J. Math. Kyoto Univ. (JMKYAZ)
38-3 (1998) 439-451

The Neumann problem on wave propagation in a 2-D external domain with cuts

By

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1. Introduction

The theory of boundary value problems for 2-D PDE's mostly deals with connected domains bounded by closed curves. A small number of investigations are devoted to the problems outside cuts in the plane. There are almost no any results concerning the well-posedness of classical problems in domains bounded by closed curves and containing cuts. It seems, that the difficulties in the analysis of these problems proceed from the different technique of the proof of the solvability theorems for domains bounded by closed curves and for plane with cuts. It is very likely, that there is no great difference between these problems in nature. In the present note we try to overcome technical difficulties for the Helmholtz equation in an external domain with cuts and therefore to suggest approach to the analysis of similar problems.

The 2-D Neumann boundary value problem for the Helmholtz equation in a multiply connected domain bounded by closed curves is considered in monographs on mathematical physics, for instance in [1]. The review on studies of the Neumann problem for this equation in the exterior of cuts is given in [4]. The present note is attempt to join these problems together and to consider both internal and external domains containing cuts. From practical stand-point such domains have great significance, because cuts model cracks, screens or wings in physical problems. We consider the case, when the parameter in the Helmholtz equation is not an eigenvalue for corresponding single connected internal domains.

The Dirichlet problem for the propagative Helmholtz equation in a 2-D external domain with cuts has been studied in [6]. The Dirichlet problem for the dissipative Helmholtz equation in both internal and external domains with cuts has been investigated in [7]. The case of strongly dissipative Helmholtz equation has been treated in [8] under weakened conditions.

The present paper is organized as follows. Formulation of the boundary value problem and the uniqueness theorem are given in the section 2. With the help of potential theory, the problem is reduced to the boundary integral equations in the section 3. The Fredholm integral equation of the 2-nd kind is derived in

Communicated by K. Ueno, June 24, 1997

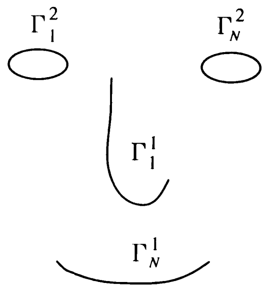

Fig. 1. An external domain

the section 4. In this section we also prove the solvability of the integral equations and formulate the existence theorem for the boundary value problem.

2. Formulation of the problem

In the plane $x = (x_1, x_2) \in R^2$ we consider the external multiply connected domain bounded by simple open curves $\Gamma_1^1, \dots, \Gamma_{N_1}^1 \in C^{2,\lambda}$ and simple closed curves $\Gamma_1^2, \dots, \Gamma_{N_2}^2 \in C^{1,\lambda}$, $\lambda \in (0, 1]$, so that the curves do not have points in common. We put

$$\Gamma^1 = \bigcup_{n=1}^{N_1} \Gamma_n^1, \quad \Gamma^2 = \bigcup_{n=1}^{N_2} \Gamma_n^2, \quad \Gamma = \Gamma^1 \cup \Gamma^2.$$

The external connected domain bounded by Γ^2 will be called $\mathscr{D}$. We assume that each curve Γ_n^k is parametricized by the arc length $s : \Gamma_n^k = \{x : x = x(s) = (x_1(s), x_2(s)),\ s \in [a_n^k, b_n^k]\}$, $n = 1, \dots, N_k$, $k = 1, 2$, so that $a_1^1 < b_1^1 < \cdots < a_{N_1}^1 < b_{N_1}^1 < a_1^2 < b_1^2 < \cdots < a_{N_2}^2 < b_{N_2}^2$ and the domain $\mathscr{D}$ is to the right when the parameter s increases on Γ_n^2. Therefore points $x \in \Gamma$ and values of the parameter s are in one-to-one correspondence except a_n^2, b_n^2, which correspond to the same point x for $n = 1, \dots, N_2$. Below the sets of the intervals on the Os axis

$$\bigcup_{n=1}^{N_1} [a_n^1, b_n^1], \quad \bigcup_{n=1}^{N_2} [a_n^2, b_n^2], \quad \bigcup_{k=1}^{2} \bigcup_{n=1}^{N_k} [a_n^k, b_n^k]$$

will be denoted by Γ^1, Γ^2 and Γ also.

The internal domain bounded by Γ_n^2 will be called $\mathscr{D}_n$, $n = 1, \dots, N_2$.

We consider Γ^1 as a set of cuts. The side of Γ^1 which is on the left, when the parameter s increases will be denoted by $(\Gamma^1)^+$ and the opposite side will be denoted by $(\Gamma^1)^-$.

