J. Math. Kyoto Univ. (JMKYAZ)
38-2 (1998), 229-240

BGG-resolution for α-stratified modules over simply-laced finite-dimensional Lie algebras

By

V. Futorny* and V. Mazorchuk

1. Introduction

This paper is a sequel of [6] where the submodule structure of α-stratified (i.e. torsion free with respect to the subalgebra corresponding to a root α) generalized Verma modules was studied. The results obtained there generalize the classical theorem of Bernstein-Gelfand-Gelfand on Verma module inclusions. The crucial role in the study is played by the generalized Weyl group W_α that acts on the space of parameters of generalized Verma modules.

Let G be a simple finite-dimensional Lie algebra over the complex numbers with a simply-laced Coxeter-Dynkin diagram (i.e. there are no multiple arrows). In the present paper for any such algebra we construct a strong BGG-resolution for α-stratified irreducible modules in the spirit of [1,10]. The non-simply-laced case is more complicated (cf. [6]). In particular, the proof of the crucial Theorem 4 is based on the fact that the diagram is simply-laced.

The structure of the paper is the following. In Section 2 we collect the notation and preliminary results. A weak generalized BGG-resolution is constructed in Section 3. Here we follow closely [1]. Section 4 contains an extension lemma for α-stratified modules which generalizes a well-known result of Rocha-Caridi for Verma modules [10]. Our proof is analogous to the one of Humphreys for Verma modules [8]. In Section 5 we study the maximal submodule of the generalized Verma module and construct a strong generalized BGG-resolution for α-stratified irreducible modules in Section 6. Finally, in Section 7 we give a character formula for certain α-stratified irreducible modules.

2. Notation and preliminary results

Let $\boldsymbol{C}$ denote the complex numbers, $\boldsymbol{Z}$ all integers, $\boldsymbol{N}$ all positive integers and $\boldsymbol{Z}_+ = \boldsymbol{N} \cup \{0\}$.

Let H be a Cartan subalgebra of G and Δ the root system of G.

Let π be a basis of Δ containing α, $\Delta_\pm = \Delta_\pm(\pi)$ the set of positive (negative) roots with respect to π. For any $S \subset \pi$ let $\Delta_\pm(S)$ be a subset generated by S (it

---

Communicated by Prof. M. Jimbo, July 15, 1996
*Regular Associate of the ICTP

consists of all the roots in $\Delta_{\pm}$ which are linear combinations of elements from S). Also let $\rho=\frac{1}{2}\sum_{\gamma\in\Delta_+}\gamma$. For $\lambda, \mu\in H^*$ we will say that $\lambda\geq_\alpha\mu$ if $\lambda-\mu=k_\alpha\alpha+\sum_{\beta\in\pi\setminus\{\alpha\}}k_\beta\beta$, $k_\alpha\in\boldsymbol{Z}$, $k_\beta\in\boldsymbol{Z}_+$. Further $(\cdot,\cdot)$ will denote the standard form on H^*. If $\beta\in\Delta_+$ then $s_\beta\in W$ will denote a corresponding reflection in H^*: $s_\beta(\lambda)=\lambda-\frac{2(\lambda,\beta)}{(\beta,\beta)}\beta$.

Fix a basis $\{h_\beta, \beta\in\pi\}$ of H normalized by the condition $\beta(h_\beta)=2$ and a non-zero element X_γ in each root subspace G_γ, $\gamma\in\Delta$ such that $[X_\beta, X_{-\beta}]=h_\beta$, $\beta\in\pi$.

Denote $N_\pm=\sum_{\gamma\in\Delta_+}G_{\pm\gamma}$, $N_\pm^\alpha=\sum_{\gamma\in\Delta_+\setminus\{\alpha\}}G_{\pm\gamma}$, $H^\alpha=\{h\in H\mid\alpha(h)=0\}$. Then we have

$$G=N_-\oplus H\oplus N_+=G^\alpha\oplus N_-^\alpha\oplus H^\alpha\oplus N_+^\alpha$$

where G^α is generated by $G_{\pm\alpha}$. Also let $H_\alpha=G^\alpha\cap H$ and thus $G^\alpha=G_\alpha\oplus H_\alpha\oplus G_{-\alpha}$.

For a Lie algebra A we will denote by $U(A)$ the universal enveloping algebra of A and by $Z(A)$ the centre of $U(A)$.

For $m\in\boldsymbol{Z}_+$ denote by $U(G)^{(m)}$ the subspace in $U(G)$ spanned by the elements of degree m (with respect to the fixed PBW-basis above).

Consider a linear space $\Omega=H^*\times C$. For (λ,p) and (μ,q) in Ω we say that $(\lambda,p)>(\mu,q)$ if $\lambda-\mu=\sum_{\beta\in\pi\setminus\{\alpha\}}n_\beta\beta$, $n_\beta\in\boldsymbol{Z}_+$ and $\lambda\neq\mu$.

