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A note on global existence of solutions to nonlinear Klein-Gordon equations in one space dimension

By

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1. Introduction

We consider the Cauchy problem for nonlinear Klein-Gordon equations

$$(1.1)\qquad \begin{cases} (\Box + 1)u = F(u, u_t, u_x, u_{tx}, u_{xx}) & \text{in } (0, \infty) \times \mathbf{R}, \\ u(0, x) = \varepsilon f(x), \quad u_t(0, x) = \varepsilon g(x) & \text{for } x \in \mathbf{R}, \end{cases}$$

where $\Box = \partial_t^2 - \partial_x^2$, $u_t = (\partial/\partial t)u$, $u_x = (\partial/\partial x)u$, etc. We suppose that the nonlinear term F is a smooth function in its arguments around the origin and satisfies

$$(1.2)\qquad F(\lambda) = O(|\lambda|^m) \quad \text{near } \lambda = 0$$

with some integer $m \geq 2$, where $\lambda = (u, u_t, u_x, u_{tx}, u_{xx})$. For simplicity, we assume that $f, g \in C_0^\infty(\mathbf{R})$. ε is a small positive parameter.

There are many studies on global existence of solutions to this type of equations, and here we recall some known results briefly. For n-space dimensional cases with nonlinear terms of mth degree as in (1.2), Klainerman–Ponce ([11]) and Shatah ([15]) showed that if $n(m-1)^2/(2m) > 1$, then there exists a unique global solution, provided that ε is sufficiently small. This condition means that $m \geq 4$ when $n = 1$, $m \geq 3$ when $n = 2, 3, 4$ and $m \geq 2$ when $n \geq 5$. For the case $n = 3, 4$ and $m = 2$, Klainerman ([9]) and Shatah ([16]) proved independently the global existence of the solution for small ε. Klainerman used the "method of invariant norms" to get a decay estimate of the solution, which was first found to be useful in the study of nonlinear wave equations (see [8]). On the other hand, Shatah used the method of normal forms to eliminate the quadratic parts of the nonlinear terms and got the sufficient decay estimate. When $n = 2$ and $m = 2$, Georgiev–Popivanov ([4]) and Kosecki ([12]) proved the global existence of the solution, assuming that the quadratic parts of the nonlinear terms satisfy certain special conditions. For general nonlinear terms with $m = 2$ in two space dimensions, Simon–Taflin ([17]) and Ozawa–Tsutaya–Tsutsumi ([14]) proved the global existence for small ε, and showed that the solution approaches a free solution as $t \to +\infty$. In [14], they combined the methods of normal forms and of the invariant norms to get the result.

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When $n = 1$ and $m = 3$, Yagi ([18]) showed that if $F = c_1F_1$ in (1.1), where

$$F_1 = 3uu_t^2 - 3uu_x^2 - u^3 \tag{1.3}$$

and $c_1 \in \mathbf{R}$ is a constant, then there exists a global solution for small ε, and that the solution approaches a free solution as $t \to +\infty$. Recently, Moriyama ([13]) proved the same result when F is a homogeneous polynomial of degree 3 which can be written as $F = \sum_{i=1}^{7} c_iF_i$, where F_1 is as in (1.3),

$$F_2 = 3u_t^2u_x - u_x^3 - 3u^2u_x + 6uu_tu_{tx}, \tag{1.4}$$

$$F_3 = uu_xu_{xx} - u^2u_x + u_t^2u_x + 2uu_tu_{tx}, \tag{1.5}$$

$$F_4 = (u_t^2 - u_x^2 - u^2)u_{xx} - 2uu_x^2, \tag{1.6}$$

$$F_5 = (u_t^2 - u_x^2 - u^2)u_{tx} - 2uu_tu_x, \tag{1.7}$$

$$F_6 = u_t^3 - 3u_x^2u_t - 3u^2u_t - 6uu_xu_{tx}, \tag{1.8}$$

$$F_7 = u_tu_x^2 + uu_tu_{xx} + 2uu_xu_{tx} \tag{1.9}$$

and $c_i \in \mathbf{R}$ $(i = 1, \dots, 7)$. In the proof, he used the normal forms to eliminate the cubic terms. Unfortunately, though the normal form to eliminate quadratic parts of the nonlinear terms for the nonlinear Klein-Gordon equations is always regular transformation, the transformation to eliminate cubic parts may be singular in general. He showed that the transformation is regular if and only if F is a linear combination of $F_1, \dots, F_7$ for the quasi-linear case. His proof of this fact suggests us that these nonlinear terms can be eliminated by transformation with polynomials.

