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The cohomology of $BO(n)$ with twisted integer coefficients

By

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Abstract

Let H^* $(BO(n), \mathbf{Z}^t)$ be the graded cohomology group of the classifying space $BO(n)$ with twisted integer coefficients. Then $H^*(BO(n);\mathbf{Z}) \oplus H^*(BO(n);\mathbf{Z}^t)$ has a structure of a $\mathbf{Z} \oplus \mathbf{Z}_2$ graded ring. In the paper this ring is described in terms of generators and relations. It extends the results on the integer cohomology ring $H^*(BO(n);\mathbf{Z})$ derived in [B] and [F].

1. Introduction

The cohomology rings of the classifying spaces for the groups $O(n)$ and $SO(n)$ with $\mathbf{Z}_2$ and $\mathbf{Z}[1/2]$ coefficients have been known for a long time, see [MS]. In 1960, E. Thomas found the group structure of $H^*(BO(n))$ with integer and $\mathbf{Z}_{2^m}$ coefficients [T]. The integer cohomology ring is much more complicated so that it lasted till the year 1982 than its structure was written down in terms of generators and relations independently by E. H. Brown [B] and M. Feshbach [F]. In a similar way the cohomology rings of $BO(n)$ and $BSO(n)$ with $\mathbf{Z}_{2^m}$ coefficients have been described in [CV].

Sometimes it is necessary to use cohomology classes of $BO(n)$ with a non-trivial system of local integer coefficients. Since $\pi_1(BO(n)) = \mathbf{Z}_2$ we have only two possible nonequivalent systems of local integer coefficients—the trivial one, denoted by $\mathbf{Z}$ and the nontrivial, which we will call twisted and denote by $\mathbf{Z}^t$.

The purpose of this note is to describe all the cohomology classes with twisted integer coefficients and relations among them. However, since the cup product of two such classes is a class with trivial integer coefficients, it is advantageous to consider cohomology classes with trivial and twisted integer coefficients together as a $\mathbf{Z} \oplus \mathbf{Z}_2$-graded ring

$$h^*(BO(n)) = H^*(BO(n);\mathbf{Z}) \oplus H^*(BO(n);\mathbf{Z}^t).$$

We will determine this ring via its generators and relations.

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2. Preliminaries and main results

We will use the definition of a system of local coefficients and singular cohomology groups with local coefficients from [Sp], exercises F in Chapter I and J in Chapter V. There one can also find the theorem on the existence of the Thom class with local integer coefficients and a version of the Thom isomorphism suitable for our purposes.

Let n be a positive integer and let $\xi = (E, X, p)$ be an n-dimensional vector bundle over a connected CW-complex X. Denote $\bar{E}$ the total space without the zero section, E_x the fiber over $x \in X$ and $i_x : E_x \to E$ the inclusion. Then $\{H_n(E_x, \bar{E}_x)\}$ form a system of local integer coefficients over X and we will denote it by $\{\mathbf{Z}_\xi\}$. An element $t \in H^n(E, \bar{E}; p^*\{\mathbf{Z}_\xi\})$ such that $i_x^* t \in H^n(E_x, \bar{E}_x; H_n(E_x, \bar{E}_x))$ corresponds to the identity in $\operatorname{Hom}(H_n(E_x, \bar{E}_x), H_n(E_x, \bar{E}_x))$ for every $x \in X$ is called the Thom class of the vector bundle ξ.

Proposition. *Let $\xi = (E, X, p)$ be an n-dimensional vector bundle over a connected CW-complex X and let $\{G\}$ be a system of local coefficients over X. Then there is just one Thom class $t \in H^n(E, \bar{E}; p^*\{\mathbf{Z}_\xi\})$ and the homomorphism*

$$\Phi_t : H^m(X; \{G\}) \to H^{n+m}(E, \bar{E}; p^*\{G\} \otimes p^*\{\mathbf{Z}_\xi\}) : \Phi_t(u) = p^*(u) \cup t$$

is an isomorphism.

