J. Math. Kyoto Univ. (JMKYAZ)
39-3 (1999), 529-556

The spectral gap of two dimensional Ising model with a hole: Shrinking effect of contours

By

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Abstract

We consider the spectral gap of the two demensional Ising model on a special graph with some special boundary conditions. The special graph is a finite square with a square hole on its center part, that is, we consider a finite square of side $2L+1$, and we remove another smaller finite square of side $2L_1+1$ $(L_1<L)$ which has the same center as the finite square of side $2L+1$, therefore there is a square hole at the center part of the finite square of side $2L+1$, we denote this special graph by $\Lambda(L,L_1)$. On this graph, the boundary of $\Lambda(L,L_1)$ is composed of "inner boundary" and "outer boundary". We will discuss two different boundary conditions of the Ising model on $\Lambda(L,L_1)$, one is that the outer boundary condition is plus and minus spins' "mixed" boundary condition and the inner boundary condition is plus boundary condition; the other is that the outer boundary condition is plus boundary condition and the inner boundary condition is an arbitrary boundary condition. On above two different boundary conditions, in the absence of an external field and at large inverse temperature β, we will show the upper bound of the spectral gap of Ising model for the first of above boundary conditions, and the lower bound of the spectral gap of Ising model for the second of above boundary conditions. These two results show that if we consider the first of above boundary conditions, and exchange this inner boundary condition with the outer boundary condition of the Ising model on $\Lambda(L,L_1)$, the spectral gap of Ising model will be greately changed. The results can be extended to some other cases, for example, we can consider some other boundary conditions and some other graphs.

1. Introduction

In this paper, we consider the spectral gap of the two dimensional Ising model on some special graph with two different special boundary conditions. This work originates in an attempt to understand relaxation phenomena of the stochastic Ising models on porous media, e.g., the lattice Sierpinski Carpet. The special graph is that, at the center part of the finite square of side $2L+1$, we remove another smaller finite square of side $2L_1+1$ $(L_1<L)$, therefore the original finite square becomes to a finite square with "a square hole" on its center part, defined as $\Lambda(L,L_1)$ (see(2.21)). There are two boundaries on this special graph $\Lambda(L,L_1)$. We call that the boundary of the inner "hole" is the "*inner boundary*" of $\Lambda(L,L_1)$ (see (2.22)),

Communicated by Prof. S.Watanabe, March 12, 1999

and the outside boundary (that is the boundary of the finite square of side $2L+1$) is the "*outer boundary*" of $\Lambda(L, L_1)$ (see (2.23)). Together with the inner boundary and the outer boundary is called the "*boundary*" of $\Lambda(L, L_1)$, that is, the "*boundary*" of $\Lambda(L, L_1)$ contains two parts, one is the inner boundary, the other is the outer boundary. Under two different "mixed" boundary conditions, we will show some estimations of the spectral gap of the Ising model on $\Lambda(L, L_1)$.

In the paper [3], in the absence of an external field and at large inverse temperature β, Higuchi and Yoshida gave an upper bound of the spectral gap for the two dimensional stochastic Ising model with a general "mixed" boundary condition on a finite square. In this paper, in $\Lambda(L, L_1)$ first we consider *Boundary Conditions of Case* 1: the inner boundary condition is "plus" boundary condition, and outer boundary condition is a special "mixed" boundary condition, that is, the plus and minus spins are arranged alternately' on the outer boundary. In other words, if we give clockwise order to the sites $\{x_1, x_2, \cdots,\}$ of the outer boundary of $\Lambda(L, L_1)$, and first put a plus spin on the site x_1, second put a minus spin on the site x_2, next put a plus spin on the site $x_3, \cdots$. Thus, we have a special "mixed" outer boundary condition, which is denote by τ. With this boundary condition τ, we can get (when β is large enough) the same upper bound of the spectral gap of the Ising model as in the paper [3], that is Theorem 1, for β large enough, there are $c(\beta)>0$ and $C>0$, we have

$$\text{gap}(\Lambda(L, L_1);\ \beta, \tau) \leq c(\beta)\exp\{-C\beta L\} \tag{1.1}$$

where $L_1 = [aL]-1$ and the constant $a<\frac{1}{7}$.

In the paper [5], in the absence of an external field and at large inverse temperature β, Martinelli gave out the lower bound of the spectral gap for the two dimensional stochastic Ising model with the "plus" boundary condition on a finite square. Now, in $\Lambda(L, L_1)$ we consider *Boundary Conditions of Case* 2: the outer boundary is "plus" boundary condition and the inner boundary is an arbitrary boundary condition. This boundary condition is also denote by τ in this paper. With this boundary condition τ, we can get (when β is large enough) the same lower bound of the spectral gap of the Ising model as in paper [5], that is Theorem 2, for β large enough, there are $\epsilon \in (0, \frac{1}{2})$ and $C>0$, we have

$$\text{gap}(\Lambda(L, L_1); \beta, \tau) \geq \exp\{-C\beta L^{\frac{1}{2}+\varepsilon}\}. \tag{1.2}$$

From above two different boundary conditions, we can see an interesting result about the spectral gap of Ising model on $\Lambda(L, L_1)$, for the *Boundary Conditions of Case* 1, when we exchange inner boundary condition with outer boundary condition, the gap is greatly changed. Intuitively, this difference can be explained as follows. From the boundary condition, these gaps are determined by the time to get to the equilibrium configuration (+phase) when the process starts from the configuration in which all spins are -1. If the boundary condition is of Case 2, then a large contour appears along the outer boundary and it starts to shrink to decrease the energy cost. When the contour shrinks to the inner boundary, the system reaches to the equilibrium. So, in this case the time evolution always goes in the direction of decreasing the energy cost. On the other hand, when the boundary condition is of Case 1, this large contour appears along the inner boundary. In order to get to the equilibrium, this contour should expand to touch the whole outer boundary, and this costs more and more energy. The system has to go through the "bottle neck", and this gives a similar estimate of spectral gap as the free boundary case.

We can consider the upper bound of the spectral gap of Ising model on $\Lambda(L, L_1)$ with another boundary condition, which is that: the outer boundary condition is the same outer boundary condition as in *Boundary Conditions of Case* 1, and the inner boundary condition is a "mixed" boundary condition. In this case, the constant 'a' in (1.1) will be changed. We also can consider the lower bound of the spectral gap of Ising model on the lattice Sierpinski carpet with the plus boundary condition, etc..

2. Notations and definitions

2.1. General definitions. Let $\boldsymbol{Z}^2$ be the usual two dimensional square lattice with sites $x=(x_1, x_2)$, equipped with the l_1-*norm*: $\|x\| \equiv \|x\|_1 = |x_1| + |x_2|$ and l_∞-*norm*: $\|x\|_\infty = \max\{|x_1|, |x_2|\}$. A set $\Lambda \subset \boldsymbol{Z}^2$ is said to be $\|\cdot\|_p$-*connected* ($p=1$ or ∞) if for each distinct $x, y \in \Lambda$, we can find some $\{x_0, \cdots, x_n\} \subset \Lambda$ with $x_0 = x, x_n = y$ and $\|x_j - x_{j-1}\|_p = 1$ $(j=1, \cdots, n)$.

Given $\Lambda \subset \boldsymbol{Z}^2$, we define the *interior* and *exterior* boundaries of Λ as:

$$\partial_{int}\Lambda \equiv \{x \in \Lambda : \exists y \notin \Lambda,\ \|x-y\| = 1\} \tag{2.1}$$

$$\partial_{ext}\Lambda \equiv \{x \notin \Lambda : \exists y \in \Lambda,\ \|x-y\| = 1\} \tag{2.2}$$

and the edge boundary $\partial\Lambda$ as:

$$\partial\Lambda = \{\{x, y\} : x \in \partial_{int}\Lambda,\ y \in \partial_{ext}\Lambda,\ \|x-y\| = 1\}. \tag{2.3}$$

We also denote by $|\Lambda|$ the cardinality of Λ.
The set $\boldsymbol{B}$ of bonds in $\boldsymbol{Z}^2$ is defined by

$$\boldsymbol{B} = \{\{x, y\} \subset \boldsymbol{Z}^2 : \|x-y\| = 1\}, \tag{2.4}$$

for a set Λ, we also define

$$\boldsymbol{B}_\Lambda = \{\{x, y\} \subset \boldsymbol{B} : (x, y) \in \Lambda \times \Lambda\}. \tag{2.5}$$

2.2. The configurations and the Gibbs states. We consider the standard two dimensional Ising model with *configuration space* $\Omega = \{-1, +1\}^{\boldsymbol{Z}^2}$, $\Omega_\Lambda = \{-1, +1\}^\Lambda$ for $\Lambda \subset \boldsymbol{Z}^2$. An element of $\Omega = \{-1, +1\}^{\boldsymbol{Z}^2}$ will usually be denoted by σ, and we use the notation $\sigma_\Lambda = \{\sigma(x), x \in \Lambda\}$ for an element of Ω_Λ. Whenever confusion does not arise we will also omit the subscript Λ in the notation σ_Λ.

Given a finite set $\Lambda \subset \boldsymbol{Z}^2$ and a *boundary condition* (b.c.) $\tau \in \Omega = \{-1, +1\}^{\boldsymbol{Z}^2}$, we consider the Hamiltonian

$$H^\tau_\Lambda(\sigma) = -\frac{1}{2} \sum_{x,y\in\Lambda,\,\|x-y\|=1} (\sigma(x)\sigma(y)-1) - \sum_{(x,y)\in\partial\Lambda} (\sigma(x)\tau(y)-1). \tag{2.6}$$

The partition function is given by

$$Z^{\beta,\tau}(\Lambda) = \sum_{\sigma\in\Omega_\Lambda} \exp[-\beta H^\tau_\Lambda(\sigma)]. \tag{2.7}$$

The Gibbs measure associated with the Hamiltonian H^τ_Λ is defined as

$$\mu^{\beta,\tau}_\Lambda(\sigma) = Z^{\beta,\tau}(\Lambda)^{-1}\exp[-\beta H^\tau_\Lambda(\sigma)], \tag{2.8}$$

where $\beta > 0$ is a parameter.

The expectation with respect to the Gibbs measure $\mu^{\beta,\tau}_\Lambda$ is denoted by $E^{\beta,\tau}_\Lambda(\cdot)$. The set of measures satisfies the DLR compatibility condition, for any two finite subsets $V \subset \Lambda \subset Z^2$

$$\mu^{\beta,\tau}_\Lambda(\sigma) = \sum_{\sigma'\in\Omega_\Lambda} \mu^{\beta,\tau}_\Lambda(\sigma')\mu^{\beta,\sigma'}_V(\sigma). \tag{2.9}$$

We introduce a partial order on Ω_Λ by saying that $\sigma \le \sigma'$ iff $\sigma(x) \le \sigma'(x)$ for all $x \in \Lambda$. A function $f: \Omega_\Lambda \mapsto R$ is called *monotone increasing* (*decreasing*) if $\sigma \le \sigma'$ implies $f(\sigma) \le f(\sigma')(f(\sigma) \ge f(\sigma'))$. An event is called *positive* (*negative*) if its characteristic function is increasing (decreasing). Given two probability measures μ, μ' on Ω_Λ, we write $\mu \le \mu'$ if $\mu(f) \le \mu'(f)$ for all increasing functions f [by $\mu(f)$ we denote the expectation with respect to the probability measure μ].

In the following sections, we will use the FKG inequalities, which state that:

(1) If $\tau \le \tau'$, then $\mu^{\beta,\tau}_\Lambda \le \mu^{\beta,\tau'}_\Lambda$.

