Certain unstable modular algebras over the mod $p$ Steenrod algebra

By

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1. Introduction

Let $p$ be an odd prime. We assume that all spaces are completed at $p$ by means of the Bousfield-Kan [4]. In this paper, a cohomology is taken with $\mathbb{Z}/p$ coefficients unless otherwise specified, and $H^*(-)$ means $H^*(-;\mathbb{Z}/p)$. Let $\mathcal{A}_p$ be the mod $p$ Steenrod algebra and $\mathcal{K}$ denote the category of unstable $\mathcal{A}_p$-algebras. The objects of $\mathcal{K}$ are called $\mathcal{K}$-algebras. For a space $X$, $H^*(X)$ is a $\mathcal{K}$-algebra. It is known, however, that a $\mathcal{K}$-algebra need not be of the form $H^*(X)$.

A $\mathcal{K}$-algebra $A$ is said to be realizable if $A$ is represented as the cohomology of some space, that is, there exists a space $X$ with $A \cong H^*(X)$ as $\mathcal{K}$-algebras. The realizability of an algebra is one of the major problems in the unstable homotopy theory. There are, indeed, many results, such as the Steenrod problem [6], the Cooke conjecture [1], and others.

In this paper we investigate the realizability of the following algebras for $n \geq 1$:

$$A_n = \mathbb{Z}/p[x_{2n}] \otimes \Lambda(y_{2n+1}, z_{2n+2p-1})$$

with Steenrod operation actions $\beta(x_{2n}) = y_{2n+1}$ and $\vartheta^1(y_{2n+1}) = z_{2n+2p-1}$. Our first result gives a necessary condition for $A_n$ to be a $\mathcal{K}$-algebra:

**Theorem A.** If $A_n$ is a $\mathcal{K}$-algebra, then $n = p^i$ for some $i \geq 0$.

By Theorem A, we concentrate on the algebras of the following form:

$$B_i = A_{p^i} = \mathbb{Z}/p[x_{2p^i}] \otimes \Lambda(y_{2p^i+1}, z_{2p^i+2p-1})$$

with $\beta(x_{2p^i}) = y_{2p^i+1}$ and $\vartheta^1(y_{2p^i+1}) = z_{2p^i+2p-1}$.

Actually, the $\mathcal{K}$-structure of $B_i$ is uniquely determined for $i > 0$ (see §2). On the other hand, $B_0$ has two $\mathcal{K}$-structures and the realizability of $B_0$ has completely determined by [2] (see Theorem 3.1). We show the $\mathcal{K}$-algebra $B_1$ is realizable as the cohomology of some $H$-spaces (see Proposition 3.2).
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The $\mathcal{A}$-algebra $B_2$ is realizable as follows: Let $X(p)$ be the $H$-space constructed by Harper [7] so that $H^*(X(p)) \simeq \Lambda(u_3, u_2 p + 1) \otimes \mathbb{Z}/p[u_2 p + 1]/(u_2 p + 1)$ with $\varphi^1(u_3) = u_2 p + 1$ and $\beta(u_2 p + 1) = u_2 p + 2$. Then the three-connective cover of $X(p)$ realizes $B_2$, namely we have

$$H^*(X(p)(3)) \simeq B_2.$$ 

Thus the realizability of $A_n$ is completely determined by the following:

**Theorem B.** If $B_i$ is realizable as the cohomology of a space, then $i = 0, 1$ or 2.

We shall prove Theorem B using the work of Lannes about his $T$-functor [8], which has been remarkable in the recent study of unstable homotopy theory.

This paper is organized as follows: In §2 and §3, we prove Theorem A and show the realizability of $B_1$, respectively. §4 is devoted to the proof of Theorem B.

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2. Proof of Theorem A

In this section we prove Theorem A, that is, if the algebra $A_n$ with the given Steenrod operation actions is a $\mathcal{A}$-algebra, then $n = p^i$ for some $i \geq 0$.