Let $j : (E, \phi) \to (E, \bar{E})$ be the inclusion. The class $e \in H^n(X; \{\mathbf{Z}_\xi\})$ such that $p^*(e) = j^*(t)$ is called the Euler class of the vector bundle ξ. Using the Thom isomorhism and the long exact sequence for the couple $(E, \bar{E})$ with coefficients $\{G\}$ we get the long exact Gysin sequence with local coefficients:

$$H^{q-1}(\bar{E}; p^*\{G\}) \xrightarrow{\Delta^*} H^{q-n}(X; \{G\} \otimes \{\mathbf{Z}_\xi\}) \xrightarrow{\cup e} H^q(X; \{G\}) \xrightarrow{p^*} H^q(\bar{E}; p^*\{G\})$$

Let $\gamma_n = (E_n, BO(n), p)$ be the universal vector bundle over the classifying space $BO(n)$ with the Euler class e_n. In this case the system of local coefficients $\{\mathbf{Z}_{\gamma_n}\}$ is equivalent to the system of twisted coefficients $\mathbf{Z}^t$ and further, $\mathbf{Z} \otimes \mathbf{Z}^t = \mathbf{Z}^t$, $\mathbf{Z}^t \otimes \mathbf{Z}^t = \mathbf{Z}$. Since E_n is homotopically equivalent to $BO(n)$, the sphere bundle SE_n is homotopically equivalent to $BO(n-1)$ for $n \geq 2$ and the inclusion $SE_n \hookrightarrow E_n$ corresponds to inclusion $i : BO(n-1) \hookrightarrow BO(n)$, the application of the Gysin sequence to this case both for trivial and twisted coefficients yields the following exact sequence which plays crucial role in our next considerations.

$$(*) \quad \longrightarrow h^{q-1}(BO(n-1)) \xrightarrow{\Delta^*} h^{q-n}(BO(n)) \xrightarrow{\cup e_n} h^q(BO(n)) \xrightarrow{i^*} h^q(BO(n-1)) \longrightarrow$$

This exact sequence can also be applied to the case $n = 1$ if we take $BO(0)$ as $SE_1 = S^\infty$.

The letters w_i and p_i will stand for the i-th Stiefel–Whitney class and the i-th Pontrjagin class of the universal vector bundle γ_n. The mapping $\rho : H^*(X, \mathbf{Z}) \to H^*(X, \mathbf{Z}_2)$ or $\rho : H^*(X, \mathbf{Z}^t) \to H^*(X, \mathbf{Z}_2)$ is induced from the reduction mod 2. The Bockstein homomorphism associated with the exact sequence $0 \to \mathbf{Z} \to$

$\mathbf{Z} \to \mathbf{Z}_2 \to 0$ will be denoted $\delta : H^i(BO(n); \mathbf{Z}_2) \to H^{i+1}(BO(n); \mathbf{Z})$ in the case of trivial integer coefficients and $\sigma : H^i(BO(n); \mathbf{Z}_2) \to H^{i+1}(BO(n); \mathbf{Z}^t)$ in the case of twisted integer coefficients. In the case of twisted coefficients the long exact sequenced induced from the short exact sequence above has the form

$$(**) \qquad \longrightarrow H^q(X; \mathbf{Z}^t) \xrightarrow{2\times} H^q(X; \mathbf{Z}^t) \xrightarrow{\rho} H^q(X; \mathbf{Z}_2) \xrightarrow{\sigma} H^{q+1}(X; \mathbf{Z}^t) \longrightarrow .$$

For the symmetric difference of two sets I and J we will use the symbol

$$\Delta(I, J) = (I \cup J) - (I \cap J).$$

Next in the ring $h^*(BO(n))$ we denote

$$w_0 = 1, \quad w_I = \prod_{i \in I} w_{2i}, \quad w_\phi = 1$$

$$p_I = \prod_{i \in I} p_i, \quad p_\phi = 1,$$

where I is a finite subset of integers.

