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Modulo odd prime homotopy normality for H-spaces

By

Kenji KUDOU and Nobuaki YAGITA

Abstract

Given an H-map $i: Y \to X$, we say that i is mod p homotpy narmal if the commutator map from $X_{(p)} \times Y_{(p)}$ to $X_{(p)}$ can be deformed into $Y_{(p)}$. In this paper, we study necessary conditions of mod p homotopy normality for the cases that X are exceptional Lie groups with odd torsion in the cohomology, by using the Morava K-theory.

1. Introduction

When X is a homotopy associative H-space of finite cohomology type, the homotopy functor $[-, X]$ takes its values in category of groups. Given an inclusion $i: Y \to X$ of H-spaces, we are interested in the property such that $i_*[Z, Y]$ are always normal subgroups of $[Z, X]$ for all finite complexes Z. If the inclusion $i: Y \subset X$ has such property, we say that the map i is homotopy normal.

I. James ([4],[5]) notices that the homotopy normality is equivalent to the fact that the commutator map $c_2: X \times Y \to X$ can be deformed into Y. James ([4], [5]) and MacCarty [8] proved many facts about non homotopy normality for classical Lie groups. For example, the standard inclusions $U(n) \subset U(n+1) \subset U(n+2) \subset \cdots$ are not homotopy normal. Furukawa [1] studied the cases including exceptional Lie groups, i.e., inclusions $G_2 \subset F_4 \subset E_6 \subset E_7 \subset E_8$ are not homotopy normal.

The above facts are proved by using the Samelson product or the Hopf algebra structure of $H^*(X; Z/p)$. In this paper, we will study these problems by using the Morava K-homology $K(n)_*(X)$ and its Pontrjagin product structure [10], [11], [12]. Since the cohomology $K(n)^*(X)$ does not have a commutative product for $p=2$, we assume that p is an odd prime througnhout this paper. Moreover we consider just the p-component. Hence we define that an H-map $i: Y \to X$ is *mod p homotpy normal* if its localization $i_{(p)}: Y_{(p)} \to X_{(p)}$ is homotopy normal. Here maps i and $i_{(p)}$ are not assumed injective.

In particular, we will study these problems when X are exceptional Lie groups. For example, suppose that $X = F_4$ and that $H^*(Y; Z/3)$ does not have any 19-dimensional primitive element. Then if an H-map $i: Y \to F_4$ is mod 3 homotopy

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normal, we will prove that $i^*H^*(F_4; Z/3)$ is isomorphic to one of the mod 3 cohomologies of F_4, $Spin(9)$, G_2 and *a point*. However we do not know yet that the natural inclusions $G_2 \subset Spin(9) \subset F_4$ are mod 3 homotopy normal or not, while they are not mod 2 homotopy normal.

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2. mod p homotopy normality

Thoughout this paper, let Y and X be simply connected homotopy associative H-spaces and $i: Y \to X$ be an H-map. Moreover we assume that X is of finite cohomology type, namely, $H^*(X) \cong H^*(Z)$ for some finite complex Z. The commutator map $c_2: X \times X \to X$ is defined by

$$c_2: X \times X \xrightarrow{d_X \times d_X} X \times X \times X \times X \xrightarrow{1 \times tw \times 1} X \times X \times X \times X$$
$$\xrightarrow{1 \times 1 \times \sigma \times \sigma} X \times X \times X \times X \xrightarrow{\mu(\mu \times \mu)} X$$

where d_X is the diagonal, tw is the twisting map, σ is the inverse and μ is the multiplication map of X. Of course, when X is a topological group, $c_2(g, h) = ghg^{-1}h^{-1}$ for $g, h \in X$.

Define an H-map $i: Y \to X$ is mod p normal if $c_2(X_{(p)} \times i(Y)_{(p)})$ is deformed into $i(Y)_{(p)}$, namely, there exist maps $f_t: X_{(p)} \times i(Y)_{(p)} \to X_{(p)}$ such that $f_0 = c_2 | X_{(p)} \times Y_{(p)}$ and $f_1(X_{(p)} \times Y_{(p)}) \subset i(Y)_{(p)}$.