In this note, we will show that $F_1, \dots, F_7$ can be actually eliminated by simple transformation. Then, it is easy to see that this transformation is compatible with the invariant norm method of Klainerman, and so we can easily show the same result as Moriyama's also when the nonlinearity involves not only the linear combinations of $F_1, \dots, F_7$, but also terms satisfying the strong null condition (see Georgiev [2]; see also Christodoulou [1] and Klainerman [10]) as well as terms of higher degree (especially, of degree 4). Our approach is similar to that of Kosecki [12]. Precisely we assume

(H) $F(\lambda)$ can be written as follows:

$$F(\lambda) = \sum_{i=1}^{10} c_iG_i(\lambda) + N(\lambda) + H(\lambda)$$

in some neighborhood of $\lambda = (u, u_t, u_x, u_{tx}, u_{xx}) = 0$, where $c_i \in \mathbf{R}$ $(i = 1, \dots, 10)$ and

(i) G_i $(1 \le i \le 10)$ are defined by

$$G_1(\lambda) = u(-u^2 + 3u_t^2 - 3u_x^2),$$

$$G_2(\lambda) = u_t(-3u^2 + u_t^2 - u_x^2) + 2u(u_t u_{xx} - u_x u_{tx}),$$

$$G_3(\lambda) = u_x(-u^2 + u_t^2 - u_x^2) + 2u(u_t u_{tx} - u_x u_{xx}),$$

$$G_4(\lambda) = u^3 - 2u^2 u_{xx} - 3uu_t^2 + 2u_t^2 u_{xx} - 2u_t u_x u_{tx} - u(u_{tx}^2 - u_{xx}^2),$$

$$G_5(\lambda) = (-u^2 + u_t^2 - u_x^2)u_{tx} - 2uu_t u_x,$$

$$G_6(\lambda) = -uu_x^2 + 2u_x(u_t u_{tx} - u_x u_{xx}) + u(u_{tx}^2 - u_{xx}^2),$$

$$G_7(\lambda) = 3u^2 u_t - 6uu_t u_{xx} - u_t^3 - 3u_t(u_{tx}^2 - u_{xx}^2),$$

$$G_8(\lambda) = u^2 u_x - 2uu_t u_{tx} - 2uu_x u_{xx} - u_t^2 u_x - u_x(u_{tx}^2 - u_{xx}^2),$$

$$G_9(\lambda) = -2uu_x u_{tx} - u_t u_x^2 + u_t(u_{tx}^2 - u_{xx}^2),$$

$$G_{10}(\lambda) = -u_x^3 + 3u_x(u_{tx}^2 - u_{xx}^2),$$

(ii) N is of the form

$$N(\lambda) = P_1(\lambda)(u_t u_{tx} - u_x u_{xx} + uu_x) + P_2(\lambda)(u_t u_{xx} - u_x u_{tx}) + P_3(\lambda)(u_{tx}^2 - u_{xx}^2 + uu_{xx})$$

with $P_i(\lambda)$ $(i = 1, 2, 3)$ which are homogeneous polynomials of degree 1,

(iii) $H(\lambda)$ is a smooth function of degree 4, i.e., $H(\lambda) = O(|\lambda|^4)$ near $\lambda = 0$.