We put $C^{0,r}(\Gamma_n^2) = \{\mathscr{F}(s) : \mathscr{F}(s) \in C^{0,r}[a_n^2, b_n^2],\ \mathscr{F}(a_n^2) = \mathscr{F}(b_n^2)\}$, $r \in [0, 1]$ and

$$C^{0,r}(\Gamma^2) = \bigcap_{n=1}^{N_2} C^{0,r}(\Gamma_n^2).$$

The tangent vector to Γ at the point $x(s)$ we denote by $\tau_x = (\cos\alpha(s), \sin\alpha(s))$, where $\cos\alpha(s) = x_1'(s)$, $\sin\alpha(s) = x_2'(s)$. Let $\mathbf{n}_x = (\sin\alpha(s), -\cos\alpha(s))$ be a normal vector to Γ at $x(s)$. The direction of $\mathbf{n}_x$ is chosen such that it will coincide with the direction of τ_x if $\mathbf{n}_x$ is rotated anticlockwise through an angle of $\pi/2$.

We say, that the function $u(x)$ belongs to the smoothness class $\mathbf{K}$ if

1) $u \in C^0(\overline{\mathscr{D}\backslash\Gamma^1}) \cap C^2(\mathscr{D}\backslash\Gamma^1)$,

2) $\nabla u \in C^0(\overline{\mathscr{D}\backslash\Gamma^2}\backslash X)$, where X is a point-set, consisting of the end-points of Γ^1:

$$X = \bigcup_{n=1}^{N_1} (x(a_n^1) \cup x(b_n^1)),$$

3) in the neighbourhood of any point $x(d) \in X$ for some constants $\mathscr{C} > 0$, $\varepsilon > -1$ the inequality holds

$$|\nabla u| \le \mathscr{C}|x - x(d)|^{\varepsilon}, \tag{1}$$

where $x \to x(d)$ and $d = a_n^1$ or $d = b_n^1$, $n = 1, \ldots, N_1$.

Let us formulate the Neumann problem for the Helmholtz equation in the domain $\mathscr{D}\backslash\Gamma^1$.

Problem U. To find a function $u(x)$ of the class $\mathbf{K}$ which satisfies the Helmholtz equation

$$u_{x_1x_1}(x) + u_{x_2x_2}(x) + \beta^2 u(x) = 0, \quad x \in \mathscr{D}\backslash\Gamma^1, \quad \beta = \text{const}, \quad \beta > 0, \tag{2a}$$

the boundary conditions

$$\left.\frac{\partial u(x)}{\partial \mathbf{n}_x}\right|_{x(s)\in(\Gamma^1)^+} = F^+(s), \quad \left.\frac{\partial u(x)}{\partial \mathbf{n}_x}\right|_{x(s)\in(\Gamma^1)^-} = F^-(s), \tag{2b}$$

$$\left.\frac{\partial u(x)}{\partial \mathbf{n}_x}\right|_{x(s)\in\Gamma^2} = F(s)$$

and the radiation conditions at infinity

$$u = O(|x|^{-1/2}), \quad \frac{\partial u}{\partial |x|} - i\beta u = o(|x|^{-1/2}), \quad |x| = \sqrt{x_1^2 + x_2^2} \to \infty. \tag{2c}$$

All conditions of the problem $\mathbf{U}$ must be satisfied in the classical sense.

On the basis of the energy equalities and the Rellich lemma [1], we can easily prove the following assertion.

Theorem 1. *If* $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{1,\lambda}$, $\lambda \in (0, 1]$, *then the problem* $\mathbf{U}$ *has at most one solution.*

We will prove the solvability theorem for the problem $\mathbf{U}$ under the additional condition $\mathbf{D}$.

We say, that the condition **D** holds if β^2 is not an eigenvalue of the internal Dirichlet problem

$$u_{x_1x_1}(x) + u_{x_2x_2}(x) + \beta^2 u(x) = 0, \quad x \in \mathscr{D}_n; \quad u(x)|_{x(s)\in\Gamma_n^2} = 0,$$

for $n = 1, \dots, N_2$.

It is well-known [1], that the condition **D** holds, if, for example, the diameter of the each domain $\mathscr{D}_n$ ($n = 1, \dots, N_2$) is small enough. More precisely [1], the condition **D** holds if

$$2\beta^2[\exp(d_n) - 1] < 1 \qquad \text{for } n = 1, \dots, N_2,$$

where d_n is the diameter of the domain $\mathscr{D}_n$.

3. Integral equations at the boundary

Below we assume that

$$F^+(s), \quad F^-(s) \in C^{0,\lambda}(\Gamma^1), \qquad F(s) \in C^{0,\lambda}(\Gamma^2), \quad \lambda \in (0,1]. \tag{3}$$

If $\mathscr{B}_1(\Gamma^1)$, $\mathscr{B}_2(\Gamma^2)$ are Banach spaces of functions given on Γ^1 and Γ^2, then for functions given on Γ we introduce the Banach space $\mathscr{B}_1(\Gamma^1) \cap \mathscr{B}_2(\Gamma^2)$ with the norm $\|\cdot\|_{\mathscr{B}_1(\Gamma^1)\cap\mathscr{B}_2(\Gamma^2)} = \|\cdot\|_{\mathscr{B}_1(\Gamma^1)} + \|\cdot\|_{\mathscr{B}_2(\Gamma^2)}$.