(2) If f and g are increasing, then $E^{\beta,\tau}_\Lambda(fg) \ge E^{\beta,\tau}_\Lambda(f)E^{\beta,\tau}_\Lambda(g)$. (2.10)

2.3. Stochastic Ising model. Now we introduce the stochastic Ising model, we will give brief definitions and notations, for the detials see in [4] or [2]. The stochastic dynamics that we want to study is defined by the Markov generator

$$(L^{\beta,\tau}_\Lambda)(\sigma) = \sum_{x\in\Lambda} c(x,\sigma)[f(\sigma^x) - f(\sigma)] \tag{2.11}$$

acting on $L^2(\Omega, d\mu^{\beta,\tau}_\Lambda)$, where

$$\sigma^x(y) = \begin{cases} \sigma(y), & \text{if } y \ne x \\ -\sigma(y), & \text{if } y = x \end{cases}.$$

The nonnegative real quantities

$$\{c(x,\sigma); x\in Z^2,\ \sigma\in\Omega\} \tag{2.12}$$

are the *transition* rates for the process.

We suppose the transition rates satisfy the following properties:

(1) Nearest neighbor interactions: If $\sigma(y)=\sigma'(y)$ for all y such that $d(x,y)\leq 1$, then $c(x,\sigma)=c(x,\sigma')$.

(2) Attractivity: If $\sigma\leq\sigma'$ and $\sigma(x)=\sigma'(x)$, then

$$\sigma(x)c(x,\sigma)\geq\sigma'(x)c(x,\sigma'). \tag{2.13}$$

(3) Detailed balance:

$$c(x,\sigma)\mu_\Lambda^{\beta,\tau}(\sigma)=c(x,\sigma^x)\mu_\Lambda^{\beta,\tau}(\sigma^x). \tag{2.14}$$

(4) Positivity and boundedness: There exist $c_m(\beta)$ and $c_M(\beta)$ such that

$$0<c_m(\beta)\leq\inf_{x,\sigma}c(x,\sigma)\leq\sup_{x,\sigma}c(x,\sigma)\leq c_M(\beta)<\infty \tag{2.15}$$

(4) guarantees that there exists a unique Markov process, and (3) implies that $\mu_\Lambda^{\beta,\tau}$ is reversible with respect to the process. (2) is essential for the coupling of Markov processes with different bouondary conditions. Finally, we define the spectral gap of the generator

$$\mathrm{gap}(\Lambda;\beta,\tau)=\mathrm{gap}(L_\Lambda^{\beta,\tau})=\inf_{f\in L^2(\Omega,d\mu_\Lambda^{\beta,\tau})}\frac{\mathscr{E}_\Lambda^{\beta,\tau}(f,f)}{\mathrm{Var}_\Lambda^{\beta,\tau}(f)}, \tag{2.16}$$

where $\mathscr{E}_\Lambda^{\beta,\tau}(f,f)$ is the Dirichlet form associated with the generator $L_\Lambda^{\beta,\tau}$

$$\mathscr{E}_\Lambda^{\beta,\tau}(f,f)=\frac{1}{2}\sum_{\sigma\in\Omega_\Lambda}\sum_{x\in\Lambda}\mu_\Lambda^{\beta,\tau}(\sigma)c(x,\sigma)[f(\sigma^x)-f(\sigma)]^2 \tag{2.17}$$

and $\mathrm{Var}_\Lambda^{\beta,\tau}(f)$ is the variance relative to the probability measure $\mu_\Lambda^{\beta,\tau}$

$$\mathrm{Var}_\Lambda^{\beta,\tau}(f)=\frac{1}{2}\sum_{\sigma,\eta\in\Omega_\Lambda}\mu_\Lambda^{\beta,\tau}(\sigma)\mu_\Lambda^{\beta,\tau}(\eta)[f(\sigma)-f(\eta)]^2. \tag{2.18}$$

Now we give the definition of the special graph $\Lambda(L,L_1)$. Let L, L_1 be any integers (L_1 can depend on L, i.e., $L_1=L_1(L)$), such that $L-L_1>2$, let $\Lambda(L)$ and $\Lambda(L_1)$ be

$$\Lambda(L)=\{(x_1,x_2): -L\leq x_1\leq L,\ \ -L\leq x_2\leq L\}. \tag{2.19}$$

$$\Lambda(L_1)=\{(x_1,x_2): -L_1\leq x_1\leq L_1,\ \ -L_1\leq x_2\leq L_1\}. \tag{2.20}$$

We define $\Lambda(L, L_1)$ as

$$\Lambda(L, L_1) = \Lambda(L) \backslash \Lambda(L_1). \tag{2.21}$$

Let $\partial_{inner}\Lambda(L, L_1)$ denote the inner boundary of $\Lambda(L, L_1)$ and $\partial_{outer}\Lambda(L, L_1)$ denote the outer boundary of $\Lambda(L, L_1)$, which are defined as in the following:

$$\partial_{inner}\Lambda(L, L_1) \equiv \{x \in \Lambda(L_1) : \exists y \in \Lambda(L, L_1), \quad \|x-y\| = 1\} \tag{2.22}$$

$$\partial_{outer}\Lambda(L, L_1) \equiv \{x \notin \Lambda(L) : \exists y \in \Lambda(L, L_1), \quad \|x-y\| = 1\} \tag{2.23}$$

Suppose τ is the boundary condition of $\Lambda(L, L_1)$, by the definitions in Section 2.2 and Section 2.3, we can similarly define $H^{\tau}_{\Lambda(L,L_1)}(\sigma)$ and $\text{gap}(\Lambda(L, L_1);\ \beta, \tau)$.

3. The upper bound of the spectral gap of Ising model with the special boundary conditions of case 1

In this section, we will consider the Ising model on $\Lambda(L, L_1)$. $\Lambda(L, L_1)$ is not a simply connected graph, it has a square hole at the center part of $\Lambda(L)$, so we will consider a special boundary condition τ.

$$\textit{Boundary Conditions of Case } 1. \tag{3.1}$$

Inner boundary condition: $\tau(x) = +1$; *for all* $x \in \partial_{inner}\Lambda(L, L_1)$.

Outer boundary condition: *Starting from* $(L+1, 0)$, *we give clockwise order to the points in* $\partial_{outer}\Lambda(L, L_1)$, *that is* $\{x_i\}$, *and set* $\tau(x_i) = (-1)^i$ *according to this order.*

Now we give the main rersults of this section:

Theoreem 1. *Let* $d=2$, *and the boundary condition* τ *be given in* (3.1), *where* $L_1 = [aL] - 1$ *and* $0 < a < \frac{1}{7}$. *There exist* $\beta_0 > 0$, $C > 0$ *and* $\{c(\beta) > 0 : \beta \geq \beta_0\}$ *such that for any* $\beta \geq \beta_0$ *and* $L \geq 1$,

$$\text{gap}(\Lambda(L, L_1); \beta, \tau) \leq c(\beta)\exp\{-C\beta L\}. \tag{3.2}$$

Remark. Comparing this Theorem 1 with the Theorem in [3], for the graphs $\Lambda(L, L_1)$ and $\Lambda(L)$, and for the same "mixed outer boundary condition", we can get the same upper bound of the spectral gaps. This means that, under this circumstances, when β is large enough, for the graph $\Lambda(L, L_1)$ the inner plus boundary condition of $\Lambda(L, L_1)$ doesn't change this upper bound $(c(\beta)\exp\{-C\beta L\})$ of the spectral gap.

3.1. Definition of contour. Before we prove Theorem 1, we will give some definitions and some Lemmas. For Z^d, the number of bonds contained in a set $\gamma \subset \boldsymbol{B}$ will be denoted by $|\gamma|$. For a bond $b = \{x, y\}$, consider a unit $(d-1)$-cell $\tilde{b} = Q(x) \cap Q(y)$, where $Q(x) = \prod_{j=1}^{d}[x_j - \frac{1}{2}, x_j + \frac{1}{2}] \subset \boldsymbol{R}^d$. Two bonds b_1 and b_1 are said to be adjacent if $\tilde{b}_1$ and $\tilde{b}_2$ have a $(d-$ 2)-cell in common. It follows that any bond b has $6(d-1)$ bonds as its adjacent neighbours. A set $\gamma \subset \boldsymbol{B}$ is said to be *connected* if for each distinct $b, b' \in \gamma$, we can find some $\{b_0, \cdots, b_n\} \subset \gamma$ with $b_0 = b, \cdots, b_n = b'$ such that b_{j-1} and b_j are adjacent for $j = 1, \cdots, n$. Next we explain the definition of contours used in this section. We will follow the definitions in [3], and restrict our attention to the case that $d=2$. A contuor is a finite subset $\gamma \subset \boldsymbol{B}$ with the following properties: there exists a finite subest $\Theta \subset \boldsymbol{Z}^2$ such that:

(i) Both Θ and Θ^c is l_1-connected,

(ii) $\gamma = \partial\Theta$. (3.3)

The set Θ is uniquely determined by γ and hence denoted by $\Theta(\gamma)$. Since a contour γ is a set of connected bonds, it follows that for each $b \in \boldsymbol{B}$ and $m = 1, 2, \cdots$,

$$\sharp\{\gamma\text{: contour with } |\gamma| = m \text{ and } b \in \gamma\} \leq 3^{m-1}. \tag{3.4}$$

If a contouor γ is a subset of $\boldsymbol{B}_{\Lambda(L,L_1)} \cup \partial\Lambda(L, L_1)$ for some $\Lambda(L, L_1)$, we say that γ is a contour in $\Lambda(L, L_1)$. For $\sigma \in \Omega_{\Lambda(L,L_1)}$, let $\epsilon = +$ or $-$, a contour γ is said to be an (ϵ)-contour in $\Lambda(L, L_1)$ at σ if it satisfies:

$$\partial_{int}\Theta(\gamma) \subset \{x \in \Lambda(L, L_1) : \sigma(x) = \epsilon 1\} \text{ and,}$$
$$\partial_{ext}\Theta(\gamma) \subset \{x \in \Lambda(L, L_1) : \sigma(x) = -\epsilon 1\} \cup \Lambda(L, L_1)^c. \tag{3.5}$$

Suppose that γ is an ϵ-contour in $\Lambda(L, L_1)$ at $\sigma \in \Omega_{\Lambda(L,L_1)}$, let S_1, S_2, S_3, S_4 be the four sides of $\partial_{int}\Lambda(L)$. Consider the case that γ doesn't intersect with all sides $\partial_{int}\Lambda(L)$, of and let $\{S'_i\}$ be let be the sides which do not intersect with γ. In this case, there must be a connected component of $(\boldsymbol{B}_{\Lambda(L,L_1)} \cup \partial\Lambda(L_1)) \cap \gamma$ which divides $\Lambda(L, L_1)$ into two connected components, one of which contains $\Theta(\gamma)$ and the other contains $\{S'_i\}$. Since the two connected components are uniquely determined by the properties alluded above, we denote the former component by $\tilde{\Theta}(\gamma)$. We decompose the sets γ and $\bar{\gamma} \overset{\text{def}}{=} \partial\Lambda(L) \cap \partial\tilde{\Theta}(\gamma)$ as follows;

$$\gamma = (\cup_{i \geq 1}\rho_i) \cup (\cup_{i \geq 1}\rho'_i) \cup (\cup_{i \geq 0}\lambda_i) \tag{3.6}$$
$$\bar{\gamma} = (\cup_{i \geq 1}\rho_i) \cup (\cup_{i \geq 1}\delta_i) \tag{3.7}$$

where $\{\rho_i\}$ are connected components of $\gamma \cap \partial\Lambda(L)$, $\{\rho'_i\}$ are connected components of $\gamma \cap \partial\Lambda(L_1)$, and $\{\delta_i\}$ are connected components of $\bar{\gamma} \backslash \gamma$. We let $\{\lambda_0, \lambda_1, \cdots\}$ denote the set of connected components of $\boldsymbol{B}_{\Lambda(L,L_1)} \cap \gamma$.