Our main result is the following:

Theorem 1.1. *Suppose that $F(\lambda)$ satisfies the assumption* (H). *Then, for any given integer $k \ge 15$ and any $f, g \in C_0^\infty(\mathbf{R})$, there exists a positive constant ε_0 such that for any $\varepsilon \in (0, \varepsilon_0]$, the Cauchy problem* (1.1) *admits a unique classical solution $u \in C^\infty([0, \infty) \times \mathbf{R})$.*

Moreover, the solution $u(t, x)$ has a free profile, i.e., there exists $(u_{+0}(x), u_{+1}(x)) \in H^{k+1}(\mathbf{R}) \times H^k(\mathbf{R})$ such that

$$\|(u - U_+)(t, \cdot)\|_{H^{k+1}(\mathbf{R})} + \|\partial_t(u - U_+)(t, \cdot)\|_{H^k(\mathbf{R})} \to 0 \text{ as } t \to +\infty, \tag{1.10}$$

where $U_+(t, x)$ is the solution to the Cauchy problem for the linear Klein-Gordon equation $(\Box + 1)v = 0$ with initial data $v(0, x) = u_{+0}(x)$ and $v_t(0, x) = u_{+1}(x)$.

Remark 1.2. (i) G_i $(i = 1, \dots, 10)$ are linearly independent as polynomials with variables in $\lambda = (u, u_t, u_x, u_{tx}, u_{xx})$. Let Λ be the set of homogeneous polynomials of degree 3 with variables in λ, and let $\tilde{\Lambda}$ be the subset of Λ, whose elements are given by $\sum_{i=1}^{10} c_i G_i(\lambda) + N(\lambda)$, where N is as in (H). Then we can show that $\dim \tilde{\Lambda} = 21$, though $\dim \Lambda = 35$.

If we restrict our attention to the quasi-linear case, we can verify by straightforward calculations that

$$\mathrm{span}\{G_1, G_2, G_3, G_4 + G_6, G_5, G_7 + 3G_9, 3G_8 + G_{10}\} = \mathrm{span}\{F_1, \dots, F_7\},$$

where $F_1, \dots, F_7$ are as in (1.3)–(1.9). In fact, we have $F_1 = G_1$, $F_2 = (3G_3 - 3G_8 - G_{10})/2$, $F_3 = (G_3 - 3G_8 - G_{10})/4$, $F_4 = (G_1 + G_4 + G_6)/2$, $F_5 = G_5$, $F_6 = (3G_2 + G_7 + 3G_9)/2$ and $F_7 = (-G_2 - G_7 - 3G_9)/4$.

(ii) If F is semilinear and satisfies (H), the cubic part of F is of the form

$$c_1 u(-u^2 + 3u_t^2 - 3u_x^2) + (c_2 u_t + c_3 u_x)(-3u^2 + u_t^2 - u_x^2)$$

with constants $c_i \in \mathbf{R}$ $(i = 1, 2, 3)$.

(iii) Recently, Yordanov ([19]) proved that if $\int_{-\infty}^{\infty} f_x(x) g(x) dx > 0$, then (1.1) with $F = u_t^2 u_x + au^3$ $(a = \text{const})$ has no global (classical) solution for any $\varepsilon > 0$. His proof is also valid for (1.1) with

$$F = u_t^2 u_x + H_1(u) + H_2(u_x) u_{xx} + H_3(u, u_t, u_x, u_{tx}, u_{xx}) u_x,$$

where $H_1 = O(|u|^2), H_2 = O(|u_x|), H_3 = O(|u|^2 + |u_t|^2 + |u_x|^2 + |u_{tx}|^2 + |u_{xx}|^2)$ and $H_3 \geq 0$. Especially, we can see that for some f and g, (1.1) has no global solution if F is of the form $F = u_t^2 u_x + bu^2 u_x + cu_x^3$ with non-negative constants b and c, while there always exists a global solution for small data when $b = -3$ and $c = -1$, according to (ii).