Let h be a commutative ring spectrum over Z/p and $h^*(-)$ be the induced generalized cohomology theory. Here we assume that, for finite complexes X and X', the Künneth formula $h^*(X \times X') \cong h^*(X) \otimes_{h^*} h^*(X')$ holds and that the Kronecker pairing induces a natural isomorphism $h_*(X) \cong \mathrm{Hom}(h^*(X), Z/p)$.

Examples for such $h_*(-)$ are the mod p ordinary homology $H_*(-; Z/p)$ and the Morava K-theory $K(n)_*(-)$ with the coefficient $K(n)_* = Z/p[v_n, v_n^{-1}]$, $|v_n| = 2(p^n - 1)$ for an odd prime p. For these theories, $h_*(X)$ are Hopf algebras with the multiplication μ_* and the comultiplication d_X^*. Hence they are cocommutative but, in general, not commutative.

Lemma 2.1 ([10], [11]). *If $x, y \in h_*(X)$ are primitive, then*

$$c_{2*}(x \otimes y) = [x, y] = xy - (-1)^{|x||y|} yx$$

Proof. Since x is primitive, $\sigma(x) = -x$ and $d_{X*}(x) = x \otimes 1 + 1 \otimes x$. Similar equations hold for y. Hence we get

$$(1 \times tw \times 1)_*(d_X \times d_X)_*(x \otimes y) = (1 \times tw \times 1)_*(x \otimes 1 + 1 \otimes x) \otimes (y \otimes 1 + 1 \otimes y)$$
$$= x \otimes y \otimes 1 \otimes 1 + (-1)^{|x||y|} 1 \otimes y \otimes x \otimes 1 + x \otimes 1 \otimes 1 \otimes y + 1 \otimes 1 \otimes x \otimes y.$$

Applying $(1\times 1\times \sigma\times \sigma)_*$, we have

$$x\otimes y\otimes 1\otimes 1-(-1)^{|x||y|}1\otimes y\otimes x\otimes 1-x\otimes 1\otimes 1\otimes y+1\otimes 1\otimes x\otimes y.$$

Also applying $\mu_*(\mu\times\mu)_*$, we have the commutator map

$$c_{2*}(x\otimes y)=-(-1)^{|x||y|}yx+xy=[x,y]$$

Corollary 2.2. *If* $i: Y\to X$ *is mod* p *homotopy normal and if* $x\in h_*(X)$ *and* $y\in h_*(Y)$ *are primitive, then* $[x,i_*(y)]\in i_*h_*(Y)$.

Corollary 2.3. *If* $x,y\in h_*(X)$ *are primitive, then so is* $c_{2*}(x\otimes y)$.

Proof. Direct computation shows that $d_*^*[x,y]=[x,y]\otimes 1+1\otimes[x,y]$.

3. *H*-spaces with one even degree generator

By the Borel theorem, the mod p cohomology $H^*(X;Z/p)$ is a tensor algebra of truncated polynomial and exterior algebras generated by even and odd dimensional elements respectively. In this section, we consider the case that the polynomial algebra in $H^*(X;Z/p)$ is generated by only one element y. By Kane [6], we know that $|y|=2(p^i+p^{i-1}+\cdots+1)$ for some i and $y^{p^2}=0$. However all known examples satify that $i=1$ and $y^p=0$. Hence we assume

$$H^*(X;Z/p)\cong Z/p[y]/(y^p)\otimes\Lambda,\qquad |y|=2p+2 \tag{3.1}$$

where Λ is an exterior algebra generated by odd degree elements. Then it is also known by Kane [6] that there exists generators $x_0, x_0'\in\Lambda$ such that

$$\mathscr{P}^1x_0=x_0' \quad\text{and}\quad \beta x_0'=y \quad\text{with}\quad |x_0|=3,\ |x_0'|=2p+1.$$