3.2. Lemmas. *Next we will show some lemmas:*

Lemma 1. *Suppose $d=2$, and γ is an (ϵ)-contour in $\Lambda(L, L_1)$ at $\sigma \in \Omega_{\Lambda(L,L_1)}$, if the contour γ doesn't intersect with the sides of $\partial_{ext}\Lambda(L_1)$, and if it does not surround the inner square hole, then we have*

$$H^{\tau}_{\Lambda(L,L_1)}(\sigma)-H^{\tau}_{\Lambda(L,L_1)}(T_\gamma\sigma)\geq\frac{2}{5}|\gamma|-2, \tag{3.8}$$

where $T_\gamma\sigma\in\Omega_{\Lambda(L,L_1)}$ *is defined by*

$$T_\gamma\sigma(x)=\begin{cases}-\sigma(x), & \text{if } x\in\Theta(\gamma)\cap\Lambda(L,L_1)\\ \sigma(x), & \text{if } x\in\Lambda(L,L_1)\backslash\Theta(\gamma)\end{cases}. \tag{3.9}$$

Proof. Since the contour γ doesn't surround the inner square hole and doesn't intersect the sides of $\partial_{ext}\Lambda(L_1)$, we can use almost the same method of the proof in the Lemma 3.1 of [3] to prove Lemma 1. Since the proof is not very long, and its method is important for this paper, we will show the proof. By the definitions in (3.6) and (3.7), we have following relations;

$$H^{\tau}_{\Lambda(L,L_1)}(\sigma)-H^{\tau}_{\Lambda(L,L_1)}(T_\gamma\sigma)\geq 2\sum_{i\geq 0}|\lambda_i|-2|\sum_{i\geq 1}\sum_{\substack{y\in\partial_{ext}\Lambda(L)\\ \{x,y\}\in\rho_i}}\tau(y)| \tag{3.10}$$

$$|\bar{\gamma}|=\sum_{i\geq 1}|\rho_i|+\sum_{i\geq 1}|\delta_i| \tag{3.11}$$

$$|\gamma|=\sum_{i\geq 1}|\rho_i|+\sum_{i\geq 0}|\lambda_i| \tag{3.12}$$

by geometric consideration,

$$|\bar{\gamma}|\leq\frac{4}{5}|\gamma|. \tag{3.13}$$

By (3.10) – (3.13),

$$|\sum_{i\geq 1}\sum_{\substack{y\in\partial_{ext}\Lambda(L)\\ \{x,y\}\in\rho_i}}\tau(y)|=|\sum_{\substack{y\in\partial_{ext}\Lambda(L)\\ \{x,y\}\in\bar{\gamma}}}\tau(y)-\sum_{i\geq 1}\sum_{\substack{y\in\partial_{ext}\Lambda(L)\\ \{x,y\}\in\delta_i}}\tau(y)|$$

$$\leq 1+\sum_{i\geq 1}|\delta_i|=1+|\bar{\gamma}|-\sum_{i\geq 1}|\rho_i|$$

$$=1+|\bar{\gamma}|-|\gamma|+\sum_{i\geq 0}|\lambda_i|$$

$$\leq 1-\frac{1}{5}|\gamma|+\sum_{i\geq 0}|\lambda_i|. \tag{3.14}$$

Note that in the proof, by the definition of τ, we use the fact that:

$$|\sum_{\substack{y\in\partial_{ext}\Lambda(L)\\ \{x,y\}\in\gamma}}\tau(y)|\leq 1.$$

By (3.10) and (3.14),

$$H^{\tau}_{\Lambda(L,L_1)}(\sigma)-H^{\tau}_{\Lambda(L,L_1)}(T_\gamma\sigma)\geq\frac{2}{5}|\gamma|-2. \tag{3.15}$$

Now we finished the proof of Lemma 1.

Lemma 2. *Suppose that L_1 satisfies $L-L_1\geq\log L$, then,*

$$\lim_{\beta\to\infty}\sup\mu^{\tau}_{\Lambda(L,L_1)}\left\{\sigma: \begin{array}{l}\text{there is a contour } \gamma \text{ in } \Lambda(L,L_1) \text{ at } \sigma\text{, which}\\ \text{intersects with sides of } \partial_{int}\Lambda(L) \text{ and } \partial_{ext}\Lambda(L_1)\end{array}\right\}=0. \tag{3.16}$$

Proof. If a contour intersects with sides of $\partial_{int}\Lambda(L)$ and $\partial_{ext}\Lambda(L_1)$, whatever it is (+)-contour or (−)-contour, there is a chain of (−)-contours $\{\gamma_i\}_{i=1}^k$ such that $\{\Theta(\gamma_i)\}_{i=1}^k$ disjoint, and $\gamma_i\cap\gamma_j\neq\emptyset$. γ_i and γ_j touch at one point, and $\cup\gamma_i$ intersects with sides of $\partial_{int}\Lambda(L)$ and $\partial_{ext}\Lambda(L_1)$, so the length of this $\cup\gamma_i$ is at least 2 $\log L$, and each contour of $\{\gamma_i\}_{i=1}^k$ doesn't surround the inner hole. So we can change inside spins of $\cup\gamma_i$, that is, let $T_{\cup\gamma_i}\sigma=\cup(T_{\gamma_i}\sigma)$. By the same argument of (3.10)–(3.15) in the proof of Lemma 1 (note that here we use the condition that the inner boundary condition is plus boundary condition) and the property of $\cup\gamma_i$-contour,

$$H^{\tau}_{\Lambda(L,L_1)}(\sigma)-H^{\tau}_{\Lambda(L,L_1)}(T_{\cup\gamma_i}\sigma)\geq\frac{2}{5}|\cup\gamma_i|-2. \tag{3.17}$$

If β is large enough,

$$\mu^{\tau}_{\Lambda(L,L_1)}\left\{\sigma: \begin{array}{l}\text{there is a contour } \gamma \text{ in } \Lambda(L,L_1) \text{ at } \sigma\text{, which}\\ \text{intersects with sides of } \partial_{int}\Lambda(L) \text{ and } \partial_{ext}\Lambda(L_1)\end{array}\right\}$$

$$\leq\mu^{\tau}_{\Lambda(L,L_1)}\left\{\sigma: \begin{array}{l}\text{there is a chain of (-)-contous } \{\gamma_i\}_{i=1}^k \text{ in } \Lambda(L,L_1)\\ \text{at } \sigma,\ \cup\gamma_i \text{ intersects with sides of } \partial_{int}\Lambda(L) \text{ and } \partial_{ext}\Lambda(L_1)\end{array}\right\}$$

$$\leq\sum_{m\geq 2\log L}4(2L+1)^2 3^{m-1}\exp\{-\beta(\tfrac{2}{5}m-2)\}\overset{\beta\to\infty}{\to}0,$$

so we complete the proof.

For $L\geq 7$, and $\epsilon=+$ or $-$, let $L_1=[aL]-1$, where $a<\frac{1}{7}$, we set

$$\Gamma_{L,L_1,\epsilon}=\left\{\sigma\in\Omega_{\Lambda(L,L_1)}: \begin{array}{l}\text{there is an } (\epsilon)\text{-contour } \gamma \text{ in } \Lambda(L,L_1)\\ \text{at } \sigma \text{ with } |\gamma|\geq 2L+2aL\end{array}\right\}. \tag{3.18}$$

Then,

Lemma 3. *Let* $\Gamma_{L,L_1,+}$, $\Gamma_{L,L_1,-}$ *be given in* (3.18) (*when* $\epsilon = +$ *and* $-$), *then*

$$\lim_{\beta\to\infty} \inf\{\mu^{\tau}_{\Lambda(L,L_1)}(\Gamma_{L,L_1,+}) + \mu^{\tau}_{\Lambda(L,L_1)}(\Gamma^{\tau}_{L,L_1,-}) : L \geq 7\} \geq 1. \tag{3.19}$$

Proof. Note that, we choose some special $x_0 \in \partial_{int}\Lambda(L)$

$$\sum_{\varepsilon=\pm} \mu^{\tau}_{\Lambda(L,L_1)}(\Gamma_{L,L_1,\varepsilon}) \geq \sum_{\varepsilon=\pm} \mu^{\tau}_{\Lambda(L,L_1)}(\{\sigma(x_0)=\epsilon 1\} \cap \Gamma_{L,L_1,\varepsilon})$$

$$= \sum_{\varepsilon=\pm} \{\mu^{\tau}_{\Lambda(L,L_1)}(\{\sigma(x_0)=\epsilon 1\}) - \mu^{\tau}_{\Lambda(L,L_1)}(\{\sigma(x_0)=\epsilon 1\} \cap \Gamma^{c}_{L,L_1,\varepsilon})\}$$

$$\geq 1 - \sum_{\varepsilon=\pm} \delta(\beta,\epsilon) \tag{3.20}$$

where

$$\delta(\beta,\epsilon) = \sup_{L,L_1} \mu^{\tau}_{\Lambda(L,L_1)}(\{\sigma(x_0)=\epsilon 1\} \cap \Gamma^{c}_{L,L_1,\varepsilon}). \tag{3.21}$$

(1) First, let $\epsilon = +$. For $\sigma \in \{\sigma(x_0) = +1\} \cap \Gamma^{c}_{L,L_1,+}$ and $x_0 \in \partial_{int}\Lambda(L)$, and by the definition of $\Gamma^{c}_{L,L_1,+}$, there is a $(+)$-contour at σ such that $x_0 \in \Theta(\gamma)$ with $|\gamma| < 2L + 2aL$.

By the choice of x_0 and $|\gamma| < 2L + 2aL$, γ cannot surround the hole, but γ may intersect with the sides of $\partial_{ext}\Lambda(L_1)$, so we consider this problem in two steps: (1)-(a). Suppose that this $(+)$-contour γ doesn't intersect with the sides of $\partial_{ext}\Lambda(L_1)$.

Since γ doesn't surround the hole and $|\gamma| < 2L + 2aL$, then γ intersects with no pair of opposite sides of $\partial_{int}\Lambda(L)$, now (3.13) is changed to be

$$|\bar{\gamma}| \leq \frac{1}{2}|\gamma|.$$

Following the proof of Lemma 1, we have

$$H^{\tau}_{\Lambda(L,L_1)}(\sigma) - H^{\tau}_{\Lambda(L,L_1)}(T_\gamma\sigma) \geq |\gamma| - 2$$

and thus,

$$\mu^{\tau}_{\Lambda(L,L_1)}\ (\gamma \text{ appears}) \leq \exp\{-\beta(|\gamma|-2)\}.$$

Then

$$\mu^{\tau}_{\Lambda(L,L_1)}\ (\text{there is a contour which satisfyies condition (1)-(a)})$$

$$\leq \sum_{\substack{\gamma:\text{contour in}\\ \Lambda((L,L_1),x_0\in\Theta(\gamma)}} \mu^{\tau}_{\Lambda(L,L_1)}(\gamma \text{ appears})$$

$$\leq \sum_{m\geq 4} 4m3^{m-1}\exp\{-\beta(m-2)\}. \tag{3.22}$$

(1)-(b). Suppose that this (+)-contour γ intersects with the sides of $\partial_{ext}\Lambda(L_1)$.

