^\#) = v$ and $\mathcal{E}(u, v) = \hat{\mathcal{E}}(u^\#, v^\#)$. Hence we can transfer the results of [9] to quasi-regular Dirichlet forms. Such procedure is called the "*transfer method*". The well-known application of the transfer method is the Fukushima decomposition as follows: for $u \in \mathcal{F}$,

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^{[u]} + N_t^{[u]}, \quad P_x\text{-a.s. for } \mathcal{E}\text{-q.e. } x \in X. \qquad (1)$$

Here $M_t^{[u]}$ is the square integrable martingale additive functional whose quadratic variation $<M^{[u]}>_t$ is the positive continuous additive functional admitting $\mathcal{E}$-exceptional set and $N_t^{[u]}$ is the continuous additive functional of zero energy (cf. §5.2 in [9]). Also the following Lyons-Zheng decomposition holds (cf. §5.7 in [9] and [6]): for $u\in\mathcal{F}$, $T>0$,

$$\tilde{u}(X_t)-\tilde{u}(X_0)=\frac{1}{2}M_t^{[u]}+\frac{1}{2}(M_{T-t}^{[u]}(r_T)-M_T^{[u]}(r^T)),\quad 0\leq t\leq T<\zeta,\quad P_m\text{-a.s.} \tag{2}$$

Here r_T is the reverse oprator on $\Omega=D([0,\infty)\rightarrow X)$:

$$r_T(\omega)(t)=\begin{cases}\omega((T-t)-) & \text{if } t\leq T\\ \omega(0) & \text{if } t>T.\end{cases}$$

For an $\mathcal{E}$-quasi-open set E and a set $\mathcal{A}$ of m-a.e. defined functions, we introduce a class of increasing sequence of $\mathcal{E}$-quasi-open sets and the local space on E of $\mathcal{A}$ denoted by Ξ_E, $\dot{\mathcal{A}}_{Eloc}$ respectively: Let $L^0(E;m)$ be the all m-a.e. defined functions on E.

$$\Xi_E=\{\{G_n\}: G_n \text{ is } \mathcal{E}\text{-quasi-open for all } n,\ G_n\subset G_{n+1}\ \ \mathcal{E}\text{-q.e. and } E=\bigcup_{n=1}^{\infty}G_n\ \ \mathcal{E}\text{-q.e.}\}.$$

$$\dot{\mathcal{A}}_{Eloc}=\{u\in L^0(E;m): \text{there exist } \{E_n\}\in\Xi_E \text{ and } u_n\in\mathcal{A} \text{ such that } u=u_n\ \ m\text{-a.e. on } E_n\}.$$

When $X-E$ is $\mathcal{E}$-polar, we simply write Ξ, $\dot{\mathcal{A}}_{loc}$ instead of Ξ_E, $\dot{\mathcal{A}}_{Eloc}$. We also use the notation $\mathcal{A}_b=\mathcal{A}\cap L^\infty(X;m)$. Another application of the transfer method is the Beurling-Deny type decomposition: there exist σ-finite signed Borel measures $\mu_{<u,v>}^{(c)}$ for u, $v\in\mathcal{F}$, a σ-finite Borel measure k on X, and σ-finite Borel measure J on $X\times X-d$ with $\mu_{<u,v>}^{(c)}(dx)$, $k(dx)$, $J(X,dx)$ charging no $\mathcal{E}$-polar set such that

$$\mathcal{E}(u,v)=\frac{1}{2}\int_X\mu_{<u,v>}^{(c)}+\int\int_{X\times X}[\tilde{u}][\tilde{v}]dJ+\int_X\tilde{u}\tilde{v}dk,\quad u,v\in\mathcal{F}. \tag{3}$$

Here $[u](x,y)=u(x)-u(y)$. Indeed, let $\hat{\mu}_{<u',v'>}^{(c)}$, $\hat{J}$, $\hat{k}$ be the measures appeared in the Beurling-Deny decomposition for $(\hat{\mathcal{E}},\hat{\mathcal{F}})$. It suffices to put $\mu_{<u,v>}^{(c)}(A)=\hat{\mu}_{<u',v'>}^{(c)}(i(A\cap Y))$, $k(A)=\hat{k}(i(A\cap Y))$ and $J(A\times B)=\hat{J}(i(A\cap Y)\times i(B\cap Y))$ for A, $B\in\mathcal{B}(X)$. In this paper, we will not use the characterization of $\mu_{<u,v>}^{(c)}$, J, K which establish the uniqueness of decomposition (3) (see [21] which gives the characterization for the uniqueness). If **M** is a diffusion process with no killing inside, namely $J=k=0$, equations (1) and (2) are extended to $u\in\dot{\mathcal{F}}_{loc}$. At this stage, $M_t^{[u]}$ is a local square integrable continuous martingale additive functional and $N_t^{[u]}$ is a continuous additive functional locally of zero energy (cf. §5.5 in [9]). Condition (**QR2**) implies that every element $u\in\mathcal{F}$ has an $\mathcal{E}$-quasi-continuous ($\mathcal{E}$-q.e. finite) m-version $\tilde{u}$([22]) and every element $u\in\dot{\mathcal{F}}_{loc}$ has so ([21]).

For an $\mathcal{E}$-quasi-open set E, we put $\mathcal{F}_E=\{u\in\mathcal{F}: \tilde{u}=0\ \mathcal{E}\text{-q.e. on } X-E\}$ and $\mathcal{E}_E(u, v)=\mathcal{E}(u, v)$ for $u, v\in\mathcal{F}_E$. Then $(\mathcal{E}_E, \mathcal{F}_E)$ is a quasi-regular Dirichlet space on $L^2(E; m)$ if $(\mathcal{E}, \mathcal{F})$ is so ([27]). In [21], we showed $\dot{\mathcal{F}}_{Eloc}=(\dot{\mathcal{F}}_E)_{loc}$. Here $(\dot{\mathcal{F}}_E)_{loc}$ is the local space of $\mathcal{F}_E$ on E.

We collect several properties of the energy measure of continuous part $\mu^{(c)}_{<u,v>}$ for $u, v\in\mathcal{F}$. The proof can be seen in [21].

Lemma 2.1. *The energy measure of continuous part $\mu^{(c)}_{<u,v>}$ satisfies the following properties.*

(Γ1) (*Schwarz inequality*) *For $u, v\in\mathcal{F}$,*

$$\left|\int_X f(x)\,g(x)\,\mu^{(c)}_{<u,v>}(dx)\right|\leq\sqrt{\textstyle\int_X f^2(x)\,\mu^{(c)}_{<u,v>}(dx)}\ \sqrt{\textstyle\int_X g^2(x)\,\mu^{(c)}_{<u>}(dx)}\,.$$

(Γ2) (*Markovian property*) *For any $u\in\mathcal{F}$ and $r>0$, $\mu^{(c)}_{<0\vee u\wedge r>}\leq\mu^{(c)}_{<u>}$.*

(Γ3) *For any $\mathbf{u}=(u_1, \cdots, u_k)$, $\mathbf{v}=(v_1, \cdots, v_l)$ with $u_i, v_j\in\mathcal{F}_b(1\leq i\leq k, 1\leq j\leq l)$ and $F\in C^1(\boldsymbol{R}^k)$, $G\in C^1(\boldsymbol{R}^l)$ with $F(0)=G(0)=0$,*

$$\mu^{(c)}_{<F(\mathbf{u}),G(\mathbf{v})>}=\sum_{i=1}^{k}\sum_{j=1}^{l}\frac{\partial F}{\partial x_i}(\tilde{\mathbf{u}})\frac{\partial G}{\partial x_j}(\tilde{\mathbf{v}})\,\mu^{(c)}_{<u_i,v_j>}.$$

(Γ4) (*Derivation property*) *For $u, v, w\in\mathcal{F}_b$, $\mu^{(c)}_{<uv,w>}=\tilde{u}\mu^{(c)}_{<u,w>}+\tilde{v}\mu^{(c)}_{<u,w>}$.*

(Γ5) *For $u, v\in\mathcal{F}$, $\mu^{(c)}_{<u>}\leq m$, $\mu^{(c)}_{<v>}\leq m$ implies $\mu^{(c)}_{<u\vee v>}\leq m$.*

(Γ6) (*Strong local property*) *For any $\mathcal{E}$-quasi-open set E and $u\in\mathcal{F}$ with $u=$constant m-a.e. on E, $I_E\,\mu^{(c)}_{<u>}=0$, In particular $I_{X-E}\mu^{(c)}_{<u>}=0$ for $u\in\mathcal{F}_E$.*

Next lemma is also shown in [21].

Lemma 2.2. (*i*) *For $u\in\mathcal{F}_{loc}$, we can define a unique σ-finite Borel measure $\mu^{(c)}_{<u>}$ on X such taht $I_{E_n}\mu^{(c)}_{<u>}=I_{E_n}\mu^{(c)}_{<u_n>}$ for $\{E_n\}\in\Xi$, $u_n\in\mathcal{F}$ satisfying $u=u_n$ m-a. e. on E_n.* (*ii*) *$\mu^{(c)}_{<u>}$, $u\in\dot{\mathcal{F}}_{loc}$ charges no $\mathcal{E}$-polar set.* (*iii*) *All assertions in Lemma 2.1 hold with the functions in $\dot{\mathcal{F}}_{loc}$ by replacing the functions in $\mathcal{F}$ except the latter assertion in (Γ6). The latter assertion is replaced by that $u\in\dot{\mathcal{F}}_{loc}$ with $\tilde{u}=0$ $\mathcal{E}$-q. e. on E^c satisfies $I_{E^c}\mu^{(c)}_{<u>}$.*

3. Girsanov transformation

Let X, m be as in Section 2 and assume that $(\mathcal{E}, \mathcal{F})$ is a quasi-regular Dirichlet space on $L^2(X; m)$. Owing to the quasi-regularity condition (**QR1**), we may treat X as a Lusinian space, namely X is homeomorphic to a Borel subset of a compact metric space. Indeed, let $\{K_l\}$ be an $\mathcal{E}$-nest of compact sets and put $Y=\bigcup_{l=1}^{\infty}K_l$, and $\bar{Y}$ is the completion of Y. Since K_l is a Borel subset of $\bar{Y}$, so Y is a Borel subset of $\bar{Y}$, hence it is a Lusinian space. We already know $m(X-Y)=0$. Thus we may take Y as the state space. Let $\mathbf{M}=(\Omega, X_t, \mathcal{F}_\infty, \mathcal{F}_t, \zeta, P_x)$ be the associated special standard process. We assume $\mathbf{M}$ is a diffusion process with no killing inside, namely the sample path of $\mathbf{M}$ is continuous until the life times ζ of $\mathbf{M}$ and $E_{hm}[f(X_{\zeta-})\ ;\ \zeta\le t]=0$ for any non-negative Borel functions f, h. At this stage the measures J and k appearing in the Beurling-Deny decomposition (3) vanish. Throughout this paper, we fix $\varphi\in\dot{\mathcal{F}}_{loc}$ with $\varphi\ge 0$. Note that every $\mathcal{E}$-quasi-open set E is an $\mathcal{E}$-q.e. finely open set, namely there exists a finely open Borel set $\tilde{E}$ such that $E=\tilde{E}$ $\mathcal{E}$-q.e. Let X_φ be the finely open Borel $\mathcal{E}$-q.e. version of $\{x\in X: 0<\tilde{\varphi}<\infty\}$. Then by [21], we have $\log\varphi\in\dot{\mathcal{F}}_{X_\varphi loc}=(\dot{\mathcal{F}}_{X_\varphi})_{loc}$. Hence by the Fukushima decomposition, we can construct the functionals $M_t^{[\log\varphi]}$ and $N_t^{[\log\varphi]}$ defined on $t<\tau_{X_\varphi}$ such that

$$\log\tilde{\varphi}(X_t)-\log\tilde{\varphi}(X_0)=M_t^{[\log\varphi]}+N_t^{[\log\varphi]},\ t<\tau_{X_\varphi}\ P_x\text{-a.s. } \mathcal{E}\text{-q.e. } x\in X. \tag{4}$$