2. Transformation with polynomials and some preliminaries

First, we will find some transformation to cancel $\sum_{i=1}^{10} c_i G_i(\lambda)$. For that purpose, let $\phi(u, u_t, u_x)$ be a homogeneous polynomial of degree 3 in its arguments and suppose that u satisfies $(\Box + 1)u = F$. Then, simple calculations give us

$$(2.1) \quad (\Box + 1)\phi(u, u_t, u_x)$$
$$= \phi + \phi_u \Box u + \phi_{u_t} \Box u_t + \phi_{u_x} \Box u_x + \phi_{u,u}(u_t^2 - u_x^2)$$
$$+ 2\phi_{u,u_t}(u_t u_{tt} - u_x u_{tx}) + 2\phi_{u,u_x}(u_t u_{tx} - u_x u_{xx})$$
$$+ \phi_{u_t,u_t}(u_{tt}^2 - u_{tx}^2) + 2\phi_{u_t,u_x}(u_{tt} u_{tx} - u_{tx} u_{xx}) + \phi_{u_x,u_x}(u_{tx}^2 - u_{xx}^2).$$

Substituting the relation $u_{tt} = u_{xx} - u + F$ into (2.1), we get

$$(2.2) \quad (\Box + 1)\phi(u, u_t, u_x)$$
$$= \phi - \phi_u u - \phi_{u_t} u_t - \phi_{u_x} u_x$$
$$+ \phi_{u,u}(u_t^2 - u_x^2) + 2\phi_{u,u_t}(u_t u_{xx} - u u_t - u_x u_{tx})$$
$$+ 2\phi_{u,u_x}(u_t u_{tx} - u_x u_{xx}) + \phi_{u_t,u_t}(u_{xx}^2 + u^2 - 2u u_{xx} - u_{tx}^2)$$
$$+ 2\phi_{u_t,u_x}(-u u_{tx}) + \phi_{u_x,u_x}(u_{tx}^2 - u_{xx}^2) + R,$$

where

$$R = \phi_u F + \phi_{u_t} F_t + \phi_{u_x} F_x + 2\phi_{u,u_t} u_t F + \phi_{u_t,u_t}(F^2 + 2u_{xx}F - 2uF) + 2\phi_{u_t,u_x} u_{tx} F. \tag{2.3}$$

Here we remark that when $F = F(\lambda)$, $\lambda = (u, u_t, u_x, u_{tx}, u_{xx})$ and $F = O(|\lambda|^3)$ near $\lambda = 0$, then R can be regarded as a function of

$$\tilde{\lambda} = (u, u_t, u_x, u_{tx}, u_{xx}, u_{txx}, u_{xxx})$$

and $R(\tilde{\lambda}) = O(|\tilde{\lambda}|^5)$ near $\tilde{\lambda} = 0$. Since we have assumed that ϕ is a homogeneous polynomial of degree 3, ϕ can be represented as follows:

$$\phi = b_1 u^3 + b_2 u^2 u_t + b_3 u^2 u_x + b_4 u u_t^2 + b_5 u u_t u_x + b_6 u u_x^2 + b_7 u_t^3 + b_8 u_t^2 u_x + b_9 u_t u_x^2 + b_{10} u_x^3 \tag{2.4}$$

with constants b_i $(i = 1, \dots, 10)$. Substituting (2.4) into (2.2), we get

$$(\Box + 1)\phi - R = 2\sum_{i=1}^{10} b_i G_i. \tag{2.5}$$

When F satisfies (H), choose $b_i = c_i/2$ for $i = 1, \dots, 10$ and we obtain

$$(\Box + 1)(u - \phi) = N(\lambda) + H(\lambda) - R(\tilde{\lambda}). \tag{2.6}$$

The cubic part in the right-hand side of (2.6) is $N(\lambda)$, but we can expect an extra decay with respect to time from $N(\lambda)$, because it satisfies the strong null condition. To explain this, following Klainerman [9], we introduce

$$Z_1 = t\partial_x + x\partial_t, \qquad Z_2 = \partial_t, \qquad Z_3 = \partial_x. \tag{2.7}$$

With multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, we write $Z^\alpha = Z_1^{\alpha_1} Z_2^{\alpha_2} Z_3^{\alpha_3}$. One can easily check that $[\Box + 1, Z_i] = 0$ holds for $i = 1, 2, 3$.