We have known that γ doesn't surround the hole. Because of the special chosen x_0, we see that γ intersects with the sides of $\partial_{int}\Lambda(L)$ and $\partial_{ext}\Lambda(L_1)$, and since γ doesn't surround the hole, we can find a chain of $(-)$-contours $\{\gamma_i\}_{i=1}^k$ such that $\{\Theta(\gamma_i)\}_{i=1}^k$ are disjointed, and $\gamma_i \cap \gamma_j \neq \emptyset$. γ_i and γ_j touch at one point, and $\cup\gamma_i$ intersects with the sides of $\partial_{int}\Lambda(L)$ and $\partial_{ext}\Lambda(L_1)$, by the same argument as in Lemma 2,

$$\mu^{\tau}_{\Lambda(L,L_1)} \text{ (there is a contour which satisfies condition (1)-(b))}$$

$$\leq \sum_{m \geq 2\log L} 4(2L+1)^2 3^{m-1} \exp\{-\beta(\frac{2}{5}m-2)\}. \tag{3.23}$$

Combining above (a) and (b), we obtain

$$\delta(\beta, +) = \sup_{L,L_1} \mu^{\tau}_{\Lambda(L,L_1)}(\{\sigma(x_0) = +1\} \cap \Gamma^c_{L,L_1,+})$$

$$\leq \sum_{m \geq 4} 4m3^{m-1} \exp\{-\beta(\frac{2}{5}m-2)\} + \sum_{m \geq 2\log L} 4(2L+1)^2 3^{m-1} \exp\{-\beta(\frac{2}{5}m-2)\} \overset{\beta\to\infty}{\to} 0. \tag{3.24}$$

(2) Second, for $\epsilon = -$, by a similar argument as above (in fact this case is easier),

$$\mu^{\tau}_{\Lambda(L,L_1)}(\{\sigma(x_0) = -1\} \cap \Gamma^c_{L,L_1,-}) \overset{\beta\to\infty}{\to} 0. \tag{3.25}$$

By (3.24) and (3.25), which imply that $\delta(\beta,\epsilon) \overset{\beta\to\infty}{\to} 0$, so we finish the proof of the Lemma 3.

By Lemma 3, there exist a $\beta_1 > 0$ and $\epsilon = \epsilon(L, L_1, \tau)$, such that

$$\inf\{\mu^{\tau}_{\Lambda(L,L_1)}(\Gamma_{L,L_1,\epsilon(L,L_1,\tau)}):\ \beta \geq \beta_1,\ \ L \geq 7\} \geq \frac{1}{3}. \tag{3.26}$$

We now set

$$\Gamma_{L,L_1} = \Gamma_{L,L_1,\epsilon(L,L_1,\tau)}. \tag{3.27}$$

For $\sigma \in \Omega_{\Lambda(L,L_1)}$, let

$$\bar{C}^{\tau}_{L,L_1}(\sigma) = \{\gamma : (\epsilon(L,L_1,\tau))\text{-contour in } \Lambda(L,L_1) \text{ at } \sigma,\ |\gamma| \geq 2L + 2aL\}. \tag{3.28}$$

Finally, for a contour γ, we define

$$\Gamma_{L,L_1}(\gamma)=\left\{\sigma\in\Gamma_{L,L_1}: \begin{array}{l}\gamma \text{ appears as the maximal element of } \bar{C}^{\tau}_{L,L_1}(\sigma) \text{ in}\\ \text{the sense that if } \lambda(\neq\gamma)\in\bar{C}^{\tau}_{L,L_1}(\sigma), \text{ then } \lambda\subset\Theta(\gamma)\end{array}\right\} \tag{3.29}$$

Lemma 4. *Let* $T_\gamma\Gamma_{L,L_1}(\gamma)=\{T_\gamma(\sigma):\sigma\in\Gamma_{L,L_1}(\gamma)\}$, *where* $L_1=[aL]-1$ *and* $a<\frac{1}{7}$ *then*

$$\lim_{\beta\to\infty}\sup\left\{\mu^{\tau}_{\Lambda(L,L_1)}(\Gamma_{L,L_1}\mid T_\gamma\Gamma_{L,L_1}(\gamma)): \begin{array}{c}\gamma \text{ is a contour in } \Lambda(L,L_1),\\ |\gamma|\geq 2L+2aL\end{array}\right\}=0, \tag{3.30}$$

where $T_\gamma(\sigma)$ *is defined in* (3.9) *of Lemma* 1 *and* Γ_{L,L_1} *is defined in* (3.27).

Proof. Note that

$$\begin{aligned}\Gamma_{L,L_1}\cap T_\gamma\Gamma_{L,L_1}(\gamma)&\subset\{T_\gamma(\sigma):\sigma\in\Gamma_{L,L_1}(\gamma),\ T_\gamma(\sigma)\in\Gamma_{L,L_1}\}\\ &\subset\left\{T_\gamma(\sigma): \begin{array}{l}\text{there is an } \epsilon(L,L_1,\tau)\text{-contour } \gamma' \text{ in } \Lambda(L,L_1) \text{ at}\\ T_\gamma(\sigma) \text{ such that } \gamma'\subset\Theta(\gamma) \text{ and } |\gamma'|\geq 2L+2aL\end{array}\right\}\end{aligned} \tag{3.31}$$

thus

$$\mu^{\tau}_{\Lambda(L,L_1)}(\Gamma_{L,L_1}\cap T_\gamma\Gamma_{L,L_1}(\gamma))\leq \sum_{\substack{\gamma':\gamma'\subset\Theta(\gamma)\\ |\gamma'|\geq 2L+2aL}}\ \sum_{\substack{\sigma:\sigma\in\Gamma_{L,L_1}(\gamma)\\ \gamma' \text{ appears at } T_\gamma(\sigma)}}\mu^{\tau}_{\Lambda(L,L_1)}(T_\gamma(\sigma)). \tag{3.32}$$

(1) Suppose that the contour γ surrounds the hole in $\Lambda(L,L_1)$ at σ. Then γ' doesn't intersect with sides of $\partial_{int}\Lambda(L)$, and we have:

(i) If γ' intersects with the sides of $\partial_{ext}\Lambda(L_1)$,

$$\begin{aligned}\mu^{\tau}_{\Lambda(L,L_1)}(T_\gamma(\sigma))&\leq\exp(-2\beta(|\gamma'|-8aL-8aL))\mu^{\tau}_{\Lambda(L,L_1)}(T_{\gamma'}T_\gamma(\sigma))\\ &=\exp(-2\beta(|\gamma'|-8aL-8aL))\mu^{\tau}_{\Lambda(L,L_1)}(T_\gamma T_{\gamma'}(\sigma)).\end{aligned} \tag{3.33}$$

(ii) If γ' doesn't intersect with the sides of $\partial_{ext}\Lambda(L_1)$,

$$\begin{aligned}\mu^{\tau}_{\Lambda(L,L_1)}(T_\gamma(\sigma))&\leq\exp(-2\beta(|\gamma'|-8aL))\mu^{\tau}_{\Lambda(L,L_1)}(T_{\gamma'}T_\gamma(\sigma))\\ &=\exp(-2\beta(|\gamma'|-8aL))\mu^{\tau}_{\Lambda(L,L_1)}(T_\gamma T_{\gamma'}(\sigma)).\end{aligned} \tag{3.34}$$

(2) Suppose that the contour γ doesn't surround the hole in $\Lambda(L,L_1)$ at σ, then

$$\begin{aligned}\mu^{\tau}_{\Lambda(L,L_1)}(T_\gamma(\sigma))&\leq\exp(-2\beta\,|\gamma'|)\mu^{\tau}_{\Lambda(L,L_1)}(T_{\gamma'}T_\gamma(\sigma))\\ &=\exp(-2\beta\,|\gamma'|)\mu^{\tau}_{\Lambda(L,L_1)}(T_\gamma T_{\gamma'}(\sigma)).\end{aligned} \tag{3.35}$$

by (3.32)–(3.35) and the fact that $T_{\gamma'}$ maps $\Gamma_{L,L_1}(\gamma)$ injectively into itself, we have

$$\sum_{\substack{\sigma:\sigma\in\Gamma_{L,L_1}(\gamma) \\ \gamma' \text{ appears at } T_\gamma(\sigma)}} \mu^{\tau}_{\Lambda(L,L_1)}(T_\gamma(\sigma)) \leq \exp(-2\beta(|\gamma'|-16aL))\mu^{\tau}_{\Lambda(L,L_1)}(T_\gamma\Gamma_{L,L_1}(\gamma)) \tag{3.36}$$

By (3.32) (3.36), and $a<\frac{1}{7}$

$$\mu^{\tau}_{\Lambda(L,L_1)}(\Gamma_{L,L_1} \mid T_\gamma\Gamma_{L,L_1}(\gamma)) \leq \sum_{\substack{\gamma':\gamma'\subset\Theta(\gamma) \\ |\gamma'|\geq 2L+2aL}} \exp(-2\beta(|\gamma'|-16aL))$$

$$\leq \sum_{m\geq 2L+2aL} 4(2L+1)^2 3^{m-1}\exp(-2\beta(m-16aL)) \xrightarrow{\beta\to\infty} 0. \tag{3.37}$$

Now we finished the proof of Lemma 4.

Let C_{L,L_1} be the set ot all contours in $\Lambda(L,L_1)$ which satisfies $|\gamma|\geq 2L+2aL$, and γ intersects with at most three sides of $\partial_{int}\Lambda(L)$.

Lemma 5. *For $\gamma\in C_{L,L_1}$, $L_1=[aL]-1$ $(a<\frac{1}{7})$, and let*

$$F(L,L_1,\gamma)=\inf\{H^{\tau}_{\Lambda(L,L_1)}(\sigma)-H^{\tau}_{\Lambda(L,L_1)}(T_\gamma(\sigma):\sigma\in\Gamma_{L,L_1}(\gamma)\}. \tag{3.38}$$

There is $\beta_2<\infty$, such that for $\beta>\beta_2$, we can find $c_1(\beta)>0$ and $C_1>0$, such that

$$\sum_{\gamma\in C_{L,L_1}} |\gamma|\exp(-\beta F(L,L_1,\gamma))\leq c_1(\beta)\exp(-C_1\beta L). \tag{3.39}$$

Proof. (1) Let C^1_{L,L_1} (a subset of C_{L,L_1}) be the set of all contours in $\Lambda(L,L_1)$ which don't intersect with sides of $\partial_{int}\Lambda(L)$. By the same argument as in (3.33) and (3.34)

$$\sum_{\gamma\in C^1_{L,L_1}} |\gamma|\exp(-\beta F(L,L_1,\gamma))\leq \sum_{m\geq 2L+2aL} 4m(2L+1)^2 3^{(m-1)}\exp(-2\beta(m-16aL)). \tag{3.40}$$

(2) Let C^2_{L,L_1} (a subset of C_{L,L_1}) be the set of all contours in $\Lambda(L,L_1)$ which intersect with sides of $\partial_{int}\Lambda(L)$, but do not intersect with sides of $\partial_{ext}\Lambda(L_1)$