Here $\tau_{X_\varphi}=\inf\{t\ge 0: X_t\notin X_\varphi\}$ is the first exit time of X_φ. We define a multiplicative functional of $\mathbf{M}$ by

$$L_t^{[\varphi]}=\exp[M_t^{[\log\varphi]}-\frac{1}{2}<M^{[\log\varphi]}>_t]I_{\{t<\tau X_\varphi\}}. \tag{5}$$

It is easy to see that $L_t^{[\varphi]}$ is a supermartingale multiplicative functional. Indeed, let $\{E_n\}\in\Xi_{X_\varphi}$ and $v_n\in\mathcal{F}$ with $\log\varphi=v_n$ m-a.e. on E_n. Then by Itô formula $L_t^{[\varphi]}=1+\int_0^t L_s^{[\varphi]}dM_s^{[v_n]}$, P_x-a.s. on $\{t<\tau_{\tilde{E}_n}]$ $\mathcal{E}$-q.e. $x\in X_\varphi$. Let $L_t^n=1+\int_0^t L_s^{[\varphi]}\,dM_s^{[v_n]}$. Then $E_x[L_t^n|\mathcal{F}_s]\le L_s^n$, P_x-a.s. $\mathcal{E}$-q.e. $x\in X_\varphi$ for s, $t\in[0,\infty)$ with $s<t$. Hence $E_x[L_t^{[\varphi]}|\mathcal{F}_s]\le\liminf_{n\to\infty}E_x[L_t^n I_{\{t<\tau_{X_\varphi}\}}|\mathcal{F}_s]\le\liminf_{n\to\infty}I_{\{s<\tau_{X_\varphi}\}}E_x[L_t^n|\mathcal{F}_s]\le\liminf_{n\to\infty}I_{\{s<\tau_{X_\varphi}\}}L_s^n=L_s^{[\varphi]}$, P_x-a.s. $\mathcal{E}$-q.e. $x\in X_\varphi$. Here $\tilde{E}_n$ is a finely open Borel $\mathcal{E}$-q.e. version of E_n. Let $\mathbf{M}^\varphi=(\Omega, X_t, \tau_{X_\varphi}, P_x^\varphi)$ be the part process on X_φ of the transformed process by this supermartingale multiplicative functional. It is well-known that $\mathbf{M}^\varphi$ is a right process in the sense of [29] and satisfies

$$E_x^\varphi[FI_{\{t<\tau_{X_\varphi}\}}]=E_x[L_t^{[\varphi]}F],\ F\in\mathcal{F}_t,\ x\in X_\varphi. \tag{6}$$

Owing to (6), we see $P_x^\varphi|_{\mathcal{F}_t\cap\{t<\tau_{X_\varphi}\}}$ and $P_x|_{\mathcal{F}_t\cap\{t<\tau_{X_\varphi}\}}$ are absolutely continuous with respect to each other for $\mathcal{E}$-q.e. $x\in X_\varphi$ by $L_t^{[\varphi]}>0$, P_x-a.s. on $\{t<\tau_{X_\varphi}\}$ $\mathcal{E}$-q.e. $x\in X_\varphi$. We then see that $\mathbf{M}^\varphi$ has the continuity of sample paths until the life time τ_{X_φ}, P_x^φ-a.s. $\mathcal{E}$-a.e. $x\in X_\varphi$. Indeed,

$$P_x^\varphi(\{X_s\ne X_{s-}\text{ for some } s\le r\}\cap\{r<\tau_{X_\varphi}\})=0$$

for any $r>0$. Hence

$$\mathrm{P}_x^\varphi(X_s\ne X_{s-}\text{ for some } s<\tau_{X_\varphi})\le\sum_{r\in Q\cap(0,\infty)}P_x^\varphi(\{X_s\ne X_{s-}\text{ for some } s\le r\}\cap\{r<\tau_{X_\varphi}\})=0.$$

We also see the $\varphi^2 m$-symmetry of $\mathbf{M}^\varphi$ by $\{t<\tau_{X_\varphi}(r_t)\}=\{t<\tau_{X_\varphi}\}$ and the similar calculation in the proof of Lemma 6.3.5 in [9]. Hence we can consider a Dirichlet space $(\mathcal{E}^\varphi, \mathcal{F}^\varphi)$ on $L^2(X_\varphi; \varphi^2 m)$ associated with $\mathbf{M}^\varphi$. However $\mathbf{M}^\varphi$ is not necessarily Borel right process, we can treat the several results of P. J. Fitzsimmons [6]. Indeed, since we treat X as a Lusinian space, so X_φ is Lusinian. According to the general results in P. J. Fitzsimmons [7], every symmetric right process on a Lusinian space in the sense of [29] generates a quasi-regular Dirichlet form, hence $(\mathcal{E}^\varphi, \mathcal{F}^\varphi)$ is quasi-regular, so $\mathbf{M}^\varphi$ has a version of Borel right process. Therefore, we may treat $\mathbf{M}^\varphi$ as a Borel right diffusion process. Since $P_x^\varphi|_{\mathcal{F}_t\cap\{t<\tau_{X_\varphi}\}}$ and $P_x|_{\mathcal{F}_t\cap\{t<\tau_{X_\varphi}\}}$ are absolutely continuous with respect to each other for $\mathcal{E}$-q.e. $x\in X_\varphi$, the fine topology of $\mathbf{M}_{X_\varphi}$($=$the part process of $\mathbf{M}$ on X_φ), coincides with the one of $\mathbf{M}^\varphi$ by deleting some $\mathcal{E}_{X_\varphi}$-polar and $\mathcal{E}^\varphi$-polar set. Thus we can apply the results of [6].

Theorem 3.1. (*Theorem* 4.11, *and Corollary* 4.12 *in* [6]). (*i*) $\dot{\mathcal{F}}^\varphi_{loc}=\dot{\mathcal{F}}_{X_\varphi,loc}$ *and the energy measure* $\mu^\varphi_{<u,v>}$ *of* u, $v\in\dot{\mathcal{F}}^\varphi_{loc}$ *is given by* $\mu^\varphi_{<u,v>}=\tilde{\varphi}^2\mu^{(c)}_{<u,v>}$.
(*ii*) $\mathcal{F}^\varphi\subset\{u\in\dot{\mathcal{F}}_{X_\varphi,loc}\cap L^2(X_\varphi; \varphi^2 m):\int_{X_\varphi}\tilde{\varphi}^2(x)\,\mu^{(c)}_{<u>}(dx)<\infty\}$ *and* $\mathcal{E}^\varphi(u, v)=\frac{1}{2}\int_{X_\varphi}\tilde{\varphi}^2(x)\,\mu^{(c)}_{<u,v>}(dx)$ *for* u, v, $\in\mathcal{F}^\varphi$. *Further if* $(\mathcal{E}^\varphi, \mathcal{F}^\varphi)$ *is conservative, the above inclusion is equality. In particular, if* $\varphi\in L^2(X; m)$ *and* $(\mathcal{E}^\varphi, \mathcal{F}^\varphi)$ *is conservative, then* $(\mathcal{E}^\varphi, \mathcal{F}^\varphi)$ *is recurrent.*

In particular, Theorem 3.1 (ii) implies that $\mathbf{M}^\varphi$ has no killing inside. An easy consequence of Theorem 3.1 in the case of $\varphi\equiv 1$ is as follows:

Corollary 3.1. $\mathcal{F}\subset\{u\in\dot{\mathcal{F}}_{loc}\cap L^2(X; m) : \int_X\mu^{(c)}_{<u>}(dx)<\infty\}$ *and* $\mathcal{E}(u, v)=\frac{1}{2}\int_X\mu^{(c)}_{<u,u>}(dx)$ *for* u, $v\in\mathcal{F}$. *Further if* $(\mathcal{E}, \mathcal{F})$ *is conservative, the above inclusion is equality. In particular, if* $1\in L^2(X; m)$ *and* $(\mathcal{E}, \mathcal{F})$ *is conservative, then* $(\mathcal{E}, \mathcal{F})$ *is recurrent.*

A simple sufficient condition ensuring $u\in\mathcal{F}$ for $u\in\dot{\mathcal{F}}_{loc}\cap L^2(X; m)$ with $\int_X\mu^{(c)}_{<u>}(dx)<\infty$ is as follows: Denote by Cap_f the fine 1-capacity of $(\mathcal{E}, \mathcal{F})$, namely, for a finely open Borel set G,

$$\mathrm{Cap}_f(G)=\inf\{\mathcal{E}_1(u, u) : u\in\mathcal{F} \quad u\geq 1 \quad m\text{-a.e. on } G\},$$

and for any set B,

$$\mathrm{Cap}_f(B)=\inf\{\mathrm{Cap}_f(G): B\subset G,\ G \text{ is finely open Borel.}\}.$$

If G is open, $\mathrm{Cap}_f(G)$ coincides with the usual 1-capacity $\mathrm{Cap}(G)$.

Theorem 3.2. *Let* G *be a finely open Borel set with finite fine* 1-*capacity and* $u\in\dot{\mathcal{F}}_{loc}\cap L^2(X, m)$ *with* $\int_X\mu^{(c)}_{<u>}(dx)<\infty$. Suppose that $\tilde{u}=0$ $\mathcal{E}$-*q.e. on* $X-G$. *Then* $u\in\mathcal{F}$.

Proof. We let $\mathcal{F}^{ref}=\{u\in\dot{\mathcal{F}}_{loc}\cap L^2(X; m): \int\mu^{(c)}_{<u>}(dx)<\infty\}$. Then $\mathcal{F}_b$

is an ideal in $\mathcal{F}_b^{ref}$, namely $\dot{\mathcal{F}}_b^{ref}\cdot\mathcal{F}_b\subset\mathcal{F}_b$. First we show this ideal property. Let $u\in\dot{\mathcal{F}}_b^{ref}$, $v\in\mathcal{F}_b$ with $v=R_1g$, $g\in L^2(X;m)\cap\mathcal{B}_b(X)$. Recall that every $\mathcal{E}$-quasi-open set E has finely open Borel $\mathcal{E}$-q.e. version, so we may assume E is finely open Borel. Since $u\in\dot{\mathcal{F}}_{locb}$, there exists $\{E_n\}\in\Xi$ and $u_n\in\mathcal{F}_b$ such that $u=u_n$ m-a.e. on E_n. Put $v_n=R_1^{E_n}g=E.[\int_0^{\sigma_{X-E_n}}e^{-s}g(X_s)\,ds]$. Then we have $uv_n\in\mathcal{F}_b$ and $\mathcal{E}_1(uv_n,uv_n)\leq\int_X\tilde{u}^2\mu^{(c)}_{<v_n>}+\|g\|^2_\infty(\mu^{(c)}_{<u>}(X)+\int_X u^2dm)$. Since $\mathcal{E}_1(v-v_n,v-u_n)\to 0\,(n\to\infty)$, we have $\{uv_n\}$ is an $\mathcal{E}_1^{1/2}$-bounded sequence, hence $uv\in\mathcal{F}$ by the Banach-Saks theorem. For general $v\in\mathcal{F}_b$, v is approximated by a sequence of the type R_1g, $g\in L^2(X;m)\cap\mathcal{B}_b(X)$. Thus we get $uv\in\mathcal{F}_b$ by the same argument as above. Next we show the assertion. Since $\mathrm{Cap}_f(G)<\infty$, there exists $e_G\in\mathcal{F}_b$ with $\tilde{e}_G=1$ $\mathcal{E}$-q.e. on G. When u is bounded, we see $u=ue_G\in\dot{\mathcal{F}}_b^{ref}\cdot\mathcal{F}_b\subset\mathcal{F}_b$. For general $u\in\dot{\mathcal{F}}^{ref}$ with $\tilde{u}=0$, $\mathcal{E}$-q.e. on $X-G$, $u^{(n)}=(-n)\vee u\wedge n\in\mathcal{F}$ is $\mathcal{E}_1^{1/2}$-bounded. So the Banach-Saks theorem tells us $u\in\mathcal{F}$.