For any sufficiently smooth function $v(t, x)$ and non-negative integer k, we define

$$|v(t, x)|_k = \sum_{|\alpha| \le k} |Z^\alpha v(t, x)| \tag{2.8}$$

and

$$\|v(t)\|_{k,p} = \|v(t, \cdot)\|_{k,p} = \| \, |v(t, \cdot)|_k \|_{L^p(\mathbf{R})} \qquad \text{for } 1 \le p \le +\infty. \tag{2.9}$$

Let u be a solution to $(\Box + 1)u = F$. Then we get

$$u_t u_{tx} - u_x u_{xx} + u u_x = u_t u_{tx} - u_x u_{tt} + u_x F = Q(u, u_t) + u_x F, \tag{2.10}$$

$$u_t u_{xx} - u_x u_{tx} = Q(u, u_x), \tag{2.11}$$

$$u_{tx}^2 - u_{xx}^2 + u u_{xx} = u_{tx}^2 - u_{tt} u_{xx} + u_{xx} F = Q(u_x, u_t) + u_{xx} F, \tag{2.12}$$

where $Q(U,V) = U_t V_x - U_x V_t$. Using Z_1, we can write

$$Q(U,V) = \frac{1}{t}(U_t(Z_1 V) - (Z_1 U)V_t), \qquad t \neq 0$$

and we also have

$$Z^\alpha Q(U,V) = \sum_{|\beta|+|\gamma|=|\alpha|} C^\alpha_{\beta,\gamma} Q(Z^\beta U, Z^\gamma V),$$

where $C^\alpha_{\beta,\gamma}$ are appropriate constants. Therefore we obtain (see Georgiev [2], Klainerman [10] and also Katayama [6], [7]):

Lemma 2.1. *Suppose that u satisfies $(\Box + 1)u = F$ and that $F(\lambda) = O(|\lambda|^3)$ near $\lambda = 0$. Let k (≥ 0) be an integer, and let $N(\lambda)$ be as in (ii) of (H). If $|u(t,x)|_{[k/2]+2} \leq 1$, then we have*

$$|N(\lambda)(t,x)|_k \leq C_k\{(1+t)^{-1}|u(t,x)|^2_{[k/2]+2}(|u(t,x)|_{k+1} + |u'(t,x)|_{k+1}) + |u(t,x)|^3_{[k/2]+2}(|u(t,x)|_k + |u'(t,x)|_{k+1})\}, \tag{2.13}$$

where $u' = (u_t, u_x)$, $[m]$ denotes the largest integer which does not exceed m, and C_k is a positive constant depending on k.

Here we remark that the left-hand sides of (2.10) and (2.12) are concerned with the unit condition introduced by Kosecki ([12]).

We conclude this section with the following decay estimate due to Hörmander [5] (see also Georgiev [3]):

Lemma 2.2. *Let u be a solution to $(\Box + 1)\, u = F$ in $(0,\infty)\times \mathbf{R}$ with initial data 0. Suppose that $\operatorname{supp} F(t,\cdot) \subset \{|x| \leq t+\rho\}$ for any $t \geq 0$ with some positive constant ρ. Then we have*

$$(1+t+|x|)^{1/2}|u(t,x)| \leq C\sum_{j=0}^{\infty}\sum_{|\alpha|\leq 3}\sup_{s\in[0,t]\cap I_j}(1+s)\|Z^\alpha F(s,\cdot)\|_{L^2(\mathbf{R})}, \tag{2.14}$$

where $I_0 = [0,2], I_j = [2^{j-1}, 2^{j+1}]$ $(j \geq 1)$ and C is a positive constant.