We consider the angular potential from [3], [4] for the equation (2a) on Γ^1

$$w_1[\mu](x) = \frac{i}{4}\int_{\Gamma^1} \mu(\sigma) V(x,\sigma)\, d\sigma. \tag{4}$$

The kernel $V(x,\sigma)$ is defined on the each curve Γ_n^1, $n = 1, \dots, N_1$ by the formula

$$V(x,\sigma) = \int_{a_n^1}^{\sigma} \frac{\partial}{\partial \mathbf{n}_y} \mathscr{H}_0^{(1)}(\beta|x - y(\xi)|)\, d\xi, \quad \sigma \in [a_n^1, b_n^1],$$

where $\mathscr{H}_0^{(1)}(z)$ is the Hankel function of the first kind

$$\mathscr{H}_0^{(1)}(z) = \frac{\sqrt{2}\exp(iz - i\pi/4)}{\pi\sqrt{z}} \int_0^\infty \exp(-t) t^{-1/2} \left(1 + \frac{it}{2z}\right)^{-1/2} dt,$$

$$y = y(\xi) = (y_1(\xi), y_2(\xi)), \quad |x - y(\xi)| = \sqrt{(x_1 - y_1(\xi))^2 + (x_2 - y_2(\xi))^2}.$$

Below we suppose that $\mu(\sigma)$ belongs to the Banach space $C_q^\omega(\Gamma^1)$, $\omega \in (0,1]$, $q \in [0,1)$ and satisfies the following additional conditions

$$\int_{a_n^1}^{b_n^1} \mu(\sigma)\, d\sigma = 0, \quad n = 1, \dots, N_1. \tag{5}$$

We say, that $\mu(s) \in C_q^\omega(\Gamma^1)$ if

$$\mu(s) \prod_{n=1}^{N_1} |s - a_n^1|^q\, |s - b_n^1|^q \in C^{0,\omega}(\Gamma^1),$$

where $C^{0,\omega}(\Gamma^1)$ is a Holder space with the index ω and

$$\|\mu(s)\|_{C_q^\omega(\Gamma^1)} = \left\|\mu(s)\prod_{n=1}^{N_1}|s-a_n^1|^q\,|s-b_n^1|^q\right\|_{C^{0,\omega}(\Gamma^1)}.$$

As shown in [3], [4] for such $\mu(\sigma)$ the angular potential $w_1[\mu](x)$ belongs to the class $\mathbf{K}$. In particular, the inequality (1) holds with $\varepsilon = -q$, if $q \in (0,1)$. Moreover, integrating $w_1[\mu](x)$ by parts and using (4) we express the angular potential in terms of a double layer potential

$$w_1[\mu](x) = -\frac{i}{4}\int_{\Gamma^1}\rho(\sigma)\frac{\partial}{\partial \mathbf{n}_y}\mathscr{H}_0^{(1)}(\beta|x-y(\sigma)|)\,d\sigma, \tag{6}$$

with the density

$$\rho(\sigma) = \int_{a_n^1}^{\sigma}\mu(\xi)\,d\xi, \quad \sigma \in [a_n^1, b_n^1], \quad n = 1,\dots,N_1.$$

Consequently, $w_1[\mu](x)$ satisfies both equation (2a) outside Γ^1 and the conditions at infinity (2c).

Let us construct a solution of the problem $\mathbf{U}$. This solution can be obtained with the help of potential theory for the Helmholtz equation (2a). We seek a solution of the problem in the following form

$$W[\nu,\mu](x) = v_1[\nu](x) + w[\mu](x), \tag{7}$$

where

$$\begin{aligned} v_1[\nu](x) &= \frac{i}{4}\int_{\Gamma^1}\nu(\sigma)\mathscr{H}_0^{(1)}(\beta|x-y(\sigma)|)\,d\sigma, \\ w[\mu](x) &= w_1[\mu](x) + w_2[\mu](x), \\ w_2[\mu](x) &= \frac{i}{4}\int_{\Gamma^2}\mu(\sigma)\mathscr{H}_0^{(1)}(\beta|x-y(\sigma)|)\,d\sigma, \end{aligned} \tag{8}$$

and $w_1[\mu](x)$ is given by (4), (6).

By $\int_{\Gamma^k}\dots d\sigma$ we mean

$$\sum_{n=1}^{N_k}\int_{a_n^k}^{b_n^k}\dots d\sigma.$$

We will look for $\nu(s)$ in the space $C^{0,\lambda}(\Gamma^1)$.

We will seek $\mu(s)$ from the Banach space $C_q^\omega(\Gamma^1)\cap C^{0,\lambda/2}(\Gamma^2)$, $\omega \in (0,1]$, $q \in [0,1)$ with the norm $\|\cdot\|_{C_q^\omega(\Gamma^1)\cap C^{0,\lambda/2}(\Gamma^2)} = \|\cdot\|_{C_q^\omega(\Gamma^1)} + \|\cdot\|_{C^{0,\lambda/2}(\Gamma^2)}$. Besides, $\mu(s)$ must satisfy conditions (5).