Let $r\in\boldsymbol{C}$. Consider a linear space $B_r=\sum_{\beta\in\pi\setminus\{\alpha\}}\boldsymbol{C}\beta+r\alpha$ with a fixed point $r\alpha$, a $\boldsymbol{Z}$-module $\tilde{B}_r=B_r\oplus\boldsymbol{Z}\alpha$ and let $\Omega_r=B_r\times\boldsymbol{C}$, $\tilde{\Omega}_r=\tilde{B}_r\times\boldsymbol{C}$.

In [6] we introduced the generalized Weyl grop W_α acting on the space Ω_r in the following way.

Consider a partition of π: $\pi=\pi_1\cup\pi_2$ where $\pi_1=\{\gamma\in\pi\mid\alpha+\gamma\in\Delta\}$, $\pi_2=\{\gamma\in\pi\mid\alpha+\gamma\notin\Delta\}$. For $(\lambda,p)\in\Omega_r$ and $\beta\in\pi_1$ denote

$$n_\beta^\pm(\lambda,p)=\frac{1}{2}(\lambda(h_\alpha+2h_\beta)\pm p)$$

and define $(\lambda_\beta,p_\beta)\in\Omega_r$, where $\lambda_\beta=\lambda-n_\beta^-(\lambda,p)\beta$, $p_\beta=n_\beta^+(\lambda,p)$.

For each $\beta\in\pi$ consider $l_\beta\in GL(\Omega_r)$ such that

$$l_\beta(\lambda,p)=\begin{cases}(\lambda,-p), & \beta=\alpha\\ (s_\beta\lambda,p), & \beta\in\pi_2\setminus\{\alpha\}\\ (\lambda_\beta,p_\beta), & \beta\in\pi_1.\end{cases}\qquad (*)$$

Then $W_\alpha=\langle l_\beta, \beta\in\pi\rangle$.

It is easy to see that W_α is isomorphic to the Weyl group W. Moreover, there exists a root system $\Delta_{\alpha,r}$ in Ω_r with respect to which W_α is the Weyl group [6]. We denote by σ_β the reflection in Ω_r corresponding to a root $\beta\in\Delta_{\alpha,r}$. Also let $(\cdot,\cdot)_r$ denote a corresponding nondegenerate form on Ω_r and $\zeta=\zeta_{\alpha,r}:\Delta\to\Delta_{\alpha,r}$ be a natural bijection.

Let pr_i, $i=1,2$ be a natural projection on the i-th component of Ω_r.

For a G-module V with a Jordan-Hölder series let $\mathcal{J}H(V)$ denote the set of all irreducible subquotients of V. A G-module V is called weight module if

$$V=\bigoplus_{\lambda\in H^*} V_\lambda$$

where all $V_\lambda=\{v\in V\,|\,hv=\lambda(h)v \text{ for all } h\in H\}$ are finite-dimensional. If $V_\lambda\neq 0$ then λ is called a weight of V. Denote by supp V the set of all weights of V. A weight λ is called a highest weight if $V_{\lambda+\beta}=0$ for all $\beta\in\Delta_+$. A weight G-module V is said to be α-stratified if X_α and $X_{-\alpha}$ act injectively on V.

Let V be a weight G-module. A non-zero element $v\in V$ is said to be α-primitive (with respect to G) if $v\in V_\lambda$ for some $\lambda\in H^*$ and $N_+^\alpha v=0$.

It is known that $c=(h_\alpha+1)^2+4X_{-\alpha}X_\alpha$ generates $Z(G^\alpha)$. Let $a,b\in \boldsymbol{C}$. Any such pair defines a unique indecomposable weight G^α-module $F(a,b)$ on which $X_{-\alpha}$ acts injectively and where a is an eigenvalue of h_α and b is an eigenvalue of c. The module $F(a,b)$ has a $\boldsymbol{Z}$-basis $\{v_i\}$ such that $X_{-\alpha}v_i=v_{i-1}$, $h_\alpha v_i=(a+2i)v_i$ and $X_\alpha v_i=\frac{1}{4}(b-(a+2i+1)^2)v_{i+1}$.

One can easily check (see [6, Lemma 2.2]) that the module $F(a,b)$ is torsion free if and only if $b\neq(a+2l+1)^2$ for all $l\in\boldsymbol{Z}$.

Set $\Omega^s=\{(\lambda,p)\in\Omega\,|\,p\neq\pm(\lambda(h_\alpha)+2l) \text{ for all } l\in\boldsymbol{Z}\}$, $\Omega_r^s=\Omega_r\cap\Omega^s$, $\tilde{\Omega}_r^s=\tilde{\Omega}_r\cap\Omega^s$. Hence, if $(\lambda,p)\in\Omega^s$ then $F((\lambda-\rho)(h_\alpha),p^2)$ is irreducible and torsion free.