4. Carathéodory metric (Intrinsic metric)

Let X, m, $(\mathcal{E},\mathcal{F})$ as in Section 3. We assume that J and k, the measures appearing in (3), vanish. Fix a subset $\mathcal{C}$ of $\mathcal{F}\cap C_b(X)$ and let $\mathcal{D}=\mathcal{C}\cup-\mathcal{C}$, the symmetrization of $\mathcal{C}$. We consider the following pseudo metric:

$$\rho(x,y)=\sup\{u(x)-u(y):u\in\mathcal{D},\ \mu^{(c)}_{<u>}\leq m\}. \tag{7}$$

Denote by $B_r(p)$, open ball by ρ with radius $r>0$ and center $p\in X$. We consider the next conditions.

(A) : ρ is a metric on X which generates the same topology endowed on X.

(B) : The 1-capacity of $B_r(p)$ with respect to $(\mathcal{E},\mathcal{F})$ is finite for any $r>0$ and $p\in X$.

We consider a 1-excessive function $g\in\mathcal{F}$ with $\tilde{g}>0$ $\mathcal{E}$-q.e. Put $\mathcal{F}^g=\{u\in L^2(X;g^2m):ug\in\mathcal{F}\}$ and $\mathcal{E}^g(u,v)=\mathcal{E}(ug,vg)$ $\mathcal{E}_1^g(u,v)=\mathcal{E}^g(u,v)+\int_X uvg^2dm$ for $u,v\in\mathcal{F}^g$. Then $(\mathcal{E}_1^g,\mathcal{F}^g)$ is a Dirichlet space on $L^2(X;g^2m)$ We then see that $\mathcal{F}_b^g\subset\dot{\mathcal{F}}_{loc}$ (Proposition 2.5 (iv) in [20]). Set $h=G_1f$ with $f\in L^1(X;m)$, $0<f\leq 1$ and $h_n=nh\wedge 1$. Here G_1 is the 1-resolvent on $L^2(X;m)$ associated with $(\mathcal{E},\mathcal{F})$. Then h and h_n satisfy the conditions on g.

Lemma 4.1. $1\in\dot{\mathcal{F}}_{loc}$ *and* $\mu^{(c)}_{<1>}=0$.

Proof. Put $E_n=\{\tilde{h}>\frac{1}{n}\}$. Since $\tilde{h}>0$ $\mathcal{E}$-q.e., $\{E_n\}\in\Xi$. So $1=h_n$ m-a.e. on E_n implies $1\in\dot{\mathcal{F}}_{loc}$. On the other hand, by $(\Gamma 6)$, $I_{E_n}\mu^{(c)}_{<1>}=I_{E_n}\mu^{(c)}_{<h_n>}=0$. Noting that $\mu^{(c)}_{<1>}$ charges no $\mathcal{E}$-polar set, we have $\mu^{(c)}_{<1>}=0$.

Lemma 4.2. *Let* $u_i\in\dot{\mathcal{F}}_{locb}$ *and put* $u=\bigvee_{i=1}^{\infty}u_i$, $v=\bigwedge_{i=1}^{\infty}u_i$. *Suppose that for* *each* $i, n\in\boldsymbol{N}$, $u_i\in\mathscr{F}_b^{h_n}$ *and* $|u_i(x)-u_i(y)|\leq\rho(x, y)$ *with* $\sup_{i\geq 1}\|u_i\|_\infty\leq r$ *and* $\mu^{(c)}_{<u_i>}\leq m\,(i\geq 1)$. *Then* $u, v\in\dot{\mathscr{F}}_{locb}\cap\mathscr{F}_b^{h_n}$ *with* $\|u\|_\infty, \|v\|_\infty\leq r$, $\mu^{(c)}_{<u>}, \mu^{(c)}_{<u>}\leq m$ *and* $|u(x)-u(y)|\leq\rho(x, y)$, $|v(x)-v(y)|\leq\rho(x, y)$.

Proof. The proof is the same as Lemma 3.1 in [20] by using properties (Γ4), (Γ5) in Lemma 2.1 and Lemma 2.2. We only to show $\mu^{(c)}_{<u>}\leq m$. Set $u^{(k)}=\bigvee_{i=1}^{k}u_i$. Then we see $u^{(k)}h_n\in\mathscr{F}$ and $\sup_{k\geq 1}\mathscr{E}(u^{(k)}h_n, u^{(k)}h_n)\leq 2r^2\mathscr{E}(h_n, h_n)+2\int_X h_n^2 dm<\infty$. Hence the Banach-Saks the orem tells us that there exists a subsequence $\{k_j\}$ of k such that the Cesàro means of $u^{(k_j)}h_n$ converge to some $uh_n\in\mathscr{F}$. Consider $A\in\mathscr{B}(X)$ with $A\subset\{\tilde{h}>1/n\}$. Then we get $\mu^{(c)}_{<u>}(A)=\mu^{(c)}_{<uh_n>}(A)=\lim_{l\to\infty}\mu^{(c)}_{<\frac{1}{l}\sum_{j=1}^{l}u^{(k_j)}h_n>}(A)$. Hence we have $\sqrt{\mu^{(c)}_{<u>}(A)}\leq\lim_{l\to\infty}\frac{1}{l}\sum_{j=1}^{l}\sqrt{\mu^{(c)}_{<u^{(k_j)}h_n>}(A)}\sqrt{m(A)}$, which implies $\mu^{(c)}_{<u>}\leq m$.

Denote by ρ_p, the distance function ρ_p: $x\mapsto\rho(p, x)$ and by $\rho_{p,r}$, the cut-off function $\rho_{p,r}$: $x\mapsto(r-\rho(p, x))\vee 0$. Next lemma is a variant showed in [20]:

Lemma 4.3. *Suppose that* (**A**) *holds. Then* $\rho_p\wedge r\in\dot{\mathscr{F}}_{loc}\cap C_b(X)$ *and* $\mu^{(c)}_{<\rho_p\wedge r>}\leq m$ *for any* $r>0$ *and* $p\in X$.

Proof. The proof is similar to the proof of Theorem 1.1 in [20]. We present its proof for completeness. Owing to Lindelöf covering theorem. there exists a countable number of points $y_i=y_i^{(n)}\in X$ such that $X=\bigcup_{n=1}^{\infty}B_{\frac{1}{n}}(y_i)$. For each $i\in\boldsymbol{N}$, we can take $\hat{\phi}_i=\hat{\phi}_i^{(n)}\in\mathscr{D}$ with $\mu^{(c)}_{<\hat{\phi}_i>}\leq m$ and $\hat{\phi}_i(p)-\hat{\phi}_i(y_i)\geq\rho_p(y_i)-\frac{1}{n}$. Since $\hat{\phi}_i$ is the admissible functions in the definition.of the ρ, it also satisfies $\hat{\phi}_i(y)\geq\hat{\phi}_i(p)-\rho_p(y)$ for all $y\in X$ as well as $\hat{\phi}_i(y)\leq\hat{\phi}_i(y_i)+\frac{1}{n}$ for all $y\in B_{\frac{1}{n}}(y_i)$. Together with the triangle inequality $\rho_p(y)\geq\rho_p(y_i)-\frac{1}{n}$ for all $y\in B_{\frac{1}{n}}(y_i)$ the latter yields $\hat{\phi}_i(y)\leq\hat{\phi}_i(p)-\rho_p(y)+\frac{3}{n}$ for all $y\in B_{\frac{1}{n}}(y_i)$. Let $\phi_i: y\mapsto 0\vee(\hat{\phi}_i(p)-\hat{\phi}_i(y))\wedge r$. Then by Lemma 4.1, $\varphi_i=\varphi_i^{(n)p}$ satisfies

$$\phi_i\in\dot{\mathscr{F}}_{loc}\cap\mathscr{F}^{h_n}\cap C_b(X) \text{ with } \mu^{(c)}_{<\varphi_i>}\leq m.$$

$$0\leq\phi_i(y)\leq\rho_p(y)\wedge r \text{ for all } y\in X. \text{ In particulaqr } \|\varphi\|_\infty\leq r.$$

$$\phi_i(y)\geq\left(\rho_p(y)-\frac{3}{n}\right)\wedge r \text{ for all } y\in B_{\frac{1}{n}}(y_i).$$

Put $\phi=\phi^{(n)p}=\bigvee_{i=1}^{\infty}\phi_i^{(n)p}$. Then $\phi\in\dot{\mathscr{F}}_{loc}\cap\mathscr{F}^{h_n}\cap C_b(X)$ with $\|\varphi\|_\infty\leq r$ and $\mu^{(c)}_{<\varphi>}\leq m$ by Lemma 4.2 and $(\rho_p-\frac{3}{n})\wedge r\leq\phi\leq\rho_p\wedge r$. Henece we have

$$\rho_p(y)\wedge r=\lim_{n\to\infty}\phi^{(n)p}(y)=\bigvee_{n=1}^{\infty}\bigwedge_{k=n}^{\infty}\phi^{(k)p}=\bigwedge_{n=1}^{\infty}\bigvee_{k=n}^{\infty}\phi^{(k)p},$$

which implies $\rho_p\wedge r\in\dot{\mathscr{F}}_{loc}\cap\mathscr{F}^{hn}\cap C_b(X)$ with $\mu^{(c)}_{<\rho_p\wedge r>}\leq m$ by Lemma 4.2.

Proposition 4.1. *Suppose that* (**A**) *holds. Then* $\rho_p\in\dot{\mathscr{F}}_{loc}\cap C(X)$ *and* $\mu^{(c)}_{<\rho_p>}\leq m$ *for any* $p\in X$.

Proof. Since $\rho_p=\rho_p\wedge r$ m-a.e. on $B_r(p)$, ρ_p is an element of the local space of $\dot{\mathscr{F}}_{loc}$ on X. By [21], we have $\rho_p\in\dot{\mathscr{F}}_{loc}$. Owing to a strong local property of $\mu^{(c)}_{<u>}$ for $u\in\dot{\mathscr{F}}_{loc}$, we have $I_{B_r(p)}\,\mu^{(c)}_{<\rho_p>}=I_{B_r(p)}\,\mu^{(c)}_{<\rho_p\wedge r>}\leq m$, hence $\mu^{(c)}_{<\rho_p>}\leq m$.

Theorem 4.1. *Suppose that* (**A**) *and* (**B**) *hold. Then* $\rho_{p,r}\in\mathscr{F}\cap C_b(X)$ *and* $\mu^{(c)}_{<\rho_{p,r}}\leq m$.

Proof. Since $1\in\dot{\mathscr{F}}_{loc}$ and $\mu^{(c)}_{<1>}=0$, we see that $\rho_{p,r}\in\dot{\mathscr{F}}_{loc}$ and $\mu^{(c)}_{<\rho_{p,r}>}\leq m$. Under (**B**), $m(B_r(p))<\infty$. Hence $\rho_{p,r}$ satisfies the condition in Theorem 3.2.