It follows from [3], that for such $\mu(s)$, $\nu(s)$ the function (7) belongs to the class $\mathbf{K}$ and satisfies all conditions of the problem $\mathbf{U}$ except the boundary condition (2b).

To satisfy the boundary condition we put (7) in (2b), use the limit formulas for the angular potential from [3] and arrive at the integral equation for the densities $\mu(s)$, $v(s)$:

$$\text{(9a)}\quad \pm\frac{1}{2}v(s)+\frac{i}{4}\int_{\Gamma^1}v(\sigma)\frac{\partial}{\partial\mathbf{n}_x}\mathscr{H}_0^{(1)}(\beta|x(s)-y(\sigma)|)\,d\sigma$$

$$-\frac{1}{2\pi}\int_{\Gamma^1}\mu(\sigma)\frac{\sin\varphi_0(x(s),y(\sigma))}{|x(s)-y(\sigma)|}\,d\sigma+\frac{i}{4}\int_{\Gamma^1}\mu(\sigma)\frac{\partial}{\partial\mathbf{n}_x}V_0(x(s),\sigma)\,d\sigma$$

$$+\frac{i}{4}\int_{\Gamma^2}\mu(\sigma)\frac{\partial}{\partial\mathbf{n}_x}\mathscr{H}_0^{(1)}(\beta|x(s)-y(\sigma)|)\,d\sigma=F^{\pm}(s),\quad s\in\Gamma^1,$$

$$\text{(9b)}\quad \frac{i}{4}\int_{\Gamma^1}v(\sigma)\frac{\partial}{\partial\mathbf{n}_x}\mathscr{H}_0^{(1)}(\beta|x(s)-y(\sigma)|)\,d\sigma$$

$$-\frac{1}{2\pi}\int_{\Gamma^1}\mu(\sigma)\frac{\sin\varphi_0(x(s),y(\sigma))}{|x(s)-y(\sigma)|}\,d\sigma+\frac{i}{4}\int_{\Gamma^1}\mu(\sigma)\frac{\partial}{\partial\mathbf{n}_x}V_0(x(s),\sigma)\,d\sigma-\frac{1}{2}\mu(s)$$

$$+\frac{i}{4}\int_{\Gamma^2}\mu(\sigma)\frac{\partial}{\partial\mathbf{n}_x}\mathscr{H}_0^{(1)}(\beta|x(s)-y(\sigma)|)\,d\sigma=F(s),\quad s\in\Gamma^2,$$

where

$$V_0(x,\sigma)=\int_{a_n^1}^{\sigma}\frac{\partial}{\partial\mathbf{n}_y}h(\beta|x-y(\xi)|)\,d\xi,\quad \sigma\in[a_n^1,b_n^1],\quad n=1,2,\dots,N_1,$$

$$h(z)=\mathscr{H}_0^{(1)}(z)-\frac{2i}{\pi}\ln\frac{z}{\beta}.$$

By $\varphi_0(x,y)$ we denote the angle between the vector $\overrightarrow{xy}$ and the direction of the normal $\mathbf{n}_x$. The angle $\varphi_0(x,y)$ is taken to be positive if it is measured anticlockwise from $\mathbf{n}_x$ and negative if it is measured clockwise from $\mathbf{n}_x$. Besides, $\varphi_0(x,y)$ is continuous in $x,y\in\Gamma$ if $x\neq y$.

Equation (9a) is obtained as $x\to x(s)\in(\Gamma^1)^{\pm}$ and comprises two integral equations. The upper sign denotes the integral equation on $(\Gamma^1)^+$, the lower sign denotes the integral equation on $(\Gamma^1)^-$.

In addition to the integral equations written above we have the conditions (5).

Subtracting the integral equations (9a) we find

$$\text{(10)}\qquad v(s)=(F^+(s)-F^-(s))\in C^{0,\lambda}(\Gamma^1).$$

We note that $v(s)$ is found completely and satisfies all required conditions. Hence, the potential $v_1[v](x)$ is found completely as well.

We introduce the function $f(s)$ on Γ by the formula

$$\text{(11)}\quad f(s)=F(s)-\frac{i}{4}\int_{\Gamma^1}(F^+(\sigma)-F^-(\sigma))\frac{\partial}{\partial\mathbf{n}_x}\mathscr{H}_0^{(1)}(\beta|x(s)-y(\sigma)|)\,d\sigma,\quad s\in\Gamma,$$

where

$$F(s) = \tfrac{1}{2}(F^+(s) + F^-(s)), \quad s \in \Gamma^1.$$

As shown in [4], if $s \in \Gamma^1$, then $f(s) \in C^{0,\lambda}(\Gamma^1)$. Hence, $f(s) \in C^{0,\lambda}(\Gamma)$.