Using this notation our main result reads as

Theorem 1. *The cohomology ring*

$$h^*(BO(n)) = H^*(BO(n); \mathbf{Z}) \oplus H^*(BO(n); \mathbf{Z}^t)$$

is the polynomial ring over $\mathbf{Z}$ *generated by the elements*

$$p_i, \quad \delta w_I, \quad \sigma w_I, \quad \sigma 1, \quad e_n$$

where $1 \le i \le (n-1)/2$ *and* I *ranges over all finite nonempty subsets of* $\{l \in \mathbf{N}; 1 \le l \le (n-1)/2\}$ *modulo the ideal generated by the following relations in which* J *can be empty.*

$$(1) \qquad 2\delta w_I = 0$$

$$(2) \qquad 2\sigma w_I = 0$$

$$(3) \qquad 2\sigma 1 = 0$$

$$(4) \qquad \delta w_I \delta w_J = \sum_{i \in I} \delta w_{2i} \delta w_{\Delta(I - \{i\}, J)} p_{(I - \{i\}) \cap J}$$

$$(5) \qquad \delta w_I \sigma w_J = \sum_{i \in I} \delta w_{2i} \sigma w_{\Delta(I - \{i\}, J)} p_{(I - \{i\}) \cap J}$$

$$(6) \qquad \sigma w_I \sigma w_J = \delta w_I \delta w_J + \sigma 1 \sigma w_{\Delta(I, J)} p_{(I \cap J)}$$

$$(7) \qquad e_n = \sigma w_{n-1} \quad \text{if } n \text{ is odd.}$$

Theorem 2. *The cohomology ring*

$$h^*(BO) = H^*(BO; \mathbf{Z}) \oplus H^*(BO; \mathbf{Z}^t)$$

is the polynomial ring over $\mathbf{Z}$ *generated by the elements*

$$p_i, \quad \delta w_I, \quad \sigma w_I, \quad \sigma 1$$

where i is a positive integer and I ranges over all finite nonempty subsets of positive integers modulo the ideal generated by the relations (1)–(6).

3. Proofs

Let X be a path connected topological space. Let a system of local coefficients $\{\mathbf{Z}\}$ over X be determined by the element $w \in H^1(X;\mathbf{Z}_2)$. The cohomology groups with such local coefficients will be denoted $H^*(X;\mathbf{Z}_w)$. In $BO(n)$ the nontrivial system of local coefficients is given by the first Stiefel-Whitney class w_1.

Lemma 1. *In* $H^1(BO(1);\mathbf{Z}^t)$ *the Euler class* e_1 *is different from zero,*

$$h^*(BO(1)) = \mathbf{Z}[e_1]/\langle 2e_1\rangle$$

and $e_1 = \sigma 1, \rho(\sigma 1) = w_1$.

Proof. From [W, VI. 3. 2] we know that $H^0(BO(1);\mathbf{Z}^t) \cong 0$. Since S^∞ is contractible, the Gysin sequence yields the short exact sequence

$$0 \longrightarrow H^0(S^\infty;\mathbf{Z}) \cong \mathbf{Z} \xrightarrow{\Delta^*} H^0(BO(1);\mathbf{Z}) \cong \mathbf{Z} \xrightarrow{\cup e_1} H^1(BO(1);\mathbf{Z}^t) \longrightarrow 0$$

and isomorphisms

$$H^k(BO(1);\mathbf{Z}^t) \xrightarrow{\cup e_1} H^{k+1}(BO(1);\mathbf{Z}), \quad H^k(BO(1);\mathbf{Z}) \xrightarrow{\cup e_1} H^{k+1}(BO(1);\mathbf{Z}^t)$$

for all $k \geq 1$. We know that $H^2(BO(1);\mathbf{Z}) \cong \mathbf{Z}_2$, consequently $H^1(BO(1);\mathbf{Z}^t) \cong \mathbf{Z}_2$ with the generator e_1 and $2e_1 = 0$. Hence

$$h^*(BO(1)) = \mathbf{Z}[e_1]/\langle 2e_1\rangle.$$

From the long exact sequence (**) for $BO(1)$ we get that $\sigma : H^0(BO(1);\mathbf{Z}_2) \to H^1(BO(1);\mathbf{Z}^t)$ is a monomorhism. Hence $\sigma 1 = e_1$ and $\rho(\sigma 1) = w_1$.

Notice that we have just proved Theorem 1 for $n = 1$.

Lemma 2. *Let* $\sigma : H^i(X;\mathbf{Z}_2) \to H^{i+1}(X;\mathbf{Z}_w)$ *be the Bockstein homomorphism. Then*

$$\rho\sigma x = wx + Sq^1 x$$

for all $x \in H^i(X;\mathbf{Z}_2)$.

