J. Math. Kyoto Univ. (JMKYAZ)
39-2 (1999) 319–330

Homotopy classification of three connected cover with modular cohomology

By

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Received May 25, 1998

§ 1. Introduction

Let p be an odd prime. We assume that all spaces are completed at p in the sense of Bousfield-Kan [4]. In this paper, a cohomology is taken with $\mathbf{Z}/p$-coefficients unless otherwise specified. Let $\mathscr{A}_p$ be the mod p Steenrod algebra, and $\mathscr{K}$ denote the category of unstable $\mathscr{A}_p$-algebras. The objects of $\mathscr{K}$ are called $\mathscr{K}$-algebras. For a space $X, H^*(X)$ is a $\mathscr{K}$-algebra. It is known, however, that a $\mathscr{K}$-algebra need not be of the form $H^*(X)$.

An algebra A is said to be *realizable* if A is a $\mathscr{K}$-algebra and is represented as the cohomology of some space, namely there exists a space X with $A \cong H^*(X)$ as $\mathscr{K}$-algebras. The realizability of an algebra is one of the major problems in the unstable homotopy theory. There are, indeed, many results, such as the Steenrod problem, the Cooke conjecture, and others (cf. [1], [2], [3], [6], [8], [10], [11], [13] and [16]).

In this paper we investigate the realizability on the following algebras for $m \geq 1$:

$$A_m = \mathbf{Z}/p[x_{2p^2}] \otimes \Lambda(y_{2p^2+1}, z_{2m+1})$$

with Steenrod operation action $\beta(x_{2p^2}) = y_{2p^2+1}$. It is known that many spaces realize the algebra A_m. In fact, there is the following result due to Cooke-Smith:

Theorem 1.1 ([6; Thm. 1.1, 1.2, 1.4]). (1) *For* $1 \leq t \leq p$, *there is a space* Y_t *so that the cohomology*

$$H^*(Y_t) \cong \Lambda(u_3, u_{2p+1}) \otimes \mathbf{Z}/p[u_{2p+2}]/(u_{2p+2}^t)$$

with $\mathscr{P}^1(u_3) = u_{2p+1}$ *and* $\beta(u_{2p+1}) = u_{2p+2}$.

(2) *For the three-connected cover of* Y_t *which is denoted by* $Y_t\langle 3\rangle$, $H^*(Y_t\langle 3\rangle) \cong A_{t(p+1)-1}$, *namely for* $1 \leq t \leq p$, *the algebra* $A_{t(p+1)-1}$ *is realizable as the cohomology.*

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* Partially supported by JSPS Research Fellowships for Young Scientists.
Communicated by Prof. A. Kono, June 3, 1998

Here we note that $Y_1 = B_1(p)$ and $Y_p = K(p)$ are H-spaces constructed by Mimura-Toda [16] and Harper [10], respectively. Then $B_1(p)\langle 3\rangle$ and $K(p)\langle 3\rangle$ also have H-structures, and so we see that A_p and $A_{p(p+1)-1}$ are realizable by the cohomology of H-spaces. Our first result is concerned with the restriction for the realizability of A_m as the cohomology of H-spaces.

Theorem A. *If A_m is realizable as the cohomology of some H-space, then $m = p$ or $p(p+1)-1$, and moreover $A_p \cong H^*(B_1(p)\langle 3\rangle)$ and $A_{p(p+1)-1} \cong H^*(K(p)\langle 3\rangle)$ as $\mathscr{K}$-algebras.*

Concerning the general case, by Theorem 1.1, it is natural to show that if A_m is realizable as the cohomology of some space, then $m = t(p+1)-1$ for $1 \le t \le p$. We have the following partial result under the assumption that $1 \le m \le 2p^3 - 1$.

Theorem B. *Assume that $1 \le m \le 2p^3 - 1$. If A_m is realizable as the cohomology of some space, then $m = t(p+1)-1$ for some $1 \le t \le p$, and moreover $A_{t(p+1)-1} \cong H^*(Y_t\langle 3\rangle)$ as $\mathscr{K}$-algebras.*

For $p = 2$, such a restriction as in Theorem B does not hold. In fact, Aguadé-Broto-Notbohm [2] have constructed a space Z such that the mod 2 cohomology $H^*(Z) \cong \mathbf{Z}/2[x_8] \otimes \Lambda(y_9)$ with $Sq^1(x_8) = y_9$, and so for any $m \ge 1$, $A_m \cong H^*(Z \times S^{2m+1})$ is realizable as the cohomology.

We guess that for $m \ge 2p^3$, the algebra A_m cannot be realizable as the cohomology, and the further study will be discussed in the forthcoming paper, in which we need more complicated computations to determine the $\mathscr{K}$-structure of A_m for $m \ge 2p^3$ (see Proposition 2.4).

From Theorem A and Theorem B, we concentrate on the following algebras for $1 \le t \le p$:

$$A_{t(p+1)-1} \cong H^*(Y_t\langle 3\rangle) \cong \mathbf{Z}/p[x_{2p^2}] \otimes \Lambda(y_{2p^2+1}, z_{2t(p+1)-1})$$

with $\beta(x_{2p^2}) = y_{2p^2+1}$. Aguadé-Broto-Santos [3] studied the homotopy uniqueness of $Y_1\langle 3\rangle$, and they have shown that if there is a space X with $H^*(X) \cong A_p$ as $\mathscr{K}$-algebras, then $X \simeq Y_1\langle 3\rangle$ up to p-completion. We can generalize their uniqueness result for any $1 \le t \le p$ as follows:

Theorem C. *For $1 \le t \le p$, if there is a space X so that $H^*(X) \cong A_{t(p+1)-1}$ as $\mathscr{K}$-algebras, then X is homotopy equivalent to $Y_t\langle 3\rangle$ up to p-completion.*

By combining Theorem A, Theorem B and Theorem C, our results show that for $m \ge 1$, any H-space which realizes A_m is homotopy equivalent to $B_1(p)\langle 3\rangle$ or $K(p)\langle 3\rangle$, and that for $1 \le m \le 2p^3 - 1$, any space which realizes A_m is homotopy equivalent to $Y_t\langle 3\rangle$ for some $1 \le t \le p$.

This paper is organized as follows: In §2, we recall the Lannes theory about the T-functor, and apply the theory to the algebras A_m. In particular, we determine the $\mathscr{K}$-structure of A_m to compute the T-functor. These results will be essential in the latter sections. In §3, we prove Theorems A and B combining the results of §2 with the spectral sequence arguments. §4 is devoted to the proof of Theorem C.

