J. Math. Kyoto Univ. (JMKYAZ)
38-3 (1998) 587-593

Generators of the cohomology of BV_3

By

Masaki KAMEKO

1. Introduction

Let $P_k = \mathbf{F}_2[x_1, \dots, x_k]$ be a polynomial algebra over the field $\mathbf{F}_2$ generated by $x_1, \dots, x_k$. By assigning degree 1 to each x_j, P_k is regarded as a graded algebra over the ground field $\mathbf{F}_2$. The mod 2 cohomology ring of the classifying space BV_k of the elementary abelian 2-group V_k with rank k, is isomorphic to P_k as a graded algebra. Through this isomorphism, we may regard P_k as an $\mathscr{A}$-module where $\mathscr{A}$ stands for the mod 2 Steenrod algebra.

From early days of algebraic topology, topologists have been studying this cohomology ring and by making use of this cohomology ring, topologists have been proving many theorems. But our knowledge of this cohomology ring is not deep enough. For instance, we do not know even the dimension of the vector space $QP_k^n = (\mathbf{F}_2 \otimes_{\mathscr{A}} P_k)^n$ for $k = 4$.

In this paper, we determine the upper bound for the dimension of the above vector space QP_3^n and give a set of generators in terms of monomials in Theorem 5.2 and we will see that $\dim QP_3^n \leq 21$. In his thesis [4], the author gave the proof for the linear independence of these monomials in QP_3^n. After his thesis was submitted, the lower bound for $\dim QP_3^n$ is provided by Ali, Crabb and Hubbuck in [1] and Boardman in [2] investigating homology of BV_3 instead of its cohomology. In this paper, we do not give the proof of linear independence of our generators but they actually form a minimal set of generators.

2. $\alpha(n)$ and $\beta(n)$

We begin with the definitions and elementary properties of $\alpha(n)$ and $\beta(n)$. $\alpha(n)$ is the number of 1's in the dyadic expansion of n and $\beta(n)$ is the smallest positive integer that satisfies the condition $\alpha(n+\beta(n)) \leq \beta(n)$.

We need the following properties of $\beta(n)$.

Proposition 2.1. *$\alpha(n+m) \leq m$ if and only if $\beta(n) \leq m$.*

Proof. The "*only if*" part is nothing but the definition of $\beta(n)$. The "*if*" part is shown as follows: if $\beta(n) \leq m$, then

Communicated by A. Kono, January 23, 1998

$$\begin{aligned}\alpha(n+m) &= \alpha(n+\beta(n)+m-\beta(n)) \\ &\leq \alpha(n+\beta(n))+\alpha(m-\beta(n)) \\ &\leq \beta(n)+m-\beta(n) \\ &= m.\end{aligned}$$

Proposition 2.2. *$n-\beta(n)$ is non-negative and even.*

Proof. Since $\alpha(n+n)=\alpha(n)\leq n$, $\beta(n)\leq n$ by definition. Assume that $n-\beta(n)$ is odd. Then $n+\beta(n)-1$ is even and non-negative. Thereby $\alpha(n+\beta(n)-1)=\alpha(n+\beta(n))-1\leq\beta(n)-1$. This contradicts the definition of $\beta(n)$. Therefore $n-\beta(n)$ must be even.

Proposition 2.3. *Suppose $\beta(n)=k$. Then $\beta\left(\frac{n-k}{2}\right)\leq k$.*

Proof.

$$\alpha\left(\frac{n-k}{2}+k\right)=\alpha\left(\frac{n+k}{2}\right)=\alpha(n+k)\leq k.$$

By Proposition 2.1, we get the desired result.

3. Monomials

In this section, we consider several properties of monomials including a theorem of Wood. Here, we adopt Boardman's notation in [2]. We assign to each letter L the monomial $x(L)$ of the form $x_1^{e_1}x_2^{e_2}\cdots x_k^{e_k}$ where e_i is 0 or 1. We need 2^k letters to make this correspondence one-to-one. We compose strings of letters in the usual way, by juxtaposition and we also consider the length of string in the usual way, by counting the number of letters in the string.