Proof. Using a universal example $\varphi : K(\mathbf{Z}_2, i) \times K(\mathbf{Z}_2, 1) \to K(\mathbf{Z}_2, i+1)$ for $\rho\sigma$, we get

$$\rho\sigma(\iota_i \otimes 1) = \varphi^*(\iota_{i+1}) = aSq^1\iota_i \otimes 1 + b\iota_i \otimes \iota_1 + c(1 \otimes \iota_1^{i+1}),$$

where ι_k is a fundamental class in $H^k(K(\mathbf{Z}_2,k);\mathbf{Z}_2)$ and $a,b,c\in\{0,1\}$. Hence also

$$\rho\sigma x = aSq^1x + bwx + cw^{i+1}.$$

Since $\rho\sigma 0 = 0$ for every w, we get immediately that $c=0$. The choice $w=0$ gives $\sigma=\delta$ and consequently $a=1$.

Next choose $X = BO(1), w = w_1, x = w^i \in H^i(BO(1);\mathbf{Z}_2)$. If i is even then, from Lemma 1 and (**) we obtain that $\sigma : H^i(BO(1);\mathbf{Z}_2)\to H^{i+1}(BO(1);\mathbf{Z}')$ is an isomorphism. Hence

$$bw_1^{i+1} = Sq^1(w_1^i) + bw_1w_1^i = \rho(\sigma w_1^i) = \rho(e_1^{i+1}) = w_1^{i+1},$$

which yields $b=1$. If i is odd, then $\sigma : H^i(BO(1);\mathbf{Z}_2)\to H^{i+1}(BO(1);\mathbf{Z}')$ is zero. Consequently,

$$(1+b)w_1^{i+1} = Sq^1(w_1^i) + bw_1w_1^i = \rho\sigma(w_1^i) = 0,$$

which again implies $b=1$.

Lemma 3. *Let $\{\mathbf{Z}\}$ be a system of local integer coefficients over X. Suppose that every torsion element of $H^*(X;\{\mathbf{Z}\})$ has order* 2. *Then the reduction $\rho : H^*(X;\{\mathbf{Z}\})\to H^*(X;\mathbf{Z}_2)$ restricted on the torsion subgroup of $H^*(X;\{\mathbf{Z}\})$ is an injection.*

Proof follows immediately from the long exact sequence (**).

Lemma 4. *In $H^*(BO(n);\mathbf{Z}_2)$ the following relations hold*

$$\rho(\delta w_I\delta w_J) = \sum_{i\in I} Sq^1w_{2i}Sq^1w_{\Delta(I-\{i\},J)}\rho p_{(I-\{i\})\cap J}$$

$$\rho(\delta w_I\sigma w_J) = \sum_{i\in I} Sq^1w_{2i}\rho\sigma w_{\Delta(I-\{i\},J)}\rho p_{(I-\{i\})\cap J}$$

$$\rho(\sigma w_I\sigma w_J) = Sq^1w_ISq^1w_J + \rho\sigma 1\rho\sigma w_{\Delta(I,J)}\rho p_{I\cap J}$$

where I and J are arbitrary subsets of $\{l\in\mathbf{N}; 1\le l\le (n-1)/2\}$.

Proof. The first formula was proved in [B] and [F]. Using this formula and the fact that $w_{2i}^2 = \rho p_i$, we can easily prove the second one:

$$\begin{aligned}\rho(\delta w_I\sigma w_J) &= Sq^1w_I(Sq^1w_J + w_1w_J) = Sq^1w_ISq^1w_J + Sq^1w_I\cdot w_1w_J\\ &= \sum_{i\in I} Sq^1w_{2i}Sq^1w_{\Delta(I-\{i\},J)}\rho p_{(I-\{i\})\cap J}\\ &\quad + \sum_{i\in I} Sq^1w_{2i}w_{\Delta(I-\{i\},J)}\rho p_{(I-\{i\})\cap J}w_1\\ &= \sum_{i\in I} Sq^1w_{2i}\rho\sigma w_{\Delta(I-\{i\},J)}\rho p_{(I-\{i\})\cap J}\end{aligned}$$