For such H-space X, the Morava K-theory $K(2)^*(X)$ is just a tensor product $K(2)^*(X)\cong H^*(X;Z/p)\otimes K(2)^*$, and the Hopf algebra structure is given in [12]

Theorem 3.1 ([12], [6]). *Let X be an H-spase satisfing* (3.1). *Then*$H^*(X;Z/p)$ *(resp.* $K(2)^*(X)$*) has a quotient Hopf algebra* $Q_K^*=K[y]/(y^p)\otimes\Lambda(x_i,x_i'\,|\,0\le i\le p-2)$ *with* $K=Z/p$ *(resp.* $K(2)^*$*),* $|x_i|=2(p+1)i+3$, $|x_i'|=2(p+1)(i+1)-1$ *such that the dual Hopf algebra* Q_{K*} *is multiplicatively generated by* z, z', y *with the relations* $ad^{p-1}(y)(z)=0$ *(resp.* $-v_2z$*),* $ad^{p-1}(y)(z')=0$ *(resp.* $=-v_2z'$*),* $y^p=0$ *(resp.* $=-v_2y$*), and* $ad(z)(z')=0$ *where* $ad(y)(z)=[y,z]$. *Moreover the K-module of primitive elements in* Q_{K*} *is generated by* $ad^i(y)(z)$, $ad^i(y)(z')$, y *which are duals of indecomposable elements* x_i, x_i', y *respectively.*

Let us write $ad^i(y)(z)=z_i$ and $ad^i(y)(z')=z_i'$. From the above theorem, $Q_{K*}\cong K[y]/(y^p)\otimes\Lambda(z_i,z_i'\,|\,0\le i\le p-2))$ additively. So we can take a K-module basis

(3.2) $$Q_{K*}\cong K(y^k(z_0)^{a_0}\cdots(z_{p-2})^{a_{p-2}}(z'_0)^{a'_0}\cdots(z'_{p-2})^{a'_{p-2}})$$

with $0\leq k\leq p-1$ and $a_i, a'_i=0$ or 1. Let F_s be the filtration of Q_{K*} generated by monomials $y^k(z_0)^{a_0}\cdots(z_{p-2})^{a_{p-2}}(z'_0)^{a'_0}\cdots(z'_{p-2})^{a'_{p-2}}$ such that $\Sigma a_i+\Sigma a'_i\geq s$.

Then it is immediate that $F_sF_t\subset F_{s+t}$ and $ad(y)F_s\subset F_s$. Let $F_{1,+}$ be the module generated by y^kz_i, $y^kz'_i$ for $k\geq 1$, $0\leq i\leq p-2$ and elements in F_2 so that $Q_*/F_{1,+}\cong K(y^s, z_i, z'_i)$. Then note that $ad(y)F_{1,+}\subset F_{1,+}$.

Theorem 3.2. *Let $i\colon Y\to X$ be mod p homotopy normal for an H-space X satisfying* (3.1). *Let $Q^*=Q^*_{K(2)}$. Suppose that the quotient map to Q^* splits, i.e., $Q^*\subset K(2)^*(X)$ as Hopf algebras. Then i^*Q^* is isomorphic to one of the following Hopf algebras*

$$Q^*,\ Q^*/(y),\ K(2)^*\otimes\Lambda(x_i\,|\,0\leq i\leq p-2),\ K(2)^*\otimes\Lambda(x'_i\,|\,0\leq i\leq p-2),\ K(2)_*.$$