(2)-(a). Suppose that the contour $\gamma\in C^2_{L,L_1}$ doesn't surround the hole in $\Lambda(L,L_1)$ at σ. We can follow the proof of Lemma 1, since we have the condition "γ intersects with at most three sides of $\partial_{int}\Lambda(L)$". So (3.13) is changed to be

$$|\bar{\gamma}|\leq\frac{3}{4}|\gamma|$$

and $|\gamma|\geq 2L+2aL$, by the same methods of (3.10), (3.14) and (3.15)

$$H^{\tau}_{\Lambda(L,L_1)}(\sigma)-H^{\tau}_{\Lambda(L,L_1)}(T_\gamma\sigma)\geq\frac{2}{4}|\gamma|-2. \tag{3.41}$$

(2)-(b). Suppose that the contour $\gamma\in C^2_{L,L_1}$ surrounds the hole in $\Lambda(L,L_1)$ at σ, and γ intersects with no pair of opposite sides of $\partial_{int}\Lambda(L)$. Then (3.13) is changed to be

$$|\bar{\gamma}|\leq\frac{1}{2}|\gamma|$$

and $|\gamma|\geq 2L+6aL$. Thus,

$$H^{\tau}_{\Lambda(L,L_1)}(\sigma)-H^{\tau}_{\Lambda(L,L_1)}(T_\gamma\sigma)\geq|\gamma|-16aL-2. \tag{3.42}$$

(2)-(c). Suppose that the contour γ surrounds the hole in $\Lambda(L,L_1)\in C^2_{L,L_1}$ at σ, and γ intersects with one pair of opposite sides of $\partial_{int}\Lambda(L)$. Then (3.13) is changed to be

$$|\bar{\gamma}|\leq\frac{3}{4}|\gamma|$$

and $|\gamma|\geq 4L+4aL$

$$H^{\tau}_{\Lambda(L,L_1)}(\sigma)-H^{\tau}_{\Lambda(L,L_1)}(T_\gamma\sigma)\geq\frac{1}{2}|\gamma|-16aL-2. \tag{3.43}$$

By (2)-(a)(b)(c)

$$\sum_{\gamma\in C^2_{L,L_1}}|\gamma|\exp(-\beta F(L,L_1,\gamma))\leq\sum_{m\geq 2L+2aL}4m(2L+1)^2 3^{(m-1)}\exp(-\beta(\frac{1}{2}m-2))$$
$$+\sum_{m\geq 2L+6aL}4m(2L+1)^2 3^{(m-1)}\exp(-\beta(m-16aL-2))$$
$$+\sum_{m\geq 4L+4aL}4m(2L+1)^2 3^{(m-1)}\exp(-\beta(\frac{1}{2}m-16aL-2)). \tag{3.44}$$

(3) Let C^3_{L,L_1} (a subset of C_{L,L_1}) be the set of all contours in $\Lambda(L,L_1)$ which intersect with sides of $\partial_{int}\Lambda(L)$ and sides of $\partial_{ext}\Lambda(L_1)$. Then we have

$$H^{\tau}_{\Lambda(L,L_1)}(\sigma)-H^{\tau}_{\Lambda(L,L_1)}(T_\gamma\sigma)\geq 2\sum_{i\geq 0}|\lambda_i|-2|\sum_{i\geq 1}\sum_{\substack{y\in\partial_{ext}\Lambda(L)\\ \{x,y\}\in\rho_i}}\tau(y)|-2\sum_{i\geq 1}|\rho'_i| \tag{3.45}$$

$$|\bar{\gamma}|=\sum_{i\geq 1}|\rho_i|+\sum_{i\geq 1}|\delta_i| \tag{3.46}$$

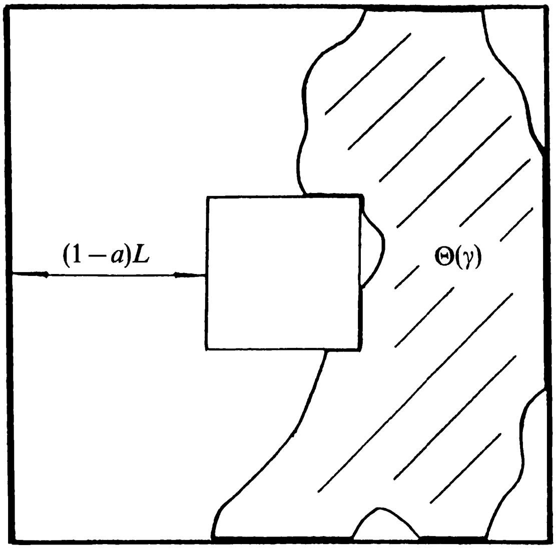

Figure 1.

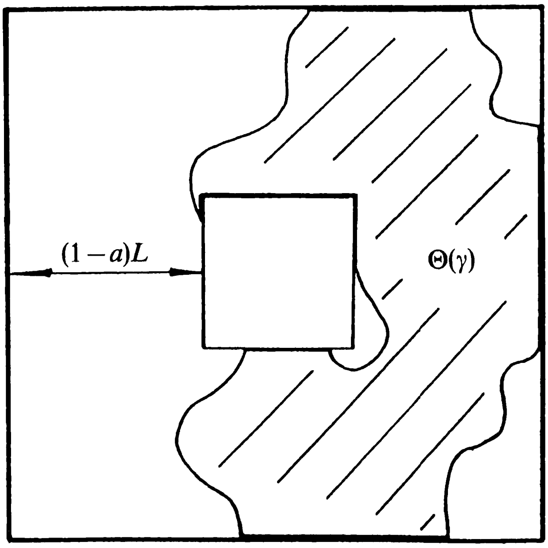

Figure 2.

$$|\gamma| = \sum_{i\geq 1}|\rho_i| + \sum_{i\geq 0}|\lambda_i| + \sum_{i\geq 1}|\rho_i'| \tag{3.47}$$

by (3.45)–(3.47),

$$|\sum_{i\geq 1}\sum_{\substack{y\in\partial_{ext}\Lambda(L)\\ \{x,y\}\in\rho_i}}\tau(y)| = |\sum_{\substack{y\in\partial_{ext}\Lambda(L)\\ \{x,y\}\in\gamma}}\tau(y) - \sum_{i\geq 1}\sum_{\substack{y\in\partial_{ext}\Lambda(L)\\ \{x,y\}\in\delta_i}}\tau(y)|$$

$$\leq 1+\sum_{i\geq 1}|\delta_i|=1+|\bar{\gamma}|-\sum_{i\geq 1}|\rho_i|. \tag{3.48}$$

(3)-(a). Suppose that $\Sigma_{i\geq 1}|\rho_i'|\leq 4aL$.

i) If γ intersects with three sides of $\partial_{int}\Lambda(L)$, then we have $|\gamma|\geq 4L+2(1-a)L$, and (see Figure 1),

$$|\bar{\gamma}|-2(1-a)L\leq\frac{|\gamma|-2(1-a)L}{2},$$

so we have

$$|\bar{\gamma}|\leq\frac{|\gamma|+2(1-a)L}{2}.$$

By (3.46)–(3.48)

$$|\sum_{i\geq 1}\sum_{\substack{y\in\partial_{ext}\Lambda(L)\\ \{x,y\}\in\rho_i}}\tau(y)|\leq 1+\frac{|\gamma|+2(1-a)L}{2}-|\gamma|+\sum_{i\geq 0}|\lambda_i|+\sum_{i\geq 1}|\rho_i'|.$$

Hence, by (3.45)

$$H^{\tau}_{\Lambda(L,L_1)}(\sigma)-H^{\tau}_{\Lambda(L,L_1)}(T_\gamma\sigma)\geq|\gamma|-2(1-a)L-4\sum_{i\geq 1}|\rho_i'|-2$$

$$\geq|\gamma|-2(1-a)L-16aL-2, \tag{3.49}$$

here we use the condition $\Sigma_{i\geq 1}|\rho_i'|\leq 4aL$ in the last inequality.

ii) If γ intersects with only one pair of opposite sides of $\partial_{int}\Lambda(L)$, then we have $|\gamma|\geq 4L$, and

$$|\bar{\gamma}|\leq\frac{|\gamma|}{2}$$

by the same method as i),

$$H^{\tau}_{\Lambda(L,L_1)}(\sigma)-H^{\tau}_{\Lambda(L,L_1)}(T_\gamma\sigma)\geq|\gamma|-16aL-2. \tag{3.50}$$

iii) If γ intersects with no pair of opposite sides of $\partial_{int}\Lambda(L)$, by the condition that $\gamma\in C^3_{L,L_1}$, we have $|\gamma|\geq 2L+2aL$, and,

$$|\bar{\gamma}|\leq\frac{|\gamma|}{2}$$

$$H^{\tau}_{\Lambda(L,L_1)}(\sigma)-H^{\tau}_{\Lambda(L,L_1)}(T_\gamma\sigma)\geq|\gamma|-16aL-2. \tag{3.51}$$

(3)-(b). Suppose that $\Sigma_{i\geq 1}|\rho_i'|>4aL$.

i) If γ intersects with three sides of $\partial_{int}\Lambda(L)$, then we have $|\gamma|\geq 4L+2(1-a)L+4aL$, and (see Figure 2)

$$|\bar{\gamma}|-2(1-a)L\leq\frac{|\gamma|-2(1-a)L}{2},$$

so we have

$$|\bar{\gamma}|\leq\frac{|\gamma|+2(1-a)L}{2}$$

and

$$|\sum_{i\geq 1}\sum_{\substack{y\in\partial_{ext}\Lambda(L)\\ \{x,y\}\in\rho_i}}\tau(y)|\leq 1-\frac{|\gamma|-2(1-a)L}{2}+\sum_{i\geq 0}|\lambda_i|+\sum_{i\geq 1}|\rho_i'|.$$

By (3.45), we have

$$H^{\tau}_{\Lambda(L,L_1)}(\sigma)-H^{\tau}_{\Lambda(L,L_1)}(T_\gamma\sigma)\geq|\gamma|-2(1-a)L-4\sum_{i\geq 1}|\rho_i'|-2$$

$$\geq|\gamma|-2(1-a)L-32aL-2, \tag{3.52}$$

here we use the condition $8aL\geq\Sigma_{i\geq 1}|\rho_i'|\geq 4aL$ in the last inequality.

ii) If γ intersects with only one pair of opposite sides of $\partial_{int}\Lambda(L)$, then we have $|\gamma|\geq 4L+4aL$, and

$$|\bar{\gamma}|\leq\frac{|\gamma|}{2}$$

so we have

$$|\sum_{i\geq 1}\sum_{\substack{y\in\partial_{ext}\Lambda(L)\\ \{x,y\}\in\rho_i}}\tau(y)|\leq 1-\frac{|\gamma|}{2}+\sum_{i\geq 0}|\lambda_i|+\sum_{i\geq 1}|\rho_i'|$$

and

$$H^{\tau}_{\Lambda(L,L_1)}(\sigma)-H^{\tau}_{\Lambda(L,L_1)}(T_\gamma\sigma)\geq|\gamma|-4\sum_{i\geq 1}|\rho_i'|-2$$

$$\geq|\gamma|-32aL-2. \tag{3.53}$$

iii) If γ intersects with no pair of opposite sides of $\partial_{int}\Lambda(L)$, but intersects with two sides of $\partial_{int}\Lambda(L)$, then we have $|\gamma|\geq 4L+4aL$, and

$$|\bar{\gamma}|\leq\frac{|\gamma|}{2}$$

by above ii)

$$H^{\tau}_{\Lambda(L,L_1)}(\sigma)-H^{\tau}_{\Lambda(L,L_1)}(T_\gamma\sigma)\geq|\gamma|-32aL-2. \quad (3.54)$$

iv) If γ intersects with only one side of $\partial_{int}\Lambda(L)$, then we have $|\gamma|\geq 2L+4aL$, and

$$|\bar{\gamma}|\leq\frac{|\gamma|-2L}{2}$$

$$H^{\tau}_{\Lambda(L,L_1)}(\sigma)-H^{\tau}_{\Lambda(L,L_1)}(T_\gamma\sigma)\geq|\gamma|+2L-32aL-2. \quad (3.55)$$