Since $H=H_\alpha\oplus H^\alpha$, any element $\lambda\in H^*$ can be written uniquely as $\lambda=\lambda_\alpha+\lambda^\alpha$ where $\lambda_\alpha\in \mathrm{H}_\alpha^*$ and $\lambda^\alpha\in(H^\alpha)^*$. Let $a,b\in\boldsymbol{C}$ and $\lambda\in H^*$ such that $(\lambda-\rho)(h_\alpha)=(\lambda_\alpha-\rho)(h_\alpha)=a$. Define an H-module structure on $F(a,b)$ by letting $hv=\lambda^\alpha(h)v$ for any $h\in H^\alpha$ and any $v\in F(a,b)$. Thus $F(a,b)$ becomes a $G^\alpha+H$-module. Moreover, we can consider $F(a,b)$ as $D=H+G^\alpha+N_+^\alpha$-module with a trivial action of N_+^α.

The generalized Verma module associated with α, λ, b is defined as follows:

$$M_\alpha(\lambda,b)=U(G)\underset{U(D)}{\otimes}F(a,b).$$

Set $M(\lambda,b)=M_\alpha(\lambda,b)$.

It will be more convenient to use a slightly different parametrization of generalized Verma modules replacing $M(\lambda,b)$ by $M(\lambda,p)$ where $p^2=b$. Thus we always have $M(\lambda,p)=M(\lambda,-p)$.

Note that module $M(\lambda,p)$ has a unique maximal submodule and it is α-stratified if and only if $(\lambda,p)\in\Omega^s$.

We will denote by $Z^*(G)$ the set of all homomorphisms from $Z(G)$ to $\boldsymbol{C}$. It follows from [3, Corollary 1.11] that module $M(\lambda,p)$ admits a central character $\theta_{(\lambda,p)}\in Z^*(G)$, i.e. $zv=\theta_{(\lambda,p)}(z)v$ for any $z\in Z(G)$ and $v\in M(\lambda,p)$.

Denote by $L(\lambda,p)$ the unique irreducible quotient of $M(\lambda,p)$.

Lemma 1. ([3, Corollary 3.4]). *$L(\lambda,p)\simeq L(\lambda+k\alpha,p)$ fo all $k\in \mathbf{Z}$.*

The following order on Ω_r was introduced in [6]: Let (λ,p), $(\mu,q)\in\Omega_r$ and $\beta\in\Delta_{\alpha,r}$. We will write $(\lambda,p)\overset{\beta}{\rightarrow}(\mu,q)$ if $(\mu,q)=\sigma_\beta(\lambda,p)$ and $(\beta,(\lambda,p))_r\in \mathbf{N}$ for $\beta\neq\zeta(\alpha)$. Then $(\mu,q)\ll(\lambda,p)$ will mean that there exists a sequence $\beta_1,\beta_2,\cdots,\beta_k$ in $\Delta_{\alpha,r}$ such that $(\mu,q)\overset{\beta_1}{\rightarrow}\sigma_{\beta_1}(\mu,q)\overset{\beta_2}{\rightarrow}\cdots\overset{\beta_{k-1}}{\rightarrow}\sigma_{\beta_{k-1}}\cdots\sigma_{\beta_1}(\mu,q)\overset{\beta_k}{\rightarrow}(\lambda,p)$.

The main result of [6, Theorem 7.6] is the following theorem which describes the structure of α-stratified generalized Verma module with respect to the order on Ω_r.

Theorem 1. *Let (λ,p) and $(\mu,q)\in\tilde{\Omega}_r^s$. The following statements are equivalent:*

1. $M(\mu,q)\subset M(\lambda,p)$;
2. $L(\mu,q)\in\mathcal{J}H(M(\lambda,p))$;
3. *There exists $k\in\mathbf{Z}$ such that $(\mu+k\alpha,q)\ll(\lambda,p)$.*

Let

$$P^{++}=\{(\lambda,p)\in\Omega_r^s\mid w(\lambda,p)\ll(\lambda,p)\text{ for all } w\in W_\alpha\}.$$

In this paper we discuss the construction of analogues of the weak and the strong BGG-resolutions for irreducible modules $L(\lambda,p)$ with $(\lambda,p)\in P^{++}$.

3. Cohomological part of the weak BGG-resolution

Let $P=\Delta_+(\pi\backslash\{\alpha\})$ and let B be a subalgebra of G generated by all root subspaces

$$G_{-\beta},\ \beta\in P.$$

An element (λ,p) will be called minimal if

$$\mathrm{pr}_1((\lambda,p)-\sigma_\beta(\lambda,p))=\beta$$

holds for every $\beta\in\pi\backslash\{\alpha\}$. In this section we fix a minimal element (λ,p). This element plays a role of the trivial highest weight in the case of Verma modules.

Consider the subalgebra B as a module over a subalgebra $A=N_+^\alpha+H$ under the following action:

$$h\cdot a=[h,a]+\lambda(h)a$$

for any $h\in H$ and $a\in B$, and

$$b\cdot a=\begin{cases}[b,a], & [b,a]\in B;\\ 0, & [b,a]\notin B.\end{cases}$$

for all $b\in N_+^\alpha$ and $a\in B$. Clearly, this action can be naturally extended to the action on the exterior powers $\bigwedge^k B$ for all $k\in\mathbf{N}$.

