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On Morava K-theory of $B(\mathbf{Z}/p)^m$ as a representation of $m \times m$ matrices ring $M_m(\mathbf{F}_p)$

By

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1. Introduction and Main Results

Let $\overline{K(n)}^*(\ )$ be the p-adic Morava $K(n)$-theory of period 2 with the coefficient ring

$$\overline{K(n)}_* = \mathbf{Z}_p[v_n, v_n^{-1}, t, t^{-1}]/(t^{p^n-1} - v_n) = \mathbf{Z}_p[t, t^{-1}].$$

Here $\deg t = 2$ and $\mathbf{Z}_p$ is the ring of p-adic integers. For any $\mathbf{Z}_p$-algebra R we define

$$\overline{K(n)}_R^*(\ ) = \overline{K(n)}^*(\ ) \otimes R.$$

If R is a finitely generated free $\mathbf{Z}_p$-module, then $\overline{K(n)}_R^*(\ )$ will be a complex orientable cohomology theory. Throughout this paper for any local field K we denote by $\mathcal{O}_K$ its ring of integers. Then we have

$$\overline{K(n)}_{\mathcal{O}_K}^0(B\mathbf{Z}/p) \cong \mathcal{O}_K[[x]]/([p]x),$$

where $[p]x$ is the p-series of the Lubin-Tate formal group law and we can choose such an orientation x, that

$$[p]x = px - x^{p^n}.$$

Thus we have that $\overline{K(n)}_{\mathcal{O}_K}^0(B\mathbf{Z}/p)$ is free of rank p^n over the coefficient ring $\mathcal{O}_K$ and $\overline{K(n)}_{\mathcal{O}_K}^1(B\mathbf{Z}/p) = 0$. Hence

$$\overline{K(n)}_{\mathcal{O}_K}^0(B\mathbf{Z}/p \times B\mathbf{Z}/p) \cong \overline{K(n)}_{\mathcal{O}_K}^0(B\mathbf{Z}/p) \otimes_{\mathcal{O}_K} \overline{K(n)}_{\mathcal{O}_K}^0(B\mathbf{Z}/p)$$

and the product map $m : \mathbf{Z}/p \times \mathbf{Z}/p \to \mathbf{Z}/p$ induces a ring homomorphism

$$(Bm)^* : \overline{K(n)}_{\mathcal{O}_K}^0(B\mathbf{Z}/p) \to \overline{K(n)}_{\mathcal{O}_K}^0(B\mathbf{Z}/p) \otimes_{\mathcal{O}_K} \overline{K(n)}_{\mathcal{O}_K}^0(B\mathbf{Z}/p).$$

So $\overline{K(n)}_{\mathcal{O}_K}^0(B\mathbf{Z}/p)$ is a bicommutative Hopf algebra over $\mathcal{O}_K$.

Now we consider a classifying space of m-times direct product of $\mathbf{Z}/p$. It is easy to see that also $\overline{K(n)}_{\mathcal{O}_K}^0(B(\mathbf{Z}/p)^m)$ is a bicommutative Hopf algebra over $\mathcal{O}_K$. The semi-group $M_m(\mathbf{F}_p)$ of $m \times m$ matrices with entries in $\mathbf{F}_p$ acts on $(\mathbf{Z}/p)^m$

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from the right. Thus we have induced left action of $M_m(\mathbf{F}_p)$ on

$$\overline{K(n)}^0_{\mathcal{O}_K}(B(\mathbf{Z}/p)^m) = \mathcal{O}_K[x_1, x_2, \dots, x_m]/([p]x_1, [p]x_2, \dots, [p]x_m).$$

An element $A = (a_{ij}) \in M_m(\mathbf{F}_p)$ acts on the generator x_i by the following way

$$Ax_i = \sum_{j=1}^{m} {}_F[a_{ij}]x_j. \tag{1}$$

Here $\sum_F$ means the summation using the formal group low F of the theory $\overline{K(n)}^*$.