By (3)-(a) and (3)-(b), we have

$$\sum_{\gamma\in C^3_{L,L_1}}|\gamma|\exp(-\beta F(L,L_1,\gamma))$$

$$\leq\sum_{m\geq 4L+2(1-a)L}4m(2L+1)^2 3^{(m-1)}\exp(-\beta(m-2(1-a)L-16aL-2))$$

$$+\sum_{m\geq 4L}4m(2L+1)^2 3^{(m-1)}\exp(-\beta(m-16aL-2))$$

$$+\sum_{m\geq 2L+2aL}4m(2L+1)^2 3^{(m-1)}\exp(-\beta(m-16aL-2))$$

$$+\sum_{m\geq 4L+2(1-a)L+4aL}4m(2L+1)^2 3^{(m-1)}\exp(-\beta(m-2(1-a)L-32aL-2))$$

$$+2\sum_{m\geq 4L+4aL}4m(2L+1)^2 3^{(m-1)}\exp(-\beta(m-32aL-2))$$

$$+\sum_{m\geq 2L+4aL}4m(2L+1)^2 3^{(m-1)}\exp(-\beta(m+2L-32aL-2)). \quad (3.56)$$

Combining (1) (2) (3), by (3.40), (3.44) and (3.56), there is a $\beta_2<\infty$, when $\beta>\beta_2$, we can find $c_1(\beta)>0$, $C_1>0$

$$\sum_{\gamma\in C_{L,L_1}}|\gamma|\exp(-\beta F(L,L_1,\gamma))\leq c_1(\beta)\exp(-C_1\beta L). \quad (3.57)$$

Now we finished the proof of Lemma 5.

3.3. Proof of Theorem 1. We will prove Theorem 1 in two steps:
(1) Suppose $L\geq 7$, $L_1=[aL]-1$ and $a<\frac{1}{7}$, for $\sigma\in\Gamma_{L,L_1}$ (see (3.27)), let $\Lambda(L,L_1,\sigma)=\{x\in\Lambda(L,L_1):\sigma^x\notin\Gamma_{L,L_1}\}\neq\emptyset$, then we have the following properties:

(i) there exists an unique $\epsilon(L,L_1,\tau)$-contour $\gamma=\gamma(\sigma)$ in $\Lambda(L,L_1)$ at σ such that

$|\gamma| \geq 2L + 2aL$,

(ii) γ intersects with at most three sides of $\partial_{int}\Lambda(L)$,

(iii) $|\Lambda(L, L_1, \sigma)| \leq 2|\gamma|$. (3.58)

If there were two contours satisfying (i), it would be impossible that $\sigma^x \notin \Gamma_{L,L_1}$ for some x. (iii) is obvious. (ii) can be seen as follows: For $x \in \Lambda(L, L_1)$, consider a set of contours in $\Lambda(L, L_1)$ at σ^x such that $|\gamma_i| < 2L + 2aL$ and $x \in \gamma_i$, say C_x, that is

$$C_x = \{\gamma_1, \cdots, \gamma_j\} = \left\{\gamma : \begin{array}{c} \gamma \text{ is an } \epsilon(L, L_1, \tau) \text{ contour in} \\ \Lambda(L, L_1) \text{ at } \sigma^x, x \in \gamma \end{array}\right\}.$$

Note that

$$\gamma \subset \cup_{i=1}^{j} \gamma_i \cup \{b \in \boldsymbol{B} : x \in b\}.$$

Then we can see, for any $x \in \Lambda(L, L_1)$, the contours in C_x intersect with at most three sides of $\partial_{int}\Lambda(L)$.

Let C_{L,L_1} be the set of all contours in $\Lambda(L, L_1)$ which satisfies $|\gamma| \geq 2L + 2aL$, and $\gamma \in C_{L,L_1}$ intersect with at most three sides of $\partial_{int}\Lambda(L)$, and let $\Gamma_{L,L_1}(\gamma)$ be given in (3.29). We want to prove that:

$$\text{gap}(\Lambda(L, L_1); \beta, \tau) \leq 6c_M(\beta) \sum_{\gamma \in C_{L,L_1}} |\gamma| \frac{\mu^{\tau}_{\Lambda(L,L_1)}(\Gamma_{L,L_1}(\gamma))}{\mu^{\tau}_{\Lambda(L,L_1)}(\Gamma^c_{L,L_1})}. \tag{3.59}$$

Let $\chi_{\Gamma_{L;L_1}} : \Omega_{\Lambda(L,L_1)} \to \{0, 1\}$ be the indicator function of Γ_{L,L_1}, by (2.16) and (3.26), there is a $\beta_1 > 0$, such that for $\beta \geq \beta_1$, we have

$$\begin{aligned}
\text{gap}(\Lambda(L, L_1), \tau) &= \inf_{f \in L^2(\Omega, d\mu^{\beta,\tau}_{\Lambda})} \frac{\mathscr{E}^{\beta,\tau}_{\Lambda}(f, f)}{\text{Var}^{\beta,\tau}_{\Lambda}(f)} \\
&\leq \frac{c_M(\beta)}{\mu^{\tau}_{\Lambda(L,L_1)}(\Gamma_{L,L_1})\mu^{\tau}_{\Lambda(L,L_1)}(\Gamma^c_{L,L_1})} \times \\
&\quad \times \sum_{x \in \Lambda(L,L_1)} \sum_{\sigma} \mu^{\tau}_{\Lambda(L,L_1)}(\sigma)\chi_{L,L_1}(\sigma) \,|\, \chi_{L,L_1}(\sigma^x) - \chi_{L,L_1}(\sigma)| \\
&\leq 3c_M(\beta) \sum_{x \in \Lambda(L,L_1)} \sum_{\sigma \in \Gamma_{L,L_1}, \sigma^x \notin \Gamma_{L,L_1}} \frac{\mu^{\tau}_{\Lambda(L,L_1)}(\sigma)}{\mu^{\tau}_{\Lambda(L,L_1)}(\Gamma^c_{L,L_1})} \\
&= 3c_M(\beta) \sum_{\sigma \in \Gamma_{L,L_1}} |\Lambda(L, L_1, \sigma)| \frac{\mu^{\tau}_{\Lambda(L,L_1)}(\sigma)}{\mu^{\tau}_{\Lambda(L,L_1)}(\Gamma^c_{L,L_1})} \\
&\leq 6c_M(\beta) \sum_{\gamma \in C_{L,L_1}} |\gamma| \frac{\mu^{\tau}_{\Lambda(L,L_1)}(\Gamma_{L,L_1}(\gamma))}{\mu^{\tau}_{\Lambda(L,L_1)}(\Gamma^c_{L,L_1})}.
\end{aligned}$$

Now we have proved the inequality (3.59).

(2) For $\gamma \in C_{L,L_1}$, let

$$F(L, L_1, \gamma) = \inf\{H^{\tau}_{\Lambda(L,L_1)}(\sigma) - H^{\tau}_{\Lambda(L,L_1)}(T_\gamma \sigma);\ \sigma \in \Gamma_{L,L_1}(\gamma)\}.$$

Then,

$$\mu^{\tau}_{\Lambda(L,L_1)}(\Gamma_{L,L_1}(\gamma)) \le \mu^{\tau}_{\Lambda(L,L_1)}(T_\gamma \Gamma_{L,L_1}(\gamma)) \exp\{-\beta F(L, L_1, \gamma)\}. \tag{3.60}$$

By Lemma 4, since $\gamma \in C_{L,L_1}$, there exists a $\beta_3 > 0$, such that

$$\inf\left\{\mu^{\tau}_{\Lambda(L,L_1)}(\Gamma^c_L \mid T_\gamma \Gamma_{L,L_1}(\gamma)) :\ \begin{matrix} \beta > \beta_3, \gamma \text{ is a contour in } \Lambda(L, L_1), \\ |\gamma| \ge 2L + 2aL \end{matrix}\right\} \ge \frac{1}{2}$$

then,

$$\begin{aligned}\frac{\mu^{\tau}_{\Lambda(L,L_1)}(\Gamma_{L,L_1}(\gamma))}{\mu^{\tau}_{\Lambda(L,L_1)}(\Gamma^c_{L,L_1})} &\le \frac{\mu^{\tau}_{\Lambda(L,L_1)}(T_\gamma \Gamma_{L,L_1}(\gamma))}{\mu^{\tau}_{\Lambda(L,L_1)}(\Gamma^c_{L,L_1})} \exp\{-\beta F(L, L_1, \gamma)\} \\ &= \frac{1}{\mu^{\tau}_{\Lambda(L,L_1)}(\Gamma^c_{L,L_1} \mid T_\gamma \Gamma_{L,L_1}(\gamma))} \exp\{-\beta F(L, L_1, \gamma)\} \\ &\le 2\exp\{-\beta F(L, L_1, \gamma)\}.\end{aligned} \tag{3.61}$$

By (3.59) and (3.61) we have

$$\operatorname{gap}(\Lambda(L, L_1);\ \beta, \tau) \le 12 c_M(\beta) \sum_{\gamma \in C_{L,L_1}} |\gamma| \exp\{-\beta F(L, L_1, \gamma)\}.$$

By Lemma 5, there is a $\beta_2 < \infty$, such that when $\beta > \beta_2$,

$$\operatorname{gap}(\Lambda(L, L_1);\ \beta, \tau) \le 12 c_M(\beta) c_1(\beta) \exp\{-C_1 \beta L\}. \tag{3.62}$$

If $\beta > \max\{\beta_1, \beta_2, \beta\}$, there are $c(\beta) > 0$, $C > 0$,

$$\operatorname{gap}(\Lambda(L, L_1);\ \beta, \tau) \le c(\beta) \exp\{-C\beta L\}. \tag{3.63}$$

We finished the proof of Theorem 1.

4. The lower bound of the spectral gap of Ising model with the special boundary conditions of case 2

4.1. Boundary conditions and main results. In this section, we will consider the Ising model on $\Lambda(L, L_1)$ with another special boundary condition τ.

Boundary Conditions of Case 2. (4.1)

Inner boundary: *an arbitrary boundary condition.*

Outer boundary: $\tau(x) = +1$; *for all* $x \in \partial_{outer}\Lambda(L, L_1)$.

In this section, we will use the same definitions and notations coming from [5], and use the argument of [5]. We let $(+)$ and $(-)$ denote the two extreme configurations in $\Omega_{\Lambda(L,L_1)}$ identically equal to plus and minus one respectively, and for any rectangle R whose sides are pararell to the x-axis and y-axis, let $\mu_R^{\tau_1,\tau_1,\tau_3,\tau_4}$ denote the Gibbs measure on R with the boundary conditions τ_1, τ_2, τ_3, τ_4 on the outer boundary of its four sides ordered clockwise starting from the bottom side. We use the usual convention that, if one of the configurations τ_i is identically equal to $+1$ or -1, then we replace it by a $+$ or $-$ sign. Thus for example τ_1, $+$, $-$, $+$ means τ_1 boundary condition on the bottom side, plus boundary condition on the vertical ones and minus boundary condition on the top one. Whenever confusion does not arise we will also omit the subscript $\Lambda(L, L_1)$ in the notation $\sigma_{\Lambda(L,L_1)}$.