3. Proof of Theorem 1.1

Now we are ready to state the proof of Theorem 1.1. Because we have the local existence theorem, what we have to do is to get some *a priori* estimates. Let $u(t,x)$ be a solution to (1.1) for $0 \leq t < T$ with F satisfying (H). We define

$$E_k(T;u) = \sup_{0\leq t<T}\Big\{\sup_{x\in\mathbf{R}}((1+t+|x|)^{1/2}|u(t,x)|_{[k/2]+3}) + \|u(t,\cdot)\|_{k,2} + \|u'(t,\cdot)\|_{k,2} + (1+t)^{-\mu}(\|u(t,\cdot)\|_{k+2,2} + \|u'(t,\cdot)\|_{k+2,2})\Big\}, \tag{3.1}$$

where $u' = (u_t, u_x)$, $0 < \mu < 1/2$ and k is an integer ≥ 15. In order to get the global existence, it suffices to prove

Proposition 3.1. *For any ε (≤ 1) and M (≤ 1), $E_k(T; u) \leq M$ implies*

$$E_k(T; u) \leq C_k(\varepsilon + M^3),$$

where C_k is a positive constant depending on k, but independent of $T > 0$, M (≤ 1) and ε (≤ 1).

Once we get this proposition, if we choose sufficiently small M and ε_0 to satisfy

$$C_k M^2 \leq \frac{1}{4}, \qquad C_k \varepsilon_0 \leq \frac{M}{4} \quad \text{and} \quad E_k(0; u) < M,$$

it follows that $E_k(T; u) \leq M$ implies $E_k(T; u) \leq M/2$ for any $\varepsilon \leq \varepsilon_0$. Then, by the continuity arguments, we can show that $E_k(t; u) \leq M$ holds as long as the solution $u(t, x)$ exists, provided that $\varepsilon \leq \varepsilon_0$. The global existence of the solution follows immediately from this *a priori* estimate and the local existence theorem.

Proof of Proposition 3.1. We assume that $E_k(T; u) \leq M$. From (2.6) we have

$$(\Box + 1) Z^\alpha (u - \phi) = Z^\alpha (N(\lambda) + H(\lambda) - R(\tilde{\lambda})) \tag{3.2}$$

for $0 \leq t < T$ and for any multi-index α, where ϕ and R are as in (2.4) with $b_i = c_i/2$ and in (2.3) respectively. In the following we write C_k for various constants which are independent of T, M and ε, but may change line by line.

Let α be a multi-index with $|\alpha| \leq [k/2] + 3$. Applying Lemma 2.2 to (3.2), we get

$$\begin{aligned} (3.3) \qquad & (1 + t + |x|)^{1/2} |(u - \phi)(t, x)|_{[k/2]+3} \\ & \leq C_k \varepsilon + C_k \sum_{j=0}^{\infty} \sum_{\beta \leq [k/2]+6} \sup_{s \in [0, t] \cap I_j} (1 + s) \{ \|Z^\beta N(\lambda)(s, \cdot)\|_{L^2} \\ & \quad + \|Z^\beta H(\lambda)(s, \cdot)\|_{L^2} + \|Z^\beta R(\tilde{\lambda})(s, \cdot)\|_{L^2} \}. \end{aligned}$$

Observing that $H(\lambda) = O(|\lambda|^4)$, by Lemma 2.1 and the assumption we get

$$\begin{aligned} (3.4) \qquad & \|Z^\beta N(\lambda)(s, \cdot)\|_{L^2} + \|Z^\beta H(\lambda)(s, \cdot)\|_{L^2} \\ & \leq C_k \{ (1 + s)^{-1} \|u(s)\|^2_{[([k/2]+6)/2]+2, \infty} (\|u(s)\|_{[k/2]+7, 2} + \|u'(s)\|_{[k/2]+7, 2}) \\ & \quad + \|u(s)\|^3_{[([k/2]+6)/2]+2, \infty} (\|u(s)\|_{[k/2]+6, 2} + \|u'(s)\|_{[k/2]+7, 2}) \} \\ & \leq C_k (1 + s)^{-3/2} M^3 \quad \text{for } |\beta| \leq [k/2] + 6 \text{ and } 0 \leq s < T. \end{aligned}$$

Here we used $\left[\frac{[k/2]+6}{2}\right] + 2 \leq [k/2] + 3$, $[k/2] + 7 \leq k$ for $k \geq 13$. Since $R(\tilde{\lambda}) =$