Let ε be the unique eigenvalue on $M(\lambda,p)$ of a quadratic Casimir operator

$$C=h_0+\sum_{\alpha\in\Delta_+} X_{-\alpha}X_\alpha,$$

where h_0 is a certain fixed element in $S(H)$. Note that this eigenvalue is determined uniquely by (λ,p) via a generalized Harish-Chandra homomorphism [5].

Define $U_\varepsilon=U(G)/(C-\varepsilon)$ and consider the following G-modules:

$$D_k=U_\varepsilon \underset{U(A)}{\otimes} \overset{k}{\bigwedge} B,$$

where $k\in \boldsymbol{Z}_+$.

Following [1], for $k\in \boldsymbol{N}$ define the homomorphisms $d_k: D_k\to D_{k-1}$ as follows:

$$d_k(X\otimes X_1\wedge X_2\wedge\cdots\wedge X_k)=$$

$$\sum_{i=1}^{k}(-1)^{i+1}XX_i\otimes X_1\wedge\cdots\wedge\hat{X}_i\wedge\cdots\wedge X_k$$

$$+\sum_{1\leqslant i<j\leqslant k}(-1)^{i-j}X\otimes[X_i,X_j]\wedge X_1\wedge\cdots\wedge\hat{X}_i\wedge\cdots\wedge\hat{X}_j\wedge\cdots\wedge X_k.$$

Since $d_k\circ d_{k+1}=0$ we immediately obtain that the sequence

$$0\leftarrow D_0/\operatorname{Im} d_1\overset{\eta}{\leftarrow} D_0\overset{d_1}{\leftarrow} D_1\overset{d_2}{\leftarrow} D_2\overset{d_3}{\leftarrow}\cdots$$

is a complex. Here η is a natural projection. We will denote this complex by $V_\alpha(\lambda,\varepsilon)$.

Theorem 2. *The complex $V_\alpha(\lambda,\varepsilon)$ is exact.*

Proof. The algebra U_ε inherits the natural gradation on $U(G)$ by the degree of the monomials. Using that we can define a gradation on D_k. For $l\geq k$ let $D_k^{(l)}$ be a subspace spanned by the elements $x\otimes y$ where x is an element in U_ε of degree less than or equal to $l-k$ and $y\in\overset{k}{\bigwedge} B$. It is clear that $d_k(D_k^{(l)})\subset D_{k-1}^{(l)}$ and thus d_k induces a homomorphism

$$d_k^{(l)}: D_k^{(l)}/D_k^{(l-1)}\to D_{k-1}^{(l)}/D_{k-1}^{(l-1)}.$$

Denote $\hat{D}_k^{(l)}=D_k^{(l)}/D_k^{(l-1)}$. Also set $M^{(l)}=\hat{D}_0^{(l)}/\operatorname{Im} d_1^{(l)}$ and let $\eta^{(l)}$ be a corresponding induced homomorphism.

It is sufficient to show for every l the exactness of the complex

$$0\leftarrow M^{(l)}\overset{\eta^{(l)}}{\leftarrow}\hat{D}_0{}^{(l)}\overset{d_1^{(l)}}{\leftarrow}\hat{D}_1{}^{(l)}\overset{d_2^{(l)}}{\leftarrow}\hat{D}_2{}^{(l)}\overset{d_3^{(l)}}{\leftarrow}\cdots \tag{1}$$

By the PBW theorem for every $k \in \boldsymbol{Z}_+$ one can write:

$$D_k = \left(U(N_-) \otimes \bigwedge^k B \right) \oplus \left(\sum_{m \geq 1} X_\alpha^m U(N_-^\alpha) \otimes \bigwedge^k B \right)$$

and hence

$$\hat{D}_k^{(l)} \simeq \left(U(N_-)^{(l-k)} \otimes \bigwedge^k B \right) \oplus \left(\sum_{m=1}^{l-k} X_\alpha^m U(N_-^\alpha)^{(l-k-m)} \otimes \bigwedge^k B \right).$$

We will denote by $s_\alpha N_-$ a subalgebra generated by N_-^α and X_α. Let N_-^B ($s_\alpha N_-^B$ resp.) be a subalgebra generated by $X_{-\beta}$, $\beta \in \Delta_+$, $\beta \notin \Delta_+(\pi \setminus \{\alpha\})$ ($\beta \in s_\alpha \Delta_+$, $\beta \notin s_\alpha \Delta_+(\pi \setminus \{\alpha\})$ resp.) and let $S_j(B)$ be a set of all homogeneous elements of degree j in the symmetric algebra of B. Then

$$\hat{D}_k^{(l)} \simeq \left(\sum_{j=0}^{l-k} U(N_-^B)^{(l-j-k)} S_j(B) \otimes \bigwedge^k B \right) \oplus \left(\sum_{j=0}^{l-k} U(s_\alpha N_-^B)^{(l-j-k)} S_j(B) \otimes \bigwedge^k B \right).$$