Now we state the main result in this Section.

Theorem 2. *Let* $\epsilon \in (0, \frac{1}{2})$ *be given, and* L, L_1 *be any integers such that* $L - L_1 > 4$ *and* $L_1 > 2L^{\frac{1}{2}+\varepsilon}$, *then there exist* $\beta_0 < \infty$ *and* $C > 0$, *when* $\beta \geq \beta_0$ *we have*

$$\text{gap}(\Lambda(L, L_1), \beta, \tau) \geq \exp(-C\beta L^{\frac{1}{2}+\varepsilon}). \tag{4.2}$$

Remark 1. Comparing this Theorem 2 with the Theorem 3.1 in [5], for the graphs $\Lambda(L, L_1)$ and $\Lambda(L)$ and for the same plus "outer boundary condition", we can get the same lower bound for the spectral gaps. This means that, under this circumstances, when β is large enough, for the graph $\Lambda(L, L_1)$ the inner arbitrary boundary condition of $\Lambda(L, L_1)$ doesn't change this lower bound $(\exp(-C\beta L^{\frac{1}{2}+\varepsilon}))$ of the spectral gap.

Remark 2. Although we consider the condition $L_1 > 2L^{\frac{1}{2}+\varepsilon}$ in Theorem 2, for $L_1 \leq 2L^{\frac{1}{2}+\varepsilon}$, by using almost the same argument, we can get the same result as that of Theorem 2. But in this case, we should do a little modification on the updatings and proofs.

4.2. Block-Glauber dynamics for Ising model. In this subsection, we will briefly introduce the notations for Block-Glauber dynamics, for the detail, see [5] [6]. Let $V \subset Z^2$ be a given finite set, $\tau \in \Omega_{Z^2}$ be the boundary condition and $\mu_V^{\beta,\tau}$ the Gibbs measure. We will also consider a more general version of the finite volume dynamics disscussed so far in which more than one spin can flip at once. Let $\mathscr{D} = \{V_1, \cdots, V_n\}$ be a covering of V, i.e., $V = \cup_i V_i$. Then we will denote by *block dynamics* with blocks $\{V_1, \cdots, V_n\}$ the continuous time Markov chain in which each block waits an exponential time of mean one and the configuration inside the block is replaced by a new configuration distributed according to the Gibbs measure of the block given the previous configuration outside the block. More precisely, the generator of the Markov process corresponding to $\mathscr{D}$ is defined as

$$(L^{\{V_i\},\beta,\tau})f(\sigma_V)=\sum_{i=1}^{n}\sum_{\eta\in\Omega_{V_i}}\mu_{V_i}^{\beta,(\tau\sigma_V)}(\eta)[f(\sigma_V^\eta)-f(\sigma_V)] \tag{4.3}$$

where $(\tau\sigma_V)$ denotes the configuration in Ω_{Z^2} equal to τ outside V and to σ_V inside V, while σ_V^η is the configuration in Ω_V equal to η in V_i and to $\sigma_{V\setminus V_i}$ in $V\setminus V_i$. We will refer to the Markov process generated by $L^{\{V_i\},\beta,\tau}$ as the $\{V_i\}$-dynamics. The operator $L^{\{V_i\},\beta,\tau}$ is self-adjoint on $L^2(\Omega, d\mu_V^\tau)$, i.e., the block dynamics is reversible with respect to the Gibbs measure $\mu_V^{\beta,\tau}$. Then,

$$\text{gap}_V(\{V_i\})=\inf_{f\in L^2(\Omega_V,d\mu_V^{\beta,\tau})}\frac{\mathscr{E}(f,f)}{Var(f)} \tag{4.4}$$

where

$$\mathscr{E}(f,f)=\frac{1}{2}\sum_i\sum_{\sigma_V}\sum_{\eta\in\Omega_{V_i}}\mu_V^{\beta,\tau}(\sigma_V)\mu_{V_i}^{\beta,(\tau\sigma_V)}(\eta)[f(\sigma_V^\eta)-f(\sigma_V)]^2 \tag{4.5}$$

$$Var(f)=\frac{1}{2}\sum_{\sigma,\eta}\mu_V^{\beta,\tau}(\sigma)\mu_V^{\beta,\tau}(\eta)[f(\sigma)-f(\eta)]^2. \tag{4.6}$$

The coupling for the $\{V_i\}$-dynamics is essential in this section, for the details, see §4 Section 1 in [5].

Next we introduce a Lemma, which comes from Lemma 3.1 in [5], we will omit the proof.

Lemma 6. *Let us call $S_N=\{t_1,\cdots,t_N\}$ an ordered sequence of updatings if for any $i=1,\cdots,N$:*

i) *at time t_i the dynamics updates the rectangle V_i,*

ii) *there are no updatings between times t_i and t_{i+1}.*

Then, for any N large enough (independent of t):

$$P(\textit{there exists no ordered sequence in } [0,t])\le\exp\left(-\frac{tN^{-N}}{2}\right). \tag{4.7}$$

The following Proposition comes from the Proposition 3.4 in [6], we also omit the proof.

Proposition 1. *Let $\mathscr{D}=\{V_1,\cdots,V_n\}$ be an arbitrary collection of finite sets and* $V=\cup_i V_i$. For any given boundary condition $\tau\in\Omega$, let $L_V^{\beta,\tau}$ be given in (2.11), and let $L^{\{V_i\},\beta,\varphi}$ *be given in* (4.3). *Then we have*

$$\text{gap}(L_V^{\beta,\tau})\ge\text{gap}(L^{\{V_i\},\beta,\tau})\inf_i\inf_{\varphi\in\Omega}\text{gap}(L_{V_i}^{\beta,\varphi})(\sup_{x\in V}\#\{i: V_i\ni x\})^{-1}. \tag{4.8}$$

Let $l=2[L^{\frac{1}{2}+\varepsilon}]$, where $\epsilon\in(0,\frac{1}{2})$, suppose that $L>L_1>l$. Without loss of generality, we can suppose that $N\equiv\frac{2(L-L_1)}{l}-1$ is an integer. For $i=1,\cdots,N$, we define seven kind of rectangles:

$$A_i=\{x\in Z^2:\ -L\le x_1\le L,\ -L+(i-1)\tfrac{l}{2}\le x_2\le -L+(i+1)\tfrac{l}{2}\},$$

$$B_{N+i}=\{x\in Z^2:\ -L\le x_1\le L,\ L-(i+1)\tfrac{l}{2}\le x_2\le L-(i-1)\tfrac{l}{2}\},$$

$$C_{2N+i}=\{x\in Z^2:\ L-(i+1)\tfrac{l}{2}\le x_1\le L+(i-1)\tfrac{l}{2},\ -L_1-l\le x_2\le L_1+l\},$$

$$D_{3N+i}=\{x\in Z^2:\ -L+(i-1)\tfrac{l}{2}\le x_1\le -L+(i+1)\tfrac{l}{2},\ -L_1-l\le x_2\le L+l\},$$

$$\bar{Q}=\{x\in Z^2:\ -L_1-l\le x_1\le L_1+l,\ -L-l\le x_2\le L_1+l\}\backslash\Lambda(L_1),$$

and

$$\bar{C}_{2N+i}=\{x\in Z^2: L-(i+1)\tfrac{l}{2}\le x_1\le L,\ -L\le x_2\le L\},$$

$$\bar{D}_{3N+i}=\{x\in Z^2: -L\le x_1\le -L+(i+1)\tfrac{l}{2},\ -L\le x_2\le L\},\tag{4.9}$$

and let $\{Q\}=\{A_i,B_i,C_i,D_i,\bar{Q},i=1,\cdots,N\}$. By the above definition, $\{Q\}$ is the covering of $\Lambda(L,L_1)$, and by (4.3), we can construct the $\{Q\}$-dynamics.

We will do the updatings in the following order:

(a) first, we do the updating of $\{A_i\}$, in the order of $A_1,A_2,\cdots,A_N$,
(b) second, we do the updating of $\{B_i\}$ in the order of $B_{N+1},B_{N+2},\cdots,B_{2N}$,
(c) third, we do the updating of $\{C_i\}$, in the order of $C_{2N+1},C_{2N+2},\cdots,C_{3N}$,
(e) next, we do the updating of $\{D_i\}$ in the order of $D_{3N+1},D_{3N+2},\cdots,D_{4N}$,
(f) at last, we do the updating of $\bar{Q}$. (4.10)

The idea why we do above updatings comes from [5], we want to enforce the $(+)$ spins and $(-)$ spins agree after the updatings, and by Lemma 6, we see that with large probability, we have this updatings. Now we introduce a Lemma, it comes from Theorem 6.4. in [6].

Lemma 7. *Let $\bar{Q}$ be defined as in* (4.9), *then*

$$\inf_{\tau}\text{gap}\ \{\bar{Q},\tau\}\ge\frac{1}{|\bar{Q}|}c_m\exp(-4\beta\cdot 2\sqrt{2}l)\tag{4.11}$$

where $l=2[L^{\frac{1}{2}+\varepsilon}]$ and the constant c_m has been defined in (2.15)

Proof. The proof is almost the same as the proof in Section 2 of [5] or the proof of Theorem 6.4 in [6]. But we will modify the proof of Theorem 6.4 in [6]; we give a new definition about the order of sites in $\Lambda(L,L_1)$.

Let $x, y \in \Lambda(L, L_1)$, we order the sites in $\Lambda(L, L_1)$ as follow:

$$x<y, \text{ iff } x_1+x_2<y_1+y_2, \text{ or } x_1+x_2=y_1+y_2, \; x_2<y_2, \tag{4.12}$$

where $x=(x_1, x_2)$, $y=(y_1, y_2)$.

The rest of the proof is almost the same as that of Theorem 6.4 in [6].

Next we intouduce an important proposition, which comes from Proposition 3.1 in [5]. We omit the proof, too. Let R be the rectangle

$$R=\{x \in Z^2: -L_2 \leq x_1 \leq L_2, \quad -L_3 \leq x_2 \leq L_3\} \tag{4.13}$$

with $L_2 \geq L_3 \geq L^{\frac{1}{2}+\varepsilon}$.