The following computation yields the third formula:

$$\begin{aligned}
\rho(\sigma w_I \sigma w_J) &= (Sq^1 w_I + w_1 w_I)(Sq^1 w_J + w_1 w_J)\\
&= Sq^1 w_I Sq^1 w_J + w_1(w_I Sq^1 w_J + w_J Sq^1 w_I) + w_1^2 w_I w_J\\
&= Sq^1 w_I Sq^1 w_J + w_1 Sq^1(w_I w_J) + w_1^2 w_{\Delta(I,J)} \rho p_{I\cap J}\\
&= Sq^1 w_I Sq^1 w_J + w_1 Sq^1(w_{\Delta(I,J)}) \rho p_{I\cap J} + w_1^2 w_{\Delta(I,J)} \rho p_{I\cap J}\\
&= Sq^1 w_I Sq^1 w_J + w_1\{Sq^1 w_{\Delta(I,J)} + w_1 w_{\Delta(I,J)}\} \rho p_{I\cap J}\\
&= Sq^1 w_I Sq^1 w_J + \rho\sigma 1 \rho\sigma w_{\Delta(I,J)} \rho p_{I\cap J}
\end{aligned}$$

Proof of Theorem 1. We will prove Theorem 1 by induction on n using the description of the ring $H^*(BO(n);\mathbf{Z})$ in [B] and [F].

Denote

$$\mathcal{R}_n = \mathbf{Z}[z_i, x_I, y_I, y_\phi, u_n]$$

where I ranges over all nonempty subsets of integers $1 \le i \le (n-1)/2$. Let $\theta_n : \mathcal{R}_n \to h^*(BO(n))$ be such a homomorphism that

$$\theta_n(z_i) = p_i, \quad \theta_n(x_I) = \delta w_I, \quad \theta_n(y_I) = \sigma w_I, \quad \theta_n(y_\phi) = \sigma 1, \quad \theta_n(u_n) = e_n$$

and let $\mathcal{I}_n$ be the ideal in $\mathcal{R}_n$ generated by relations (1)–(7) where $p_i, \delta w_I, \sigma w_I, \sigma 1$ and e_n are replaced by z_i, x_I, y_I, y_ϕ and u_n, respectively.

The ring $\mathcal{R}_n$ can also be considered as a $\mathbf{Z}_2$ graded ring $\mathcal{R}_n = \mathcal{R}_n^0 \oplus \mathcal{R}_n^1$, where x_I and z_i have graduation 0 and y_I and u_n have graduation 1.

We will show by induction on n that $\theta_n(\mathcal{I}_n) = 0$ and that the induced homomorphism $\bar{\theta}_n : \mathcal{R}_n/\mathcal{I}_n \to h^*(BO(n))$ is an isomorphism.

For $n = 1$ the proof has been carried out in Lemma 1.

Suppose that Theorem 1 holds for $n-1$ and n is even. In this case $\mathcal{R}_n/\mathcal{I}_n = (\mathcal{R}_{n-1}/\mathcal{I}_{n-1})[u_n]$. Let $j : \mathcal{R}_{n-1} \to \mathcal{R}_n$ be the inclusion. By the inductive hypothesis $\theta_{n-1} = i^*\theta_n j$ is an epimorphism, hence $i^* : h^*(BO(n)) \to h^*(BO(n-1))$ is also an epimorphism and from $(*)$ we get a short exact sequence

$$(8) \qquad 0 \longrightarrow h^{q-n}(BO(n)) \xrightarrow{\cup e_n} h^q(BO(n)) \xrightarrow{i^*} h^q(BO(n-1)) \longrightarrow 0$$

That is why the induction on q yields that $\theta_n : \mathcal{R}_n \to h^*(BO(n))$ is an epimorphism.

We will show that all the torsion elements of $h^*(BO(n))$ are of order 2. Denote $\operatorname{Tor} h^*(BO(n))$ the subgroup of its torsion elements. The inductive hypothesis implies that $\mathbf{Z}[p_1, p_2, \dots, p_{(n-1)/2}]$ is a subring of $h^*(BO(n))$. The short exact sequence (8) yields that the multiplication by e_n is injective. Hence $\mathbf{Z}[p_1, p_2, \dots, p_{(n-1)/2}, e_n]$ is also a subring of $h^*(BO(n))$. Since θ_n is an epimorphism, we get a direct sum decomposition of $h^*(BO(n))$ as a group

$$h^*(BO(n)) = \mathbf{Z}[p_1, p_2, \dots, p_{(n-1)/2}, e_n] \oplus \operatorname{Tor} h^*(BO(n)).$$

Since $2\delta = 2\sigma = 0$, all the torsion elements are of order 2. Using Lemma 4 we obtain $\rho\theta_n(\mathcal{I}_n) = 0$. Now Lemma 3 ensures that $\theta_n(\mathcal{I}_n) = 0$.