For any homogeneous element $u \in U(N_-^B)$ ($u \in U(s_\alpha N_-^B)$ resp.) of degree $l-j-k$ we have that $d_k^{(l)}(uS_j(B) \otimes \wedge^k B) \subset uS_{j+1}(B) \otimes \wedge^{k-1} B$. Therefore it induces a complex which is in fact the Koszul complex [2] and hence is exact. Using the PBW theorem we conclude that the complex (1) decomposes into a direct sum of exact complexes and therefore is exact. The theorem is proved.

For a weight G-module V consider a formal character

$$\operatorname{ch} V = \sum_{\mu \in H^*} (\dim V_\mu) e^\mu.$$

Corollary 1.

$$\operatorname{ch} D_0 / \operatorname{Im} d_1 = \sum_{i \geq 0} (-1)^i \operatorname{ch} D_i.$$

4. Extension lemma

In this section we prove an analogue of the extension lemma ([8, 10]) for α-stratified generalized Verma modules.

Recall that α-stratified generalized Verma modules are the important objects in the category O^α which was studied in [3, 7]. This category has properties similar to those of the classical category O. It was shown, in particular, that O^α has enough projective objects.

Theorem 3. *Let* (λ,p), $(\mu,q) \in \Omega_r^s$. *If*

$$\operatorname{Ext}_{O^\alpha}(M(\mu,q), M(\lambda,p)) \neq 0$$

then $(\mu,q)\ll(\lambda,p)$.

Proof. The proof is based on the properties of the category O^α [7] and is analogous to the proof of the extension lemma in [8].

Consider a subgroup $W_\alpha^+\subset W_\alpha$ generated by all l_β, $\beta\in\pi\setminus\{\alpha\}$. Since W_α^+ is a Coxeter group we have a well-defined notion of the length $l(w)$ for any $w\in W_\alpha^+$.

Corollary 2. *For* $(\lambda,p)\in P^{++}$ *and* w_1, $w_2\in W_\alpha^+$ *with* $l(w_1)=l(w_2)$ *holds*

$$\mathrm{Ext}_{O^\alpha}(M(w_1(\lambda,p)),\ M(w_2(\lambda,p)))=0.$$

5. The structure of the maximal submodule of $M((\lambda,p)$

The main result of this section is the following

Theorem 4. *The module* $D_0/\mathrm{Im}\,d_1$ *is irreducible.*

To prove Theorem 4 we will need several lemmas.

Let $K=\Delta_-(\pi)\setminus(-P)$ and $K(G)$ be a subalgebra of $U(G)$ generated by X_β, $\beta\in K$.

Let M be a G-module. A non-zero weight vector $v\in M$ will be said to be quasi-primitive if there exists a proper submodule $F\subset M$ such that v becomes α-primitive in the quotient M/F.

Lemma 2. *Let* $(\mu,q)\in\Omega_r^s$, *F is a proper submodule of* $M(\mu,q)$, $0\neq v\in M(\mu,q)_{\mu-\rho}$ *and* $0\neq v'\in K(G)v\cap F$ *is a weight vector with weight* ν. *Then* $K(G)v$ *contains a quasi-primitive vector of weight* λ *with* $\mu-\rho>_\alpha\lambda\geq_\alpha\nu$.

Proof. Since module F is α-stratified and finitely generated one can choose a set of quasi-primitive generators $w_1,\cdots,w_l$ of F such that $w_i\in U(N_-)v$ for all i and

$$X_{-\alpha}^k v'\in\sum_i U(N_-)w_i$$

for sufficiently large $k>0$. It immediately follows from the PBW theorem that there exists i such that $w_i\in K(G)v$. Also note that if λ_i is a weight of w_i then $\lambda_i\geq_\alpha\nu$. This completes the proof of lemma.

Lemma 3. *Let* $(\mu,q)\in\Omega_r^s$ *and* $0\neq v\in M(\mu,q)_{\mu-\rho}$. *Then* $K(G)v$ *has no quasi-primitive elements except* $CX_{-\alpha}^k v$, $k\geq0$.

Proof. A direct calculation shows that for any $\tau\in H^*$ the existence of a non-zero α-primitive element in $K(G)v$ of weight $\mu-\tau$ is equivalent to the system of linear equations on $\mu(h_\beta)$, $\beta\in\pi$, and does not depend on q. But this contradicts Theorem 1. It implies that the only α-primitive elements in $K(G)v$ are $CX_{-\alpha}^k v$, $k\geq0$.