Let us consider the additive group $M_{m,n}(\mathbf{F}_p)$ of $m \times n$ matrixes with entries in $\mathbf{F}_p$. The semi-group $M_m(\mathbf{F}_p)$ acts on the group ring $\mathcal{O}_K[M_{m,n}(\mathbf{F}_p)]$ by the left matrix multiplication. We can regard $\mathcal{O}_K[M_{m,n}(\mathbf{F}_p)]$ as a Hopf algebra by putting $\Psi(A) = A \otimes A$ for $A \in M_{m,n}(\mathbf{F}_p)$.

In Section 2 we shall prove the following

Theorem 1.1. *Let K be a splitting field of the $\mathbf{Q}_p$-algebra*

$$\overline{K(n)}^0_{\mathbf{Q}_p}(\ )(B\mathbf{Z}/p) \cong \mathbf{Q}_p[x]/(px - x^{p^n}),$$

such that the residue field of K is $\mathbf{F}_{p^n}$. Then there exists a Hopf algebra isomorphism

$$f : K[M_{m,n}(\mathbf{F}_p)] \to \overline{K(n)}^0_K(B(\mathbf{Z}/p)^m)$$

which commutes with the $M_m(\mathbf{F}_p)$-action.

Let M be a semi-group. Then one can think of a representation of M over a ring $\mathcal{O}$ and one can define the Grothendieck group $R_{\mathcal{O}}(M)$ of the category of $\mathcal{O}$-representations of semi-group M. If V is such a representation, the corresponding element in $R_{\mathcal{O}}(M)$ is denoted by $[V]$.

Let $K(n)^*_{\mathbf{F}_{p^n}}(\ )$ be the Morava K-theory with coefficient ring $\mathbf{F}_{p^n}[t, t^{-1}]$. As in the p-adic case, in $K(n)^*_{\mathbf{F}_{p^n}}(B(\mathbf{Z}/p)^m)$ we also have a left action of $M_m(\mathbf{F}_p)$ induced by the right action of $M_m(\mathbf{F}_p)$ on $(\mathbf{Z}/p)^m$.

Theorem 1.2. *In $R_{\mathbf{F}_{p^n}}(M_m(\mathbf{F}_p))$ we have an equality*

$$[K(n)^0_{\mathbf{F}_{p^n}}(B(\mathbf{Z}/p)^m)] = [\mathbf{F}_{p^n}[M_{m,n}(\mathbf{F}_p)]].$$

Remark. Two $\mathbf{F}_{p^n}$-modules $K(n)^0_{\mathbf{F}_{p^n}}(B(\mathbf{Z}/p^m)$ and $\mathbf{F}_{p^n}[M_{m,n}(\mathbf{F}_p)]$ need not be isomorphic. The above theorem means that they have the same composition factors.

In the polynomial ring $\mathbf{F}_{p^n}[x_1, x_2, \dots, x_m]$ we have a natural action of $M_m(\mathbf{F}_p)$ given by the following formula

$$Ax_i = \sum_{j=1}^{m} a_{ij}x_j. \tag{2}$$

This action can be also thought as follows. First the action using the additive group low in the equation (1), and second the action defined on the polynomial part of the mod p cohomology ring $H^*((B\mathbf{Z}/p)^m; \mathbf{F}_p)$.