§2. Lannes T-functor of $\mathcal{K}$-algebras

In this section, we compute the Lannes T-functor of algebra A_m for $m \geq 1$. The results of this section will be used in §3 and §4. Let us now recall some properties of this functor. The Lannes functor $T : \mathcal{K} \to \mathcal{K}$ is a left adjoint of the functor $H^*(B\mathbf{Z}/p) \otimes -$, that is, there is the adjoint isomorphism $\mathrm{Hom}_{\mathcal{K}}(T(A), B) \cong \mathrm{Hom}_{\mathcal{K}}(A, H^*(B\mathbf{Z}/p) \otimes B)$ for $\mathcal{K}$-algebras A and B.

For a $\mathcal{K}$-map $f : A \to H^*(B\mathbf{Z}/p)$, its adjoint restricts to a $\mathcal{K}$-map $T(A)^0 \to \mathbf{Z}/p$, where $T(A)^0$ is the subalgebra of $T(A)$ of elements of degree 0. The connected component of $T(A)$ corresponding to f is defined by $T_f(A) = T(A) \otimes_{T(A)^0} \mathbf{Z}/p$, and there is the natural $\mathcal{K}$-map $\varepsilon_f : A \to T_f(A)$.

The evaluation map $e : B\mathbf{Z}/p \times \mathrm{Map}(B\mathbf{Z}/p, X) \to X$ induces a $\mathcal{K}$-map e^*, and taking the adjoint of this yields a $\mathcal{K}$-map $\lambda : T(H^*(X)) \to H^*(\mathrm{Map}(B\mathbf{Z}/p, X))$. On the component level, for a map $\phi : B\mathbf{Z}/p \to X$, there is a $\mathcal{K}$-map $\lambda_{\phi^*} : T_{\phi^*}(H^*(X)) \to H^*(\mathrm{Map}(B\mathbf{Z}/p, X)_\phi)$. The composite $\lambda_{\phi^*} \varepsilon_{\phi^*}$ is induced by the evaluation at the base point $e_\phi : \mathrm{Map}(B\mathbf{Z}/p, X)_\phi \to X$. The following theorem is due to Lannes:

Theorem 2.1 ([15; Thm. 3.2.1]). *Let X be a space and $\phi : B\mathbf{Z}/p \to X$ be a map. If $T_{\phi^*}(H^*(X))^1 = 0$, then $\lambda_{\phi^*} : T_{\phi^*}(H^*(X)) \to H^*(\mathrm{Map}(B\mathbf{Z}/p, X)_\phi)$ is an isomorphism.*

Let P_sX denote the s-th stage of the Postnikov decomposition of X, and $j_s : X \to P_sX$ be the induced map. Then $\{\mathrm{Map}(B\mathbf{Z}/p, P_sX)_{\phi_s}\}$ is a tower with $\mathrm{Map}(B\mathbf{Z}/p, X)_\phi \simeq \varprojlim_s \mathrm{Map}(B\mathbf{Z}/p, P_sX)_{\phi_s}$, where $\phi_s = j_s\phi$. The following theorem is due to Dror Farjoun-Smith:

Theorem 2.2 ([7; Thm. 1.1]). *If X is a nilpotent space, then there exists an isomorphism $T_{\phi^*}(H^*(X)) \cong \varinjlim_s H^*(\mathrm{Map}(B\mathbf{Z}/p, P_sX)_{\phi_s})$.*

Moreover, T_f can be considered as a functor from $\mathcal{K}(A)$ to $\mathcal{K}(T_f(A))$, where $\mathcal{K}(A)$ denotes the subcategory of $\mathcal{K}$ whose objects have A-module structure compatible with its underlying $\mathcal{K}$-structure.

We also regard $T_f(M)$ as an object of $\mathcal{K}(A)$ through the natural $\mathcal{K}$-map $\varepsilon_f : A \to T_f(A)$ for any $\mathcal{K}(A)$-algebra M, and $\varepsilon_f : M \to T_f(M)$ becomes a morphism of $\mathcal{K}(A)$-algebras. It is well known that T_f is exact, and commutes with suspensions and tensor products.

From now on, we apply the Lannes theory to the algebras A_m. For the cohomology of an H-space, there is the following result due to Dwyer-Wilkerson:

Proposition 2.3 ([9; Thm. 3.2, Lemma 4.5]). *If X is an H-space with finitely generated cohomology, and $f : H^*(X) \to H^*(B\mathbf{Z}/p)$ is a $\mathcal{K}$-map, then $\varepsilon_f : H^*(X) \to T_f(H^*(X))$ is an isomorphism.*

For the general case, we need to determine the $\mathscr{K}$-structure of A_m to compute the T-functor.

Proposition 2.4. *Assume that* $1 \leq m \leq 2p^3 - 1$. *If* A_m *is a* $\mathscr{K}$*-algebra, then the following hold:*

(1)

$$\mathscr{P}^{p^i}(y_{2p^2+1}) = \begin{cases} \kappa z_{2p^2+2p-1} & \text{for } i = 0,\ m = p^2 + p - 1 \text{ and } \kappa = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

(2) $\beta(z_{2m+1}) = 0$.

(3)

$$\mathscr{P}^{p^i}(z_{2m+1}) = \begin{cases} \lambda x_{2p^2}^{p(p-1)} z_{2m+1} \bmod I & \text{for } i = 3 \text{ and } \lambda = 0, 1 \\ 0 \bmod I & \text{for } i \neq 3, \end{cases}$$

where $I = (y_{2p^2+1})$ *denotes the ideal of* A_m *generated by* y_{2p^2+1}.

Proof. First we prove (1). If $\mathscr{P}^{p^i}(x_{2p^2}) \neq 0$ for $i = 0, 1$, then

$$\mathscr{P}^{p^i}(x_{2p^2}) = \begin{cases} y_{2p^2+1} z_{2p-3} & \text{for } i = 0 \text{ and } m = p - 2 \\ y_{2p^2+1} z_{2p^2-2p-1} & \text{for } i = 1 \text{ and } m = p^2 - p - 1 \end{cases} \tag{2.5}$$

up to unit. For the dimensional reason, if $\mathscr{P}^p(y_{2p^2+1}) \neq 0$, then $m = 2p^2 - p$, $p^2 - p$ for which $\mathscr{P}^p(y_{2p^2+1}) = z_{4p^2-2p+1}$, $\mathscr{P}^p(y_{2p^2+1}) = x_{2p^2} z_{2p^2-2p+1}$ up to unit. However, using the Adem relation $\mathscr{P}^p\beta = \mathscr{P}^1\beta\mathscr{P}^{p-1} + \beta\mathscr{P}^p$ and by (2.5), we obtain that $\mathscr{P}^p(y_{2p^2+1}) = \mathscr{P}^1\beta\mathscr{P}^{p-1}(x_{2p^2}) + \beta\mathscr{P}^p(x_{2p^2}) = 0$.