Now suppose that $v'\in(K(G)v)_\nu$ is quasi-primitive and $(K(G)v)_\xi$ has no quasi-primitive elements if $\xi >_\alpha \nu$. Consider a basis T in $\Delta_+ \setminus \{\alpha\}$ containing $\pi \setminus \{\alpha\}$. Then $X_\gamma v' = 0$ for all $\gamma \in \pi \setminus \{\alpha\}$, by Lemma 2. If $\gamma \in T \setminus \pi$ then $(\gamma, \alpha) \neq 0$. Let $Q \simeq sl(2, C)$ be a subalgebra generated by $X_{\pm\gamma}$ and F be a Q-module generated by v'. Suppose that $X_\gamma v' \neq 0$. Since v' is quasi-primitive it implies that $v' \in F'$, where F' is a Q-module generated by $X_\gamma v'$. Then F'_ν contains a non-zero element v'' such that $X_\gamma v'' = 0$ and hence F' has a finite-dimensional quotient. Since $M(\mu, q)$ is α-stratified then $v_k = X^k_{-\alpha} v'$ is quasi-primitive for al $k > 0$. Note that $X_\gamma v_k = 0$ for all k. Indeed, if $X_\gamma v_k \neq 0$ for some $k > 0$ then we can apply to v_k the same arguments as above and conclude that a Q-module generated by $X_\gamma v_k$ also has a finite-dimensional quotient of the same dimension. But $(\alpha, \gamma) \neq 0$ and hence these finite-dimensional modules have different highest weights which is a contradiction from the $sl(2)$-theory. Therefore, $X_\gamma v_k = 0$ for all $k > 0$. Using the fact that the root system Δ is finite we find $m \geq 0$ such that $X_\beta v_m = 0$ for all $\beta \in T$. Hence, v_m is α-primitive and thus belongs to $CX^k_{-\alpha} v$ for some $k \geq 0$. We conclude that v' is α-primitive and belongs to $CX^k_{-\alpha} v$ for some $k \geq 0$.

Lemma 4. *Let V be a quotient of $M(\mu, q)$, $0 \neq v \in M(\mu, q)_{\mu - \rho}$ and $\nu \in H^*$ a weight of V. Then* $\dim V_\nu \geq \dim (K(G)v)_\nu$ *where* $(K(G)v)_\nu = K(G)v \cap M(\mu, q)_\nu$. *Moreover, if* $\dim V_\nu = \dim (K(G)v)_\nu$ *for all ν such that* $\dim (K(G)v)_\nu \neq 0$, *then module V is irreducible.*

Proof. It follows immediately from Lemma 3 that $\dim V_\nu \geq \dim (K(G)v)_\nu$ for all ν. Suppose that $\dim V_\nu \geq \dim (K(G)v)_\nu$ for all ν such that $\dim (K(G)v)_\nu \neq 0$. Now let ν be such that $V_\nu \neq 0$ and $0 \neq w \in V_\nu$. Since V is α-stratified, $\nu - k\alpha \in \operatorname{supp} V$ for all $k \geq 0$. Clearly, there exists $m \geq 0$ for which $\nu - m\alpha \in \operatorname{supp} K(G)v$. Applying Lemmas 2 and 3 we conclude that $X^m_{-\alpha} w$, and hence w, generates V. It follows that V is irreducible.

Proof of Theorem 4. Let $0 \neq v \in M(\lambda, p)_{\lambda - \rho}$. It follows from Corollary 1 that

$$\dim (D_0 / \operatorname{Im} d_1)_\nu = \dim (K(G)v \cap M(\lambda, p)_\nu)$$

for all weights $\nu \in \operatorname{supp} K(G)v$. Using Lemma 4 we conclude that $D_0 / \operatorname{Im} d_1$ is irreducible which completes the proof.

6. Strong BGG-resolution

In this section we follow [1, 10] to construct the strong BGG-resolution for irreducible α-stratified module $L(\lambda, p)$ with $(\lambda, p) \in P^{++}$.

Let $(\lambda, p) \in P^{++}$. For $k \geq 0$ denote

$$(W_\alpha^+)^k = \{ w \in W_\alpha^+ \mid l(w) = k \}$$

and set

$$C_k = \sum_{w \in (W_\alpha^+)^k} M(w(\lambda, p)).$$

Define a map $\delta_i : C_i \to C_{i-1}$ using the matrix $(d^i_{w_1 w_2})$, $w_1 \in (W_\alpha^+)^i$, $w_2 \in (W_\alpha^+)^{i-1}$ where $d^i_{w_1 w_2} = s(w_1, w_2)$ if $w_1 > w_2$ (with respect to the Bruhat order) and zero otherwise. Here the numbers $s(w_1, w_2)$ are defined as in [1, Lemma 10.4]. Set $m = |\Delta_+(\pi \setminus \{\alpha\})|$.

Theorem 5. *Let $\eta : M(\lambda,p) \to L(\lambda,p)$ be a natural projection. Then the sequence*

$$0 \leftarrow L(\lambda,p) \stackrel{\eta}{\leftarrow} C_0 \stackrel{\delta_1}{\leftarrow} C_1 \stackrel{\delta_2}{\leftarrow} \cdots \stackrel{\delta_m}{\leftarrow} C_m \leftarrow 0$$

is exact.