Adding the integral equations (9a) and taking into account (9b) we obtain the integral equation for $\mu(s)$ on Γ

$$\begin{aligned} (12) \qquad & -\frac{1}{2\pi}\int_{\Gamma^1} \mu(\sigma)\frac{\sin\varphi_0(x(s), y(\sigma))}{|x(s) - y(\sigma)|}\,d\sigma \\ & + \frac{i}{4}\int_{\Gamma^1} \mu(\sigma)\frac{\partial}{\partial \mathbf{n}_x} V_0(x(s), \sigma)\,d\sigma - \frac{1}{2}\delta(s)\mu(s) \\ & + \frac{i}{4}\int_{\Gamma^2} \mu(\sigma)\frac{\partial}{\partial \mathbf{n}_x}\mathscr{H}_0^{(1)}(\beta|x(s) - y(\sigma)|)\,d\sigma = f(s), \quad s \in \Gamma, \end{aligned}$$

where $\delta(s) = 0$ if $s \in \Gamma^1$ and $\delta(s) = 1$ if $s \in \Gamma^2$, $f(s)$ is given in (11).

Let us show that any integrable on Γ^1 and continuous on Γ^2 solution of equation (12) belongs to $C^{0,\lambda/2}(\Gamma^2)$. Indeed, it follows from [3], [4] that if $s \in \Gamma^2$, then the kernel of the integral term in (12) can be expressed in the form

$$\frac{I_0(s,\sigma)}{s - \sigma} + I_1(s,\sigma),$$

where $I_0(s,\sigma) \in C^{0,\lambda}(\Gamma^2 \times \Gamma)$, $I_1(s,\sigma) \in C^{0,\lambda/2}(\Gamma^2 \times \Gamma)$ and $I_0(s,s) = 0$. From [5] we obtain

$$\frac{I_0(s,\sigma)}{s - \sigma} + I_1(s,\sigma) = \frac{I_2(s,\sigma)}{|s - \sigma|^{1-\lambda/2}} + I_1(s,\sigma),$$

where $I_2(s,\sigma) \in C^{0,\lambda/2}(\Gamma^2 \times \Gamma)$. In accordance with [5], due to this representation the integral term from (12) belongs to $C^{0,\lambda/2}(\Gamma^2)$ in s. Since $f(s) \in C^{0,\lambda}(\Gamma^2)$, the solution $\mu(s)$ of (12) belongs to $C^{0,\lambda/2}(\Gamma^2)$.

Thus, if $\mu(s)$ is a solution of equations (5), (12) from the space $C_q^{\omega}(\Gamma^1) \cap C^0(\Gamma^2)$, $\omega \in (0,1]$, $q \in [0,1)$, then $\mu(s) \in C_q^{\omega}(\Gamma^1) \cap C^{0,\lambda/2}(\Gamma^2)$ and the potential (7) satisfies all conditions of the problem $\mathbf{U}$.

The following theorem holds.

Theorem 2. *If* $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{1,\lambda}$ *conditions* (3) *hold and the system of equations* (12), (5) *has a solution* $\mu(s)$ *from the Banach space* $C_q^{\omega}(\Gamma^1) \cap C^0(\Gamma^2)$, $\omega \in (0,1]$, $q \in [0,1)$, *then a solution of the problem* $\mathbf{U}$ *is given by* (7), *where* $v(s)$ *is defined in* (10).

Below we look for $\mu(s)$ in the Banach space $C_q^{\omega}(\Gamma^1) \cap C^0(\Gamma^2)$.

If $s \in \Gamma^2$, then (12) is an equation of the second kind. If $s \in \Gamma^1$, then (12) is a singular equation [5].

Our further treatment will be aimed to the proof of the solvability of the system (5), (12) in the Banach space $C_q^{\omega}(\Gamma^1) \cap C^0(\Gamma^2)$. Moreover, we reduce the system (5), (12) to a Fredholm equation of the second kind, which can be easily computed by classical methods.

Equation (12) on Γ^2 we rewrite in the form

$$\mu(s) + \int_{\Gamma} \mu(\sigma) A_2(s,\sigma)\, d\sigma = -2f(s), \quad s \in \Gamma^2, \tag{13}$$

where

$$A_2(s,\sigma) = -\left\{\frac{i}{2}(1-\delta(\sigma))\frac{\partial}{\partial \mathbf{n}_x} V(x(s),\sigma) + \frac{i}{2}\delta(\sigma)\frac{\partial}{\partial \mathbf{n}_x}\mathscr{H}_0^{(1)}(\beta|x(s)-y(\sigma)|)\right\}$$
$$= \frac{I_2(s,\sigma)}{|s-\sigma|^{1-\lambda/2}} + I_1(s,\sigma),$$

$V(x,\sigma)$ is the kernel of the angular potential (4) and $I_j(s,\sigma) \in C^{0,\lambda/2}(\Gamma^2 \times \Gamma)$, $j = 1, 2$, as shown above.