If $\mathscr{P}^1(y_{2p^2+1}) \neq 0$, then we have that $m = p - 1$ or $m = p^2 + p - 1$ for which $\mathscr{P}^1(y_{2p^2+1}) = \nu x_{2p^2} z_{2p-1}$ or $\mathscr{P}^1(y_{2p^2+1}) = \kappa z_{2p^2+2p-1}$ for $\kappa, \nu \in \mathbf{Z}/p$. However, if $m = p - 1$, then applying the Adem relation $\mathscr{P}^{p^2}\mathscr{P}^1 = \mathscr{P}^1\mathscr{P}^{p^2} + \mathscr{P}^p\mathscr{P}^{p^2-p+1}$ to y_{2p^2+1} and by (2.5), we have the equation $\nu x_{2p^2}^p z_{2p-1} = \mathscr{P}^{p^2-p+1}(y_{2p^2+1}) = 0$, which implies that $\nu = 0$. When $m = p^2 + p - 1$, if $\kappa \neq 0$, then we replace κz_{2p^2+2p-1} as z_{2p^2+2p-1}.

Using the Adem relation $\mathscr{P}^{p^2}\beta = \mathscr{P}^1\beta\mathscr{P}^{p^2-1} + \beta\mathscr{P}^{p^2}$ and by (2.5), we obtain that $\mathscr{P}^{p^2}(y_{2p^2+1}) = \mathscr{P}^1\beta\mathscr{P}^{p^2-1}(x_{2p^2}) = 0$. For $i \geq 3$, $\mathscr{P}^{p^i}(y_{2p^2+1}) = 0$ by the unstable condition.

Next we prove (2). We assume that $\beta(z_{2m+1}) \neq 0$, and deduce a contradiction from this assumption. Let $J = (y_{2p^2+1}, z_{2m+1})$ denote the ideal of A_m generated by y_{2p^2+1} and z_{2m+1}. For the dimensional reason, $\beta(z_{2m+1}) \notin J$, and applying the Adem relation $\mathscr{P}^1\beta\mathscr{P}^m = \mathscr{P}^{m+1}\beta + m\beta\mathscr{P}^{m+1}$ to z_{2m+1}, we have that $\mathscr{P}^1\beta\mathscr{P}^m(z_{2m+1}) = (\beta(z_{2m+1}))^p \notin J$. We see that $\beta\mathscr{P}^m(z_{2m+1}) \notin J$ since J is closed under the action of $\mathscr{P}^1$, and thus for some $k \geq 1$, we can write $\beta\mathscr{P}^m(z_{2m+1}) = x_{2p^2}^k$ up to unit. However, we see that $\mathscr{P}^1\beta\mathscr{P}^m(z_{2m+1}) = \mathscr{P}^1(x_{2p^2}^k) \in J$ and this causes a contradiction. Therefore we conclude that $\beta(z_{2m+1}) = 0$.

Finally we prove (3). For the dimensional reason, $\mathscr{P}^{p^i}(z_{2m+1}) = 0 \bmod I$ for $i = 0, 1$. If $\mathscr{P}^{p^2}(z_{2m+1}) \neq 0$, then $m \geq p^2$, and we can write $\mathscr{P}^{p^2}(z_{2m+1}) = \theta x_{2p^2}^{p-1} z_{2m+1} \bmod I$ for some $\theta \in \mathbf{Z}/p$. Applying the Adem relation $\mathscr{P}^{p^2}\beta =$

$\mathscr{P}^1\beta\mathscr{P}^{p^2-1}+\beta\mathscr{P}^{p^2}$ to z_{2m+1}, and by (1), (2), we obtain that $\theta=0$, and so $\mathscr{P}^{p^2}(z_{2m+1})=0 \bmod I$.

If $\mathscr{P}^{p^3}(z_{2m+1})\neq 0$, then $m\geq p^3$, and so the ideal I is closed under the action of $\mathscr{A}_p$ since $\mathscr{P}^{p^i}(y_{2p^2+1})=0$ for $i\geq 0$ by (1). We put $\mathscr{P}^{p^3}(z_{2m+1})=\lambda x_{2p^2}^{p(p-1)}z_{2m+1} \bmod I$ for $\lambda\in \mathbf{Z}/p$. Then applying the Adem relation

$$\mathscr{P}^{p^3}\mathscr{P}^{p^3}=\sum_{i=0}^{p^2}(-1)^{p^3+i}\binom{(p-1)(p^3-i)-1}{p^3-pi}\mathscr{P}^{2p^3-i}\mathscr{P}^i$$

to z_{2m+1}, we have the equation $\lambda(\lambda-1)x_{2p^2}^{2p(p-1)}z_{2m+1}=0 \bmod I$, where we used the assumption that $m\leq 2p^3-1$ to show that $\mathscr{P}^{2p^3}(z_{2m+1})=0$. This implies that $\lambda=0,1$. By the unstable condition, $\mathscr{P}^{p^i}(z_{2m+1})=0$ for $i\geq 4$. This completes the proof.

For the $\mathscr{K}$-algebra A_m with $\mathscr{P}^{p^3}(z_{2m+1})=\lambda x_{2p^2}^{p(p-1)}z_{2m+1} \bmod I$, we use the notation $A_m^{(\lambda)}$ for $\lambda=0,1$. As is known, $H^*(B\mathbf{Z}/p)\cong \Lambda(\omega_1)\otimes \mathbf{Z}/p[\omega_2]$ with $\beta(\omega_1)=\omega_2$. Now we define a $\mathscr{K}$-map $f: A_m^{(\lambda)}\to H^*(B\mathbf{Z}/p)$ as $f(x_{2p^2})=\omega_2^{p^2}$ and $f(y_{2p^2+1})=f(z_{2m+1})=0$.