Proof. Recall that x_i and x'_i are duals in (3.2) of z_i and z'_i respectively. Suppose that $i^*(x_j)\neq 0\in K(2)^*(Y)$ for some j. Then we see that there exists $\tilde{z}_j\in K(2)_*(Y)$ such that $i_*(\tilde{z}_j)=z_j\ mod(F_{1,+})$ because if $k\neq j$, then $\langle i^*(x_k), w\rangle=0$ and $\langle i^*(x'_k), w\rangle=0$ for any $w\in K(2)_{2(p+1)j+3}(Y)$ by dimensional reason. From Corollary 2.2, the homotopy normality of i implies that there exists $\tilde{z}_k\in K(2)_*(Y)$ with $i_*(\tilde{z}_k)=z_k\ mod(F_{1,+})$ for all $k\geq j$. Let $\tilde{z}_0=v_2^{-1}\tilde{z}_{p-1}$. Then we get $\tilde{z}_k$ for all $0\leq k\leq p-2$. So the composition of maps

$$K(2)_*(Y)\to K(2)_*(X)\to Q_*/(y,z')$$

is epic. Therefore we have proved that if $i^*(x_j)\neq 0$ for some j then

$$K(2)_*\otimes\Lambda(i^*(x_i)\,|\,0\leq i\leq p-2)\subset K(2)_*(Y).$$

Similar fact holds for the cases $i^*(x'_j)\neq 0$.

Next suppose that $i^*(y)\neq 0$. Then there is $\tilde{y}\in K(2)_*(Y)$ with $i_*(\tilde{y})=y$. Hence $[y,z]=[i_*(\tilde{y}),z]\in i_*K(2)_*(Y)$ for all $z\in K(2)_*(X)$. This implies $i^*Q^*\cong Q^*$.

To consider the mod p ordinary homology version of the above theorem, we recall the connective Morava K-theory $k(n)_*(-)$ with the coefficient $k(n)_*=Z/p[v_n]$. The usual Morava K-theory is just the localization $K(n)_*(-)=[v_n^{-1}]k(n)_*(-)$. Moreover we know that the condition $K(n)_*(X)\cong K(n)_*\otimes H_*(X;Z/p)$ implies $k(n)_*(X)\cong k(n)_*\otimes H_*(X;Z/p)$ by the naturality of the Atiyah-Hirzebruch spectral sequence. Since $k(n)$ is a connective spectrum, there is the natural Thom map for reduced theories $\tilde{k}(n)_*(X)\to\tilde{H}_*(X;Z/p)$ which is an isomorphism if $*<2(p^n-1)+2$ for spaces X in (3.1), since $|v_n|=2(p^n-1)$.

Theorem 3.3. *Let $i\colon Y\to X$ be mod p homotopy normal for an H-space X satisfying* (3.1). *Let $Q^*=Q^*_{Z/p}$. Suppose that there does not exist any primitive* element of degree $2(p^2-1)+3$ *nor* $2(p^2-1)+2p+1$ *in* $H^*(Y;Z/p)$ *and that* $Q^* \subset H^*(X;Z/p)$ *as Hopf algebras. Then* $i^*(Q^*)$ *is isomorphic to one of the following Hopf algebras*

$$Q^*,\ Q^*/(y),\ \Lambda(x_i \mid 0 \le i \le p-2),\ Z/p.$$

Proof. We consider the mod p-cohomology version of the proof of Theorem 4.2. Suppose first that the there is x'_j such that $i^*(x'_j) \ne 0$ in $H^*(Y;Z/p)$. Then we see that there exists $\tilde{z}'_j \in H_*(Y;Z/p)$ such that $i_*(\tilde{z}'_j) = z'_j$ $mod(F_{1,+})$. Moreover $|\tilde{z}'_j| < 2(p^2-1) = |v_2|$ implies that we can identify $\tilde{z}'_j \in k(2)_*(Y)$. The mod p homotopy normality for i implies that there exists $\tilde{z}'_k \in k(2)_*(Y)$ for all $k \ge j$ with $i_*(\tilde{z}'_k) = z'_k = ad(y)^k(z')$ $mod(F_{1,+})$. For $p-2 \ge k \ge j$, by the dimensional reason such that $|ad^k(y)(z')| < |v_2|$, we can take $\tilde{z}'_k$ also in $H_*(Y;Z/p)$.