It can be easily proved that

$$\frac{\sin \varphi_0(x(s), y(\sigma))}{|x(s) - y(\sigma)|} - \frac{1}{\sigma - 2} \in C^{0,\lambda}(\Gamma^1 \times \Gamma^1)$$

(see [3], [4] for details). Therefore we can rewrite (12) on Γ^1 in the form

$$\frac{1}{\pi}\int_{\Gamma^1} \mu(\sigma)\frac{d\sigma}{\sigma - s} + \int_{\Gamma} \mu(\sigma)\, Y(s,\sigma)\, d\sigma = -2f(s), \quad s \in \Gamma^1, \tag{14}$$

where

$$Y(s,\sigma) = \left\{(1-\delta(\sigma))\left[\frac{1}{\pi}\left(\frac{\sin\varphi_0(x(s),y(\sigma))}{|x(s)-y(\sigma)|} - \frac{1}{\sigma - s}\right) - \frac{i}{2}\frac{\partial}{\partial \mathbf{n}_x} V_0(x(s),\sigma)\right]\right.$$
$$\left. - \frac{i}{2}\delta(\sigma)\frac{\partial}{\partial \mathbf{n}_x}\mathscr{H}_0^{(1)}(\beta|x(s)-y(\sigma)|)\right\} \in C^{0,p_0}(\Gamma^1 \times \Gamma),$$

$p_0 = \lambda$ if $0 < \lambda < 1$ and $p_0 = 1 - \varepsilon_0$ for any $\varepsilon_0 \in (0,1)$ if $\lambda = 1$.

4. The Fredholm integral equation and the solution of the problem

Inverting the singular integral operator in (14) we arrive at the following integral equation of the second kind [5]:

$$\mu(s) + \frac{1}{Q_1(s)}\int_{\Gamma} \mu(\sigma) A_0(s,\sigma)\, d\sigma + \frac{1}{Q_1(s)}\sum_{n=0}^{N_1-1} G_n s^n$$
$$= \frac{1}{Q_1(s)}\Phi_0(s), \quad s \in \Gamma^1, \tag{15}$$

where

$$A_0(s,\sigma) = -\frac{1}{\pi}\int_{\Gamma^1} \frac{Y(\xi,\sigma)}{\xi - s} Q_1(\xi)\, d\xi,$$

$$Q_1(s) = \prod_{n=1}^{N_1} \left|\sqrt{s-a_n^1}\sqrt{b_n^1 - s}\right| \operatorname{sign}(s - a_n^1),$$

$$\Phi_0(s) = \frac{1}{\pi}\int_{\Gamma^1} \frac{2Q_1(\sigma)f(\sigma)}{\sigma - s}\, d\sigma,$$

$G_0, \ldots, G_{N_1-1}$ are arbitrary constants.

To derive equations for $G_0, \ldots, G_{N_1-1}$ we substitute $\mu(s)$ from (15) in the conditions (5), then we obtain

$$\int_\Gamma \mu(\sigma) l_n(\sigma)\, d\sigma + \sum_{m=0}^{N_1-1} B_{nm} G_m = H_n, \quad n = 1, \ldots, N_1, \tag{16}$$

where

$$\begin{aligned} l_n(\sigma) &= -\int_{\Gamma_n^1} Q_1^{-1}(s) A_0(s,\sigma)\, ds, \\ B_{nm} &= -\int_{\Gamma_n^1} Q_1^{-1}(s) s^m\, ds, \\ H_n &= -\int_{\Gamma_n^1} Q_1^{-1}(s)\Phi_0(s)\, ds. \end{aligned} \tag{17}$$

By B we denote the $N_1 \times N_1$ matrix with the elements B_{nm} from (17). As shown in [4], the matrix B is invertible. The elements of the inverse matrix will be called $(B^{-1})_{nm}$. Inverting the matrix B in (16) we express the constants $G_0, \ldots, G_{N_1-1}$ in terms of $\mu(s)$

$$G_n = \sum_{m=1}^{N_1} (B^{-1})_{nm} \left[H_m - \int_\Gamma \mu(\sigma) l_m(\sigma)\, d\sigma \right].$$

We substitute G_n in (15) and obtain the following integral equation for $\mu(s)$ on Γ^1

$$\mu(s) + \frac{1}{Q_1(s)}\int_\Gamma \mu(\sigma) A_1(s,\sigma)\, d\sigma = \frac{1}{Q_1(s)}\Phi_1(s), \quad s \in \Gamma^1, \tag{18}$$

where

$$A_1(s,\sigma) = A_0(s,\sigma) - \sum_{n=0}^{N_1-1} s^n \sum_{m=1}^{N_1} (B^{-1})_{nm} l_m(\sigma),$$

$$\Phi_1(s) = \Phi_0(s) - \sum_{n=0}^{N_1-1} s^n \sum_{m=1}^{N_1} (B^{-1})_{nm} H_m.$$

It can be shown using the properties of singular integrals [2], [5], that $\Phi_0(s)$, $A_0(s,\sigma)$ are Holder functions if $s \in \Gamma^1$, $\sigma \in \Gamma$. Therefore, $\Phi_1(s)$, $A_1(s,\sigma)$ are also Holder functions if $s \in \Gamma^1$, $\sigma \in \Gamma$. Consequently, any solution of (18) belongs to $C^{\omega}_{1/2}(\Gamma^1)$ and below we look for $\mu(s)$ on Γ^1 in this space.