First we show that the ideal $I = (y_{2n+1}, z_{2n+2p-1})$ generated by $y_{2n+1}$ and $z_{2n+2p-1}$ is closed under the action of $\mathcal{A}_p$. If $x \in I$, then $\beta(x), \varphi^p(x) \in I$ for $i \geq 0$ since $\beta(y_{2n+1}) = \beta(z_{2n+2p-1}) = 0$ and $(\varphi^p(x)) = \varphi^{p^i}(x^p) = 0$. Hence $\mathbb{Z}/p[x_{2n}] \cong A_n/I$ has a $\mathcal{A}$-structure, and this implies that $n = p^i r$ for some $i \geq 0$ and $r | (p - 1)$. Thus, to complete the proof, we have only to show that $r = 1$.

We remark that the generator $x_{2p^r}$ can be taken to satisfy

$$(2.1) \varphi^p(x_{2p^r}) = rx_2^{s+1}$$

for $s = (p - 1)/r$. In fact, using the variation of a result of Adams-Wilkerson as in [3, Th. 4.2] (see also [1, Th. 2.1]), $\mathbb{Z}/p[x_{2p^r}]$ is isomorphic to $\mathbb{Z}/p[t_{2p^r}]^{\mathbb{Z}}$ with $\varphi^p(t_{2p}) = t_{2p}^{s} p$ as $\mathcal{A}$-algebras, where $\mathbb{Z}/r$ acts as ring automorphisms and as the usual multiplication on $t_{2p^r}$.

Now we divide the proof into two cases for $i > 0$ and $i = 0$. First assume that $i > 0$. Then, there is an Adem relation

$$(2.2) \varphi^p \beta = \varphi^1 \beta \varphi^{p-1} + \beta \varphi^p.$$

Using (2.1) and applying the operations of (2.2) on $x_{2p^r}$, we have

$$(2.3) \varphi^p(y_{2p^{r+1}}) = (r - 1)x_2^{s} y_{2p^{r+1}}.$$

For the dimensional reason, we can put $\varphi^p(z_{2p^{r+2p-1}} = ax_2^{s} z_{2p^{r+2p-1}}$ for some $a \in \mathbb{Z}/p$. Then applying (2.2) to $z_{2p^{r+2p-1}}$, we have $a = 0$. Thus
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(2.4) \[ P^p(z_{2p^0+2p-1}) = 0. \]

When \( i > 1 \), there is an Adem relation \( P^P P^{p^0} + P^1 P^1 = P^P P^1 \), and we apply these on \( y_{2p^0+1} \). Then, using also (2.3) and (2.4), we have \( P^1((r-1)x_{2p^0}y_{2p^0+1}) = P^P(z_{2p^0+2p-1}) = 0 \). Since \( P^1(x_{2p^0}y_{2p^0+1}) = x_{2p^0}^2 y_{2p^0+2p-1} \neq 0 \), we can conclude that \( r = 1 \). When \( i = 1 \), applying the operations in the Adem relation \( P^P P^{p^0} = P^{2p^0+1} + P^2 P^1 \) on \( y_{2p^0+1} \), we obtain \( P^1 P^P(y_{2p^0+1}) = -(r-1)x_{2p^0}^2 y_{2p^0+2p-1} \). On the other hand, using the Adem relation \( P^P P^P = 2P^{2p^0} + P^{2p^0-1} P^1 \), we get \( P^1 P^P(y_{2p^0+1}) = ((r-1)(r-2)/2)x_{2p^0}^2 y_{2p^0+2p-1} \). Thus we also have the result \( r = 1 \) in this case, which completes the proof for \( i > 0 \).

Next consider the case \( i = 0 \). Applying the Adem relation

(2.5) \[ 2P^1 P^1 = P^1 P^1 + P^1 P^1 \]
on \( x_{2r} \), we have

(2.6) \[ P^1(x_{2r+2p-1}) = 2(r-1)x_{2r} x_{2r+2p-1} - r(r-1)x_{2r}^2 y_{2r+1} \].