5. Presentation of theorem and its proof

Let X, m, $(\mathscr{E},\mathscr{F})$, $\mathbf{M}$, φ, $\mathbf{M}^{\varphi}$ be as in Section 3 and ρ, $B_r(p)$ defined as in Section 4. $\mathbf{M}^{\varphi}$, is said to be recurrent if the associated Dirichlet space $(\mathscr{E}^{\varphi},\mathscr{F}^{\varphi})$ is recurrent in the sense of [9] (p.48 in [9]). We denote $v_p(r)=v_{p,\varphi}(r)=\int_{B_r(p)}\varphi^2dm$ and $r_0=v_p^{-1}(e)$. We prepare next conditions.

(**E**): For any $r>0$, $p\in X$, $\mu^{(c)}_{<\varphi>}(B_r(p))<\infty$ and $\varphi\in L^2(B_r(p);m)$

(**R**): $\displaystyle\int_1^{\infty}\frac{r}{v_p(r)}\,dr=\infty$ for some $p\in X$.

(**C**): $\displaystyle\int_{r_0}^{\infty}\frac{r}{\log v_p(r)}\,dr=\infty$ for some $p\in X$.

Theorem 5.1. *Assume* (**A**), (**B**) and (**E**).

(*i*) *Suppose that* (**R**) *holds. Then* $\mathbf{M}^{\varphi}$ *is recurrent.*

(*ii*) *Suppose that* (**C**) *holds. Then* $\mathbf{M}^{\varphi}$ *is conservative in the sense of* $P^{\varphi}_{\varphi^2m}(\tau_{X_\varphi}<\infty)=0$.

When $\varphi\equiv1$, condition (**B**) implies (**E**), hence we have the following.

Corollary 5.1. *Assume* (**A**), (**B**) *and* $\varphi\equiv1$.

(*i*) *Suppose that* (**R**) *holds. Then* **M** *is recurrent.*

(*ii*) *Suppose that* (**C**) *holds. Then* **M** *is conservative.*

Lemma 5.1. *Suppose that* (**A**), (**B**) *and* (**E**) *hold. Then there exists* $\varphi_{p,r}\in\mathscr{F}$ *such that* $\varphi=\varphi_{p,r}$ *m-a.e. on* $B_r(p)$.

Proof. Put $\varphi_{p,r}=\varphi(\rho_{p,r+1}\wedge1)$. Then $\int_X\varphi_{p,r}^2dm\leq\int_{B_{r+1}(p)}\varphi^2dm<\infty$ and

$\int_X d\mu^{(c)}_{<\varphi_{p,r}>} \leq 2\int_{B_{r+1}(p)}(\rho_{p,r+1}\wedge 1)^2 d\mu^{(c)}_{<\varphi>} + 2\int_{B_{r+1}(p)}\tilde{\varphi}^2 d\mu^{(c)}_{<\rho_{p,r+1}\wedge 1>} \leq 2\mu^{(c)}_{<\varphi>}(B_{r+1}(p)) + 2\int_{B_{r+1}(p)}\varphi^2 dm < \infty$. Thus Theorem 3.2 tells us the desired assertion.

Since $\varphi \in \mathcal{F}$ is bounded away from zero and infinity on the set $\{\frac{1}{2^n} < \varphi < 2^n\}$, Lemma 2.2 and Theorem 2.1 in Takeda [35] hold by using Theorem 4.3 in [6] in our situation. Hene $\varphi \in \mathcal{F}$ implies that $\mathbf{M}^\varphi$ is recurrent. when $\mathbf{M}$ is recurrent, $\mathbf{M}^\varphi$ is so for $\varphi \in \mathcal{F}_e$. Here $\mathcal{F}_e$ is the extended Dirichlet space (see [9]). Put $\rho_{p,r,R} = (\frac{\rho_{p,R}}{R-r}) \wedge 1$ for $0 < r < R$.

Lemma 5.2. *Suppose that* (**A**) *and* (**B**) *hold and* $\varphi \in \mathcal{F}$. *Then* $\rho_{p,r}|_{X_\varphi}, \rho_{p,r,R}|_{X_\varphi} \in \mathcal{F}^\varphi$.

Proof. Since $\mathbf{M}^\varphi$ is recurrent, we can use Theorem 3.1 (ii). So $\mathcal{F}^\varphi = \{u \in \dot{\mathcal{F}}_{X_\varphi loc} \cap L^2(X_\varphi;\ \varphi^2 m) : \int_{X_\varphi}\tilde{\varphi}^2(x)\,\mu^{(c)}_{<u>}(dx) < \infty\}$. We see $\rho_{p,r}|_{X_\varphi}, \rho_{p,r,R}|_{X_\varphi} \in \dot{\mathcal{F}}_{X_\varphi loc} \cap L^2(X_\varphi;\ \varphi^2 m)$ by $\dot{\mathcal{F}}_{loc}|_{X_\varphi} \subset \dot{\mathcal{F}}_{X_\varphi loc}$ and $\int_X \rho^2_{p,r}\varphi^2 dm \leq r^2 \int_X \varphi^2 dm < \infty$, $\int_X \rho^2_{p,r,R}\varphi^2 dm \leq \int_X \varphi^2 dm < \infty$. On the other hand $\int_X \tilde{\varphi}^2 d\mu^{(c)}_{<\rho_{p,r}>} \leq \int_X \varphi^2 dm < \infty$, $\int_X \tilde{\varphi}^2 d\mu^{(c)}_{<\rho_{p,r,R}>} \leq \frac{1}{(R-r)^2}\int_X \varphi^2 dm < \infty$. Thus we have $\rho_{p,r}|_{X_\varphi}$, $\rho_{p,r,R}|_{X_\varphi} \in \mathcal{F}^\varphi$.

Proposition 5.1. *Suppose* (**A**), (**B**) *and* (**E**) *hold. Then* $\rho_{p,r}|_{X_\varphi}$, $\rho_{p,r,R}|_{X_\varphi} \in \mathcal{F}^\varphi$.

Proof. By Lemma 5.1, there exists $\varphi_{p,r} \in \mathcal{F}$ *such* that $\varphi = \varphi_{p,r}$ m-a.e. on $B_r(p)$. Note that $X_\varphi \cap B_r(p) = X_{\varphi_{p,r}} \cap B_r(p)$. Let $p_t^{\varphi B_r(p) \cap X_\varphi}$ be the transition kernel associated with the part process of $\mathbf{M}^\varphi$ on $B_r(p) \cap X_\varphi$. We wee easily $p_t^{\varphi B_r(p) \cap X_\varphi}u(x) = p_t^{\varphi_{p,r} B_r(p) \cap X_{\varphi_{p,r}}}u(x)$ $\mathcal{E}$-q.e. $x. \in X$. Let $\mathcal{F}^\varphi_{B_r(p)\cap X_\varphi}$ be the part space of $\mathcal{F}^\varphi$ on $B_r(p) \cap X_\varphi$. Owing to the transfer method, we can apply Theorem 4.4.1 in [9]. Hence we have $\mathcal{F}^\varphi_{B_r(p)\cap X_\varphi} = \mathcal{F}^{\varphi_{p,r}}_{B_r(p)\cap X_{\varphi_{p,r}}}$. Thus we have $\rho_{p,r}|_{X_\varphi} \in \mathcal{F}^{\varphi_{p,r}}_{B_r(p)\cap X_{\varphi_{p,r}}} = \mathcal{F}^\varphi_{B_r(p)\cap X_\varphi}$. Similarly $\rho_{p,r,R}|_{X_\varphi} \in \mathcal{F}^{\varphi_{p,R}}_{B_R(p)\cap X_{\varphi_{p,R}}} = \mathcal{F}^\varphi_{B_R(p)\cap X_\varphi}$.

Proof of Theorem 5.1 (*i*). Condition (**R**) implies $\sum_{n=1}^\infty \frac{4^n}{v_p(2^n)} = \infty$, hence $\sum_{n=1}^\infty \frac{4^n}{v_p(2^{n+1}) - v_p(2^n)} = \infty$. Put $p_{k,n} = \frac{4^k}{v_p(2^{k+1}) - v_p(2^k)} \Big/ \sum_{i=1}^\infty \frac{4^i}{v_p(2^{i+1}) - u_p(2^i)}$ and $u_n = \sum_{k=1}^n p_{k,n}\,\rho_{p,2^k,2^{k+1}}$. Then $u_n \in \mathcal{F}^\varphi$ by Proposition 5.1. Since $\mathcal{E}^\varphi(\rho_{p,2^k,2^{k+1}}, \rho_{p,2^l,2^{l+1}}) = 0$ for $l \neq k$, we have

$$\mathcal{E}^\varphi(u_n, u_n) = \sum_{k=1}^n p^2_{k,n}\int_{B_{2^{k+1}}(p) - B_{2^k}(p)} \tilde{\varphi}^2 d\mu^{(c)}_{<\rho_{p,2^k,2^{k+1}}>}$$

$$\leq \sum_{k=1}^n p^2_{k,n}\frac{1}{4^k}\int_{B_{2^{k+1}}(p) - B_{2^k}(p)} \varphi^2 dm$$

$$=\sum_{k=1}^{n} p_{k,n}^2 \frac{1}{4^k}(v_p(2^{k+1})-v_p(2^k))$$

$$=1/\left(\sum_{k=1}^{n}\frac{4^k}{v_p(2^{k+1})-v_p(2^k)}\right)\to 0\,(n\to\infty).$$

On the other hand $0\leq u_n\leq 1$ and $\lim_{n\to\infty} u_n=1$ by $\lim_{n\to\infty}\rho_{p,2^n,2^{n+1}}=1$, $\lim_{n\to\infty} p_{k,n}=0$ for each k. The recurrence of $\mathbf{M}^{\varphi}$ holds by Theorem 1.6.5 in [9].

Proof of Theorem 5.1 (*ii*). By Proposition 5.1, we have the finiteness of the fine 1-capacity of $B_r(p)\cap X_{\varphi}$ with respect to $(\mathscr{E}^{\varphi}, \mathscr{F}^{\varphi})$. Hence, for $R_{\alpha}^{\varphi}1=E^{\varphi}_{\cdot}[\int_0^{\tau_{X_{\varphi}}} e^{-\alpha t}dt]$, there exists $u_{p,r}\in\mathscr{F}^{\varphi}$ such that $R_{\alpha}^{\varphi}1=u_{p,r}$ m-a.e. on $B_r(p)\cap X_{\varphi}$. Indeed, it suffices to set $u_{p,r}=R_{\alpha}^{\varphi}1\wedge e^{\varphi}_{B_r(p)\cap X_{\varphi}}$. Here $e^{\varphi}_{B_r(p)\cap X_{\varphi}}$ is the 1-equilibrium potential of $B_r(p)\cap X_{\varphi}$ with respect to the fine 1-capacity of $(\mathscr{E}^{\varphi}, \mathscr{F}^{\varphi})$. So, for $v\in\mathscr{F}^{\varphi}_{B_r(p)\cap X_{\varphi}}$, $\mathscr{E}_{\alpha}^{\varphi}(R_{\alpha}^{\varphi}1, v)$ has a meaning and $\mathscr{E}_{\alpha}^{\varphi}(1-\alpha R_{\alpha}^{\varphi}1, v)=0$. Put $u=1-\alpha R_{\alpha}^{\varphi}1$ and $w(t)=ue^{\alpha t}-\|u\|_{\infty}$. Condition (**C**) implies $t_k=\sum_{n=1}^{k}\frac{R_n^2}{\log u_p(R_n)}\to\infty\,(k\to\infty)$ for $R_k=2^k r\,(r\geq r_0)$. So for each $T>0$, ther exists $N\in\boldsymbol{N}$ with $t_N\geq T$. We let $\phi_k(t)=w^+(t)(\frac{2}{R_k}\rho_{p,2R_k}\wedge 1)^2\exp[-\frac{\rho_{p,R_k}}{4(T-t_k-t)}]$. By the similar method as in the proof of Theorem 4 in [30], we see $w^+(t)=0$ m-a.e. for any $t>0$, namely $w(t)\leq 0$ m-a.e. for any $t>0$. Hance $u=1-\alpha R_{\alpha}^{\varphi}1=0$ m-a.e.