Proposition 2. *Let $m>0$ and $\epsilon \in (0, \frac{1}{2})$ be given, then there exists $\beta_0 \equiv \beta_0(\epsilon, m)$ independent of R such that for all $\beta \geq \beta_0$ and all $x \in R$ with $x_2 \leq L_3 - \frac{1}{2}L^{\frac{1}{2}+\varepsilon}$, we have:*

$$\mu_R^{+,+,+,+}(\sigma(x)=1) - \mu_R^{+,+,-,+}(\sigma(x)=1) \leq \exp(-mL_2^{2\varepsilon}). \tag{4.14}$$

4.3. Probability estimate of special sequence of updatings. Let us use the following convention:

$$V_i = \begin{cases} A_i, & 1 \leq i \leq N, \\ B_i, & N+1 \leq i \leq 2N, \\ C_i, & 2N+1 \leq i \leq 3N, \\ D_i, & 3N+1 \leq i \leq 4N, \\ \bar{Q}, & i=4N+1. \end{cases}$$

Let $S_{4N+1}=\{t_1, \cdots, t_N, t_{N+1}, \cdots, t_{2N}, t_{2N+1}, \cdots, t_{3N}, t_{3N+1}, \cdots, t_{4N}, t_{4N+1}\}$ be a fixed ordered sequence with $t_1=0$, let $\sigma_{t_i}^{\{Q\},\tau}$ be the configuration of $\{Q\}$-dynamics (see Section 4.2) at time t_i starting from the initial configuration σ, and the i-th updating occurs in the box V_i. For $m=1, \cdots, N$,

$$\begin{aligned} &R_m^A = \{x \in \cup_{j \leq m} A_j : x_2 \leq -L+(m+1)\tfrac{l}{2} - [\tfrac{l}{4}]\}, \\ &R_{N+m}^B = \{x \in \cup_{j \leq m} B_{N+j} : x_2 \geq L-(m+1)\tfrac{l}{2} + [\tfrac{l}{4}]\} \cup R_N^A, \\ &R_{2N+m}^C = \{x \in \cup_{j \leq m} C_{2N+j} : x_1 \geq L-(m+1)\tfrac{l}{2} + [\tfrac{l}{4}]\} \cup R_{2N}^B, \\ &R_{3N+m}^D = \{x \in \cup_{j \leq m} D_{3N+j} : x_1 \leq -L+(m+1)\tfrac{l}{2} - [\tfrac{l}{4}]\} \cup R_{3N}^C, \\ &R_{4N+1}^{\bar{Q}} = \bar{Q} \cup R_{4N}^D = \Lambda(L, L_1). \end{aligned} \tag{4.15}$$

For $i=1, \cdots, 4N+1$, let

$$R_i \in \{R_1^A, \cdots, R_N^A, R_{N+1}^B, \cdots, R_{2N}^B, R_{2N+1}^C, \cdots, R_{3N}^C, R_{3N+1}^D, \cdots, R_{4N}^D, R_{4N+1}^{\bar{Q}}\},$$

for example, $R_{2N+1} = R_{2N+1}^C$. Next, we define the events:

$$F_i(x) = \{(+)_{t_i}^{\{Q\},\tau}(x) \neq (-)_{t_i}^{\{Q\},\tau}(x)\}, \qquad i = 1, \cdots, 4N+1, \tag{4.16}$$

$$F_i = \bigcup_{\{x \in R_i\}} F_i(x), \qquad i = 1, \cdots, 4N+1 \tag{4.17}$$

In particular, we have

$$F_{4N+1} = \bigcup_{x \in \Lambda(L, L_1)} F_{4N+1}(x). \tag{4.18}$$

Let $q_i = P(F_i)$, $i = 1, 2, \cdots, 4N+1$, then we have for every $n \leq 4N$,

$$q_{i+1} \leq q_i + P(F_{i+1} \cap F_i^c) \leq \sum_{n=1}^{4N} P(F_{n+1} \cap F_n^c) + P(F_1).$$

Hence by induction, we have

$$q_{4N+1} \leq \sum_{n=1}^{N-1} P(F_{n+1} \cap F_n^c) \leq \sum_{n=N}^{2N-1} P(F_{n+1} \cap F_n^c) + \sum_{n=2N}^{3N-1} P(F_{n+1} \cap F_n^c)$$
$$+ \sum_{n=3N}^{4N-1} P(F_{n+1} \cap F_n^c) + P(F_1) + P(F_{4N+1} \cap F_{4N}^c) \tag{4.19}$$

Next we will show that

$$q_{4N+1} \leq 4N(2L+1)^2 \exp(-mL^{2\varepsilon}). \tag{4.20}$$

First, we consider the third term, $\Sigma_{n=2N}^{3N-1} P(F_{n+1} \cap F_n^c)$.

$$P(F_{n+1} \cap F_n^c) \leq \sum_{\substack{x \in R_{n+1} \cap C_{n+1} \\ \sigma \in \Omega_{\Lambda(L, L_1)}}} \mu_{\Lambda(L, L_1)}^{\tau}(\sigma)$$
$$\times P(F_{n+1}(x) \cap [\cap_{y \in R_n} \{(+)_{t_n}^{\{Q\},\tau}(y) = (-)_{t_n}^{\{Q\},\tau}(y) = \sigma_{t_n}^{\{Q\},\tau}(y)\}]), \tag{4.21}$$

where $n \in \{2N, \cdots, 3N-1\}$. Then the summand in the right hand side of (4.21) can be estimated from above by:

$$\mu_{\Lambda(L, L_1)}^{\tau}(\sigma) E[\mu_{C_{n+1}}^{\sigma_{t_n}^{\{Q\},\tau}, +, \sigma_{t_n}^{\{Q\},\tau}, \sigma_{t_n}^{\{Q\},\tau}}(\eta(x) = 1) - \mu_{C_{n+1}}^{\sigma_{t_n}^{\{Q\},\tau}, -, \sigma_{t_n}^{\{Q\},\tau}, \sigma_{t_n}^{\{Q\},\tau}}(\eta(x) = 1)], \tag{4.22}$$

where E is the expectation over the random configuration $\sigma_{t_n}^{\{Q\},\tau}$. Since the dynamics is reversible with respect to $\mu_{\Lambda(L, L_1)}^{\tau}(\sigma)$ and by the DLR property,

$$\sum_{\sigma\in\Omega_{A(L,L_1)}} \mu^{\tau}_{\Lambda(L,L_1)}(\sigma) E\mu_{C_{n+1}}^{\sigma^{\{Q\},\tau}_{t_n},+,\sigma^{\{Q\},\tau}_{t_n},\sigma^{\{Q\},\tau}_{t_n}}(\eta(x)=1)$$

$$= \sum_{\sigma\in\Omega_{A(L,L_1)}} \mu^{\tau}_{\Lambda(L,L_1)}(\sigma)\mu_{C_{n+1}}^{\sigma,+,\sigma,\sigma}(\eta(x)=1)$$

$$\leq \sum_{\sigma\in\Omega_{\bar{C}_{n+1}}} \mu_{\bar{C}_{n+1}}^{+,+,+,+}(\sigma)\mu_{C_{n+1}}^{\sigma,+,\sigma,\sigma}(\eta(x)=1)$$

$$=\mu_{\bar{C}_{n+1}}^{+,+,+,+}(\eta(x)=1). \tag{4.23}$$

Similarly we obtain following;

$$\sum_{\sigma\in\Omega_{A(L,L_1)}} \mu^{\tau}_{\Lambda(L,L_1)}(\sigma) E\mu_{C_{n+1}}^{\sigma^{\{Q\},\tau}_{t_n},-,\sigma^{\{Q\},\tau}_{t_n},\sigma^{\{Q\},\tau}_{t_n}}(\eta(x)=1)$$
$$\geq \mu_{\bar{C}_{n+1}}^{+,-,+,+}(\eta(x)=1). \tag{4.24}$$

By Proposition 2, we have

$$\sum_{x\in R_{n+1}\cap C_{n+1}} [\mu_{\bar{C}_{n+1}}^{+,+,+,+}(\eta(x)=1)-\mu_{\bar{C}_{n+1}}^{+,-,+,+}(\eta(x)=1)]\leq(2L+1)^2\exp(-mL^{2\varepsilon}), \tag{4.25}$$

where $m\equiv m(\beta)$, which diverges as $\beta\to\infty$. Thus, for $2N\leq n\leq 3N-1$, we have

$$P(F_{n+1}\cap F_n^c)\leq(2L+1)^2\exp(-mL^{2\varepsilon}), \tag{4.26}$$

and

$$\sum_{n=2N}^{3N-1} P(F_{n+1}\cap F_n^c)\leq N(2L+1)^2\exp(-mL^{2\varepsilon}). \tag{4.27}$$

We can use the same method to estimate $\Sigma_{n=3N}^{4N-1}P(F_{n+1}\cap F_n^c)$,

$$\sum_{n=3N}^{4N-1} P(F_{n+1}\cap F_n^c)\leq N(2L+1)^2\exp(-mL^{2\varepsilon}). \tag{4.28}$$

Note that, by the definition of $\{Q\}$-dynamics, we have $P(F_{4N+1}\cap F_{4N}^c)=0$. We can follow the proof in Section 3 of [5], or use a similar argument of above proofs, we can get

$$\sum_{n=1}^{N-1} P(F_{n+1}\cap F_n^c)\leq N(2L+1)^2\exp(-mL^{2\varepsilon}). \tag{4.29}$$

$$\sum_{n=N}^{2N-1} P(F_{n+1}\cap F_n^c)\leq N(2L+1)^2\exp(-mL^{2\varepsilon}). \tag{4.30}$$

where $m\equiv m(\beta)$. Similarly we can estimate $P(F_1)$. Thus, we finally obtain (4.20)

$$q_{4N+1} \leq 4N(2L+1)^2 \exp(-mL^{2\varepsilon}).$$

4.4. Proof of Theorem 2. Given a sequence $S_{4N+1}=\{t_1,\cdots,t_{4N+1}\}$ of updatings we say that S_{4N+1} is good sequence iff S_{4N+1} is ordered and the event F^c_{4N+1} occurs at the end of the sequence. Because of (4.20) we know that the probability that an ordered sequence of updating S_{4N+1} is also a good sequence is larger than

$$1-(4N+1)(2L+1)^2 \exp(-mL^{2\varepsilon}) > \frac{1}{2} \tag{4.31}$$

for L large enough. Thus, using Lemma 6, we get that if $T=\exp(L^{\frac{1+\varepsilon}{2}})$ and L is large enough:

$$P(\text{there exists a good sequence in } [0,T]) \geq \frac{1}{3} \tag{4.32}$$

We conclude by observing that, if there exists a good sequence in $[0,t]$, then, by monotonicity, at the end of the sequence, the configurations $(+)_t^{\{Q\},\tau}$ and $(-)_t^{\{Q\},\tau}$ will be identical. Therefore we can estimate

$$P((+)_t^{\{Q\},\tau} \neq (-)_t^{\{Q\},\tau}) \leq \left(\frac{2}{3}\right)^{[\frac{t}{T}]} \tag{4.33}$$

which immediately implies that

$$\operatorname{gap}(\{Q\},\tau) \geq T^{-1} \log\left(\frac{3}{2}\right) = \exp(-L^{\frac{1+\varepsilon}{2}}) \log\left(\frac{3}{2}\right). \tag{4.34}$$

By Proposition 1, we want to estimate the term "$(\sup_{x\in V}\#\{i: V_i \ni x\})^{-1}$", by the construction of covering defined in (4.9), we have $(\sup_{x\in V}\#\{i: V_i \ni x\}) \leq 3$, so by (4.8), (4.11), (4.32) and (4.34), we have

$$\operatorname{gap}(\Lambda(L,L_1);\ \beta,\tau) \geq \frac{1}{3} \inf_i \inf_\varphi \operatorname{gap}(L_{V_i}^{\beta,\varphi}) \operatorname{gap}(\{Q\},\tau)$$

$$\geq \frac{1}{3}(2L+1)^{-2} c_m \exp(-8\sqrt{2}\beta 2L^{\frac{1}{2}+\varepsilon}) \exp(-L^{\frac{1+\varepsilon}{2}}) \log\left(\frac{3}{2}\right) \tag{4.35}$$

so we can find some $C>0$, when β large enough

$$\geq \exp(-C\beta L^{\frac{1}{2}+\varepsilon}) \tag{4.36}$$

Now we complete the proof.

Acknowledgment. The author would like to thank Professor Yasunari Higuchi for his kind helping and guidance in this work, and also thanks for his important comments and a careful reading of the manuscript.

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