We apply (2.5) on \( y_{2r+1} \), and see that \( P^1 P^1(x_{2r+2p-1}) = 0 \). By (2.6), we also have \( P^1 P^1(x_{2r+2p-1}) = 2(r-1)xy_{2r+1} x_{2r+2p-1} \). From these equations, we can conclude that \( r = 1 \) since \( s \neq 0 \). Hence we have completed the proof of Theorem A.

3. Realization of \( B_0 \) and \( B_1 \)

By Theorem A, the realizability of \( A_n \) is concentrated on the following cases:

\[ B_i = A_p = Z/p[y_{2p^0+1}, z_{2p^0+2p-1}] \quad \text{for } i \geq 0 \]

with \( \beta(x_{2p^0}) = y_{2p^0+1} \) and \( P^1(y_{2p^0+1}) = z_{2p^0+2p-1} \).

First we consider the realizability of \( B_0 \). By (2.6) we have \( P^1(x_{2p^0+1}) = 0 \), and for the dimensional reason and unstability, we see that the \( A_p \) actions on \( B_0 \) are completely determined except for \( P^P(z_{2p^0+1}) \). Let \( B(p) \) be the \( H \)-space introduced by Mimura-Toda [9] so that \( H^*(B(p)) \cong \Lambda(u_3, u_{2p+1}) \) with \( P^1(u_3) = u_{2p+1} \), and \( B(p)B(3; p) \) denote the homotopy fiber of the map of degree \( p \)

\[ [p]: B(p) \to K(Z,3). \]

Then the following results of Aguadé-Broto-Santos [2] completely determine the realizability of \( B_0 \), by which it turns out that there are just two \( K \)-structures on \( B_0 \):

**Theorem 3.1 ([2])**

(1) On the \( K \)-algebra \( B_0 \), \( P^P(z_{2p^0+1}) = 0 \) or \( x_{2p^0}^2 z_{2p^0+1} \).

(2) If \( P^P(z_{2p^0+1}) = x_{2p^0}^2 z_{2p^0+1} \), then the \( K \)-algebra \( B_0 \) cannot be realizable as a cohomology of some space.

(3) If \( P^P(z_{2p^0+1}) = 0 \), then the \( K \)-algebra \( B_0 \) is realizable as the cohomology of \( B(p)B(3; p) \), namely

\[ H^*(B(p)B(3; p)) \cong B_0. \]
If there is a space $X$ so that $H^*(X) \cong B_0$ as $\mathcal{K}$-algebras, then $X \cong B(p)\langle 3; p \rangle$ up to $p$-completion.

For $i > 0$, if we impose the unstability condition on $B_i$, the $\mathfrak{p}$-actions on $B_1$ are completely determined except for $\mathfrak{p}^p(y_{2p^{i+1}})$ and $\mathfrak{p}^p(z_{2p^{i+2}p^{-1}})$ by dimensional reason. But it follows $\mathfrak{p}^p(y_{2p^{i+1}}) = \mathfrak{p}^p(z_{2p^{i+2}p^{-1}}) = 0$ from (2.3) and (2.4). Thus, $B_i$ for $i > 0$ has a unique $\mathcal{K}$-structure.

For the realizability of $B_1$, we have the following:

**Proposition 3.2.** The $\mathcal{K}$-algebra $B_1$ is realizable as the cohomology of an $H$-space.

**Proof.** There is an $H$-space $Y(p)$ satisfying $H^*(Y(p)) = \Lambda(u_3, u_4 p^{-1})$. In fact, $Y(3) = G_2$, the exceptional Lie group, if $p = 3$. For $p \geq 5$, as a special case of [5], we have an $H$-space $Y(p)$ which contains the cell complex

$$S^3 \cup_{\alpha} e^{4p^{-1}},$$

where $\alpha \in \pi_{4p^{-2}}(S^3) \cong Z/p$ is the generator. Computing the Serre spectral sequence, we see that the three-connective cover $Y(p)\langle 3 \rangle$ of $Y(p)$ realizes $B_1$, namely we have

$$H^*(Y(p)\langle 3 \rangle) \cong B_1,$$

which completes the proof.