Let S be a string $L_1L_2\cdots L_r$ where $L_1,\ldots,L_r$ are letters. We define the monomial $x(S)$ by

$$x(S)=x(L_1)x(L_2)^2\cdots x(L_r)^{2^{r-1}}.$$

Let us define two sequences associated with strings. Let $w_i(S)=\deg x(L_i)$ where L_i is the i-th letter in the string S. Let $e_i(S)=e_i$ if $x(S)=x_1^{e_1}\cdots x_k^{e_k}$. We define $w(S)$ and $e(S)$ as follows:

$$w(S)=(w_1(S),\ldots,w_r(S),0,0,\ldots),$$

$$e(S)=(e_1(S),\ldots,e_k(S),0,0,\ldots)$$

where r is the length of S. $w(S)$ is the weight vector of $x(S)$.

We consider the lexicographic order on the set of sequences of non-negative integers. Let $\ell=(\ell_1,\ell_2,\ldots,\ell_r,\ldots)$, $\ell'=(\ell'_1,\ell'_2,\ldots,\ell'_r,\ldots)$ be sequences of non-negative integers. We say $\ell < \ell'$ if and only if there is a positive integer t such that $\ell_s = \ell'_s$ for all $s < t$ and $\ell_t < \ell'_t$.

Now we are ready to define the order on the set of monomials.

Let $x(S)$, $x(S')$ be monomials in P_k. We say $x(S) < x(S')$ if and only if one of the following holds:

1. $w(S) < w(S')$
2. $w(S) = w(S')$ and $e(S) < e(S')$

For the proof of the following lemma, we refer the reader to the proof of Lemma 2.2 in [3].

Lemma 3.1. *For any monomial $x(S)$ and $a \in \mathscr{A}_+$,*

$$ax(S) = \sum_{S'} x(S')$$

where $w(S') < w(S)$ for each S'. Here $\mathscr{A}_+$ is the set of positive degree elements in $\mathscr{A}$.

Definition 3.2. A monomial $x(S)$ is said to be inadmissible if there exists a finite set of monomials $x(S') < x(S)$ such that

$$x(S) \equiv \sum_{S'} x(S')$$

modulo the image of the Steenrod algebra. A monomial $x(S)$ is said to be admissible if it is not inadmissible.

It is clear that the set of admissible monomials in P_k is a minimal set of generators of P_k.

In addition, we define *strictly inadmissible* monomials.

Definition 3.3. We say $x(S)$ is strictly inadmissible if and only if there exists a finite set of monomials $x(S') < x(S)$ such that

$$x(S) \equiv \sum_{S'} x(S')$$

modulo the image of $\mathscr{A}(r)$ where r is the length of string S and $\mathscr{A}(r)$ is the subalgebra of $\mathscr{A}$ generated by $Sq^1, \ldots, Sq^{2^{r-1}}$.

The advantage of considering strictly inadmissible monomials is displayed by the following Theorem 3.4, which is our principal tool.

Theorem 3.4. *Let $S = S_0S_1S_2$ be a string of letters. If $x(S_1)$ is strictly inadmissible, then $x(S)$ is inadmissible.*

Proof. Let r_0 be the length of S_0 and let r_1 be the length of S_1. The assumption is that $x(S_1)$ is a sum of monomials $x(S'_1)$ where $x(S'_1) < x(S_1)$ and polynomials $ax(S''_1)$ where $a \in \mathscr{A}_+(r_1)$. Here $\mathscr{A}_+(r_1)$ stands for the set of positive degree elements in $\mathscr{A}(r_1)$. Thus $x(S)$ is a sum of monomials $x(S_0S'_1S_2)$ and polynomials

$$x(S_0)(ax(S_1''))^{2^{r_0}}x(S_2)^{2^{r_1+r_0}}$$

where $a \in \mathscr{A}_+(r_1)$. It is clear that $x(S_0S_1'S_2) < x(S)$. So it suffices to show that

$$x(S_0)(ax(S_1''))^{2^{r_0}}x(S_2)^{2^{r_1+r_0}}$$

is congruent to a sum of monomials $x(S'') < x(S)$ modulo the image of the Steenrod algebra.