We put

$$Q(s) = (1-\delta(s))Q_1(s) + \delta(s), \quad s \in \Gamma.$$

Instead of $\mu(s) \in C^{\omega}_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$ we introduce the new unknown function $\mu_*(s) = \mu(s)Q(s) \in C^{0,\omega}(\Gamma^1) \cap C^0(\Gamma^2)$ and rewrite (13), (18) in the form of one equation

$$\mu_*(s) + \int_{\Gamma} \mu_*(\sigma)Q^{-1}(\sigma)A(s,\sigma)\,d\sigma = \Phi(s), \quad s \in \Gamma, \tag{19}$$

where

$$A(s,\sigma) = (1-\delta(s))A_1(s,\sigma) + \delta(s)A_2(s,\sigma),$$

$$\Phi(s) = (1-\delta(s))\Phi_1(s) - 2\delta(s)f(s).$$

Thus, the system of equations (12), (5) for $\mu(s)$ has been reduced to the equation (19) for the function $\mu_*(s)$. It is clear from our consideration that any solution of (19) gives a solution of system (12), (5).

As noted above, $\Phi_1(s)$ and $A_1(s,\sigma)$ are Holder functions if $s \in \Gamma^1$, $\sigma \in \Gamma$. More precisely (see [4], [5]), $\Phi_1(s) \in C^{0,p}(\Gamma^1)$, $p = \min\{1/2, \lambda\}$ and $A_1(s,\sigma)$ belongs to $C^{0,p}(\Gamma^1)$ in s uniformly with respect to $\sigma \in \Gamma$.

We arrive at the following assertion.

Lemma. *If* $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{1,\lambda}$, $\lambda \in (0,1]$, $\Phi(s) \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$, $p = \min\{\lambda, 1/2\}$, *and* $\mu_*(s)$ *from* $C^0(\Gamma)$ *satisfies the equation* (19), *then* $\mu_*(s) \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$.

The condition $\Phi(s) \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$ holds if $f(s) \in C^{0,\lambda}(\Gamma)$.

Hence below we will seek $\mu_*(s)$ from $C^0(\Gamma)$.

Since $A_1(s,\sigma) \in C^0(\Gamma^1 \times \Gamma)$ and due to the special representation for $A_2(s,\sigma)$ from (13), the integral operator from (19):

$$A\mu_* = \int_{\Gamma} \mu_*(\sigma)Q^{-1}(\sigma)A(s,\sigma)\,d\sigma$$

is a compact operator mapping $C^0(\Gamma)$ into itself. Therefore, (19) is a Fredholm equation of the second kind in the Banach space $C^0(\Gamma)$.

Let us show that homogeneous equation (19) has only a trivial solution. Then, according to Fredholm's theorems, the inhomogeneous equation (19) has a unique solution for any right-hand side. We will prove this by a contradiction. Let $\mu_*^0(s) \in C^0(\Gamma)$ be a non-trivial solution of the homogeneous equation (19). According to the lemma $\mu_*^0(s) \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$, $p = \min\{\lambda, 1/2\}$. Therefore the function $\mu^0(s) = \mu_*^0(s)Q^{-1}(s) \in C^p_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$ converts the homogeneous equations (13), (18) into identities. Using the homogeneous identity (18) we check, that $\mu^0(s)$ satisfies conditions (5). Besides, acting on the homogeneous identity (18) with a singular operator with the kernel $(s-t)^{-1}$ we find that $\mu^0(s)$ satisfies the homogeneous equation (14). Consequently, $\mu^0(s)$ satisfies the homogeneous equation (12). On the basis of theorem 2, $W[0,\mu^0](x) \equiv w[\mu^0](x)$ is a solution of the homogeneous problem **U**. According to theorem 1: $w[\mu^0](x) \equiv 0$, $x \in \mathscr{D} \backslash \Gamma^1$. Using the limit formulas for tangent derivatives of an angular potential [3], we obtain

$$\lim_{x \to x(s) \in (\Gamma^1)^+} \frac{\partial}{\partial \tau_x} w[\mu^0](x) - \lim_{x \to x(s) \in (\Gamma^1)^-} \frac{\partial}{\partial \tau_x} w[\mu^0](x) = \mu^0(s) \equiv 0, \quad s \in \Gamma^1.$$

Hence, $w[\mu^0](x) = w_2[\mu^0](x) \equiv 0$, $x \in \mathscr{D}$, and $\mu^0(s)$ satisfies the following homogeneous equation

$$(20) \qquad -\frac{1}{2}\mu^0(s) + \frac{i}{4}\int_{\Gamma^2} \mu^0(\sigma) \frac{\partial}{\partial \mathbf{n}_x} \mathscr{H}_0^{(1)}(\beta|x(s) - y(\sigma)|)\, d\sigma = 0, \quad s \in \Gamma^2.$$

The Fredholm equation (20) is well-known in classical mathematical physics. We arrive at (20) when solving the Neumann problem for the Helmholtz equation (2a) in the domain $\mathscr{D}$ by the single layer potential. According to [1], if the condition **D** holds, then the equation (20) has only the trivial solution $\mu^0(s) \equiv 0$ in $C^0(\Gamma^2)$. To prove this we note, that $w_2[\mu^0](x)$ satisfies the following Dirichlet problem

$$\Delta w_2 + \beta^2 w_2 = 0 \ \text{ in } \ \mathscr{D}_n, \quad w_2|_{\Gamma^2_n} = 0,$$

for $n = 1, \dots, N_2$, where Δ is Laplacian. If the condition **D** holds, then $w_2[\mu^0](x) \equiv 0$ in $\mathscr{D}_n$ $(n = 1, \dots, N_2)$ and due to the jump of the normal derivative of the single layer potential $w_2[\mu^0](x)$ on Γ^2 we obtain $\mu^0(s) \equiv 0$, $s \in \Gamma^2$.