Remark 5.1. (i) If φ does not satisfy (**E**), we have the possibility of the attainability to $X_{\Delta}-X_{\varphi}$ of $\mathbf{M}^{\varphi}$. However $\{\tilde{\varphi}=\infty\}$ is $\mathscr{E}$-polar, it can be non-$\mathscr{E}^{\varphi}$-polar for some $\varphi\in\dot{\mathscr{F}}_{loc}$(see Example 6.2).

(ii) In [30], he assumed the irreducibility of $(\mathscr{E}, \mathscr{F})$ to show the L^p-Liouville property. However, to show the recurrence of $(\mathscr{E}, \mathscr{F})$, the irreducibility is not needed as in [23].

6. Examples

Example 6.1. (Locally compact state spaces). Let X be a locally compact separable metric space and m a positive Radon measure on X with full topological support. Let $(\mathscr{E}, \mathscr{F})$ be a C_0-regular strong local Dirichlet space with a special standard core $\mathscr{C}$([9]). For an open set G, we denote $\mathscr{C}_G=\{u\in\mathscr{C}\colon \mathrm{supp}[u]\subset G\}$. We consider the Carathéodory metric used in [30] as follows:

$$\rho(x, y)=\sup\{u(x)-u(y)\colon\ u\in\mathscr{F}\cap C_0(X),\ \mu_{<u>}^{(c)}\leq m\}.$$

We assume that ρ generates the same topology endowed on X and any open balls $B_r(p)=\{x\in X\colon \rho(x, p)<r\}$ with radius $r>0$, center $p\in X$ are relatively compact. Then assumptions (**A**), (**B**) are satisfied and (**E**) holds for $\varphi\in\mathcal{F}_{loc}$ wity $\varphi\geq 0$. Here $\mathcal{F}_{loc}=\{u\in L^2_{loc}(X;m)$: for any relatively compact open set G, there exists $u_G\in\mathcal{F}$ such that $u=u_G$ m-a.e. on $G\}$. So (**C**) implies the conservativeness of $\mathbf{M}^{\varphi}$. In particular, if there exist $p\in X$, c_1, c_2, $r_0>0$ such that $v_p(r)\leq c_1\exp[c_2r^2]$ for any $r\geq r_0$, then (**C**) holds. In this case, the conservativeness of $\mathbf{M}^{\varphi}$ is proved in M. Takeda [35].

Next we consider the situation which does not necessarily satisfy (**A**), (**B**) and (**E**). But the criteria in [30] are applicable. Suppose that $\varphi\in\dot{\mathcal{F}}_{loc}$ with $\varphi>0$ m-a.e. satisfies $\int_X\tilde{\varphi}^2d\mu^{(c)}_{<u>}<\infty$for any $u\in\mathscr{C}$. Owing to Lemma 4.5 in [6], we have $\mathscr{C}|_{X_\varphi}\subset\mathcal{F}^{\varphi}$. Since $\varphi>0$ m-a.e., we get $m(X-X_\varphi)=0$ and supp $[\varphi^2m]=X$. So we may treat $(\mathscr{E}^{\varphi},\mathcal{F}^{\varphi})$ is a Dirichlet space on $L^2(X;\varphi^2m)$ and $\mathscr{C}\subset\mathcal{F}^{\varphi}$. Let $(\mathscr{E}^{\varphi},\mathcal{F}^{\varphi}_0)$ be the closure of $(\mathscr{E}^{\varphi},\mathscr{C})$ on $L^2(X;\varphi^2m)$. Then $\mathcal{F}^{\varphi}_0\subset\mathcal{F}^{\varphi}$ and $\mathscr{C}$ is a core of $\mathcal{F}^{\varphi}_0$. If $\mathcal{F}^{\varphi}_0=\mathcal{F}^{\varphi}$, the conservativeness and recurrence of $\mathbf{M}^{\varphi}$ follow the criteria of K. Th. Sturm [30]. When $\varphi\in\mathcal{F}_{loc}$ with $\varphi\geq 0$ m-a.e. and $d\mu^{(c)}_{<u>}=\Gamma(u,u)\,dm$ for any $u\in\mathscr{C}$, and (**A**), (**B**) are satisfied, we can see $\mathcal{F}^{\varphi}_0=\mathcal{F}^{\varphi}$ under the conditions (1.13) and (1.14) in A. Eberle [4]. In particular, if φ is bounded on $B_r(p)$, (1.13) and (1.14) in A. Eberle [4] are satisfied. We give another observation without assuming $\varphi\in\mathcal{F}_{loc}$, $d\mu^{(c)}_{<u>}=\Gamma(u,u)dm$ for $u\in\mathscr{C}$, and (**A**), (**B**).

Proposition 6.1. *Suppose that $\varphi\in\dot{\mathcal{F}}_{loc}\cap C(X)$ with $\varphi>0$ m-a.e. satisfies $\int_X\tilde{\varphi}^2d\mu^{(c)}_{<u>}<\infty$ for any $u\in\mathscr{C}$. Then $\mathcal{F}^{\varphi}_0=\mathcal{F}^{\varphi}$.*

Proof. Put $F_n=\{1/n\leq\varphi\leq n\}\cap\bar{G}_n$, then $\{F_n\}$ is an $\mathscr{E}^{\varphi}$-nest with respect to $(\mathscr{E}^{\varphi},\mathcal{F}^{\varphi})$. Here $\{G_n\}$ is an increasing sequence of relatively compact open sets with $\bar{G}_n\subset G_{n+1}$. Take $u\in\mathcal{F}^{\varphi}_{F_n}$. Then we get by ($\Gamma$6) and Theorem 3.1 (ii), $u\in\dot{\mathcal{F}}_{X_\varphi,loc}\cap L^2(X;m)$ and $\mu^{(c)}_{<u>}(X)<\infty$. Note that the 1-capacity of the fine interior of F_n with respect to the part space $(\mathscr{E}_{X_\varphi},\mathcal{F}_{X_\varphi})$ is finite. So by Theorem 3.2, we get $u\in\mathcal{F}_{F_n}\subset\mathcal{F}_{\{\frac{1}{n+1}<\varphi<n+1\}}$. We know $\mathscr{C}_{\{\frac{1}{n+1}<\varphi<n+1\}}$ is a core of $\mathcal{F}_{\{\frac{1}{n+1}<\varphi<n+1\}}$ (Theorem 4.4.3 (i) in [9]). Hence we get $u\in\mathcal{F}^{\varphi}_{0F_n}$. By the definition of $\mathscr{E}^{\varphi}$-nest , we have $\mathcal{F}^{\varphi}\subset\mathcal{F}^{\varphi}_0$, so $\mathcal{F}^{\varphi}=\mathcal{F}^{\varphi}_0$.

Under the conditions in Proposition 6.1, the recurrence and conservative criteria obeys [30].

Consider the case that $\mathbf{M}$ is the d-dimensional Brownian motion on $X=\boldsymbol{R}^d$. The associated Dirichlet space is given by $(\mathscr{E},\mathcal{F})=(1/2\mathbf{D},H^1(\boldsymbol{R}^d))$. Here $\mathbf{D}(u,v)=\int_{\boldsymbol{R}^d}\nabla u(x)\cdot\nabla v(x)\,dx$ and $H^1(\boldsymbol{R}^d)=\{u\in L^2(\boldsymbol{R}^d)$: all distribution derivatives of u are in $L^2(\boldsymbol{R}^d)\}$. Suppose that $\varphi\in H^1_{loc}(\boldsymbol{R}^d)$ with $\varphi>0$ a.e. and there exist constants $\hat{c}_1$, $\hat{c}_2$,$\hat{r}_0>1$ such that $v_0(r)\leq\hat{c}_1\exp[\hat{c}_2r^2\log^+r]$ for any $r\geq\hat{r}_0$, then (**C**) holds. In this case, the conservativeness of $\mathbf{M}^{\varphi}$ holds by Theorem 4.2 in M. Takeda [36] and the criteria in [30].

Example 6.2. Consider the d-dimensional Euclidean space $\boldsymbol{R}^d$ with d-dimensional Lebesgue measure $dx\,(d\geq 3)$ and $\varphi_\delta(x)=1/|x|^\delta$, $x\in\boldsymbol{R}^d\,(d/2-1\leq\delta<d/2)$. Then $\varphi_\delta\notin H^1_{loc}(\boldsymbol{R}^d)$ but $\varphi_\delta\in L^2_{loc}(\boldsymbol{R}^d)\cap H^1_{loc}(\boldsymbol{R}^d-\{0\})\cap \dot{H}^1_{loc}(\boldsymbol{R}^d)$, so (**R**) holds, (**E**) does not hold for Dirichlet integral and balls with center 0 by Euclidean metric. We let

$$\mathscr{E}^{\varphi_\delta}(u,v)=\frac{1}{2}\int_{\boldsymbol{R}^d}\nabla u(x)\cdot\nabla u(x)\varphi_\delta^2(x)dx,\ u,v\in C_0^\infty(\boldsymbol{R}^d).$$

Then $(\mathscr{E}^{\varphi_\delta}, C_0^\infty(\boldsymbol{R}^d-\{0\}))$ is closable on $L^2(\boldsymbol{R}^d;\varphi_\delta^2dx)$ and denote by $(\mathscr{E}^{\varphi_\delta},\mathscr{F}_0^{\varphi_\delta})$ its closure on $L^2(\boldsymbol{R}^d;\varphi_\delta^2dx)$ (see Example 3.3.2 in [9]). Denote by $H^1(\boldsymbol{R}^d)^{\varphi_\delta}$ the Dirichlet space on $L^2(\boldsymbol{R}^d;\varphi_\delta^2dx)$ associated with the transformed process by $L_t^{\varphi_\delta}$. Note that $X_{\varphi_\delta}=\boldsymbol{R}^d$q.e. Here "q.e." means that it holds off exceptional set with respect to the Newtonian capacity. Theorem 3.1 (ii) implies $H^1(\boldsymbol{R}^d)^{\varphi_\delta}\subset\{u\in\dot{H}^1(\boldsymbol{R}^d)_{loc}\cap L^2(\boldsymbol{R}^d;\varphi_\delta^2dx):\int_{\boldsymbol{R}^d}|\nabla u(x)|^2\varphi_\delta^2(x)dx<\infty\}$. Applying Lemma 4.5 in [6], we have $C_0^\infty(\boldsymbol{R}^d-\{0\})\subset H^1(\boldsymbol{R}^d)^{\varphi_\delta}$. Then $\mathscr{F}_0^{\varphi_\delta}\subset H^1(\boldsymbol{R}^d)^{\varphi_\delta}$. We know that the 1-capacity of the origin with respect to $(\mathscr{E}^{\varphi_\delta},\mathscr{F}_0^{\varphi_\delta})$ is positive if and only if $\delta\neq d/2-1$ (see Theorem 4.11 in [1] and Example 2 in [34], Example 3.3.2 in [9]).