Recall that

$$(ax(S_1''))^{2^{r_0}} = a'(x(S_1'')^{2^{r_0}})$$

for some $a' \in \mathscr{A}_+(r_1 + r_0)$, that if $a' \in \mathscr{A}_+(r_1 + r_0)$, then $\chi(a')$ is also in $\mathscr{A}_+(r_1 + r_0)$ and that for each $\chi(a') \in \mathscr{A}_+(r_1 + r_0)$, $\chi(a')x^{2^{r_1+r_0}} = 0$. Hence, by Peterson's lemma,

$$\begin{aligned} x(S_0)(ax(S_1''))^{2^{r_0}}x(S_2)^{2^{r_1+r_0}} &\equiv x(S_0)(a'(x(S_1'')^{2^{r_0}}))x(S_2)^{2^{r_1+r_0}} \\ &\equiv (\chi(a')x(S_0))x(S_1'')^{2^{r_0}}x(S_2)^{2^{r_1+r_0}} \end{aligned}$$

modulo the image of the Steenrod algebra. By Lemma 3.1, each monomial $x(S_0'')$ in $(\chi(a')x(S_0))$ is smaller than $x(S_0)$. Hence, each monomial in

$$(\chi(a')x(S_0))x(S_1'')^{2^{r_0}}x(S_2)^{2^{r_1+r_0}}$$

is also smaller than $x(S)$. This completes the proof.

We end this section by recalling a theorem of Wood.

Theorem 3.5 (Wood). *Let $x(S)$ be a monomial of degree n. If*

$$\alpha(n + w_1(S)) > w_1(S),$$

then $x(S)$ is in the image of the Steenrod algebra.

For the proof of Theorem 3.5, we refer the reader to [5]. For our purpose the following modification is convenient.

Theorem 3.6. *Let $x(S)$ be a monomial of degree n. If $\beta(n) > k$ then $x(S)$ is in the image of the Steenrod algebra.*

4. Reduction

In this section we define functions ϕ and ψ which assign monomials of P_k to monomials of P_k. By making use of these functions we reduce the problem of finding a basis of the vector space QP_k^n to the cases $\beta(n) < k$. Let us assign to the letter A the monomial $x(A) = x_1 \cdots x_k$ of degree k.

Definition 4.1. Let $x(S)$ be a monomial. Define monomials $\phi(x(S))$ and $\psi(x(S))$ by

$$\phi(x(S)) = x(AS),$$
$$\psi(x(S)) = x(S') \quad (\text{if } S = AS'),$$
$$\psi(x(S)) = 0. \qquad (\text{otherwise}).$$

It is clear that ϕ and ψ induce homomorphisms between vector spaces P_k^n and $P_k^{(n-k)/2}$ as long as these vector spaces are well-defined. The following is the theorem of this section.

Theorem 4.2. *If $\beta(n) = k$, ψ induces an isomorphism of vector spaces*

$$Q\psi\colon QP_k^n \to QP_k^{(n-k)/2}$$

with the inverse $Q\phi$.

Proof. Suppose that ψ and ϕ induce homomorphisms of vector spaces

$$Q\psi\colon QP_k^n \to QP_k^{(n-k)/2}$$

and

$$Q\phi\colon QP_k^{(n-k)/2} \to QP_k^n.$$

Then, $Q\psi \circ Q\phi$ is the identity and $Q\psi$ is onto. On the other hand, by Theorem 3.6, $Q\phi$ is also onto. Therefore the theorem holds. Thus we may complete the proof by proving that $Q\psi$ and $Q\phi$ are well-defined under the above conditions.

Let $a \in \mathscr{A}_+$ and let S be a string. Then, if $S = AS'$, then

$$ax(S) = x(A)(a'x(S'))^2 + \sum_{S''} x(S'')$$

for some $a' \in \mathscr{A}_+$ or

$$ax(S) = \sum_{S''} x(S'')$$

where $w_1(S'') < k$. If $w_1(S) < k$, then $ax(S)$ is a sum of monomials $x(S'')$ with $w_1(S'') < k$. Since ψ maps above $x(S'')$'s to zero, ψ maps the image of the Steenrod algebra to the image of the Steenrod algebra. So, $Q\psi$ is well-defined.

On the other hand, if $a \in \mathscr{A}_+$, then

$$x(A)(ax(S))^2 = a'x(AS) + \sum_{S''} x(S'')$$

for some $a' \in \mathscr{A}_+$ and $w_1(S'') < k$. Since $x(S'')$'s are in the image of the Steenrod algebra by Theorem 3.6, ϕ also maps the image of the Steenrod algebra to the image of the Steenrod algebra. So, $Q\phi$ is also well-defined. This completes the proof.