The ideal $(x_1^{p^n}, x_2^{p^n}, \ldots, x_m^{p^n})$ is invariant under this action of $M_m(\mathbf{F}_p)$. Thus we have an induced action on the truncated polynomial ring

$$\mathbf{F}_{p^n}[x_1, x_2, \ldots, x_m]/(x_1^{p^n}, x_2^{p^n}, \ldots, x_m^{p^n}).$$

Then using Theorem 1.2 we can prove the following

Theorem 1.3. *Let* $\mathbf{F}_{p^n}[x_1, x_2, \ldots, x_m]/(x_1^{p^n}, x_2^{p^n}, \ldots, x_m^{p^n})$ *be a representation of the semi-group* $M_m(\mathbf{F}_p)$ *with the action given by* (2). *Then in* $R_{\mathbf{F}_{p^n}}(M_m(\mathbf{F}_p))$ *we have*

$$[\mathbf{F}_{p^n}[x_1, x_2, \ldots, x_m]/(x_1^{p^n}, x_2^{p^n}, \ldots, x_m^{p^n})] = [\mathbf{F}_{p^n}[M_{m,n}(\mathbf{F}_p)]].$$

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2. Proofs of the Theorems

Definition 2.1. *Let A be Hopf algebra. An element $b \in A$ is called a group like element of A, if $\psi b = b \otimes b$ for the coproduct Ψ.*

It is well known, see for example [5], Lemma 1.4, that the set of group like elements in A is a group with respect to the multiplication in A. Let us denote the group of group like elements in $\overline{K(n)}^0_{\mathcal{O}_K}(B(\mathbf{Z}/p)^m)$ by $S_{m,n}(\mathcal{O}_K)$.

For $C \in M_m(\mathbf{F}_p)$ we have that $BC : B(\mathbf{Z}/p)^m \to B(\mathbf{Z}/p)^m$ is an H-map of H-spaces. So

$$(BC)^* : \overline{K(n)}^0_{\mathcal{O}_K}(B(\mathbf{Z}/p)^m) \to \overline{K(n)}^0_{\mathcal{O}_K}(B(\mathbf{Z}/p)^m)$$

is a Hopf algebra homomorphism. Hence $M_m(\mathbf{F}_p)$ acts on $\overline{K(n)}^0_{\mathcal{O}_K}(B(\mathbf{Z}/p)^m)$ as Hopf algebra homomorphisms and thus $M_m(\mathbf{F}_p)$ acts on $S_{m,n}(\mathcal{O}_K)$ as group homomorphisms. If $f(x) \in \overline{K(n)}^0_{\mathcal{O}_K}(B(\mathbf{Z}/p)^m)$ is group like element, by definition $f(F(x,y)) = f(x)f(y)$. So $(f(x))^p = f([p]x) = 0$ and it follows that $S_{m,n}(\mathcal{O}_K)$ is an elementary abelian p-group. Hence we have

Lemma 2.2. *$M_m(\mathbf{F}_p)$ acts on $S_{m,n}(\mathcal{O}_K)$ as an $\mathbf{F}_p$-linear mapping.*

Let K be a splitting field of the $\mathbf{Q}_p$-algebra $A \cong \mathbf{Q}_p[x]/(px - x^{p^n})$, such that the residue field of K is $\mathbf{F}_{p^n}$. It is shown in [4], equation (6), that the dual algebra A^* is also a polynomial ring of one variable y divided by the ideal generated by a polynomial

$$f(y) = p^{p^{n-1}+\cdots+p^2+1}y - y^{p^n}.$$

Thus K is also a splitting field of the algebra A^*. For such a field K in [5] is proved the following

Theorem 2.3. *a) The group of group like elements $S_{1,n}(\mathcal{O}_K)$ is isomorphic to $(\mathbf{Z}/p)^n$; b) $K(n)^0_K(B\mathbf{Z}/p) \cong K[S_{1,n}(\mathcal{O}_K)]$ as a Hopf algebra, where in $K[S_{1,n}(\mathcal{O}_K)]$ a Hopf algebra structure is given in the usual way.*

Since

$$\overline{K(n)}^0_{\mathcal{O}_K}(B(\mathbf{Z}/p)^m) \cong \overline{K(n)}^0_{\mathcal{O}_K}(B(\mathbf{Z}/p)) \otimes_{\mathcal{O}_K} \cdots \otimes_{\mathcal{O}_K} \overline{K(n)}^0_{\mathcal{O}_K}(B(\mathbf{Z}/p)),$$

it is easy to see that

$$S_{m,n}(\mathcal{O}_K) \cong (S_{1,n}(\mathcal{O}_K))^m \cong (\mathbf{Z}/p)^{mn}.$$