Proposition 6.2. *$\mathbf{M}^{\varphi_\delta}$ is recurrent if $\delta=d/2-1$ and non-conservative if $\delta\neq d/2-1$.*

Proof. First we show $\mathscr{F}_0^{\varphi_\delta}=H^1(\boldsymbol{R}^d)^{\varphi_\delta}$. Consider the case $X=\boldsymbol{R}^d-\{0\}$ and $(\mathscr{E},\mathscr{F})=(1/2\mathbf{D}, H^1(\boldsymbol{R}^d-\{0\}))$, $\mathscr{C}=C_0^\infty(\boldsymbol{R}^d-\{0\})$. Since $\varphi_\delta\in H^1_{loc}(\boldsymbol{R}^d-\{0\})$ is positive continuous, we get $\mathscr{F}_0^{\varphi_\delta}=H^1(\boldsymbol{R}^d-\{0\})^{\varphi_\delta}$ by Proposition 6.1. Noting that $\{0\}$ is polar for Brownian motions, we have $\mathscr{F}_0^{\varphi_\delta}=H^1(\boldsymbol{R}^d-\{0\})^{\varphi_\delta}=H^1(\boldsymbol{R}^d)^{\varphi_\delta}$. When $\delta\neq d/2-1$, $\{0\}$ is non-$\mathscr{E}^{\varphi_\delta}$-polar, which implies the non-conservativness of $\mathbf{M}^{\varphi_\delta}$. Note that the state space of $\mathbf{M}^{\varphi_\delta}$ is $\boldsymbol{R}^d-\{0\}$ and the $\mathscr{E}$-polarity is equivalent to the $\mathscr{E}_1$-polarity in the framework of C_0-regular Dirichlet forms. Suppose $\delta=d/2-1$. To show the recurrence of $\mathbf{M}^{\varphi_\delta}$, we should show the C_0-regularity of $(\mathscr{E}^{\varphi_\delta}, H^1(\boldsymbol{R}^d)^{\varphi_\delta})$ on $L^2(\boldsymbol{R}^d;\varphi_\delta^2dx)$. It suffices to show that 1-capacity of $Br(0)$ is always finite with respect to $(\mathscr{E}^{\varphi_\delta}, H^1(\boldsymbol{R}^d)^{\varphi_\delta})=(\mathscr{E}^{\varphi_\delta},\mathscr{F}_0^{\varphi_\delta})$. Then Theorem 3.2 tells us $C_0^\infty(\boldsymbol{R}^d)\subset H^1(\boldsymbol{R}^d)^{\varphi_\delta}$, which implies the C_0-regularity on $\boldsymbol{R}^d$. So the condition (**R**) and the criterion in [30] give the recurrence of $\mathbf{M}^{\varphi_\delta}$. We emphasize that Lemma 4.5 in [6] is not applicable to show $C_0^\infty(\boldsymbol{R}^d)\subset H^1(\boldsymbol{R}^d)^{\varphi_\delta}$, since $\varphi_\delta\notin H^1_{loc}(\boldsymbol{R}^d)$. Next we show the finiteness of 1-capacity. Take annuli $A_{\varepsilon,r}(0)=\{x\in\boldsymbol{R}^d:\ \varepsilon<|x|<r\}$. Note that $C_0^{\mathrm{Lip}}(\boldsymbol{R}^d-\{0\})$, totality of Lipschitz continuous functions with compact support in $\boldsymbol{R}^d-\{0\}$, is a core of $(\mathscr{E}^{\varphi_\delta},\mathscr{F}_0^{\varphi_\delta})$. Consider a piesewise linear function g_ε on $[0,\infty]$ with $g_\varepsilon=0$ on $[0,\varepsilon]\cup[2r,\infty)$ and $g_\varepsilon=1$ on $[2\varepsilon, r]$. Set $f_\varepsilon(x)=g_\varepsilon(|x|)$. Then $\mathrm{Cap}^{\varphi_\delta}(A_{\varepsilon,r}(0))\leq\mathscr{E}_1^{\varphi_\delta}(f_\varepsilon,f_\varepsilon)$. This is bounded with respect to $\varepsilon\in(0,r)$, because $\int_\varepsilon^{2\varepsilon}\left(\frac{1}{\varepsilon}\right)^2\frac{t^{d-1}}{t^{2\delta}}dt\leq\frac{2^{d-2\delta}}{d-2\delta}\varepsilon^{d-2\delta-2}=2$.

Hence $\mathrm{Cap}^{\varphi_\delta}(B_r(0)-\{0\})<\infty$, so $\mathrm{Cap}^{\varphi_\delta}(B_r(0))<\infty$ by $\mathscr{E}_1^{\varphi_\delta}$-polarity of $\{0\}$.

Example 6.3. (Recurrence of $\mathbf{M}^\varphi$ not covered by [30], [35]). Let $G=\{x\in\boldsymbol{R}^d: x=(\bar{x}, x_d), \bar{x}\in\boldsymbol{R}^{d-1}, |x_d|<1\}$ $(d\geq 3)$ and $m(dx)=\sigma^b(x)dx$ $(\sigma(x)=1-|x_d|, -1<b<0)$.
Consider

$$\mathscr{E}(u, v)=\frac{1}{2}\int_G \nabla u(x)\cdot\nabla v(x)\sigma^{b+4}(x)(dx),\ u, v\in C_0^\infty(G)$$

Then $(\mathscr{E}, C_0^\infty(G))$ is colsable on $L^2(G; m)$ and denote by $(\mathscr{E}, \mathscr{F})$ its closure on $L^2(G; m)$. The Carathéodory metric is given by

$$\rho^0(x, y)=\sup\{u(x)-u(y): u\in\mathscr{F}\cap C_0(G), |\nabla u(x)|^2\leq\sigma(x)^{-4} a.e.\ x\}.$$

We also let

$$\rho(x, y)=\sup\{u(x)-u(y): u\in\mathscr{F}_{loc}\cap C(G), |\nabla u(x)|^2\leq\sigma(x)^{-4} a.e.\ x\}.$$

For an open subset D of G with $\bar{D}\subset G$, we put

$$\rho_D(x, y)=\sup\{u(x)-u(y): u\in\mathscr{F}_{Dloc}\cap C(D), |\nabla u(x)|^2\leq\sigma(x)^{-4} a.e.\ x\}$$

and

$$\rho_D^0(x, y)=\sup\{u(x)-u(y): u\in\mathscr{F}_D\cap C_0(D), |\nabla u(x)|^2\leq\sigma(x)^{-4} a.e.\ x\}.$$

Then $\rho_D^0(x, y)\leq\rho^0(x, y)\leq\rho(x, y)\leq\rho_D(x, y)$ for $x, y\in D$. We let $d(x, y)=|x-y|$. Suppose that $\{z\in\boldsymbol{R}^d: d(x, z)\leq d(x, y)\}\subset D$. Then $f(\cdot)=(d(x, y)-d(x, \cdot))\vee 0\in H_0^1(D)\cap C_0(D)$ with $|\nabla f|\leq 1$a.e. So we see by $\mathscr{F}_D\cap C_0(D)=H_0^1(D)\cap C_0(D)$, $d(x, y)\leq\rho_D^0(x, y)\leq(1-c)^2 d(x, y)$, where $c=\inf_{x\in D}|x_d|$. Hence ρ_D^0 makes the same topology on D. So by Proposition 1 a) in [30], we get ρ_D makes the same topology on D. On the other hand, $\rho_D^0(x, y)\leq\rho^0(x, y)\leq\rho(x, y)\leq\rho_D(x, y)$ for $x, y\in D$. So (**A**) holds for ρ^0 and ρ (Proposition 1 a) in [30]). We take $\psi_1(x)=\sigma(x)$, $\psi_2(x)=|\bar{x}|$and $\psi_3(x)=|x|$. Then $\psi_i\in\mathscr{F}_{loc}\cap C(G)$ satisfy $|\nabla\psi_i(x)|^2\leq\sigma(x)^{-4}$ $(i=1, 2, 3)$, hence $\{x\in G: \rho(x, y)<r\}\subset\{x\in G: |\bar{x}|<r, |x|<r, |x_d|<\frac{r}{r+1}\}$, so (**B**) holds for ρ, hence holds for ρ^0 and $\rho^0=\rho$ (Proposition 1 c) in [30]). Then we have

$$\int_{Br(0)}\sigma^b(x)dx\leq\int_{\{|\bar{x}|<r, |x_d|<\frac{r}{r+1}\}}(1-|x_d|)^b dx\leq r^{d-1}\frac{2r}{r+1}\int_{S^{d-2}}d\omega\ (r\to\infty).$$

Hence the associated process $\mathbf{M}$ is conservative, further $\mathbf{M}$ is recurrent if $d=1, 2, 3$. We let $\varphi(x)=1/|x|^\delta$ $(d/2-(b+3)/2\leq\delta<d/2-1)$. Then we see

$$\int_{Br(0)}|\nabla\varphi(x)|^2\sigma^{b+4}(x)dx\leq\int_{\{|x|<r\}}\frac{1}{|x|^{2\delta+2}}dx<\infty$$

and

$$\int_G |\nabla\varphi(x)|^2\sigma^{b+4}(x)\,dx=\int_{S^{d-2}}d\omega\int_0^1 t^{d-2\delta-3}(1-t)^{b+4}dt\int_0^\infty\frac{u^{d-2}}{(1+u^2)^{\delta+1}}=\infty.$$

Hence $\varphi\in\mathcal{F}_{loc}$, but $\varphi\notin\mathcal{F}_e$. On the other hand,

$$\int_{B_r(0)}\varphi^2(x)\sigma^b(x)\,dx\le\int_{\{|\bar{x}|<r,|x_d|<\frac{r}{r+1}\}}\frac{1}{|\bar{x}|^{2\delta}}(1-|x_d|)^b dx\sim r^{d-2\delta-1}\le r^{b+2}\le r^2\ (r\to\infty).$$

Thus (**E**) and (**R**) hold for φ. Since φ is continuous on $\boldsymbol{R}^d-\{0\}$ and $\varphi>0$, Proposition 6.1 is applicable. Hence $\mathbf{M}^\varphi$ is recurrent by the criteria in K. Th. Sturm [30]. Let $\{r_i\}_{i\in N}$ be the totality of rational points in $\{x\in\boldsymbol{R}^d\colon |x|\le 1\}$ with $r_1=0$ and put $\varphi_i(x)=1/|x-r_i|^\delta$ and $\psi=\sum_{i=1}^\infty\frac{1}{2^i}\varphi_i$. Then $\nabla\psi=\sum_{i=1}^\infty\frac{1}{2^i}\nabla\varphi_i$ in distribution sense. Hence we see $\psi^2\le\sum_{i=1}^\infty\frac{1}{2^i}\varphi_i^2$ and $|\nabla\psi|^2\le\sum_{i,j=1}^\infty\frac{1}{2^{i+j}}\frac{\delta^2}{|x-r_i|^{\delta+1}|x-r_3|^{\delta+1}}=(\sum_{i=1}^\infty\frac{1}{2^i}|\nabla\varphi_i|)^2$ by $\nabla\varphi_i(x)\cdot\nabla\varphi_j(x)=\frac{-\delta(x-r_i)/|x-r_i|}{|x-r_i|^{\delta+1}}\cdot\frac{-\delta(x-r_j)/|x-r_j|}{|x-r_j|^{\delta+1}}=\frac{\delta^2(x-r_i)\cdot(x-r_j)}{|x-r_i|^{\delta+2}|x-r_j|^{\delta+2}}$.
Since $|r_i|\le 1$ and $|\bar{r}_i|\le 1$, we get

$$\int_{B_r(0)}\psi^2(x)\sigma^{b+4}(x)\,dx\le\sum_{i=1}^\infty\frac{1}{2^i}\int_{\{|x-r_i|<r+1\}}\frac{1}{|x-r_i|^{2\delta}}dx<\infty$$

and

$$\int_{B_r(0)}|\nabla\psi(x)|^2\sigma^{b+4}(x)\,dx\le\sum_{i=1}^\infty\frac{1}{2^i}\int_{\{|x-r_i|<r+1\}}\frac{1}{|x-r_i|^{2\delta+2}}dx<\infty$$

So $\psi\in\mathcal{F}_{loc}$, but $\psi\notin\mathcal{F}_e$ by $\frac{1}{2}|\nabla\varphi|\le|\nabla\psi|$. On the other hand,

$$\int_{B_r(0)}\psi^2(x)\,\sigma^b(x)\,dx\le\sum_{i=1}^\infty\frac{1}{2^i}\int_{\{|\bar{x}-\bar{r}_i|<r+1,|x_d|<\frac{r}{r+1}\}}\frac{1}{|\bar{x}-\bar{r}_i|^{2\delta}}(1-|x_d|)^b\,dx\sim r^{d-2\delta-1}$$
$$\le r^{b+2}\le r^2.$$

Hence (**E**) and (**R**) hold for ψ. Therefore $\mathbf{M}^\psi$ is recurrent by our criteria. Note that ψ is non-bounded on $B_r(0)$. So we can not easily check the conditions (1.13) and (1.14) in [4]. Also ψ is not continuous, so we can not apply the latter observation in Example 6.1 for ψ.