By Theorem 3.6, if $\beta(n) > k$, then

$$\dim QP_k^n = 0$$

and by Theorem 4.2 and Proposition 2.3, we may reduce the rest of the problem of finding the (minimal set of) generators of P_k to the cases $\beta(n) < k$.

5. Generators of H^*BV_3

We assign to each letter L from A through H the monomial $x(L)$ according to the table:

L	A	B	C	D	E	F	G	H
$x(L)$	$x_1x_2x_3$	x_1x_2	x_1x_3	x_2x_3	x_1	x_2	x_3	1

One may verify the following lemma by direct computation of the action of the Steenrod algebra on P_3 up to degree 22. So we leave the proof of Lemma 5.1 below to the reader.

Lemma 5.1. *The monomials correspond to the following strings are strictly inadmissible.*

weight vector $w(S)$	*string* S
(0, 1)	HE, HF, HG
(0, 2)	HB, HC, HD
(0, 3)	HA
(1, 1)	FE, GE, GF
(1, 2)	EB, EC, ED FB, FC, FD GB, GC, GD
(1, 3)	EA, FA, GA
(2, 1)	DE
(2, 2)	CB, DB, DC
(2, 3)	BA, CA, DA
(1, 1, 1)	EEF, EEG, FFG
(2, 2, 1)	CCF
(2, 2, 2)	BBC, BBD, CCD
(2, 1, 1, 1)	$BEFG$, $CEFF$, $CEFG$,
(2, 2, 1, 1)	$BBEG$, $BBFG$, $BCEF$, $CCEF$
(2, 2, 2, 1)	$BBBG$

By looking at the strictly inadmissible monomials of length 2 in the table above, it is clear that if $x(S)$ is admissible then the letters in the string S is in the alphabetical order. Thus, by additional observation of the strictly inadmissible monomials of length greater than 2 and $x(DE)$, we have Theorem 5.2 below. The expression of our main theorem in this form is due to Boardman in [2].

Theorem 5.2. *The set of monomials $x(S)$ that correspond to all strings S that match any of the following nine patterns generates QP_3.*

$$\begin{array}{lll} A^*B^*E^*, & A^*B^*[E]F^*, & A^*BBG^*, \\ A^*[B]C^*E^*, & A^*CEF, & A^*[B]C^*[E]G^*, \\ A^*[B][C]D^*F^*, & A^*[B][C]D^*[F]G^*, & \\ A^*[E][F]G^*. & & \end{array}$$

where

L^* matches any sequence (0 or more) of copies of L;

$[L]$ matches L or the null string.

From Theorem 5.2, we have the following table of the upper bound for $\dim QP_3^n$ for $\beta(n) \leq 2$, which is, as a matter of fact, $\dim QP_3^n$ itself.

	$s=0$	$s=1$	$s=2$	$s=3$	$s=4$	$s \geq 5$
$t=0$	1	3	7	10	13	14
$t=1$	3	8	15	14	14	14
$t=2$	6	14	21	21	21	21
$t \geq 3$	7	14	21	21	21	21

Table of (the upper bound of) $\dim QP_3^n$ where $n = 2^{s+t} + 2^t - 2$.

DEPARTMENT OF GENERAL EDUCATION
TAKAMATSU NATIONAL COLLEGE OF TECHNOLOGY

References

[1] Mohamed Ali Alghamdi, M. C. Crabb and J. R. Hubbuck, Representations of the homology of BV and the Steenrod algebra I, London Math. Soc. Lecture Note Ser., **176** Cambridge Univ. Press (1992) 217–234.

[2] J. M. Boardman, Modular representations on the homology of powers of real projective space, Contemp. Math., **146** Amer. Math. Soc. (1993) 49–70.

[3] D. P. Carlisle and R. M. Wood, The boundedness conjecture for the action of the Steenrod algebra on polynomials, London Math. Soc. Lecture Note Ser., **176** Cambridge Univ. Press (1992) 203–216.

[4] M. Kameko, Thesis, The Johns Hopkins University, May 1990.

[5] R. M. W. Wood, Steenrod squares of polynomials and the Peterson conjecture, Math. Proc. Cam. Phil. Soc., **105** (1989) 307–309.