Proposition 2.6. *Assume that* $1\leq m\leq 2p^3-1$. *Then the following hold:*
(1) $\varepsilon_f: A_m^{(0)}\to T_f(A_m^{(0)})$ *is an isomorphism.*
(2) $T_f(A_m^{(1)})\cong \mathbf{Z}/p[x_{2p^2}]\otimes\Lambda(y_{2p^2+1},w_{2m-2p^3+1})$, *and* $\varepsilon_f: A_m^{(1)}\to T_f(A_m^{(1)})$ *is given as*

$$\begin{cases} \varepsilon_f(x_{2p^2})=x_{2p^2},\\ \varepsilon_f(y_{2p^2+1})=y_{2p^2+1},\\ \varepsilon_f(z_{2m+1})=x_{2p^2}^p w_{2m-2p^3+1}.\end{cases}$$

Proof. The quotient map $q: A_m^{(\lambda)}\to \mathbf{Z}/p[x_{2p^2}]$ is a $\mathscr{K}$-map since the ideal $J=(y_{2p^2+1},z_{2m+1})$ of $A_m^{(\lambda)}$ generated by y_{2p^2+1} and z_{2m+1} is closed under the action of $\mathscr{A}_p$. We see that $q^*:\mathrm{Hom}_{\mathscr{K}}(\mathbf{Z}/p[x_{2p^2}],H^*(B\mathbf{Z}/p))\to \mathrm{Hom}_{\mathscr{K}}(A_m^{(\lambda)},H^*(B\mathbf{Z}/p))$ becomes an isomorphism, and by [1; Lemma 3.2], $T_f(\mathbf{Z}/p[x_{2p^2}])\cong T_g(\mathbf{Z}/p[x_{2p^2}])\cong \mathbf{Z}/p[x_{2p^2}]$ for a non-trivial $\mathscr{K}$-map $g:\mathbf{Z}/p[x_{2p^2}]\to H^*(B\mathbf{Z}/p)$.

If $\mathscr{P}^1(y_{2p^2+1})\neq 0$, then by Proposition 2.4, $m=p^2+p-1$ and $\mathscr{P}^1(y_{2p^2+1})=z_{2p^2+2p-1}$. In this case, the required conclusion holds by [13; Prop. 4.2]. Thus we assume that $\mathscr{P}^1(y_{2p^2+1})=0$. Then the ideal $I=(y_{2p^2+1})$ is closed under the action of the Steenrod algebra. First we compute the T-functor of $B_m^{(\lambda)}=A_m^{(\lambda)}/I=\mathbf{Z}/p[x_{2p^2}]\otimes\Lambda(z_{2m+1})$. Since T_f is exact, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & z_{2m+1}\mathbf{Z}/p[x_{2p^2}] & \longrightarrow & B_m^{(\lambda)} & \longrightarrow & \mathbf{Z}/p[x_{2p^2}] & \longrightarrow & 0\\ & & \downarrow{\scriptstyle \varepsilon_f} & & \downarrow{\scriptstyle \varepsilon_f} & & \| & & \\ 0 & \longrightarrow & T_f(z_{2m+1}\mathbf{Z}/p[x_{2p^2}]) & \longrightarrow & T_f(B_m^{(\lambda)}) & \longrightarrow & \mathbf{Z}/p[x_{2p^2}] & \longrightarrow & 0,\end{array}$$

where the horizontal arrows are exact sequences of $\mathscr{K}(A_m^{(\lambda)})$-algebras.

If $\lambda=0$, then we have $T_f(z_{2m+1}\mathbf{Z}/p[x_{2p^2}]) \cong z_{2m+1}T_f(\mathbf{Z}/p[x_{2p^2}]) \cong z_{2m+1}\mathbf{Z}/p[x_{2p^2}]$ since $z_{2m+1}\mathbf{Z}/p[x_{2p^2}] \cong \Sigma^{2m+1}\mathbf{Z}/p[x_{2p^2}]$ as $\mathscr{K}(A_m^{(0)})$-algebras and T_f commutes with suspensions. By the above diagram, $\varepsilon_f : B_m^{(0)} \to T_f(B_m^{(0)})$ is an isomorphism.

For $\lambda=1$, because $z_{2m+1}\mathbf{Z}/p[x_{2p^2}] \cong \Sigma^{2m-2p^3+1}x_{2p^2}^p\mathbf{Z}/p[x_{2p^2}]$ as $\mathscr{K}(A_m^{(1)})$-algebras, we have that $T_f(z_{2m+1}\mathbf{Z}/p[x_{2p^2}]) \cong w_{2m-2p^3+1}T_f(x_{2p^2}^p\mathbf{Z}/p[x_{2p^2}])$, where w_{2m-2p^3+1} is an element which has trivial $\mathscr{A}_p$-actions. If we apply T_f to the following exact sequence:

$$x_{2p^2}^p\mathbf{Z}/p[x_{2p^2}] \to \mathbf{Z}/p[x_{2p^2}] \to \mathbf{Z}/p[x_{2p^2}]/x_{2p^2}^p\mathbf{Z}/p[x_{2p^2}],$$

then since $T_f(\mathbf{Z}/p[x_{2p^2}]/x_{2p^2}^p\mathbf{Z}/p[x_{2p^2}])=0$ by [9; Prop. 2.3], we see that $T_f(x_{2p^2}^p\mathbf{Z}/p[x_{2p^2}]) \cong T_f(\mathbf{Z}/p[x_{2p^2}]) \cong \mathbf{Z}/p[x_{2p^2}]$. By the above diagram, $T_f(B_m^{(1)}) \cong \mathbf{Z}/p[x_{2p^2}] \otimes \Lambda(w_{2m-2p^3+1})$ and $\varepsilon_f : B_m^{(1)} \to T_f(B_m^{(1)})$ is given as

$$\begin{cases} \varepsilon_f(x_{2p^2}) = x_{2p^2}, \\ \varepsilon_f(z_{2m+1}) = x_{2p^2}^p w_{2m-2p^3+1}. \end{cases}$$

Next we have the following commutative diagram whose horizontal arrows are exact sequences of $\mathscr{K}(A_m^{(\lambda)})$-algebras:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & y_{2p^2+1}B_m^{(\lambda)} & \longrightarrow & A_m^{(\lambda)} & \longrightarrow & B_m^{(\lambda)} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \varepsilon_f} & & \Big\downarrow{\scriptstyle \varepsilon_f} & & \Big\downarrow{\scriptstyle \varepsilon_f} & & \\
0 & \longrightarrow & T_f(y_{2p^2+1}B_m^{(\lambda)}) & \longrightarrow & T_f(A_m^{(\lambda)}) & \longrightarrow & T_f(B_m^{(\lambda)}) & \longrightarrow & 0.
\end{array}$$

Using the same argument as above, and by Proposition 2.4, we have the required conclusions. This completes the proof.

