J. Math. Kyoto Univ. (JMKYAZ)
39-3 (1999), 421–434

On the area of the complement of the invariant component of certain b-groups and on sequences of terminal regular b-groups

By

Hideki MIYACHI

Introduction

Let G be a finitely generated Fuchsian group of the first kind, and $\partial T(G)$ the Bers boundary of the Teichmüller space of G. Let χ_φ be the canonical isomorphism from G to the b-group corresponding to $\varphi \in \partial T(G)$ with suitable normalizations (cf. Section 1.2), and Δ_φ the invariant component of $\chi_\varphi(G)$. We know that *any* $\varphi \in \partial T(G)$ has a sequence $\{\varphi_m\}_{m=1}^{\infty}$ corresponding to terminal regular b-groups in $\partial T(G)$ such that φ_m converges to φ_0 and that the area of $C \backslash \Delta_{\varphi_m}$ tends to zero (cf. Remark (2) in Section 3.3). The main result of this paper is the following.

Theorem 1. *Let* $\{\varphi_n\}_{n=1}^{\infty} \subset \partial T(G)$ *be a sequence corresponding to terminal regular b-groups such that*

(a) *For any hyperbolic element* $g \in G$, *there exist* $\epsilon(g)$, $N(g) > 0$ *such that for* $n > N(g)$, *if* $\chi_{\varphi_n}(g)$ *is loxodronmic, then* $|\mathrm{tr}^2(\chi_{\varphi_n}(g)) - 4| \geq \epsilon(g)$, *and*

(b) *The Euclidean area of* $C \backslash \Delta_{\varphi_n}$ *tends to* 0 *as* $n \to \infty$.

Then every accumulation point of the sequence corresponds to a totally degenerate group.

This paper is organized as follows: In section 1, we fix our notations and recall some basic definitions and facts. Section 2 deals with a lower estimate of the complement of the invariant component of a b-group which contains triangle groups as component subgroups. This class of b-groups, by definition, involves the set of terminal regular b-groups. In Section 3, we give the proof of Theorem 1 and several remarks about our result.

The author would like to express his hearty gratitude to Professor Yoichi Imayoshi for his constant encouragement. He would like to thank Professor Hiromi Ohtake, Professor Hiroshige Shiga, and Professor Masahiko Taniguchi for their useful advices and conversations. He also thanks Professor Yohei Komori for useful and stimulating conversations.

Communicated by Prof K. Ueno, September 18, 1998

1. Preliminaries

1.1. For $E \subset \hat{C}$, we denote by Möb(E) the group of Möbius transformations g satisfying that $g(E)=E$. Through this paper, all discrete groups in Möb($\hat{C}$) are *torsion free*. A finitely generated non-elementary Kleinian group Γ is called a b-group if it has precisely one simply connected invariant component Δ_Γ of its region of discontinuity $\Omega(\Gamma)$. By Ahlfors' finiteness theorem, a b-group represents a finitely many Riemann surfaces, each with a finite Poincaré area. By Bers' second area inequality (cf.[18]), the total Poincaré area of $\Omega(\Gamma)/\Gamma$ is at most twice the Poincaré area of Δ_Γ/Γ. If equality holds, Γ is called *regular.*

Let Γ be a b-group. For an accidental parabolic transformation (A.P.T.) $g \in \Gamma$, we denote by A_g the axis of g (cf. [14,p.611] and [14, Lemma 1]). Let $\{g_i\}_{i=1}^s$ be a basis for A.P.T.s in Γ. (cf. [14, p.612]). Let π be a projection mapping from Δ_Γ to $R:=\Delta_\Gamma/\Gamma$. Then, a system $C_\Gamma:=\{\pi(A_g)\}_{g:\text{A.P.T.}}$ becomes a partition on A, that is, C_Γ is the system of mutually disjoint simple closed geodesics (cf. [14, p.613]). The system C_Γ and a components of $R\backslash C_\Gamma$ are called the *partition with respect to* Γ and a *block of* Γ respectively.