Example 6.4. (Balls of finite capacity without relative compactness). Let $X=\boldsymbol{R}$ and $m(dx)=e^{-2x^2}dx$. Consider a pre-Dirichlet form $\mathcal{E}(u,v)=\frac{1}{2}\int_{-\infty}^\infty u'(x)v'(x)\,dx$ for $u,v\in C_0^\infty(\boldsymbol{R})$. Then $(\mathcal{E},C_0^\infty(\boldsymbol{R}))$ is closable on $L^2(\boldsymbol{R};m)$. We denote $(\mathcal{E},\mathcal{F})$ its closure on $L^2(\boldsymbol{R};m)$. The Carathéodory metric of $(\mathcal{E},\mathcal{F})$ is given by

$$\rho(x,y)=\sup\{u(x)-u(y)\colon u\in C_0^\infty(\boldsymbol{R}),\ |u'(x)|\le e^{-x^2}a.e.\ x\}.$$

We see

$$\rho(x, y)=\int_x^y e^{-t^2}dt \text{ for } x\leq y.$$

Hence $B_r(p)=\boldsymbol{R}$ for $r\geq\sqrt{\pi}$. On the other hand, we see $1\in\mathcal{F}$, thus 1-capacity of $\boldsymbol{R}$ is finite. So (**A**), (**B**) are satisfied. (**E**) holds if and only if $\varphi\in\mathcal{F}$. In this case, recurrence holds by Theorem 2.1 in [35].

Example 6.5 (Infinite dimensional state space-Banach spaces). Let B be a real separable Banach space with dual space B^* and H a real separable Hilbert space with dual space H^* such that H is densely and contiunously embedded in B. In this situation, there exist a constant $c>0$ with $\|h\|_B\leq c\|h\|_H$, $h\in H$ and $\|e\|_{H^*}\leq c\|e\|_{B^*}$, $e\in B^*$. Let $FC_b^\infty(B)$ be the totality of cylindrical smooth bounded functions on B as follows:

$$FC_b^\infty(B)=\{u\colon B\mapsto\boldsymbol{R} : \text{ there exists } n\in\boldsymbol{N} \text{ and } f\in C_b^\infty(\boldsymbol{R}^n),\ l_1, l_2, \cdots, l_n\in B^* \text{ such that } u(z)=f({}_{B^*}\langle l_1, z\rangle_B, {}_{B^*}\langle l_2, z\rangle_B, \cdots, {}_{B^*}\langle l_n, z\rangle_B)\}$$

Here $C_b^\infty(\boldsymbol{R}^n)$ is the totality of smooth functions on $\boldsymbol{R}^n$ such that all derivatives are bounded. For $u, v\in FC_b^\infty(B)$ with $u(z)=f({}_{B^*}\langle l_1, z\rangle_B, {}_{B^*}\langle l_2, z\rangle_B, \cdots, {}_{B^*}\langle l_n, z\rangle_B)$ and $v(z)=g({}_{B^*}\langle \hat{l}_1, z\rangle_B, {}_{B^*}\langle \hat{l}_2, z\rangle_B, \cdots, {}_{B^*}\langle \hat{l}_m, z\rangle_B)$, we let

$$\Gamma(u, v)=\sum_{i=1}^n\sum_{j=1}^m\frac{\partial f}{\partial x_i}({}_{B^*}\langle l_1, z\rangle_B, {}_{B^*}\langle l_2, z\rangle_B, \cdots, {}_{B^*}\langle l_n, z\rangle_B)$$
$$\times\frac{\partial g}{\partial x_j}({}_{B^*}\langle \hat{l}_1, z\rangle_B, {}_{B^*}\langle \hat{l}_2, z\rangle B, \cdots, {}_{B^*}\langle \hat{l}_m, z\rangle_B)\,(l_i, \hat{l}_j)_{H^*}.$$

Let μ be a Borel probability measure on B with $\text{supp}[\mu]=B$. We consider

$$\mathcal{E}(u, v)=\frac{1}{2}\int_B\Gamma(u, v)d\mu \text{ for } u, v\in FC_b^\infty(B)$$

and assume its closability on $L^2(B;\mu)$. Then its colsure $\mathcal{F}$ is a quasi-regular Dirichlet space on $L^2(B;\mu)$ (see [22], [27]). Then $1\in FC_b^\infty(B)$ and $\mathcal{E}(1, 1)=0$, hence $(\mathcal{E}, \mathcal{F})$ is recurrent and the 1-capacity of total space B is always finite. According to a Hahn-Banach theorem, we can take a countable set $\{e_i\}_{i\in N}$ of B^* with $\|e_i\|_{B^*}=1$ such that $\|z\|_B=\sup_{i\in N}{}_{B^*}\langle e_i, z\rangle_B=\sup_{i\in N}|{}_{B^*}\langle e_i, z\rangle_B|$ for any $z\in B$ (*IV* Proposition 4.2 in [22]). We define

$$\mathcal{C}=\{f({}_{B^*}\langle e_i, z\rangle_B)/c : i\in\boldsymbol{N}, f\in C_b^\infty(\boldsymbol{R}) \text{ with } |f'|\leq 1\}.$$

Then the Carathéodory metric

$$\rho(x, y)=\sup\{u(x)-u(y): u\in\mathcal{C}\cup-\mathcal{C}, \Gamma(u, u)\leq 1 \quad \mu\text{-a.e.}\}$$

coincides with $\|x-y\|_B/c$. So ρ satisfies conditions (**A**) and (**B**). Fix an $e\in B^*$. We consider $f_1(x)=|x|$ or $=|x|\sqrt{\log^+ x}$ and $f_2(x)=\exp[x^2]$ or $=$

$\exp[x^2 \log^+ x]$ and put $\varphi_i(z) = f_i({}_{B^*}\langle e, z\rangle_B)$ $(i=1, 2)$. Then $\varphi_1, \varphi_2 \in \dot{\mathcal{F}}_{loc}$ and they satisfy the condition (**E**) with $d\mu^{(c)}_{\langle\varphi_i\rangle} = \Gamma(\varphi_i)d\mu$ and φ_1(resp. φ_2) satisfies (**R**) (resp. (**C**)). Hence $\mathbf{M}^{\varphi_1}$ is recurrent and $\mathbf{M}^{\varphi_2}$ is conservative. Also $\varphi_3(z) = f_2(\|z\|_B)$ satisfies (**E**) and (**C**). When (B, H, μ) is an abstract Wiener space, we see $\varphi_2 \notin \mathcal{F}_e$.

Example 6.6 (Infinite dimensional state space-the free loop space on $\mathbf{R}^d$: the case of finite capacity of balls with infinite capacity of whole space) **.** We follow the notations in [27]. Let $g = (g_{ij})$ be a uniformly elliptic Riemannian metric with bounded derivative over $\boldsymbol{R}^d$ and $\Delta_g = (det\ g)^{-1/2} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}[(det\ g)^{1/2} g^{ij} \frac{\partial}{\partial x_j}]$ the corresponding Laplace-Beltrami operator. Let $p_t(x, y)$, $x, y \in \boldsymbol{R}^d$, $t \geq 0$, be the associated heat kernel with respect to the volume element with regards to g. Let $W(\boldsymbol{R}^d)$ be the totality of the continuous paths ω: $[0, 1] \to \boldsymbol{R}^d$ and $\mathcal{L}(\boldsymbol{R}^d) = [\omega \in W(\boldsymbol{R}^d) : \omega(0) = \omega(1)\}$, namely $\mathcal{L}(\boldsymbol{R}^d)$ is the free loop space over $\boldsymbol{R}^d$. $\mathcal{L}(\boldsymbol{R}^d)$ is a Banach space equipped with the uniform norm $\|\omega\|_\infty = \sup\{|\omega(t)| : t \in [0, 1]\}$. Let P_1^x be the pinned measure on $\{\omega \in \mathcal{L}(\boldsymbol{R}^d) : \omega(0) = x\}$, namely the finite dimensional distribution of P_1^x is given by

$$P_1^x(\omega(t_i) \in dx_1, \omega(t_2) \in dx_2, \cdots, \omega(t_n) \in dx_n)$$
$$= p_{t_1}(x, x_1)\ p_{t_2-t_1}(x_1, x_2) \cdots p_{t_n-t_{n-1}}(x_{n-1}, x_n)\ p_{1-t_n}(x_n, x)/p_1(x, x)\ dx_1\, dx_2 \cdots dx_n$$

for $0 < t_1 < t_2 < \cdots < t_n < 1$. Let $\mu = \int_{R^d} p_1(x, x) P_1^x dx$. Then

$$\mu(\omega(t_1) \in dx_1, \omega(t_2) \in dx_2, \cdots, \omega(t_n) \in dx_n)$$
$$= \int_{R^d} p_{t_1}(x, x_1) p_{t_2-t_1}(x_1, x_2) \cdots p_{t_n-t_{n-1}}(x_{n-1}, x_n) p_{1-t_n}(x_n, x)\, dx_1 dx_2 \cdots dx_n dx$$

for $0 < t_1 < t_2 < \cdots < t_n < 1$. In particular, $\mu(\omega(t) \in dy) = p_1(y, y)\, dy$, hence there exist constants $c_1, c_2 > 0$ independent of t such that $c_1 r^d \leq \mu(|\omega(t)| \leq r) \leq c_2 r^d$ by virtue of the uniform ellipticity of g and Theorem 5.5.2 and Theorem 5.6.1 in [3]. μ is called the "*Bismut*" measure on $\mathcal{L}(\boldsymbol{R}^d)$. Note that μ is not finite but σ-finite. Let H be the totality of the absolutely continuous maps h: $[0, 1] \to T_{\omega(0)}\, \boldsymbol{R}^d \equiv \boldsymbol{R}^d$ such that

$$(h, h)_H = \int_0^1 g_{\omega(0)}\ (\dot{h}(s), \dot{h}(s))\, ds + \int_0^1 g_{\omega(0)}(h(s), h(s))\, ds < \infty.$$