The crucial case is $k=p-1$ where $i_*(\tilde{z}'_{p-1}) = ad^{p-1}(y)(z') = v_2 z'$ $mod(F_{1,+})$. Hence there are two possibilities; there exists $\tilde{z}'$ such that $i_*(\tilde{z}') = z'$ $mod(F_{1,+})$ or there exists a $k(2)_*$-module generator $\tilde{z}''$ such that $i_*(\tilde{z}'') = v_2 z'$ $mod(F_{1,+})$. For the later case, by the dimensional reason, $\tilde{z}''$ is also in $H_*(Y;Z/p)$.

Suppose that $\tilde{z}''$ is a $k(2)_*$-algebra generator. Taking the dual, we know that there exists a primitive element in $H^*(Y;Z/p)$ of degree $|\tilde{z}''| = 2(p^2-1)+2p+1$. By the assumption of this theorem, there does not exist such an element and so we get $\tilde{z}' \in H_*(Y;Z/p)$. Next suppose that $\tilde{z}'' = \Sigma uw$ in $k(2)_*(Y)$ with $u \ne 0$, $w \ne 0$ in $H_*(Y;Z/p)$. In the projection image to $k(2)_* \otimes Q_*$, we see $\Sigma i_*(u) i_*(w) = v_2 z' \ne 0$ $\in F_1/F_2$. Since $F_1 F_1 \subset F_2$, there is u and w such that $i_*(u)$ (*or* $i_*(w)$) is not zero in F_0/F_1, namely, is y^k $mod(F_1)$ for $1 \le k \le p-1$. This means $i^*(y^k) \ne 0$ in $H^*(Y;Z/p)$. So $i^*(y) \ne 0$ and $i^*(x'_0) \ne 0$ since $\beta x'_0 = y$. Hence for all cases we have $z' \in H^*(Y;Z/p)$. Thus from corollary 2.2, there is $\tilde{z}'_k \in H^*(Y;Z/p)$ for all $0 \le k \le p-2$.

Here we note that $\mathscr{P}^1(x_0) = x'_0$ and that $i^*(x'_0) \ne 0$ implies $i^*(x_0) \ne 0$ also. Hence we get also the existence of $\tilde{z}_k$ for all $0 \le k \le p-2$ by the mod p homotopy normality. So the composition of maps $H_*(Y;Z/p) \to H_*(X;Z/p) \to Q_*/(y)$ is epic. Therefore we have proved if $i^*(x'_j) \ne 0$, then

$$\Lambda(i^*(x_i), i^*(x'_i) \mid 0 \le i \le p-2) \subset H^*(Y;Z/p).$$

If $i^*(x'_k) = 0$ for all $0 \le k \le p-2$ and $i^*(x_j) \ne 0$ for some j, then by the non existence of primitive element of degree $2(p^2-1)+3$, we get similarly

$$\Lambda(i^*(x_i) \mid 0 \le i \le p-2) \cong i^*Q^*.$$

Thus we have shown the theorem.

For the cases $p=3,5$, the quotient Hopf algebra Q^* are isomorphic to $H^*(F_4;Z/3)$, $H^*(E_8;Z/5)$ respectively [9],[7].

$$H^*(F_4;Z/3)\cong Z/3[y_8]/(y_8^3)\otimes\Lambda(x_3,x_7,x_{11},x_{15})$$

$$H^*(E_8;Z/5)\cong Z/5[y_{12}]/(y_{12}^5)\otimes\Lambda(x_3,x_{11},x_{15},x_{23},x_{27},x_{35},x_{39},x_{47})$$

where subscript means the degree, i.e., $|x_i|=i$. The reduced powers are also known

$\mathscr{P}^1x_i=x_{i+2(p-1)}$ (i.e., $\mathscr{P}^1x_3=x_7$, $\mathscr{P}^1x_{11}=x_{15}$ for $p=3$).
Note that this means $\mathscr{P}^1x_i=x_i'$ in the notations in Theorem 3.1.