4. **Proof of Theorem B**

We use the Lannes theory concerning the $T$-functor in the proof of Theorem B. Thus, we recall the theory first. The functor $T: \mathcal{K} \rightarrow \mathcal{K}$ is the left adjoint of the functor $H^*(BZ/p) \otimes -$, that is, there is an adjoint isomorphism $\text{Hom}_\mathcal{K}(T(A), B) \cong \text{Hom}_\mathcal{K}(A, H^*(BZ/p) \otimes B)$ for $\mathcal{K}$-algebras $A$ and $B$.

For a $\mathcal{K}$-map $f: A \rightarrow H^*(BZ/p)$, its adjoint restricts to a $\mathcal{K}$-map $T(A)^0 \rightarrow Z/p$, where $T(A)^0$ is the subalgebra of $T(A)$ of elements of degree 0. The connected component $T_\pi(A)$ of $T(A)$ corresponding to $f$ is defined by $T_\pi(A) = T(A) \otimes_{T(A)^0} Z/p$, and there is a natural $\mathcal{K}$-map $e_f: A \rightarrow T_f(A)$.

The evaluation map $e: BZ/p \times \text{Map}(BZ/p, X) \rightarrow X$ induces a $\mathcal{K}$-map $e^*$, and taking the adjoint of this yields a $\mathcal{K}$-map $\lambda^*: T(H^*(X)) \rightarrow H^*(\text{Map}(Z/p, X))$. For a map $\phi: BZ/p \rightarrow X$, there is a $\mathcal{K}$-map $\lambda^*_\phi: T_\phi(H^*(X)) \rightarrow H^*(\text{Map}(BZ/p, X)\phi)$ considering componentwise. Then, by definition, the composite $\lambda^*_\phi e^*$ is induced by the evaluation $e\phi: \text{Map}(BZ/p, X)\phi \rightarrow X$ at the base point. The following theorem is due to Lannes:

**Theorem 4.1** ([8]). For a map $\phi: BZ/p \rightarrow X$, if $T_\phi(H^*(X))^1 = 0$, then $\lambda^*_\phi: T_\phi(H^*(X)) \rightarrow H^*(\text{Map}(BZ/p, X)\phi)$ is an isomorphism.

Moreover, for each $\mathcal{K}$-algebra $A$, $T_f$ can be considered as a functor from $\mathcal{K}(A)$
to $\mathcal{K}(T_f(A))$, where $\mathcal{K}(A)$ denotes the subcategory of $\mathcal{K}$ each of whose objects has an $A$-module structure compatible with its $\mathcal{K}$-structure.

We also regard $T_f(M)$ as an object of $\mathcal{K}(A)$ through the natural $\mathcal{K}$-map $\varepsilon_f: A \to T_f(A)$ for any object $M$ of $\mathcal{K}(A)$, and $\varepsilon_f: M \to T_f(M)$ becomes a morphism of $\mathcal{K}(A)$-algebras. It is well known that $T_f$ is exact, and commutes with suspensions and tensor products.

To prove Theorem B, we need the $T$-functor for $B_i$. As is known, $H^*(BZ/p) \cong \Lambda(w_1) \otimes \mathbb{Z}/p[w_2]$ with $\beta(w_1) = w_2$. Now we define a $\mathcal{K}$-map $f: B_i \to H^*(BZ/p)$ as $f(x_{2^p}) = w_2^2$ and $f(y_{2^p+1}) = f(z_{2^p+2^{p-1}}) = 0$.