Proof. It follows from the construction that this sequence is a complex.

To show the exactness in each term we will follow the proof of [10, Corollary 10.6].

Let R be the category of all weight G-modules having central character. Clearly every module $V \in R$ has a decomposition

$$V = \sum_{\chi \in Z^*(G)} V(\chi),$$

where $V(\chi)$ is a component with central character χ. Let $\theta \in Z^*(G)$ be a central character of $M(\lambda,p)$ and let $F_\theta : R \to R$ be a functor such that $F_\theta(V) = V(\theta)$ for all $V \in R$.

Obviously, there exists a minimal element $(\mu,q) \in P^{++}$ and a finite-dimensional G-module U such that $Y = F_\theta(L(\mu,q) \otimes U)$ contains an α-primitive element with parameters (λ,p). Moreover, the dimension of $Y_{\lambda-\rho}$ equals 1.

We will show that in fact $Y \simeq L(\lambda,p)$. Suppose that Y is not irreducible and F is some non-trivial submodule of Y. Then it follows from Lemma 4 that the dimension growth of Y/F is strictly less than the dimension growth of any irreducible module $L(\lambda',p')$ in R. The obtained contradiction implies that $Y \simeq L(\lambda,p)$.

Let ε be an eigenvalue of C on $L(\mu,q)$. Consider an exact complex $V_\alpha(\mu,\varepsilon)$. Applying the functor $F_\theta(\cdot \otimes U)$ to $V_\alpha(\mu,\varepsilon)$ we obtain the following exact complex:

$$0 \leftarrow L(\lambda,p) \stackrel{\eta}{\leftarrow} B_0 \stackrel{d_1}{\leftarrow} B_1 \stackrel{d_2}{\leftarrow} B_2 \stackrel{d_3}{\leftarrow} \cdots$$

where $B_i = F_\theta(D_i \otimes U)$, $i \geq 0$.

Using [1, Proposition 9.6] and Theorem 3 we coclude that

$$B_i \simeq C_i, \quad i \geq 0.$$

Following [10, Lemmas 10.2, 10.5] there exists a sequence of isomorphisms $v^i : B_i \to C_i$ which makes the following diagram commutative:

$$\begin{array}{ccccccccc}
\cdots \to & B_2(\lambda,p) & \stackrel{d_2}{\to} & B_1(\lambda,p) & \stackrel{d_1}{\to} & B_0(\lambda,p) & \stackrel{\eta}{\to} & L(\lambda,p) & \to 0 \\
& {\scriptstyle v^2}\downarrow & & {\scriptstyle v^1}\downarrow & & {\scriptstyle v^0}\downarrow & & {\scriptstyle 1}\downarrow & \\
\cdots \to & C_2(\lambda,p) & \stackrel{\delta_2}{\to} & C_1(\lambda,p) & \stackrel{\delta_1}{\to} & C_0(\lambda,p) & \stackrel{\eta}{\to} & L(\lambda,p) & \to 0.
\end{array}$$

This completes the proof of the theorem.

Corollary 3. *If* $(\lambda,p)\in P^{++}$ *and* M *is the maximal submodule of* $M(\lambda,p)$ *then*

$$M=\sum_{\gamma\in\pi\backslash\{\alpha\}} M(\sigma_\gamma(\lambda,p)).$$

Proof. Follows immediately from Theorem 5.

7. Character formula

In this section we use the strong BGG-resolution to obtain a character formula for a G-module $L(\lambda,p)$ with $(\lambda,p)\in P^{++}$.

For $\nu\in H^*$ let

$$H(\nu)=\nu+\sum_{\beta\in\pi\backslash\{\alpha\}} \mathbf{Z}\beta.$$

Set for any $\nu\in\operatorname{supp} V$

$$\operatorname{ch}^{\alpha,\nu}(V)=\sum_{\mu\in H(\nu)} (\dim V_\mu)e^\mu.$$

Lemma 5. *Let* V *be an* α*-stratified G-module and* $\nu\in\operatorname{supp} V$ *then*

$$\operatorname{ch}(V)=\left(\sum_{i=-\infty}^{+\infty} e^{i\alpha}\right)\operatorname{ch}^{\alpha,\nu}(V).$$

Proof. Follows from the fact that $X_{\pm\alpha}$ act injectively on V.

Let $\varphi:H^*\to H(0)$ be a natural projection along the root α. Set $\Delta'=\{\varphi(\beta)\mid\beta\in\Delta_+\}$. It is easy to see (see for example [9]) that for any $(\mu,q)\in\Omega$

$$\operatorname{ch}^{\alpha,\mu-\rho}(M(\mu,q))=e^{\mu-\rho}\prod_{\beta\in\Delta'}(1-e^{-\beta})^{-1}$$

and thus

$$\operatorname{ch}(M(\mu,q))=e^{\mu-\rho}\prod_{\beta\in\Delta_+\backslash\{\alpha\}}(1-e^{-\beta})^{-1}\left(\sum_{i=-\infty}^{+\infty} e^{i\alpha}\right)$$

by Lemma 5.