Corollary 3.4. *Let $i\colon Y\to F_4$ (resp. E_8) be a mod 3 (resp. 5) homotopy normal map. Suppose that there does not exist any primitive element in $H^*(Y;Z/p)$ of degree 19 (resp. 51). Then $i^*H^*(F_4;Z/3)$ (resp. $i^*H^*(E_8;Z/5)$) is isomorphic to one of the following Hopf algebras*

$$H^*(F_4;Z/3),\ H^*(F_4;Z/3)/(y_8),\ \Lambda(x_3,x_{11}),\ Z/3$$
$$(\textit{resp. } H^*(E_8;Z/5),\ H^*(E_8;Z/5)/(y_{12}),\ \Lambda(x_3,x_{15},x_{27},x_{39}),\ Z/5).$$

Proof. Suppose that $i^*(x_j')\neq 0$. Then $i^*(x_j)\neq 0$ since $\mathscr{P}^1x_j=x_j'$. Since there does not exist any primitive element in $H^*(Y;Z/p)$ of degree $2(p^2-1)+3$, we get $i^*(x_0)\neq 0$ from the argument in the proof of Theorem 3.3. Therefore $i^*(x_0')\neq 0$ from $\mathscr{P}^1x_0=x_0'$. Thus we have the corollary (without the assumpion for the degree $2(p^2-1)+2p+1$).

The advantage of using the Morava K-theory for $X=H^*(F_4;Z/3)$ is just to exclude the Hopf algebras $\Lambda(x_{11})$, $\Lambda(x_{11},x_{15})$ which seem not to be proved by only using reduced powers and the Hopf algebra structure of $H^*(F_4;Z/3)$. The homotopy group $\pi_{11}(G_2)$ is isomorphic to $Z/3$ and it defines the generator x_{11} in $H^*(G_2;Z/3)$. This induces $\pi_{12}(BG_{2(3)})\cong Z/3$. So there is a map $S^{12}\to BG_2$ which represents a generator of $\pi_{12}(BG_2)$. Then we have a map of loop spaces

$$(\Omega S^{12})_{(3)}\cong S^{11}_{(3)}\times(\Omega S^{23})_{(3)}\to G_{2(3)}=\Omega BG_{2(3)}\subset F_{4(3)}.$$

We know that $i^*H^*(F_4;Z/3)=\Lambda(x_{11})$ and hence this map is not mod 3 homotopy normal.

Next consider the cases exceptional Lie groups E_6, E_7 for $p=3$. The cohomologies are known

$$H^*(E_6;Z/3)\cong H^*(F_4;Z/3)\otimes\Lambda(x_9,x_{17})$$

$$H^*(E_7;Z/3)\cong H^*(F_4;Z/3)\otimes\Lambda(x_{19},x_{27},x_{35})$$

with $\mathscr{P}^3=x_{19}$, $\mathscr{P}^1x_{15}=x_{27}$, $\mathscr{P}^1x_{15}=ex_{19}$ $(e=\pm 1)$. Denote also by z_i the dual of x_i. The Pontrijagin product structre in $H_*(E_6;Z/3)$ (resp. $H_*(E_7;Z/3)$) is given by

$$ad(y)(z_9)=z_{17}\ (\text{resp. } ad(y)\ (z_{19})=z_{27},\ ad(y)(z_{27})=z_{35}).$$

Lemma 3.5. *The Pontrjagin product structure in $K(2)_*(E_6)$ (resp. $K(2)_*(E_7)$) is given by $ad(y)(z_{17}) = -v_2 z_9$ (resp. $ad(y)(z_{35}) = -v_2 z_{27}$).*

Proof. Since $y^3 = -v_2 y$, we always have

$$ad(y)^3(z_i) = ad(y^3)(z_i) = -v_2 ad(y)(z_i).$$

Since ad(y) (z_{19}) is primitive, we see that $ad(y)(z_{17}) = \lambda v_2 z_9$, $\lambda \in Z/3$, from the dimenisional reason. Then

$$\begin{aligned} v_2 z_{17} &= v_2 ad(y)(z_9) = -ad(y)^3(z_9) = -ad(y)(ad(y)(z_{17})) \\ &= -ad(y)(\lambda v_2 z_9) = -\lambda v_2 z_{17}. \end{aligned}$$

Thus we know $\lambda = -1$. The case E_7 is proved similarly.