Induction on q and in (8) and 5-lemma yield that $\bar{\theta}_n : \mathcal{R}_n/\mathcal{I}_n \to h^*(BO(n))$ is an isomorphism.

Moreover, we will show that $p_{n/2} = e_n^2$ in $h^*(BO(n))$ if n is even. Consider the inclusion $k : SO(n) \hookrightarrow O(n)$. It induces the ring homomorphism $k^* : h^*(BO(n)) \to H^*(BSO(n); \mathbf{Z})$. The classes k^*w_i, k^*p_i and k^*e_n are the Stiefel-Whitney, Pontrjagin and Euler classes of the universal oriented vector bundle over $BSO(n)$, respectively. From the knowledge of $h^*(BO(n))$ and $H^*(BSO(n); \mathbf{Z})$ we can conclude that k^* restricted on the free part of $H^*(BO(n); \mathbf{Z})$ is a monomorphism and that $k^*(p_{n/2} - e_n^2) = 0$. Hence, $p_{n/2} = e_n^2$.

Now, suppose that Theorem 1 holds for $n-1$ and n is odd. Let t_n be the Thom class of γ_n. Then ρt_n is the Thom class with $\mathbf{Z}_2$ coefficients and hence $\rho e_n = w_n$. According to Lemma 2,

$$\rho\sigma w_{n-1} = w_1 w_{n-1} + Sq^1 w_{n-1} = w_n = \rho e_n,$$

and consequently,

$$e_n = \sigma w_{n-1} + 2v \tag{9}$$

where $v \in H^n(BO(n); \mathbf{Z}^t)$. We will show that $2v = 0$. *2n* is not a multiple of 4 hence $H^{2n}(BO(n); \mathbf{Z})$ contains only elements od order two. Particullary, $2ve_n = 0$. From the long exact sequence $(*)$, we get that $2v = \Delta^*s$ where $s \in H^{2n-1}(BO(n-1); \mathbf{Z}^t)$. From the inductive hypothesis we know that all elements in $H^{2n-1}(BO(n-1); \mathbf{Z}^t)$ are of order 2, consequently $2s = 0$. That is why $4v = \Delta^*(2s) = 0$ and also $2e_n = 2\sigma w_{n-1} + 4v = 0$.

All elements of $H^n(BO(n-1); \mathbf{Z}^t)$ are of order 2 as well. Hence $2i^*v = 0$, and using again the exact sequence $(*)$, we have $2v = me_n$ for some $m \in \mathbf{Z}$. Substituting to (9) we obtain $(1-m)e_n = \sigma w_{n-1}$. Since

$$\rho((1-m)e_n) = \rho\sigma w_{n-1} = Sq^1 w_{n-1} + w_1 w_{n-1} = w_n \neq 0,$$

m has to be of the form *2l* and, consequently, $2v = 2le_n = 0$. So, we have proved that

$$e_n = \sigma w_{n-1}. \tag{10}$$

To prove that $\theta_n : \mathcal{R}_n \to h^*(BO(n))$ is an epimorphism we will show that $\operatorname{im} i^*\theta_n = \ker \Delta^* = \operatorname{im} i^*$. We get

$$\operatorname{im} i^*\theta_n = \mathbf{Z}[p_1, \ldots, p_{(n-3)/2}, e_{n-1}^2] \oplus \operatorname{Tor} h^*(BO(n-1))$$

from the following computations which use the knowledge of $h^*(BO(n-1))$ and Lemma 3.