Set $\rho'=\frac{1}{2}\sum_{\beta\in P}\beta$.

Theorem 6. *Let* $(\lambda,p)\in P^{++}$. *Then there exists an element* $a(\lambda,p)\in H^*$ *such that*

$$\operatorname{ch}(L(\lambda,p))=\left(\sum_{i=-\infty}^{+\infty} e^{i\alpha}\right)\left(\prod_{\beta\in -K\backslash\{\alpha\}}(1-e^{-\beta})^{-1}\right)$$
$$\times\left(\sum_{w\in W_\alpha^+}(-1)^{l(w)}e^{w(\lambda+a(\lambda,p)+\rho')-a(\lambda,p)}\right)\left(\sum_{w\in W_\alpha^+}(-1)^{l(w)}e^{w(\rho')}\right)^{-1}$$

Proof. It follows from Theorem 5, that the character $\operatorname{ch} L(\lambda,p)$ satisfies the following alternating formula:

$$\operatorname{ch} L(\lambda,p) = \sum_{i\geq 0} (-1)^i \sum_{w\in(W_\alpha^+)^{(i)}} \operatorname{ch} M(w(\lambda,p)).$$

Thus using the character formula for $M(\mu,q)$ above we obtain

$$\operatorname{ch} L(\lambda,p) = \left(\sum_{i=-\infty}^{+\infty} e^{i\alpha}\right)\left(\prod_{\beta\in -K\setminus\{\alpha\}} (1-e^{-\beta})^{-1}\right)$$
$$\times \sum_{i\geq 0} (-1)^i \sum_{w\in(W_\alpha^+)^{(i)}} e^{\mathrm{pr}_1(w(\lambda,p))-\rho} \prod_{\beta\in P} (1-e^{-\beta})^{-1}.$$

Since the group W_α^+ is an affine reflection group in every Ω_r the result follows from the classical Weyl character formula for finite-dimensional modules [4, Theorem 7.5.9].

Note that the element $a(\lambda,p)$ in Theorem 6 is determined uniquely by the element in Ω_r with respect to which the group W_α^+ is linear.

Acknowledgment. This paper was completed during the visit of the first author to The Erwin Schroedinger International Institute for Mathematical Physics whose hospitality and support are greatly appreciated. The authors also would like to thank the referee for the very careful review of the paper and many useful remarks.

INSTITUTO DE MATEMATICA E ESTATISTICA
UNIVERSIDADE DE SAO PAULO
CAIXA POSTAL 66281–CEP 05315–970
SAO PAULO, BRASIL
e-mail: futorny@ime.usp.br

DEPARTMENT OF MATHEMATICS
KIEV UNIVERSITY
64 VLADIMIRSKAYA ST.
KIEV, 252033, UKRAINE
e-mail: mazorchu@uni-alg.kiev.ua

References

[1] I. F. Bernstein, I. M. Gelfand and S. I. Gelfand, Differential operators on the base affine space and the study of G-modules, in: I. M. Gelfand, ed., Publ. of 1971 Summer School in Math., Janos Bolyai Math. Soc., Budapest, 21–64.

[2] H. Cartan and S. Eilenberg, Homological algebra, Princeton University Press, Princeton, 1956.

[3] A. J. Coleman and V. M. Futorny, Stratified L-modules, J. Algebra, **163**(1994), 219–234.

[4] J. Dixmier, Algebres Enveloppantes, Paris, 1974.

[5] Yu. A. Drozd, V. M. Futorny and S. A. Ovsienko, S-homomorphism of Harish-Chandra and G-modules generated by semiprimitive elements, Ukrainian Math. J., **42**(1990), 1032–1037.

[6] V. Futorny and V. Mazorchuk, Structure of α-stratified modules for finite-dimensional Lie algebras I, J. Algebra, **183** (1996), 456–482.

[7] V. Futorny and D. Pollack, A new category of Lie algebra modules satisfying the BGG-reciprocity principle, Comm. in Algebra, **22**-(1)(1994), 213–227.

[8] J. Humphreys, Highest weight modules for semisimple Lie algebras, in: V. Dlab, P. Gabriel, ed., Representation Theory I, Lecture notes in Math. 831, Springer-Verlag, 1980, 72–103.

[9] V. Mazorchuk. Structure of the generalized Verma modules. Ph. D. Thesis, Kiev University, Kiev, 1996.

[10] A. Rocha-Caridi, Splitting criteria for G-modules induced from a parabolic and a Bernstein - Gelfand - Gelfand resolution of a finite-dimensional, irreducible G-module. Trans. Amer. Math. Soc., **262**(1980), 335–366.