Corollary 3.6. *Let $i: Y \to E_6$ (resp. E_7) be a mod 3 homotopy normal map. Suppose that $K(2)_*(Y) \cong K(2)_* \otimes H_*(Y; Z/3)$ and there does not exist any primitive element in $H^*(Y; Z/3)$ of degree 19 nor 25 (resp. 19 nor 43). Then $i^*H^*(E_6; Z/3)$ (resp. $i^*H^*(E_7; Z/3)$ is isomorphic to one of the following Hopf algebras.*

$H^*(E_6; Z/3)$, $H^*(E_6; Z/3)/(y)$, $\Lambda(x_9, x_{17})$, $H^*(F_4; Z/3)/(y)$, $\Lambda(x_3, x_{11})$, $Z/3$

(*resp.* $H^*(E_7; Z/3)/(y, x_{19})$, $H^*(F_4; Z/3)/(y)$, $\Lambda(x_3, x_{11})$, $Z/3$)

Proof. By the assumption $K(2)^*(Y) \cong K(2)^* \otimes H^*(Y; Z/3)$ it follows that $i^*(x) \neq 0$ in $H^*(Y; Z/3)$ implies $i^*(x) \neq 0$ in $k(2)^*(Y)$. For the case $X = E_6$, if $i^*(y) \neq 0$, then $-[z_9, y] = z_{17}$ shows $i^*(x_{17}) \neq 0$. The non existence of any $k(2)_*$-algebra generator $z'' \in k(2)_*(Y)$ such that $ad(y)(z_{11}) = i^*(z'')$ nor $ad(y)(z_{17}) = i_*(z'')$ implies the corollary for this case. When $X = E_7$, facts that $i^*(x_{19}) = 0$ and $\mathscr{P}^3 x_{15} = x_{27}$ can prove the corollary. Here we use the nonexisence of any primitive element of degree $|ad(y)(z_{11})|$ and $|ad(y)(z_{35})|$.

Corollary 3.7 ([1]). *The naturad inclusions $F_4 \subset E_6 \subset E_7$ are not mod 3 homotopy normal.*

4. *H*-spaces with two even degree generators

In this section, we consider a simply connected homotopy associative *H*-space X such that

$$H^*(X; Z/p) \cong Z/p[y, u]/(y^p, u^p) \otimes \Lambda, \quad |y| \neq |u|. \tag{4.1}$$

However the known example is only the case $p = 3$ and $X = E_8 \times X'$ for some X' such that $H^*(X'; Z/3)$ is isomorphic to an exterior algegra. Therefore we only consider the case $p = 3$ and $X = E_8$ hereafter. The ordinary mod 3 cohomology of E_8 is ([9], [7])

$$H^*(E_8; Z/3) \cong Z/3[y_8, u_{20}]/(y_8^3, u_{20}^3) \otimes \Lambda(x_3, x_7, x_{15}, x_{19}, x_{27}, x_{35}, x_{39}, x_{47})$$

The reduced powers are also known

$$\mathscr{P}^1 x_3 = x_7,\ \mathscr{P}^1 x_{15} = e x_{19},\ \mathscr{P}^1 x_{35} = x_{39}\ (e = \pm 1)$$

$$\mathscr{P}^3 x_7 = x_{19},\ \mathscr{P}^3 x_{15} = x_{27},\ \mathscr{P}^3 x_{27} = -x_{39},\ \mathscr{P}^3 x_{35} = x_{47}.$$

$$\beta x_7 = x_8,\ \beta x_{19} = \mathscr{P}^3 y_8 = u_{20}$$

The Morava K-theory $K(3)_*(E_8)$ is given in [12].