For $E \subset R$, a stabilizer group of a component of $\pi^{-1}(E)$ in Γ is called a *covering group of E* in Γ (cf. [16, p.251]). A covering group of a block is called a *structure subgroup.* We say that a set of structure subgroups $\{H_j\}_{j=1}^s$ of Γ is a *basis of structure subgroups* of Γ if each H_i are not mutually conjugate in Γ and every structure subgroup is conjugate some H_i in Γ. A stabilizer subgroup of a component of $\Omega(\Gamma)-\Delta_\Gamma$ in Γ is called a *component subgroup.* We say that a Kleinian group Γ is a *b-group with no moduli* if Γ is a b-group satisfying either $\Omega(\Gamma)$ is connected or each component subgroup of Γ is a triangle group (cf.[7]), where a Kleinian group is called a triangle group if it is conjugate in Möb($\hat{C}$) to the principal congruence subgroup of level 2:

$$\langle z \mapsto z+2,\ z \mapsto z/(-2z+1)\rangle.$$

A b-group Γ is called *terminal regular* if Γ is regular and has no moduli. A b-group Γ is called *totally degenerate* if $\Omega(\Gamma)$ is connected.

1.2. Let G be a finitely generated Fuchsian group of the first kind acting on $\Sigma:=\{z \in \hat{C} \mid |z|>1\}$. Let $B(G)$ be the complex Banach space of holomorphic functions $\varphi(z)$, $z \in \Sigma$ with norm $\|\varphi\|=\sup(|z|^2-1)^2|\varphi(z)|/4<\infty$ which satisfy the functional equation of quadratic differentials $\varphi(g(z))g'(z)^2=\varphi(z)$, $g \in G$. It is well-known that $\dim B(G)<\infty$ and that for every $\varphi \in B(G)$, there exists the local univalent function $W_\varphi(z)$, $z \in \Sigma$ such that the Schwarzian derivative $\{W_\varphi, -\}$ of W_φ is equal to φ and that W_φ forms

$$W_\varphi(z)=z+\sum_{n=1}^{\infty} b_n z^{-n}$$

near $z=\infty$. For $\varphi \in B(G)$, we denote by χ_φ the homomorphism from G into Möb($\hat{C}$) defined by the equation $\chi_\varphi(g) \circ W_\varphi = W_\varphi \circ g$, $g \in G$.

The Teichmüller space $T(G)$ of G can be identified with a bounded domain in $B(G)$. We know that for $\varphi \in \partial T(G)$, W_φ is univalent and $G_\varphi := \chi_\varphi(G)$ is a b-group with the invariant component $\Delta_\varphi := W_\varphi(\Sigma)$. We call $\partial T(G)$ the *Bers boundary of* $T(G)$ (see [7], [9], and [12]). Let Γ be a b-group and f a conformal mapping from Σ to Δ_Γ, if $\{f, -\} \in \partial T(f^{-1}\Gamma f)$, Γ is called a *boundary group.*

1.3. Assume that $f(z) = z + b_0 + \Sigma_{k=1}^{\infty} b_k z^{-k}$ is univalent on Σ. Then the following inequality, called the *Golusin's inequality*, holds:

$$\sum_{k=1}^{\infty} k \left| \sum_{l=1}^{N} b_{kl} \lambda_l \right|^2 \leq \sum_{l=1}^{N} \frac{|\lambda_l|^2}{l}, \tag{1}$$

for any $\lambda_l \in C$, $(l = 1, 2, \cdots, N)$, and

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = -\sum_{k,l=1}^{\infty} b_{kl} z^{-k} \zeta^{-1} \qquad (z, \zeta \in \Sigma).$$

(cf. [3,p.91]). The coefficients $\{b_{kl}\}_{k,l=1}^{\infty}$ are called the *Grunsky coefficients* of f. This inequality induces the following:

$$\left| \sum_{k,l=1}^{\infty} b_{kl} \lambda_k \mu_l \right|^2 \leq \sum_{l=1}^{\infty} \frac{|\lambda_l|^2}{l} \sum_{l=1}^{\infty} \frac{|\mu_l|^2}{l}. \tag{2}$$

for λ_k, $\mu_k \in C$ $(k = 1, 2, \cdots)$ such that $\{k^{-1/2}\lambda_k\}_{k=1}^{\infty}$ $\{k^{-1/2}\mu_k\}_{k=1}^{\infty} \in l^2$.

2. The area of the complement of an invariant component

In this section, we will give a lower estimate of the Euclidean area of the complement of the invariant component of a b-group containing triangle groups as component groups.