§ 3. Proofs of Theorems A and B

In this section we prove Theorem A and Theorem B using the spectral sequence arguments. First, we study the non-realizability for the algebras $A_m^{(1)}$, which will be used in the proof of Theorem B.

Proposition 3.1. *For $1 \le m \le 2p^3-1$, $A_m^{(1)}$ cannot be realizable as the cohomology.*

Proof. We assume that there is a space X so that $H^*(X) \cong A_m^{(1)}$, and deduce a contradiction from this assumption. By a result of Lannes [15; Thm. 3.1.1], there is a map $\phi : B\mathbf{Z}/p \to X$ such that $\phi^* = f$. By Theorem 2.2, we can choose a sufficient large $s>0$ so that $H^*(P_sX) \cong A_m^{(1)}$ and $H^*(\mathrm{Map}(B\mathbf{Z}/p, P_sX)_{\phi_s}) \cong T_f(A_m^{(1)})$ up to dimension $2m+2$. Then, $\varepsilon_f : A_m^{(1)} \to T_f(A_m^{(1)})$ can be realized by the evaluation map

$$e_{\phi_s} : \mathrm{Map}(B\mathbf{Z}/p, P_sX)_{\phi_s} \to P_sX$$

up to dimension $2m+2$, and thus we have the induced homomorphism for the Bockstein spectral sequences:

$$e_{\phi_s}^* : \{B_i(P_sX), \beta_i\} \to \{B_i(\mathrm{Map}(B\mathbf{Z}/p, P_sX)_{\phi_s}), \beta_i\}.$$

We see that $\{z_{2m+1}\} \in B_i(P_sX)$ becomes a permanent cycle. On the other hand, for $e_{\phi_s}^*(\{z_{2m+1}\}) = \{x_{2p^2}^p w_{2m-2p^3+1}\} \in B_i(\mathrm{Map}(B\mathbf{Z}/p, P_sX)_{\phi_s})$, by [5; Thm. 5.4], we have $\beta_2(\{x_{2p^2}^p w_{2m-2p^3+1}\}) = \{x_{2p^2}^{p-1} y_{2p^2+1} w_{2m-2p^3+1}\} \neq 0$. This causes a contradiction, and thus $A_m^{(1)}$ cannot be realizable as the cohomology.

Now we prove Theorem B as follows:

Proof of Theorem B. We assume that A_m is realizable, that is, there exists a space X such that $H^*(X) \cong A_m$. This forces $A_m = A_m^{(0)}$ by Proposition 3.1. A result of Lannes [15; Thm. 3.1.1] implies that there is a map $\phi : B\mathbf{Z}/p \to X$ such that $\phi^* = f$. We see that the evaluation map $e_\phi : \mathrm{Map}(B\mathbf{Z}/p, X)_\phi \to X$ is a homotopy equivalence by Theorem 2.1 and Proposition 2.6. Let $\iota : B\mathbf{Z}/p \to \mathrm{Map}(B\mathbf{Z}/p, X)_\phi$ be the adjoint of $\phi\mu$, where μ is the multiplication of an H-structure of $B\mathbf{Z}/p$. Then we have the following commutative diagram of fibrations:

$$\begin{array}{ccccccc} B\mathbf{Z}/p & = & B\mathbf{Z}/p & \longrightarrow & EB\mathbf{Z}/p & \longrightarrow & B^2\mathbf{Z}/p \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \iota} & & \downarrow & & \| \\ X & \underset{\simeq}{\xleftarrow{e_\phi}} & \mathrm{Map}(B\mathbf{Z}/p, X)_\phi & \longrightarrow & X_1 & \xrightarrow{j} & B^2\mathbf{Z}/p, \end{array} \tag{3.2}$$

where $X_1 = (\mathrm{Map}(B\mathbf{Z}/p, X)_\phi)_{hB\mathbf{Z}/p}$ denotes the Borel construction.

We consider the Serre spectral sequence for the bottom fibration whose E_2-term is given as

$$E_2^{*,*} = H^*(B^2\mathbf{Z}/p) \otimes A_m.$$

As is known, $H^*(B^2\mathbf{Z}/p) \cong \mathbf{Z}/p[\eta_2, \beta\mathscr{P}^{\Delta_i}\beta\eta_2 | i \geq 0] \otimes \Lambda(\beta\eta_2, \mathscr{P}^{\Delta_i}\beta\eta_2 \mid i \geq 0)$, where $\mathscr{P}^{\Delta_i} = \mathscr{P}^{p^i} \cdots \mathscr{P}^1$ and η_2 denotes the fundamental class. From the diagram (3.2), we have that $\tau(x_{2p^2}) = \mathscr{P}^{\Delta_1}\beta\eta_2 + \delta_{2p^2+1}$ for some decomposable element $\delta_{2p^2+1} \in H^*(B^2\mathbf{Z}/p)$, since $f(x_{2p^2}) = \omega_2^{p^2}$ and $\tau(\omega_2^{p^2}) = \mathscr{P}^{\Delta_1}\beta\eta_2$, where τ denotes the transgression.

We set $\gamma_{2p^2+2p} = (\beta\mathscr{P}^1\beta\eta_2)^p + \mathscr{P}^1\beta(\delta_{2p^2+1})$. Then, $\mathscr{P}^1\beta(\delta_{2p^2+1})$ does not contain the term $(\beta\mathscr{P}^1\beta\eta_2)^p$, and since $j^*(\gamma_{2p^2+2p}) = \mathscr{P}^1\beta(j^*(\mathscr{P}^{\Delta_1}\beta\eta_2 + \delta_{2p^2+1})) = 0$, there exists an element of total degree $2p^2+2p-1$ which kills γ_{2p^2+2p} in the spectral sequence.