Theorem 4.1 ([12]). *There is a $K(3)_*$-algebra isomorphism $K(3)^*(E_8) \cong K(3)^* \otimes H^*(E_8; Z/3)$. Let z_i (resp. $y, u) \in K(3)_*(E_8)$ be the dual elements of x_i (resp. y_8, $u_{20}) \in K(3)^*(E_8)$. The Pontrjagin ring $K(3)_*(E_8)$ is generated by two elements, $u, z' = z_{19}$, with the relations $u^9 = 0$, $ad(u)^8(z') = 0$. $(z')^2 = 0$ (and $u^3 = -v_3 y$). The adjoint map is given by the following arrows, i.e., $z \to [u, z]$,*

$$z_{19} \to z_{39} \to -v_3 z_7, \qquad z_7 \to z_{27} \to z_{47} \to -v_3 z_{15}$$

$$z_{15} \to z_{35} \to -v_3 z_3, \qquad z_3 \to 0.$$

Theorem 4.2. *Let $i: Y \to E_8$ be a mod 3 homotopy normal map. Let $(a_0, \cdots, a_7)$ be the ordered set (19, 39, 7, 27, 47, 15, 35, 3). Then $i^* K(3)^*(E_8)$ is isomorphic to one of the following Hopf algebras*

$K(3)^* \otimes \Lambda(x_{a_j}, x_{a_{j+1}}, \cdots, x_{a_7})$ for $0 \le j \le 7$,

$K(3)^*[y]/(y^3) \otimes \Lambda(x_{a_j}, \cdots, x_{a_7})$ for $0 \le j \le 2$,

$K(3)^*$ and $K(3)^*(E_8)$.

Proof. The a_j is ordered so that $ad(u)(z_{a_j}) = z_{a_{j+1}}$ or $-v_3 z_{a_{j+1}}$ in $K(3)_*(E_8)$. By the arguments similar to the proof of Theorem 3.2 and the facts $\beta(x_7) = y$ and $x_7 = x_{a_2}$, we can prove the theorem.

By arguments quite similar to the proof of Theorem 3.3, we get the following theorem.

Theorem 4.3. *Let $i: Y \to E_8$ be mod 3 homotopy normal map. Suppose that there does not exist any primitive element of degree 55,59 nor 67. Then $i^* H^*(E_8; Z/3)$ is isomorphic to one of the following Hopf algebras*

$\Lambda(x_{a_j}, x_{a_{j+1}}, \cdots, x_{a_7})$ for $0 \le j \le 7$,

$Z/p[y]/(y^3) \otimes \Lambda(x_{a_j}, \cdots, x_{a_7})$ for $0 \le j \le 2$

$Z/3$ and $H^*(E_8; Z/3)$.

Using the Hopf algebra structure of $H^*(E_8; Z/3)$ and the reduced power operations, we get the similar result as above, however it seems difficult to exclude the case $\Lambda(x_{15}, x_{35})$.

There is well known chain of inclusions of simple Lie groups $SU(3) \subset G_2 \subset Spin(7) \subset Spin(8) \subset Spin(9) \subset F_4 \subset E_6 \subset E_7 \subset E_8$. Furukawa [1] showed that any $H \subset G$ above is not homotopy normal.

Corollary 4.4 ([1]). *Let $i: H \subset E_8$ be any inclusion of above except for $H = SU(3)$ nor G_2. Then i is not mod 3 homotopy normal.*

Proof. For each subgroup H, there is not any primitive element in $H^*(H; Z/3)$ of degree 55, 59 nor 67. For the case $H = E_7$, $i^*(x_{19}) \neq 0$ in $H^*(E_7; Z/3)$ but $i^*(x_{47}) = 0$. This contradicts the theorem. For other cases, $i^*(x_7) \neq 0$ but $i^*(x_{35}) = 0$ implies the non mod 3 homotopy normality.

Faculty of Education
Ibaraki University
e-mail: pmkudou@mito.ipc.ibaraki.ac.jp, yagita@mito.ipc.ibaraki.ac.jp

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