**Proposition 4.2.** $\varepsilon_f: B_i \to T_f(B_i)$ is an isomorphism.

**Proof.** Let $C_i = \mathbb{Z}/p[x_{2^p}] \otimes \Lambda(y_{2^p+1})$, and $k: B_i \to C_i$ be the quotient map. Then it is obvious that $k^*: \text{Hom}_\mathcal{K}(C_i, H^*(BZ/p)) \to \text{Hom}_\mathcal{K}(B_i, H^*(BZ/p))$ is an isomorphism. Thus, by the results of Aguadé-Broto-Notbohm [1], $T_f(C_i) \cong T_f(B_i)$ for a non trivial map $g: C_i \to H^*(BZ/p)$, and $\varepsilon_f: C_i \to T_f(C_i)$ is an isomorphism. Since $T_f$ is exact, we have the following commutative diagram whose horizontal arrows are exact sequences of $\mathcal{K}(B_i)$-algebras:

$$
\begin{array}{cccc}
0 & \to & z_{2^p+2^{p-1}} C_i & \to & B_i & \to & C_i & \to & 0 \\
& & \varepsilon_f & & \varepsilon_f & & \varepsilon_f & & \cong \\
0 & \to & T_f(z_{2^p+2^{p-1}} C_i) & \to & T_f(B_i) & \to & T_f(C_i) & \to & 0.
\end{array}
$$

(4.1)

Since $z_{2^p+2^{p-1}} C_i \cong \Sigma^{2^p+2^{p-1}} C_i$ as $\mathcal{K}(B_i)$-algebras and $T_f$ commutes with suspensions, we have $T_f(z_{2^p+2^{p-1}} C_i) \cong z_{2^p+2^{p-1}} C_i$. Hence we can conclude that $\varepsilon_f: B_i \to T_f(B_i)$ is an isomorphism by the diagram (4.1), which completes the proof.

**Proof of Theorem B.** We assume that $B_i$ is realizable, that is, $B_i \cong H^*(X)$ for some space $X$. A result of Lannes [8] implies that there is a map $\phi: BZ/p \to X$ such that $\phi^* = f$, and then the evaluation map $\phi: \text{Map}(BZ/p, X) \to X$ is a homotopy equivalence by Theorem 4.1 and Proposition 4.2. Let $i: BZ/p \to \text{Map}(BZ/p, X)$ be the adjoint of $\phi\omega$, where $\omega$ is the multiplication map for the $H$-structure of $BZ/p$. We have the following commutative diagram of fibrations:

$$
BZ/p = BZ/p \to EBZ/p \to B^2Z/p
$$

\[\begin{array}{ccc}
\phi & \downarrow & j \\
X & \xrightarrow{i} & M \\
\cong & & \xrightarrow{=} j \\
M_{BZ/p} & \xrightarrow{j} & B^2Z/p,
\end{array}\]

(4.2)

where $M = \text{Map}(BZ/p, X)$ and $M_{BZ/p} = EBZ/p \times_{BZ/p} M$ is the Borel construction. We consider the Serre spectral sequence of the bottom fibration whose $E_2$-term is given as $E_2^{*,*} = H^*(B^2Z/p) \otimes B_i$.

As is known, $H^*(B^2Z/p) \cong \mathbb{Z}/p[\eta_2, \beta, p^{A}\beta \eta_2 | j \geq 0] \otimes \Lambda(\beta \eta_2, p^{A}\beta \eta_2 | j \geq 0)$, where
\[ \mathcal{P}^i = \mathcal{P}^i \ldots \mathcal{P}^1 \] and \( \eta_2 \) denotes the fundamental class. We fix the basis \( \Gamma \) of the vector space \( H^*(B^2 \mathbb{Z}/p) \) by taking all monomials of \( \eta_2, \beta \mathcal{P}^j \beta \eta_2, \beta \eta_2 \) and \( \mathcal{P}^j \beta \eta_2 \) for \( j \geq 0 \). For the \( \mathcal{A}_p^j \)-actions on indecomposables, by definition and unstability, we have \( \mathcal{P}^{j+1}(\mathcal{P}^j \beta \eta_2) = \mathcal{P}^{j+1} \beta \eta_2 \) and \( \mathcal{P}^1(\mathcal{P}^j \beta \eta_2) = 0 \). Furthermore, we need the following:

\textbf{Lemma 4.3 ([1])}.

\begin{enumerate}
\item \( \mathcal{P}^1(\beta \mathcal{P}^j \beta \eta_2) = \begin{cases} 
0 & \text{if } j = 0, \\
(\beta \mathcal{P}^j \beta \eta_2)^p & \text{if } j > 0.
\end{cases} \)
\item \( \mathcal{P}^{j+1}(\beta \mathcal{P}^j \beta \eta_2) = \beta \mathcal{P}^{j+1} \beta \eta_2 \) for \( j \geq 0 \).
\item \( \mathcal{P}^k(\mathcal{P}^j \beta \eta_2) = \mathcal{P}^k(\beta \mathcal{P}^j \beta \eta_2) = 0 \) for \( k \neq 0, j + 1 \).
\end{enumerate}

From the diagram (4.2), we have \( \tau(x_{2p}) = \mathcal{P}^{j-1} \beta \eta_2 + \delta_{2p+1} \) since \( \phi(x_{2p}) = w_{2p} \) and \( \tau(w_{2p}) = \mathcal{P}^{j-1} \beta \eta_2 \), where \( \tau \) denotes the transgression and \( \delta_{2p+1} \) is some decomposable element in \( H^*(B^2 \mathbb{Z}/p) \). From now on, we assume that \( i \geq 3 \), and deduce a contradiction from this assumption.

We set
\[ \theta_{2p+1} = (\beta \mathcal{P}^{j-1} \beta \eta_2)^p + \mathcal{P}^1 \beta \delta_{2p+1} \]
in \( H^{2p+2} \mathbb{Z}/p \). Since \( j^*(\theta_{2p+2}) = \mathcal{P}^{j-1} \beta (j^*(\mathcal{P}^{j-1} \beta \eta_2 + \delta_{2p+1})) = 0 \), there exists an element of total degree \( 2p+2 \) which kills \( \theta_{2p+2} \) in the spectral sequence. On the other hand, we shall show that \( \theta_{2p+2} \) cannot be killed in the spectral sequence, which causes a contradiction.

First, we remark the following:

\textbf{Lemma 4.4.} \textit{When we represent} \( \theta_{2p+2} \) \textit{as a linear combination with basis} \( \Gamma \), \textit{it must contain the term} \( (\beta \mathcal{P}^{j-1} \beta \eta_2)^p \).

\textbf{Proof.} If \( i \neq 4 \), then we have the conclusion since we can see that \( \mathcal{P}^{j-1} \beta (\delta_{2p+1}) \) does not contain the term \( (\beta \mathcal{P}^{j-1} \beta \eta_2)^p \) by the \( \mathcal{X} \)-structure of \( H^*(B^2 \mathbb{Z}/p) \). Thus we assume that \( i = 4 \). We set
\[ \alpha_{2p+1} = (\beta \mathcal{P}^{j-1} \beta \eta_2)(\beta \mathcal{P}^1 \beta \eta_2)^p - p - 2(\mathcal{P}^{j-1} \beta \eta_2)(\beta \mathcal{P}^1 \beta \eta_2), \]
\[ \beta_{2p+1} = (\beta \mathcal{P}^{j-1} \beta \eta_2)(\beta \mathcal{P}^1 \beta \eta_2)^p - p - 1(\mathcal{P}^1 \beta \eta_2), \]
and
\[ \gamma_{2p+1} = (\mathcal{P}^{j-1} \beta \eta_2)(\beta \mathcal{P}^1 \beta \eta_2)^p - p - 1(\mathcal{P}^1 \beta \eta_2). \]
Then, for the dimensional reason, we can put \( \delta_{2p+1} = a\alpha_{2p+1} + b\beta_{2p+1} + c\gamma_{2p+1} + d_{2p+1} \) for some \( a, b, c \in \mathbb{Z}/p \), where \( d_{2p+1} \) is an element which does not contain
the term $\alpha_{2^p+1}$, $\beta_{2^p+1}$ or $\gamma_{2^p+1}$. We note that $\mathcal{P}^1(\beta x_{2^p+1})$, $\mathcal{P}^1(\beta_{2^p+1})$ and $\mathcal{P}^1(\gamma_{2^p+1})$ contain the term $(\beta^{\mathcal{P}1}\beta\eta_2)^p$ while $\mathcal{P}^1(\delta_{2^p+1})$ does not contain this term.