But for the dimensional reason, the differentials d_{2p^2+1} and d_{2p^2+2} cannot kill the element γ_{2p^2+2p}, since $\tau(x_{2p^2}) = \mathscr{P}^{\Delta_1}\beta\eta_2 + \delta_{2p^2+1}$ and $\tau(y_{2p^2+1}) = \beta\mathscr{P}^{\Delta_1}\beta\eta_2 + \beta(\delta_{2p^2+1})$. The only possibility is that $m = t(p+1)-1$ for some $1 \leq t \leq p$, and the differential $d_{2t(p+1)}$ kills γ_{2p^2+2p}. Here $z_{2t(p+1)-1}$ is transgressive, and

$$\tau(z_{2t(p+1)-1}) = (\beta\mathscr{P}^1\beta\eta_2)^t + \zeta_{2t(p+1)}$$

for some $\zeta_{2t(p+1)} \in H^*(B^2\mathbf{Z}/p)$ which does not contain the term $(\beta\mathscr{P}^1\beta\eta_2)^t$. Since the transgression is commutative with the Steenrod operation, we see that

$$\text{(3.3)} \qquad \begin{cases} \mathscr{P}^p(z_{2p+1}) = y_{2p^2+1} & \text{for } t = 1 \\ \mathscr{P}^1(y_{2p^2+1}) = z_{2p(p+1)-1} & \text{for } t = p, \end{cases}$$

and the other operations act trivially on $A_{t(p+1)-1}$. This concludes that $A_{t(p+1)-1} \cong H^*(Y_t\langle 3\rangle)$ as $\mathscr{K}$-algebras by [6; Thm. 1.4]. This completes the proof of Theorem B.

The proof of Theorem A is obtained by the modification of the proof of Theorem B as follows. We remark that the proof does not need the assumption that $1 \leq m \leq 2p^3 - 1$ since we do not use Proposition 2.6.

Proof of Theorem A. We assume that there exists an H-space X such that $H^*(X) \cong A_m$. For the dimensional reason, we see that $f : H^*(X) \to H^*(B\mathbf{Z}/p)$ is a Hopf algebra map. Then, the map $\phi : B\mathbf{Z}/p \to X$ becomes an H-map by [15; Thm. 3.1.1] in the proof of Theorem B. By Theorem 2.1 and Theorem 2.3, we have that the evaluation map $e_\phi : \mathrm{Map}(B\mathbf{Z}/p, X)_\phi \to X$ is a homotopy equivalence, and moreover the bottom fibration of the diagram (3.2) becomes an H-fibration by [14; Prop. 3.3]. Hence the Serre spectral sequence for this fibration has a differential Hopf algebra structure, and by the DHA lemma [12; Lemma 1–6], $\tau(z_{2t(p+1)-1}) = (\beta\mathscr{P}^1\beta\eta_2)^t + \zeta_{2t(p+1)}$ must be a primitive element of $H^*(B^2\mathbf{Z}/p)$. This implies that $t = 1, p$ and $\zeta_{2t(p+1)} = 0$. By (3.3), we have that $A_p \cong H^*(B_1(p)\langle 3\rangle)$ and $A_{p(p+1)-1} \cong H^*(K(p)\langle 3\rangle)$ as $\mathscr{K}$-algebras. This completes the proof of Theorem A.

§4. Proof of Theorem C

In this section we prove Theorem C. Aguadé-Broto-Santos [3] proved that the homotopy type of $Y_1\langle 3\rangle$ is determined by the $\mathscr{K}$-structure of its cohomology. We will generalize their argument, and show that the same result holds for $Y_t\langle 3\rangle$ for any $1 \leq t \leq p$. First, we are concerned with the homotopy uniqueness of Y_t itself, which will be used in the proof of Theorem C.

Proposition 4.1. *For $1 \leq t \leq p$, if there is a space X so that $H^*(X) \cong H^*(Y_t)$ as $\mathscr{K}$-algebras, then $X \simeq Y_t$ up to p-completion.*

First, we can prove the following lemma using the killing methods for the cohomology of Y_t:

Lemma 4.2. *The homotopy groups $\pi_j(Y_t)$ for $j < 2t(p+1)+2$ are described as follows:*

$$\pi_j(Y_t) \cong \begin{cases} \hat{\mathbf{Z}}_p & (j = 3, 2t(p+1)-1), \\ \mathbf{Z}/p & (j = 2p^2, 2p^2+2p-3), \\ 0 & (\textit{otherwise } j < 2t(p+1)+2). \end{cases}$$

Proof of Proposition 4.1. From the cohomology of Y_t as in Theorem 1.1, the cell structure of X is represented as

$$X \simeq S^3 \cup_{\alpha_1} e^{2p+1} \cup_{[p]} e^{2p+2} \cup_{\xi} e^{2p+4} \cup \cdots \cup e^{2t(p+1)+2},$$

where $\alpha_1 \in \pi_{2p-3}^S(S^0) \cong \mathbf{Z}/p$ denotes the generator. Since Y_t has the same cell structute, there exists a map $\rho : X^{(2p+2)} \to Y_t$ such that $\rho^* : H^*(Y_t) \to H^*(X^{(2p+2)})$ is an isomorphism up to dimension $2p+2$, where $X^{(k)}$ denotes the k-th skeleton of X. Now we can extend ρ to $X^{(2p+4)}$ since $X^{(2p+4)}$ is the cofiber of some attaching map $\xi : S^{2p+3} \to X^{(2p+2)}$ and $\pi_{2p+3}(Y_t) = 0$ by Lemma 4.2. Iterating this argument, we can extend ρ to $X^{(2p^2)}$, and thus get the required conclusion for $1 \le t < p$.

When $t = p$, if $\zeta : S^{2p^2} \to X^{(2p^2)}$ denotes the attaching map of $X^{(2p^2+1)}$, then using the killing methods, we see that $\pi_{2p^2}(X^{(2p^2)}) \cong \hat{\mathbf{Z}}_p \oplus \mathbf{Z}/p$ and $\zeta_* : \pi_{2p^2}(S^{2p^2}) \to \pi_{2p^2}(X^{(2p^2)})$ is the inclusion on the first factor. But we see that $\rho_* : \pi_{2p^2}(X^{(2p^2)}) \to \pi_{2p^2}(Y_p)$ becomes a projection on the second factor, and so $\rho\zeta$ is null homotopic. This ensures that ρ is extended to $X^{(2p^2+1)}$, and the same arguments as above establish the map $\rho : X \to Y_p$ which induces an isomorphism in cohomology. This completes the proof.