$O(|\tilde{\lambda}|^5)$, we have

$$(3.5)\qquad \|Z^\beta R(\tilde{\lambda})\|_{L^2} \le C_k \|u(s)\|^4_{[([k/2]+6)/2]+3,\infty}(\|u(s)\|_{[k/2]+6,2} + \|u'(s)\|_{[k/2]+8,2})$$
$$\le C_k(1+s)^{-2}M^5 \qquad \text{for } |\beta| \le [k/2]+6 \quad \text{and} \quad 0 \le s < T,$$

because $\left[\frac{[k/2]+6}{2}\right] + 3 \le [k/2]+3$ and $[k/2]+8 \le k$ hold for $k \ge 15$. Summing up, we obtain

$$(3.6)\qquad (1+t+|x|)^{1/2}|(u-\phi)(t,x)|_{[k/2]+3}$$
$$\le C_k\left(\varepsilon + \sum_{j=0}^{\infty} \sup_{s\in[0,T]\cap I_j}(1+s)^{-1/2}M^3\right)$$
$$\le C_k\left(\varepsilon + M^3\left(1+\sum_{j=1}^{\infty}(1+2^{j-1})^{-1/2}\right)\right)$$
$$\le C_k(\varepsilon + M^3) \qquad \text{for } 0 \le t < T.$$

Next, let $|\alpha| \le k$ in (3.2). Applying the energy estimate for $(\square + 1)$, we obtain

$$(3.7)\qquad \|(u-\phi)(t,\cdot)\|_{k,2} + \|(u-\phi)'(t,\cdot)\|_{k,2}$$
$$\le C_k\left\{\varepsilon + \int_0^t (\|N(\lambda)\|_{k,2} + \|H(\lambda)\|_{k,2} + \|R(\tilde{\lambda})\|_{k,2})(s,\cdot)ds\right\}.$$

Again from Lemma 2.1, we have

$$(3.8)\qquad \|N(\lambda)(s)\|_{k,2} + \|H(\lambda)(s)\|_{k,2}$$
$$\le C_k(1+s)^{-1}\|u(s)\|^2_{[k/2]+2,\infty}(\|u(s)\|_{k+1,2} + \|u'(s)\|_{k+1,2})$$
$$+ C_k\|u(s)\|^3_{[k/2]+2,\infty}(\|u(s)\|_{k,2} + \|u'(s)\|_{k+1,2})$$
$$\le C_k(1+s)^{\mu-3/2}M^3 \qquad \text{for } 0 \le s < T.$$

Since $R(\tilde{\lambda}) = O(|\tilde{\lambda}|^5)$, Hölder's inequality gives us

$$(3.9)\qquad \|R(\tilde{\lambda})\|_{k,2} \le C_k\|u(s)\|^4_{[k/2]+3,\infty}(\|u(s)\|_{k,2} + \|u'(s)\|_{k+2,2})$$
$$\le C_k(1+s)^{\mu-2}M^5 \qquad \text{for } 0 \le s < T.$$

From (3.7)–(3.9) we obtain

$$(3.10)\qquad \|(u-\phi)(t)\|_{k,2} + \|(u-\phi)'(t)\|_{k,2} \le C_k\left(\varepsilon + M^3\int_0^t (1+s)^{\mu-3/2}\,ds\right)$$
$$\le C_k\left(\varepsilon + M^3\int_0^\infty (1+s)^{\mu-3/2}\,ds\right)$$
$$\le C_k(\varepsilon + M^3) \qquad \text{for } 0 \le t < T,$$

since $\mu - 3/2 < -1$.