Using $\mathcal{P}(x_{2^p}) = \mathcal{P}(x_{2^p}) = 0$ and the $\mathcal{K}$-structure of $H^*(B^2Z/p)$, we can show that $a = b = c = 0$ by a routine calculations. Then $\mathcal{P}^1(\beta_{2^p+1}) = \mathcal{P}^1(\beta_{2^p+1})$ does not contain the term $(\beta^{\mathcal{P}1}\beta\eta_2)^p$, and we have the required conclusion.

For the dimensional reason, the element which hits $\theta_{2^p+2}$ must have one of the following forms:

$$\lambda_{2^p-1} \otimes x_{2^p}, \quad \kappa_{2^p-2} \otimes y_{2^p+1}, \quad \nu_{2^p-2} \otimes Z_{2^p+2}.$$

If $i \geq 4$, then any element of the above form cannot hit $\theta_{2^p+2}$ by Lemma 4.4 and the dimensional reason.

For $i = 3$, the only possible case $\theta_{2^p+2}$ can be hit is that $\kappa_{2^p-2} = (\beta^{\mathcal{P}1}\beta\eta_2)^p - 1 + \kappa_{2^p+2}(p-1)$ and $\tau(y_{2^p+1})$ contain the term $(\beta^{\mathcal{P}1}\beta\eta_2)^p - p + 1$, where $\kappa_{2^p+2}(p-1) \in H^*(B^2Z/p)$ is some element which does not contain the term $(\beta^{\mathcal{P}1}\beta\eta_2)^p$. But we have the following:

**Lemma 4.5.** When we represent $\tau(y_{2^p+1})$ as a linear combination with basis $\Gamma$, it does not contain the term $(\beta^{\mathcal{P}1}\beta\eta_2)^p - p + 1$.

**Proof.** Since $\tau(y_{2^p+1}) = \beta^{\mathcal{P}1}\beta\eta_2 + \beta(\delta_{2^p+1})$, it is sufficient to show that $\delta_{2^p+1}$ does not contain the term $(\beta^{\mathcal{P}1}\beta\eta_2)^p - p + 1$. For the dimensional reason, we can put $\delta_{2^p+1} = d(\beta^{\mathcal{P}1}\beta\eta_2)^p - p + \mathcal{P}(\delta_{2^p+1})$ for some $d \in Z/p$. Then we have $\mathcal{P}(\tau(x_{2^p})) = d(\mathcal{P}^1\beta\eta_2)(\beta^{\mathcal{P}1}\beta\eta_2)^p - p + \mathcal{P}(\delta_{2^p+1})$, where $\mathcal{P}(\delta_{2^p+1})$ does not contain the term $(\beta^{\mathcal{P}1}\beta\eta_2)(\beta^{\mathcal{P}1}\beta\eta_2)^p - p$. This implies that $d = 0$ since $\mathcal{P}(x_{2^p}) = 0$, and we have the required conclusion.

Then, this causes a contradiction, and we have completed the proof of Theorem B.