For $j \ge 1$, we set an algebra

$$C_j = \mathbf{Z}/p[u_2] \otimes \Lambda(u_3, u_{2p+1}) \otimes \mathbf{Z}/p[u_{2p+2}]/(u_{2p+2}^t)$$

with $\beta_j(u_2) = u_3, \mathscr{P}^1(u_3) = u_{2p+1}$ and $\beta(u_{2p+1}) = u_{2p+2}$, where β_j denotes the j-th Bockstein operation. Then we have the following:

Proposition 4.3. *The $\mathscr{K}$-structure of C_j is determined by $\mathscr{P}^p(u_{2p+1})$. In particular, $\mathscr{P}^p(u_{2p+1}) = \lambda u_2^{p(p-1)} u_{2p+1}$ for $\lambda = 0, 1$.*

Proof. Using the Adem relations $\mathscr{P}^1\mathscr{P}^1 = 2\mathscr{P}^2$ and $\mathscr{P}^1\beta\mathscr{P}^1 = \beta\mathscr{P}^2 + \mathscr{P}^2\beta$, we have $\mathscr{P}^1(u_{2p+1}) = \mathscr{P}^1(u_{2p+2}) = 0$. Since $\mathscr{P}^p(u_{2p+2}) = \beta\mathscr{P}^p(u_{2p+1})$ using the Adem relation $\mathscr{P}^p\beta = \mathscr{P}^1\beta\mathscr{P}^{p-1} + \beta\mathscr{P}^p$, it is sufficient to determine $\mathscr{P}^p(u_{2p+1})$ to establish the $\mathscr{K}$-structure of C_j. We can put

$$\mathscr{P}^p(u_{2p+1}) = \lambda u_2^{p(p-1)} u_{2p+1} + \sum_{i=1}^{t-1} \rho_i u_2^{p(p-1)-i(p+1)} u_{2p+1} u_{2p+2}^i$$
$$+ \sum_{i=0}^{t-1} \kappa_i u_2^{p^2-1-i(p+1)} u_3 u_{2p+2}^i$$

for some $\lambda, \rho_i, \kappa \in \mathbf{Z}/p$. Since $\mathscr{P}^1\mathscr{P}^p(u_{2p+1}) = \mathscr{P}^{p+1}(u_{2p+1}) = 0$, we have $\rho_i = 0$ for $1 \le i \le t-1$ and $\kappa_i = 0$ for $0 \le i \le t-1$. Finally, applying the Adem relation $\mathscr{P}^p\mathscr{P}^p = 2\mathscr{P}^{2p} + \mathscr{P}^{2p-1}\mathscr{P}^1$ to u_{2p+1}, we conclude that $\lambda = 0, 1$, and this completes the proof.

We use the notation $C_j^{(\lambda)}$ for $\lambda = 0, 1$ for the $\mathscr{K}$-algebras C_j with $\mathscr{P}^p(u_{2p+1}) = \lambda u_2^{p(p-1)} u_{2p+1}$ for $\lambda = 0, 1$, respectively. For $j \ge 1$, we define a $\mathscr{K}$-map $f_j : C_j^{(\lambda)} \to H^*(B\mathbf{Z}/p)$ as $f_j(u_2) = \omega_2$ and $f_j(u_3) = f_j(u_{2p+1}) = f_j(u_{2p+2}) = 0$. Then, we have the following results whose proof is almost the same as Proposition 2.6.

Proposition 4.4. (1) *$\varepsilon_{f_j} : C_j^{(0)} \to T_{f_j}(C_j^{(0)})$ is an isomorphism.*

(2) *$T_{f_j}(C_j^{(1)}) \cong \mathbf{Z}/p[u_2] \otimes \Lambda(u_3, v_1) \otimes \mathbf{Z}/p[v_2]/(v_2^t)$, and $\varepsilon_{f_j} : C_j^{(1)} \to T_{f_j}(C_j^{(1)})$ is given as*

$$\begin{cases} \varepsilon_{f_j}(u_2) = u_2, \\ \varepsilon_{f_j}(u_3) = u_3, \\ \varepsilon_{f_j}(u_{2p+1}) = u_2^p v_1, \\ \varepsilon_{f_j}(u_{2p+2}) = u_2^p v_2. \end{cases}$$

For $j \geq 1$, let $Y_t\langle 3; p^j\rangle$ denote the homotopy fiber of the map of degree p^j :

$$Y_t\langle 3; p^j\rangle \longrightarrow Y_t \xrightarrow{[p^j]} K(\hat{\mathbf{Z}}_p, 3).$$

Then, for $1 \leq t \leq p, H^*(Y_t\langle 3; p^j\rangle) \cong C_j^{(0)}$ as $\mathscr{K}$-algebras, namely $C_j^{(0)}$ is realizable as the cohomology. On the other hand, we have the next result for the non-realizability of $\mathscr{K}$-algebras $C_j^{(1)}$, which will be essential in the proof of Theorem C.

Proposition 4.5. *For $j \geq 1$, the $\mathscr{K}$-algebra $C_j^{(1)}$ cannot be realizable as the cohomology.*

Proof. We assume that there exists a space X such that $H^*(X) \cong C_j^{(1)}$, and deduce a contradiction from this assumption. A result of Lannes [15; Thm. 3.1.1] implies that there is a map ϕ_j such that $\phi_j^* = f_j$. By Theorem 2.2, we can choose a sufficient large $s > 0$ such that $H^*(P_sX) \cong C_j^{(1)}$ and $H^*(\mathrm{Map}(B\mathbf{Z}/p, P_sX)_{\phi_s}) \cong T_{f_j}(C_j^{(1)})$ up to dimension $2p^2 + 2p$. Then, $\varepsilon_{f_j} : C_j^{(1)} \to T_{f_j}(C_j^{(1)})$ can be realized by the map

$$e_{\phi_s} : \mathrm{Map}(B\mathbf{Z}/p, P_sX)_{\phi_s} \to P_sX$$

up to dimension $2p^2 + 2p$, and we have the induced homomorphism for the Bockstein spectral sequences:

$$e_{\phi_s}^* : \{B_i(P_sX), \beta_i\} \to \{B_i(\mathrm{Map}(B\mathbf{Z}/p, P_sX)_{\phi_s}), \beta_i\}.$$

We see that $\{u_{2p+1}u_{2p+2}^{t-1}\} \in B_i(P_sX)$ becomes a permanent cycle, while for $e_{\phi_s}^*(\{u_{2p+1}u_{2p+2}^{t-1}\}) = \{u_2^{pt}v_1v_2^{t-1}\} \in B_i(\mathrm{Map}(B\mathbf{Z}/p, P_sX)_{\phi_s})$, by [5; Thm. 5.4], we have $\beta_{j+1}(\{u_2^{pt}v_1v_2^{t-1}\}) \neq 0$ for $1 \leq t < p$, and $\beta_{j+2}(\{u_2^{p^2}v_1v_2^{p-1}\}) \neq 0$. This causes a contradiction, and we have the required conclusion.