Finally, let $|\alpha| \le k+2$. From (1.1), we have

$$(3.11)\qquad (\Box + 1)Z^\alpha u - \frac{\partial F}{\partial u_{tx}}\partial_t\partial_x Z^\alpha u - \frac{\partial F}{\partial u_{xx}}\partial_x^2 Z^\alpha u$$
$$= Z^\alpha F - \frac{\partial F}{\partial u_{tx}}\partial_t\partial_x Z^\alpha u - \frac{\partial F}{\partial u_{xx}}\partial_x^2 Z^\alpha u.$$

From the commutative properties of ∂_t, ∂_x and Z_i $(i = 1,2,3)$, we can estimate the L^2-norm of the right-hand side of (3.11) by

$$C_k\|u(t,\cdot)\|^2_{[(k+2)/2]+2,\infty}(\|u(t,\cdot)\|_{k+2,2} + \|u'(t,\cdot)\|_{k+2,2}).$$

Because $\left[\frac{k+2}{2}\right] + 2 \le [k/2] + 3$, this is bounded by $C_k(1+t)^{\mu-1}M^3$ for $0 \le t < T$. Therefore, applying the energy inequality for the equation of the form

$$(\Box + 1)v - \gamma_1(t,x)\partial_t\partial_x v - \gamma_2(t,x)\partial_x^2 v = \psi(t,x)$$

to (3.11) with $v = Z^\alpha u$, we obtain

$$(3.12)\qquad \|u(t,\cdot)\|_{k+2,2} + \|u'(t,\cdot)\|_{k+2,2} \le C_k\left(\varepsilon + M^3\int_0^t (1+s)^{\mu-1}ds\right)$$
$$\le C_k(1+t)^\mu\left(\varepsilon + \frac{M^3}{\mu}\right)$$

for $0 \le t < T$.

Now, since $|\phi| = O(|u|^3 + |u'|^3)$, Hölder's inequality and Sobolev's embedding theorem imply that

$$(3.13)\qquad |\phi(t,x)|_{[k/2]+3} \le C_k|u(t,x)|^2_{[([k/2]+3)/2]+1}|u(t,x)|_{[k/2]+4}$$
$$\le C_k\|u(t,\cdot)\|^2_{[k/2]+3,\infty}\|u(t,\cdot)\|_{[k/2]+5,2}$$
$$\le C_k(1+t)^{-1}M^3 \qquad \text{for } 0 \le t < T,$$

and

$$(3.14)\qquad \|\phi(t)\|_{k,2} + \|\phi'(t)\|_{k,2} \le C_k\|u(t)\|^2_{[[k/2]+2,\infty}(\|u(t)\|_{k,2} + \|u'(t)\|_{k+1,2})$$
$$\le C_k M^3(1+t)^{\mu-1} \le C_k M^3 \qquad \text{for } 0 \le t < T.$$

Therefore from (3.6), (3.10), (3.12), (3.13) and (3.14), we obtain

$$E_k(T;u) \leq C_k(\varepsilon + M^3). \tag{3.15}$$

This completes the proof of Proposition 3.1.

Now we prove the existence of a free profile. Since the solution $u(t,x)$ satisfies $E_k(\infty;u) \leq M$, we can show as in the proof of Proposition 3.1 that

$$(\Box + 1)(u - \phi) = N(\lambda) + H(\lambda) - R(\tilde{\lambda})$$

and

$$\|(N(\lambda) + H(\lambda) - R(\tilde{\lambda}))(t, \cdot)\|_{H^k(\mathbf{R})} \in L^1(0,\infty). \tag{3.16}$$

Therefore, there exists $(u_{+0}(x), u_{+1}(x)) \in H^{k+1}(\mathbf{R}) \times H^k(\mathbf{R})$ such that

$$\|((u-\phi) - U_+)(t, \cdot)\|_{H^{k+1}} + \|\partial_t((u-\phi) - U_+)(t, \cdot)\|_{H^k} \to 0$$

at $t \to +\infty$, where $U_+(t,x)$ is as in Theorem 1.1. Since we can see from (3.14) that

$$\|\phi(\tilde{\lambda})(t, \cdot)\|_{H^{k+1}} + \|\partial_t(\phi(\tilde{\lambda}))(t, \cdot)\|_{H^k} \leq C_k M^3 (1+t)^{\mu-1} \to 0$$

as $t \to +\infty$, we obtain

$$\|(u - U_+)(t, \cdot)\|_{H^{k+1}} + \|\partial_t(u - U_+)(t, \cdot)\|_{H^k} \to 0$$

as $t \to +\infty$. This completes the proof of Theorem 1.1.

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