Lemma 2.4. *One can choose an isomorphism $S_{m,n}(\mathcal{O}_K) \cong M_{m,n}(\mathbf{F}_p)$, such that the action of $M_m(\mathbf{F}_p)$ on $S_{m,n}(\mathcal{O}_K)$ is regarded as the usual matrix action on $M_{m,n}(\mathbf{F}_p)$.*

Proof. If we consider $S_{m,n}(\mathcal{O}_K)$ as an mn dimensional $\mathbf{F}_p$-vector space, by Lemma 2.2, the action of $A \in M_m(\mathbf{F}_p)$ on $S_{m,n}(\mathcal{O}_K)$ can be described as an $nm \times nm$ matrix. Let $f_1(x), f_2(x), \dots, f_n(x)$ be a basis of the n dimensional vector space $S_{1,n}(\mathcal{O}_K)$. Then we can choose $f_i(x_j)$, $i = 1, \dots, n$; $j = 1, \dots, m$, as a basis of $S_{m,n}(\mathcal{O}_K)$. Considering the action of $A = (a_{ij}) \in M_m(\mathbf{F}_p)$ on the basis element $f_k(x_j) \in \overline{K(n)}^0_{\mathcal{O}_K}(B(\mathbf{Z}/p)^m)$, according to formula (1) we get

$$Af_k(x_j) = f_k\left(\sum_{i=1}^{m}{}_F [a_{ji}]x_i\right).$$

Since $f(F(x,y)) = f(x)f(y)$, it follows that

$$f_k\left(\sum_{i=1}^{m}{}_F [a_{ji}]x_i\right) = \prod_{i=1}^{m} f_k(x_i)^{a_{ji}}.$$

If we write group structure of $S_{m,n}(\mathcal{O}_K)$ additively, we have

$$Af_k(x_j) = a_{j1}f_k(x_1) + a_{j2}f_k(x_2) + \cdots + a_{jm}f_k(x_m).$$

Now if we order basis $f_i(x_j)$, $i = 1, \dots, n$; $j = 1, \dots, m$, in the following way

$$f_1(x_1), \dots, f_1(x_m), f_2(x_1), \dots, f_2(x_m), \dots, f_n(x_1), \dots, f_n(x_m),$$

then the action of A can be written as an $nm \times nm$ matrix which consists of n^2 blocks, each $m \times m$ size. All matrices on the diagonal are A itself and all other blocks are 0. Therefore, if we choose isomorphism $S_{m,n}(\mathcal{O}_K) \cong M_{m,n}(\mathbf{F}_p)$ by arranging basis of $S_{m,n}(\mathcal{O}_K)$ as a matrix

$$\begin{pmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_n(x_1) \\ f_1(x_2) & f_2(x_2) & \cdots & f_n(x_2) \\ \cdots & \cdots & \cdots & \cdots \\ f_1(x_m) & f_2(x_m) & \cdots & f_n(x_m) \end{pmatrix}, \tag{3}$$

then the action of A on $S_{m,n}(\mathcal{O}_K)$ will be the usual matrix action from the left.

Let M be a semi-group. A Hopf algebra A is called an M-Hopf algebra if M acts on A as a Hopf algebra homomorphisms.