Denote by $\tau_t(\omega) : T_{\omega(0)}\, \boldsymbol{R}^d \to T_{\omega(t)}\, \boldsymbol{R}^d$ the stochastic parallel transport associated with the Levi-Civita connection of $(\boldsymbol{R}^d, g)$. We let $H_0 = \{h \in H: \tau_1(\omega) h(1) = h(0)\}$ and $T_\omega \mathcal{L}(\boldsymbol{R}^d)$ be the tangent space at a loop ω defined by $T_\omega \mathcal{L}(\boldsymbol{R}^d) = \{X = (\tau_t(\omega) h(t))_{t \in [0,1]} : h \in H_0\}$. The element $X \in T_\omega \mathcal{L}(\boldsymbol{R}^d)$ is a totality of periodical vector fields $X_t(\omega) = \tau_t(\omega)\, h(t)$. H_0 is a closed subspace of H. We also consider the space $H_\omega \mathcal{L}(\boldsymbol{R}^d)$ $(\supset T_\omega \mathcal{L}(\boldsymbol{R}^d))$ defined by $H_\omega \mathcal{L}(\boldsymbol{R}^d) = \{(\tau_t(\omega) h(t))_{t \in [0,1]} : h \in H\}$. Then $T_\omega \mathcal{L}(\boldsymbol{R}^d)$ (resp. $H_\omega \mathcal{L}(\boldsymbol{R}^d)$)

is a Hilbert space equipped with the inner product defined by $(\tau.(\omega)h_1,\tau.(\omega)h_2)_{T_\omega\mathscr{L}(\boldsymbol{R}^d)}=(h_1,h_2)_H$ for $h_1,h_2\in H_0$ (resp. $(\tau.(\omega)h_1,\tau.(\omega)h_2)_{H_\omega\mathscr{L}(\boldsymbol{R}^d)}=(h_1,h_2)_H$ for $h_1,h_2\in H$). Let FC_0^∞ be the totality of the cylindrical functions on $\mathscr{L}(\boldsymbol{R}^d)$, namely

$$FC_0^\infty=\{u:\mathscr{L}(\boldsymbol{R}^d)\mapsto\boldsymbol{R};\ \text{there exists } n\in\boldsymbol{N} \text{ and } f\in C_0^\infty((\boldsymbol{R}^d)^n),\ t_1,t_2,\cdots,t_n\in[0,1] \\ \text{such that } u(\omega)=f(\omega(t_1),\omega(t_2),\cdots,\omega(t_n)),\ \omega\in\mathscr{L}(\boldsymbol{R}^d)\}.$$

Note that FC_0^∞ is dense in $L^2(\mathscr{L}(\boldsymbol{R}^d),\mu)$. FC^∞ (resp. FC_b^∞) is similarly defined by $C^\infty((\boldsymbol{R}^d)^n)$ (resp. $C_b^\infty((\boldsymbol{R}^d)^n)$) replacing $C_0^\infty((\boldsymbol{R}^d)^n)$. We also need the space FC_0^{Lip} similary defined by $C_0^{\mathrm{Lip}}((\boldsymbol{R}^d)^n)$, the totality of the Lipschitz continuous functions on $(\boldsymbol{R}^d)^n$ with compact support. We define the directional derivative of $u\in FC^\infty$ at $\omega\in\mathscr{L}(\boldsymbol{R}^d)$ with respect to $X=(\tau.(\omega)h)\in H_\omega\mathscr{L}(\boldsymbol{R}^d)$ by

$$\begin{aligned}\partial_h u(\omega)=\partial_X u(\omega)&=\sum_{k=1}^n d_k f(\omega(t_1),\omega(t_2),\cdots,\omega(t_n))\ X_{t_k}(\omega)\\&=\sum_{k=1}^n g_{\omega(t_k)}(\nabla_k f(\omega(t_1),\omega(t_2),\cdots,\omega(t_n)),\tau_{t_k}(\omega)h(t_k))\end{aligned}$$

where d_k (resp. ∇_k) denotes the differential (resp. the gradient with respect to g) relative to the k-th coordinate of f. Note that

$$\partial_X u(\omega)=\frac{d}{ds}u(\omega+sX(\omega))|_{s=0},\ \omega\in\mathscr{L}(\boldsymbol{R}^d).$$

Let for $u\in FC^\infty$ with $u(\omega)=f(\omega(t_1),\omega(t_2),\cdots,\omega(t_n))$, $f\in C^\infty((\boldsymbol{R}^d)^n)$ and $\omega\in\mathscr{L}(\boldsymbol{R}^d)$, $\tilde{D}u(\omega)$ be the unique element in H such that $(\tilde{D}u(\omega),h)=\partial_h u(\omega)$ for any $h\in H$. Then

$$\tilde{D}u(\omega)\ (s)=\sum_{k=1}^n G(s,t_k)\tau_{t_k}^{-1}(\omega)\ \nabla_k f(\omega(t_1),\omega(t_2),\cdots,\omega(t_n)),$$

where G is the Green function of $-\frac{d^2}{dt^2}+1$ with Neumann boundary conditions on $[0,1]$, namely

$$G(s,t)=\frac{e}{2(e^2-1)}(e^{s+t-1}+e^{1-s-t}+e^{|s-t|-1}+e^{1-|s-t|}).$$

We let $Du(\omega)=\mathrm{Pr}_{H_0}\tilde{D}u(\omega)$. Here Pr_{H_0} is the projection onto H_0. We define for $u,v\in FC_0^\infty$

$$\mathscr{E}(u,v)=\frac{1}{2}\int_{\mathscr{L}(\boldsymbol{R}^d)}(Du(\omega),Dv(\omega))_H\mu(d\omega).$$

Then by uniform ellipticity of g and the above expression of $\tilde{D}u(\omega)$, we have $\mathscr{E}(u, u) < \infty$ for $u \in FC_0^\infty$ and $(\mathscr{E}, FC_0^\infty)$ is closable on $L^2(\mathscr{L}(\boldsymbol{R}^d); \mu)$, and the closure $(\mathscr{E}, \mathscr{F})$ on $L^2(\mathscr{L}(\boldsymbol{R}^d); \mu)$ is a quasi-regular local Dirichlet form (see Röckner-Schmuland [27]). Let $\mathbf{M}$ be the diffusion process associated with $(\mathscr{E}, \mathscr{F})$. The energy measure of continuous part is given by

$$\mu_{<u,v>}^{(c)}(d\omega) = (Du(\omega), Dv(\omega))_H \mu(d\omega), \ u, v \in \mathscr{F}.$$

Note that $FC_0^{\mathrm{Lip}} \subset \mathscr{F}$. Indeed, put $u \in FC_0^{\mathrm{Lip}}$ with $u(\omega) = f(\omega(t_1), \omega(t_2), \cdots, \omega(t_n))$, $f \in C_0^{\mathrm{Lip}}((\boldsymbol{R}^d)^n)$ and the molifier $\rho_\delta(x) = \prod_{k=1}^{dn} \frac{1}{\delta}\rho(\frac{x_k}{\sigma})$ with $\rho(t) = C \exp[-\frac{1}{1-t^2}]$, $|t| \le 1, =0, |t|>1$. Here $C = \int_{-1}^{1} \exp[-\frac{1}{1-t^2}]dt$. Then $u_\delta \in FC_0^\infty$ for $u_\delta(\omega) = \rho_\delta * f(\omega(t_1), \omega(t_2), \cdots, \omega(t_n))$. Since f is Lipschitz continuous with compact support, the derivative of $\rho_\delta * f$ is uniformly bounded and the support of $\rho_\delta * f$ is contained in a bounded ball in $(\boldsymbol{R}^d)^n$ uniformly. Hence we see that $\{u_\delta\}$ is an $\mathscr{E}_1^{1/2}$-bounded sequence in $\mathscr{F}$ by using $\mu(|\omega(t)| \le r) \le c_2 r^d$. So the Banach-Saks theorem tells us $u \in \mathscr{F}$. For a countable dense subset $\{s_l: l \in \boldsymbol{N}\}$ of $[0, 1]$ and a countable dense subset $\{\omega_i: i \in \boldsymbol{N}\}$ of $\mathscr{L}(\boldsymbol{R}^d)$, and $\varphi_k(t) = (k - |k-t|) \vee 0$, $t \in \boldsymbol{R}$, we let $u_{i,j,k}(\omega) = \bigvee_{l=1}^{j} \varphi_k(|\omega(s_l) - \omega_i(s_l)|)$. Then $u_{i,j,k} \in FC_0^{\mathrm{Lip}}$ with $\bigvee_{l=1}^{j} \varphi_k(|\bullet - \omega_i(s_l)|) \in C_0^{\mathrm{Lip}}(\boldsymbol{R}^d)$. On the other hand, $|Du_{i,j,k}(\omega)|_H^2 \le |\tilde{D}u_{i,j,k}(\omega)|_H^2 \le c^2$ for some constant c independent of $i, j, k \in \boldsymbol{N}$. We put

$$\mathscr{C} = \{u_{i,j,k}/c \in FC_0^{\mathrm{Lip}}: i, j, k \in \boldsymbol{N}\}.$$

Hence the Carathéodory metric

$$\rho(\omega, \tilde{\omega}) = \sup\{u(\omega) - u(\tilde{\omega}): u \in \mathscr{C} \cup -\mathscr{C}, |Du(\omega)|_H \le 1, \mu\text{-a.e.}\omega\}$$

coincides wityh $\|\omega - \tilde{\omega}\|_\infty / c$. Note that the cut-off function $\rho_{0,r}(\omega) = (r - \|\omega\|_\infty / c) \vee 0$ belongs to $\mathscr{F}$. Indeed, $u_j(\omega) = \bigwedge_{l=1}^{j} (r - |\omega(s_l)|/c) \vee 0$ belongs to FC_0^{Lip} and $\{u_j\}$ is $\mathscr{E}_1^{1/2}$-bounded.

Hence we see that the conditions (**A**) and (**B**) are satisfied. But $\mu(\mathscr{L}(\boldsymbol{R}^d)) = \infty$ implies Cap $(\mathscr{L}(\boldsymbol{R}^d)) = \infty$. Owing to Corollary 5.1, $\mu(\|\omega\|_\infty \le r) \le \mu(|\omega(t)| \le r) \le c_2 r^d$ implies the conservativeness of $\mathbf{M}$, and the recurrence of $\mathbf{M}$ if $d = 1, 2$. On the other hand, for any fixed $\omega_0 \in \mathscr{L}(\boldsymbol{R}^d)$,

$$\begin{aligned}
\mathrm{Cap}(\{\omega_0\}) &\le \mathrm{Cap}(\{\bullet(t) = \omega_0(t)\}) \\
&\le \mathscr{E}_1((1 - |\omega_0(t) - \bullet(t)|/cr) \vee 0) \\
&\le (1/r^2 + 1)\mu(|\omega_0(t) - \bullet(t)| \le cr) \\
&\le (1/r^2 + 1) \int_{\{|y - \omega_0(t)| \le cr\}} p_1(y, y) dy \\
&\le (1/r^2 + 1) c_2 r^d \to 0
\end{aligned}$$

as $r \to 0$ if $d \geq 3$. Set $\varphi_\delta(\omega) = 1/\|\omega\|_\infty^\delta$ and $\varphi_\delta^t(\omega) = 1/|\omega(t)|^\delta$ with $\delta < d/2 - 1$. Then $\varphi_\delta \in (\dot{\mathscr{F}}_{\{0\}^c})_{loc}$ and $\varphi_\delta^t \in (\mathscr{F}_{\{\omega: \dot{\omega}(t)=0\}^c})_{loc}$ by Proposition 4.1, and the results in [21], in particular, φ_δ, $\varphi_\delta^t \in \dot{\mathscr{F}}_{loc}$ if $d \geq 3$. Note that $\varphi_\delta^t \notin \mathscr{F}_e$. In this case, φ_δ and φ_δ^t satisfy (**E**) and (**C**). Hence the conservativeness of $\mathbf{M}^{\varphi_\delta}$ and $\mathbf{M}^{\varphi_\delta}$ holds.

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