$$i^*\theta_n(x_I) = \delta w_I, \quad i^*\theta_n(y_I) = \sigma w_I \quad \text{if } (n-1)/2 \notin I$$

$$i^*\theta_n(x_I) = \delta w_I e_{n-1}, \quad i^*\theta_n(y_I) = \sigma w_I e_{n-1} \quad \text{if } (n-1)/2 \in I$$

$$i^*\theta_n z_i = p_i, \quad i^*\theta_n z_{(n-1)/2} = p_{(n-1)/2} = e_{n-1}^2$$

$$i^*\theta_n u_n = i^*e_n = i^*(1 \cup e_n) = 0.$$

So all the elements of $h^q(BO(n-1))$ for $q \leq n-2$ are simultaneously in $\operatorname{im} i^*\theta_n$ and $\operatorname{im} i^* = \ker \Delta^*$. Further,

$$h^{n-1}(BO(n-1)) = \{me_{n-1}; m \in \mathbf{Z}\} \oplus (\operatorname{im} i^*\theta_n \cap h^{n-1}(BO(n-1))).$$

Since $e_n \neq 0$ and $2e_n = 0$, from $(*)$ we get the existence of an element $v \in h^{n-1}(BO(n-1))$ such that $\Delta^* v = 2$. Consequently, $\Delta^* e_{n-1} = \pm 2$. So every element of $h^*(BO(n-1))$ has the form $v = v_1 + v_2 e_{n-1}$, where $v_1, v_2 \in \operatorname{im} i^*\theta_n$. Now,

$$\Delta^*(v_1 + v_2 e_{n-1}) = v_2 \Delta^* e_{n-1} = \pm 2v_2.$$

Hence $v \in \ker \Delta^*$ if and only if $2v_2 = 0$, which implies $\ker \Delta^* = \operatorname{im} i^*\theta_n$.

Now, we can show that $\theta_n : \mathscr{R}_n \to h^*(BO(n))$ is an epimorphism by induction on q. Let $v \in h^q(BO(n))$ be arbitrary. There is $v_1 \in \mathscr{R}_n$ such that $i^*(v - \theta_n v_1) = 0$. Then the exact sequence $(*)$ implies the existence of $v_2 \in h^{q-n}(BO(n))$ such that $v_2 e_n = v - \theta_n v_1$. By the inductive hypothesis, $v_2 = \theta_n v_3$ for some $v_3 \in \mathscr{R}_n$. Consequently, $v = \theta_n v_1 + v_2 e_n = \theta_n(v_1 + v_3 u_n)$.

We have already proved that

$$i^*(\mathbf{Z}[p_1, \ldots, p_{(n-3)/2}, p_{(n-1)/2}]) = \mathbf{Z}[p_1, \ldots, p_{(n-3)/2}, e_{n-1}^2]$$

is a subring of $h^*(BO(n-1))$, hence

$$h^*(BO(n)) = \mathbf{Z}[p_1, p_2, \ldots, p_{(n-1)/2}] \oplus \operatorname{Tor} h^*(BO(n)).$$

So every torsion element of $h^*(BO(n))$ is of order 2. Applying Lemma 3, Lemma 4 and (10), we obtain $\theta_n(\mathscr{I}_n) = 0$.

It remains to prove that $\bar{\theta}_n : \mathscr{R}_n/\mathscr{I}_n \to h^*(BO(n))$ is injective. Since θ_n is an epimorphism, $h^*(BO(n))$ as a $\mathbf{Z}$ graded group is a direct sum

$$h^*(BO(n)) = \mathbf{Z}[p_1, p_2, \ldots, p_{(n-1)/2}] \oplus \operatorname{Tor} H^*(BO(n); \mathbf{Z}) \oplus \operatorname{Tor} H^*(BO(n); \mathbf{Z}').$$

The inverse images of these subgroups in the homomorphism $\bar{\theta}_n$ give the group $\mathscr{R}_n/\mathscr{I}_n$ as a direct sum

$$\mathscr{R}_n/\mathscr{I}_n = \mathbf{Z}[z_1, z_2, \ldots, z_{(n-1)/2}] \oplus \operatorname{Tor}^0(\mathscr{R}_n/\mathscr{I}_n) \oplus \operatorname{Tor}^1(\mathscr{R}_n/\mathscr{I}_n).$$

From the description of $H^*(BO(n); \mathbf{Z})$ in [B] and [F] we know that $\bar{\theta}_n$ is a group isomorphism onto $H^*(BO(n); \mathbf{Z})$ when restricted on

$$\mathbf{Z}[z_1, z_2, \ldots, z_{(n-1)/2}] \oplus \operatorname{Tor}^0(\mathscr{R}_n/\mathscr{I}_n).$$

Due to Lemma 3, it suffices to show that $\rho\bar{\theta}_n$ restricted on $\operatorname{Tor}^1(R_n/\mathscr{I}_n)$ is a group monomorphism into $H^*(BO(n); \mathbf{Z}_2)$.