Lemma 2.5. *There exists an $M_m(\mathbf{F}_p)$-Hopf algebra homomorphism*

$$f : \mathcal{O}_K[M_{m,n}(\mathbf{F}_p)] \to \overline{K(n)}^0_{\mathcal{O}_K}(B(\mathbf{Z}/p)^m).$$

Proof. The group of group like elements $S_{m,n}(\mathcal{O}_K)$ is a subset of $\overline{K(n)}^0_{\mathcal{O}_K}(B(\mathbf{Z}/p)^m)$ closed under the action of $M_m(\mathbf{F}_p)$. Hence extending on the group ring linearly we have an $M_m(\mathbf{F}_p)$-Hopf algebra homomorphism

$$\mathcal{O}_K[S_{m,n}(\mathcal{O}_K)] \to \overline{K(n)}^0_{\mathcal{O}_K}(B(\mathbf{Z}/p)^m).$$

By Lemma 2.4, $S_{m,n}(\mathcal{O}_K) \cong M_{m,n}(\mathbf{F}_p)$ as groups with the action of $M_m(\mathbf{F}_p)$. Hence

$$\mathcal{O}_K[S_{m,n}(\mathcal{O}_K)] \cong \mathcal{O}_K[M_{m,n}(\mathbf{F}_p)]$$

as $M_m(\mathbf{F}_p)$-Hopf algebras. This shows the lemma.

Proof of Theorem 1.1. Since group like elements are linearly independent, see for example [7], page 15, we see that f is a monomorphism. Now let us extend f to the monomorphism

$$f : K[M_{m,n}(\mathbf{F}_p)] \to \overline{K(n)}^0_K(B(\mathbf{Z}/p)^m).$$

Hopf algebras $K[M_{m,n}(\mathbf{F}_p)]$ and $\overline{K(n)}^0_K(B(\mathbf{Z}/p)^m)$ both have rank p^{nm} over K. So f is an isomorphism.

Let K be a local field with ring of integers $\mathcal{O}_K$, maximal ideal m and residue field k. Let M be a finite semi-group. We will construct a map $R_K(M) \to R_k(M)$ in the following way. Let E be a K-representation of M. In E we choose a lattice T, i.e. a finitely generated $\mathcal{O}_K$-submodule of E, which generates E as a K-module. Replacing T by the sum of its images under the action of M, we can assume that T is invariant under M. Then the reduction $\overline{T} = T/mT$ is a k-representation of M.

Lemma 2.6. *The image of $\overline{T}$ in $R_k(M)$ is independent of a choice of an $\mathcal{O}_K$-sub-module T.*

Proof. The case when M is a finite group is proved in [6], Theorem 32. We follow Serre's proof.

Let T_1 and T_2 be lattices of E stable under M. We must show that $[\overline{T}_1] = [\overline{T}_2]$ in $R_k(M)$. First we consider special case $mT_1 \subset T_2 \subset T_1$. Then $mT_2 \subset mT_1 \subset T_2$ and we have two short exact sequences of k-modules

$$0 \to T_2/mT_1 \to T_1/mT_1 \to T_1/T_2 \to 0$$

and

$$0 \to mT_1/mT_2 \to T_2/mT_2 \to T_2/mT_1 \to 0.$$

Hence in $R_k(M)$ we have two equations

$$[T_1/mT_1] = [T_2/mT_1] + [T_1/T_2]$$

and

$$[T_2/mT_2] = [mT_1/mT_2] + [T_2/mT_1].$$

Subtracting we get

$$[T_1/mT_1] - [T_2/mT_2] = [T_1/T_2] - [mT_1/mT_2].$$

Now we need to show that $[T_1/T_2] = [mT_1/mT_2]$. Let Π be the homomorphism obtained from the multiplication by a generator π of the ideal m

$$\Pi : T_1/T_2 \to mT_1/mT_2.$$

It is clear that Π is an epimorphism. Since T_1 and T_2 are lattices of E, they contain no torsion elements. Hence it follows that Π is monomorphism. This proves our special case.