Consequently, if $s \in \Gamma$, then $\mu^0(s) \equiv 0$, $\mu^0_*(s) = \mu^0(s)Q^{-1}(s) \equiv 0$ and we arrive at the contradiction to the assumption that $\mu^0_*(s)$ is a non-trivial solution of the homogeneous equation (19). Thus, the homogeneous Fredholm equation (19) has only a trivial solution in $C^0(\Gamma)$.

We have proved the following assertion.

Theorem 3. *If* $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{1,\lambda}$, $\lambda \in (0,1]$, *then* (19) *is a Fredholm equation of the second kind in the space* $C^0(\Gamma)$. *Moreover, if the condition* **D** *holds, then equation* (19) *has a unique solution* $\mu_*(s) \in C^0(\Gamma)$ *for any* $\Phi(s) \in C^0(\Gamma)$.

As a consequence of the theorem 3 and the lemma we obtain the corollary.

Corollary. *If* $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{1,\lambda}$, $\lambda \in (0,1]$, *the condition* **D** *holds and* $\Phi(s) \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$, *where* $p = \min\{\lambda, 1/2\}$, *then the unique solution of* (19) *in* $C^0(\Gamma)$, *ensured by theorem* 3, *belongs to* $C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$.

We recall that $\Phi(s)$ belongs to the class of smoothness required in the corollary if $f(s) \in C^{0,\lambda}(\Gamma)$. As mentioned above, if $\mu_*(s) \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$ is a solution of (19), then $\mu(s) = \mu_*(s)Q^{-1}(s) \in C^p_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$ is a solution of system (12), (5). We obtain the following statement.

Proposition. *If $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{1,\lambda}$, $f(s) \in C^{0,\lambda}(\Gamma)$, $\lambda \in (0,1]$ and the condition* **D** *holds, then the system of equations* (12), (5) *has a solution $\mu(s) \in C^p_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$, $p = \min\{1/2, \lambda\}$, which is expressed by the formula $\mu(s) = \mu_*(s)Q^{-1}(s)$, where $\mu_*(s) \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$ is the unique solution of the Fredholm equation* (19) *in $C^0(\Gamma)$.*

We remind, that if conditions (3) hold, then $f(s) \in C^{0,\lambda}(\Gamma)$ and the solution of equations (5), (12) ensured by the proposition belongs to $C^p_{1/2}(\Gamma^1) \cap C^{0,\lambda/2}(\Gamma^2)$. On the basis of the theorem 2 we arrive at the final result.

Theorem 4. *If $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{1,\lambda}$, and conditions* (3) *and* **D** *hold, then the solution of the problem* **U** *exists and is given by* (7), *where $v(s)$ is defined in* (10) *and $\mu(s)$ is a solution of equations* (12), (5) *from $C^p_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$, $p = \min\{1/2, \lambda\}$ ensured by the proposition. More precisely, $\mu(s) \in C^p_{1/2}(\Gamma^1) \cap C^{0,\lambda/2}(\Gamma^2)$.*

It can be checked directly that the solution of the problem **U** satisfies condition (1) with $\varepsilon = -1/2$. Explicit expressions for singularities of the solution gradient at the end-points of the open curves can be easily obtained with the help of formulas presented in [4].

Theorem 4 ensures existence of a classical solution of the problem **U** when $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{1,\lambda}$ and conditions (3) and **D** hold. The uniqueness of the classical solution follows from the theorem 1. On the basis of our consideration we suggest the following scheme for solving the problem **U**. First, we find the unique solution $\mu_*(s)$ of the Fredholm equation (19) from $C^0(\Gamma)$. This solution automatically belongs to $C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$, $p = \min\{\lambda, 1/2\}$. Second, we construct the solution of equations (12), (5) from $C^p_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$ by the formula $\mu(s) = \mu_*(s)Q^{-1}(s)$. This solution automatically belongs to $C^p_{1/2}(\Gamma^1) \cap C^{0,\lambda/2}(\Gamma^2)$. Finally, substituting $v(s)$ from (10) and $\mu(s)$ in (7) we obtain the solution of the problem **U**.

Modern methods for numerical analysis of integral equations with singular integrals are presented in [9].

Acknowledgements. The author is thankful to Professor R. Sakamoto for the opportunity to carry out research in Japan. The research was supported by the JSPS Fellowship and by the RFBR grant No. 96-01-01411.

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