Consider the following slight modification of the Stiefel-Whitney classes

$$v_1 = w_1, \quad v_{2i} = w_{2i}, \quad v_{2i+1} = w_{2i+1} + w_1 w_{2i}.$$

We have $Sq^1 v_1 = v_1^2$, $Sq^1 v_{2i+1} = 0$ if $i \geq 1$, $Sq^1 v_{2i} = v_{2i+1}$ if $i \leq (n-1)/2$. The monomials $\prod_{i=1}^{n} v_i^{k_i}$ form a basis of $H^*(BO(n); \mathbf{Z}_2)$. For $\phi \neq I \subseteq \{1, 2, \ldots, (n-1)/2\}$ put $v_I = w_I$.

We will show that $\mathrm{Tor}^1(\mathscr{R}_n/\mathscr{I}_n)$ has the following generators

$$y_\phi^{2l} \prod_{i=1}^{(n-1)/2} z_i^{m_i} \prod_{i=1}^{(n-1)/2} x_{\{i\}}^{k_i} y_I, \quad I \neq \phi$$

$$y_\phi^{2l+1} \prod_{i=1}^{(n-1)/2} z_i^{m_i} \prod_{i=1}^{(n-1)/2} x_{\{i\}}^{k_i}.$$

The successive use of relation (6) for products $y_K y_J$, relation (4) for products $x_K x_J$ and relation (5) for products $x_K y_J$ or $x_K y_\phi$ decomposes any monomial in $\mathrm{Tor}^1(\mathscr{R}_n/\mathscr{I}_n)$ into a sum of the monomials described above.

The group homomorphism $\rho\bar{\theta}_n$ maps these generators of $\mathrm{Tor}^1(\mathscr{R}_n/\mathscr{I}_n)$ into elements

$$v_1^{2l} \prod_{i=1}^{(n-1)/2} v_{2i}^{2m_i} \prod_{i=1}^{(n-1)/2} v_{2i+1}^{k_i} (v_1 v_I + Sq^1 v_I), \quad I \neq \phi$$

$$v_1^{2l+1} \prod_{i=1}^{(n-1)/2} v_{2i}^{2m_i} \prod_{i=1}^{(n-1)/2} v_{2i+1}^{k_i}$$

We will prove that these elements are linearly independent in $H^*(BO(n); \mathbf{Z}_2)$. All the elements of the second kind are monomials containing an odd power of v_1 and even powers of all v_{2i}. Every element of the first kind is a sum of monomials from which just one contains an odd power of v_1. This monomial

$$v_1^{2l+1} \prod_{i=1}^{(n-1)/2} v_{2i}^{2m_i} \prod_{i=1}^{(n-1)/2} v_{2i+1}^{k_i} v_I$$

is uniquely determined by the numbers l, m_i, k_i and by the nonempty set I and contains odd powers of v_{2i} for every $i \in I$. So any sum of some elements of the both kinds is different from zero.

This completes the proof that $\bar{\theta}_n$ is an isomorphism.

Proof of Theorem 2. The standard inclusions $O(n) \hookrightarrow O(m) \hookrightarrow O$ yield the fibrations

$$V_{m,m-n} \longrightarrow BO(n) \xrightarrow{p_{n,m}} BO(m), \quad V \longrightarrow BO(n) \xrightarrow{p_n} BO$$

where V is the inductive limit of the Stiefel manifolds $V_{m,m-n}$ for $m \to \infty$.

Since $V_{m,m-n}$ and V are $(n-1)$-connected, the local coefficient Serre spectral sequence (see [S]) implies that

$$p_n^*: H^i(BO; \mathbf{Z}^t) \to H^i(BO(n); \mathbf{Z}^t), \quad p_n^*: H^i(BO; \mathbf{Z}) \to H^i(BO(n); \mathbf{Z})$$

and also $p_{n,m}^*$ are isomorphisms for $i < n$. Hence

$$h^*(BO) = \varprojlim_{n \to \infty} h^*(BO(n)),$$

which completes the proof of Theorem 2.

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