Replacing T_2 by a scalar multiple does not effect $[\bar{T}_2]$. So in general case we can assume that $T_2 \subset T_1$. There exist integer $q \geq 0$ such that $m^q T_1 \subset T_2 \subset T_1$. Now we proceed by induction on q. Assume that if $i < q$ and $m^i T_1 \subset T_2 \subset T_1$, then $[\bar{T}_1] = [\bar{T}_2]$. Let $T_3 = m^{q-1}T_1 + T_2$. Then $m^i T_1 \subset T_3 \subset T_1$ and $mT_3 \subset T_2 \subset T_3$. Thus by induction we get $[\bar{T}_1] = [\bar{T}_3] = [\bar{T}_2]$

In the K-module $K[M_{m,n}(\mathbf{F}_p)] = \overline{K(n)}^0_K(B(\mathbf{Z}/p)^m)$ we can choose two $M_m(\mathbf{F}_p)$ invariant $\mathcal{O}_K$-submodules $\mathcal{O}_K[M_{m,n}(\mathbf{F}_p)]$ and $\overline{K(n)}^0_{\mathcal{O}_K}(B(\mathbf{Z}/P)^m)$. Now Theorem 1.1 directly implies Theorem 1.2.

On the ring

$$K(n)^0_{F_{p^n}}(B(\mathbf{Z}/p)^m) = \mathbf{F}_{p^n}[x_1, x_2, \ldots, x_m]/(x_1^{p^n}, x_2^{p^n}, \ldots, x_m^{p^n})$$

we have two actions of $M_m(\mathbf{F}_p)$ defined by (1) and (2). We denote this ring with the action (1) by Q_1 and with the action (2) by Q_2.

Proposition 2.7. $[Q_1] = [Q_2]$ *in* $R_{p^n}(M_m(\mathbf{F}_p))$.

This proposition is easy consequence of the next two lemmas.

Let us assume that $\deg x_i = 2,\ i = 1, \ldots, m$. In Q_1 we consider the following filtration

$$F_i = \{p(x_1, \ldots, x_m) \mid p(x_1, \ldots, x_m) \text{ consists of monomials with deg} \geq 2i\}.$$

Then we have the following

Lemma 2.8. *F_i is invariant under the $M_m(\mathbf{F}_p)$-action defined by* (1).

Proof. For $A = (a_{ij}) \in M_m(\mathbf{F}_p)$ we have

$$(a_{ij}) \cdot x_e = \sum_{j=1}^{m}{}_F [a_{ej}]x_j \equiv a_{e1}x_1 + a_{e2}x_2 + \cdots + a_{em}x_m \text{ (mod deg} > 2).$$

Let $q(x_1, x_2, \ldots, x_m) = cx_1^{i_1}x_2^{i_2} \cdots x_m^{i_m}$ be a monomial of degree $2i$, where $i_1 + i_2 + \cdots + i_m = 2i$, $c \in \mathbf{F}_{p^n}$. Since $A = (a_{ij})$ acts on Q_1 as a ring homomorphism we have

$$(a_{ij}) \cdot q(x_1, x_2, \ldots, x_m) = c((a_{ij}) \cdot x_1)^{i_1}((a_{ij}) \cdot x_2)^{i_2} \cdots ((a_{ij}) \cdot x_m)^{i_m})$$
$$\equiv c(a_{11}x_1 + \cdots + a_{1m}x_m)^{i_1} \cdots (a_{m1}x_1 + \cdots + a_{mm}x_m)^{i_m} \pmod{\deg > 2i} \quad (4)$$

So $(a_{ij}) \cdot q(x_1, x_2, \ldots, x_m) \in F_i$.

Let us denote by R_i the degree $2i$ part of Q_2. We have a map which takes each polynomial to its degree $2i$ part. This map induces $\mathbf{F}_{p^n}$-module isomorphism

$$h : F_i/F_{i+1} \to R_i.$$

From formula (4) we have the following

Lemma 2.9. *Action* (1) *in* F_i/F_{i+1} *corresponds to action* (2) *in* R_i *by the* $\mathbf{F}_{p^n}$-*module isomorphism* h.

Now from Proposition 2.7 easily we get Theorem 1.3.

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