Proof of Theorem C. We assume that there exists a space X such that $H^*(X) \cong A_{t(p+1)-1}$ as $\mathscr{K}$-algebras. In the proof of Theorem A, we considered the Serre spectral sequence for the following fibration:

$$X \xleftarrow{\simeq} \mathrm{Map}(B\mathbf{Z}/p, X)_\phi \to X_1 \to B^2\mathbf{Z}/p,$$

where $X_1 = (\mathrm{Map}(B\mathbf{Z}/p, X)_\phi)_{hB\mathbf{Z}/p}$ denotes the Borel construction, in which the transgression was described as $\tau(x_{2p^2}) = \mathscr{P}^{\Delta_1}\beta\eta_2 + \delta_{2p^2+1}$ for some decomposable element $\delta_{2p^2+1} \in H^*(B^2\mathbf{Z}/p)$. Now we compute the cohomology of X_1. We set

$$\delta_{2p^2+1} = \sum_{i=0}^{p-2} \theta_i(\beta\mathscr{P}^1\beta\eta_2)^{p-1-i}(\beta\eta_2)\eta_2^{i(p+1)} + \kappa(\beta\eta_2)\eta_2^{p^2-1}$$

$$+ \sum_{i=0}^{p-3} \sigma_i(\beta\mathscr{P}^1\beta\eta_2)^{p-2-i}(\mathscr{P}^1\beta\eta_2)\eta_2^{2+i(p+1)} - \lambda(\mathscr{P}^1\beta\eta_2)\eta_2^{p^2-p}$$

for $\theta_i, \kappa, \sigma_i, \lambda \in \mathbf{Z}/p$. Since $\mathscr{P}^1(x_{2p^2}) = 0$, we have $\theta_i = \kappa = \sigma_i = 0$ for $0 \leq i \leq p-2$, and so $\tau(x_{2p^2}) = \mathscr{P}^{\Delta_1}\beta\eta_2 - \lambda(\mathscr{P}^1\beta\eta_2)\eta_2^{p^2-p}$. Then $\tau(\mathscr{P}^p(x_{2p^2})) = -\lambda((\mathscr{P}^{\Delta_1}\beta\eta_2)\eta_2^{p^2-p} - (\mathscr{P}^1\beta\eta_2)\eta_2^{2p^2-2p})$. Since $\mathscr{P}^p(x_{2p^2}) = 0$, we have $\lambda = 0, 1$. Computing the spectral sequence, we have $H^*(X_1) \cong C_1^{(\lambda)}$, this forces $\lambda = 0$ by Proposition 4.5. A result of Lannes [15; Thm. 3.1.1] implies that there is a map $\phi_1 : B\mathbf{Z}/p \to X_1$ such that $\phi_1^* = f_1$. The evaluation map $e_{\phi_1} : \mathrm{Map}(B\mathbf{Z}/p, X_1)_{\phi_1} \to X_1$ is a homotopy equivalence by Theorem 2.1 and Proposition 4.4. Let $\imath_1 : B\mathbf{Z}/p \to \mathrm{Map}(B\mathbf{Z}/p, X_1)_{\phi_1}$ be the adjoint of $\phi_1\mu$, where μ is the multiplication of an H-structure of $B\mathbf{Z}/p$. Then, we have the following fibration by the same construction as above:

$$X_1 \xleftarrow{\simeq} \mathrm{Map}(B\mathbf{Z}/p, X_1)_{\phi_1} \to X_2 \to B^2\mathbf{Z}/p,$$

where $X_2 = (\mathrm{Map}(B\mathbf{Z}/p, X_1)_{\phi_1})_{hB\mathbf{Z}/p}$ denotes the Borel construction. Computing the spectral sequence as above, we conclude that $H^*(X_2) \cong C_2^{(0)}$. Iterating this process, we have the following sequence of spaces and maps:

$$X \xrightarrow{i_0} X_1 \xrightarrow{i_1} X_2 \xrightarrow{i_2} \cdots$$

satisfying $H^*(X) \cong A_{t(p+1)-1}, H^*(X_j) \cong C_j, i_0^* = 0$ and

$$\begin{cases} i_j^*(u_2) = 0, \\ i_j^*(u_3) = u_3, \\ i_j^*(u_{2p+1}) = u_{2p+1}, \\ i_j^*(u_{2p+2}) = u_{2p+2} \end{cases}$$

for $j \geq 1$. If we set $X_\infty = \varinjlim_j X_j$, then there is the Milnor exact sequence

$$0 \to \varprojlim_j{}^1 H^{*+1}(X_j) \to H^*(X_\infty) \to \varprojlim_j H^*(X_j) \to 0.$$

Since $\varprojlim_j{}^1 H^{*+1}(X_j) = 0$ by the Mittag-Leffler condition, we have $H^*(X_\infty) \cong \varprojlim_j H^*(X_j) \cong H^*(Y_t)$, and so $X_\infty \simeq Y_t$ by Proposition 4.1. Let F be the homotopy fiber of the composite $X_1 \to Y_t$, then $H^*(F) \cong H^*(K(\hat{\mathbf{Z}}_p, 2))$ by the spectral sequence argument, and this implies that $F \simeq K(\hat{\mathbf{Z}}_p, 2)$. By the cohomology, X_1 is homotopy equivalent to the homotopy fiber of $[p] : Y_t \to K(\hat{\mathbf{Z}}_p, 3)$, namely we have $X_1 \simeq Y_t\langle 3; p\rangle$. Therefore, we have the following commutative diagram of fibrations:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, equal] & Y_t\langle 3; p\rangle \arrow[r] \arrow[d] & B^2\mathbf{Z}/p \arrow[d] \\
Y_t\langle 3\rangle \arrow[r] \arrow[d] & Y_t \arrow[r] \arrow[d, "{[p]}"] & K(\hat{\mathbf{Z}}_p, 3) \arrow[d, "{[p]}"] \\
* \arrow[r] & K(\hat{\mathbf{Z}}_p, 3) \arrow[r, equal] & K(\hat{\mathbf{Z}}_p, 3),
\end{tikzcd}$$

from which we have the required conclusion, and this completes the proof of Theorem C.

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