2.1. Let $F < \text{Möb}(\hat{C})$ be a triangle group so that $\infty \in \Omega(F)$. Let $\{A, B\}$ be a generator of F such that A, B, and AB are parabolic. Then, we have

Lemma 1. *For $g(z) = (az + b)/(cz + d)$, $(ad - bc = 1)$, let $c_g = |c|$. Then the Euclidian area of the bounded inuariant component of F is more than $4\pi(c_A^2 + c_B^2 + c_{AB}^2)^{-1}$.*

Proof. The direct calculation gives that the interior of the circumscribing circle of the triangle whose edges has lengths x,y, and z has the area

$$\pi \left\{ 2\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \right) - \left(\frac{y^2}{z^2 x^2} + \frac{z^2}{x^2 y^2} + \frac{x^2}{y^2 z^2} \right) \right\}^{-1}.$$

Let a, b and c be fixed points of A, B and AB respectively. By Proposition 12.1 in [11, p.571],

$$c_A = \frac{2|c-b|}{|c-a||a-b|}, \quad c_B = \frac{2|a-c|}{|a-b||b-c|}, \quad \text{and} \quad c_{AB} = \frac{2|b-a|}{|b-c||c-a|}.$$

Therefore, we have the assertion.

2.2. We have the following lemma (cf. [22, p.372, Section 4]).

Lemma 2. *Let* $A \in \text{Möb}(\Sigma)$ *with* $A(\infty) \neq \infty$ *and* $\text{Möb}(\hat{C})$ *a parabolic element. Suppose that there exists a univalent function* f *from* Σ *into* $\hat{C}$ *with normalization* $f(z) = z + O(1)$ *near* $z = \infty$ *such that* $g \circ f = f \circ A$. *Then*

$$c_g^2 \leq 4(1 - |A'(0)|)/\text{tr}^2(A)|A'(0)|^3 \tag{3}$$

Proof. Let $\{b_{kl}\}_{k,l=1}^{\infty}$ be the Grunsky coefficient of f. By definition,

$$\begin{aligned} g(\infty) - g^{-1}(\infty) &= f(A(\infty)) - f(A^{-1}(\infty)) \\ &= (A(\infty) - A^{-1}(\infty)) \exp\left\{ - \sum_{k,l=1}^{\infty} b_{kl} A(\infty)^{-k} (A^{-1}(\infty))^{-l} \right\}. \end{aligned}$$

Since $A(\infty) = \{\overline{A(0)}\}$, $|A(0)| < 1$, and $|A(0)| = |A^{-1}(0)|$, $\{k^{-1/2}\overline{A(0)}^k\}_{k=1}^{\infty}$, $\{k^{-1/2}\overline{A^{-1}(0)}^k\}_{k=1}^{\infty}$ are contained in l^2. By (2), we have

$$\begin{aligned} \frac{|\text{tr}^2(g)|}{c_g^2} &= |g(\infty) - g^{-1}(\infty)|^2 \\ &= |A(\infty) - A^{-1}(\infty)|^2 \exp\left\{ -2\text{Re}\left(\sum_{k,l=1}^{\infty} b_{kl} \overline{A(0)}^k (\overline{A^{-1}(0)})^l \right) \right\} \\ &\geq \frac{\text{tr}^2(A)|A'(0)|}{1 + |A'(0)|} \exp\left\{ -2\left(\sum_{k=1}^{\infty} \frac{|A(0)|^{2k}}{k} \sum_{k=1}^{\infty} \frac{|A^{-1}(0)|^{2k}}{k} \right)^{1/2} \right\} \\ &= \frac{\text{tr}^2(A)|A'(0)|}{1 - |A'(0)|} \exp\{2 \log(1 - |A(0)|^2)\} = \frac{\text{tr}^2(A)|A'(0)|^3}{1 - |A'(0)|}. \end{aligned}$$

Since g is parabolic, we conclude (3).

2.3. For a Fuchsian group G acting on Σ and $\epsilon > 0$, by the *ϵ-thick part* $\text{thick}_\varepsilon(G)$ for G we mean that the set of points $z \in \Sigma$ such that the hyperbolic distance $d(z, g(z))$ between z and $g(z)$ is more than ϵ for all parabolic $g \in G$. For b-group Γ, let f be a conformal mapping from Σ to Δ_Γ. We define the ϵ-thick part $\text{thick}_\varepsilon(\Gamma)$ for Γ by $f(\text{thick}_\varepsilon(f^{-1}\Gamma f))$.

For a parabolic $A \in \text{Möb}(\Sigma)$ and $\epsilon > 0$, the *ϵ-horocycle* $C_{A,\varepsilon}$ of A is, by definition, the circle C in $\overline{\Sigma}$ through the fixed point of A such that for $d(z, A(z)) = \epsilon$ for $z \in C$. For a hyperbolic $A \in \text{Möb}(\Sigma)$, we denote by L_A the axis of A.

We say that a closed curve in Σ/G is the *ϵ-horocycle* if there exists a

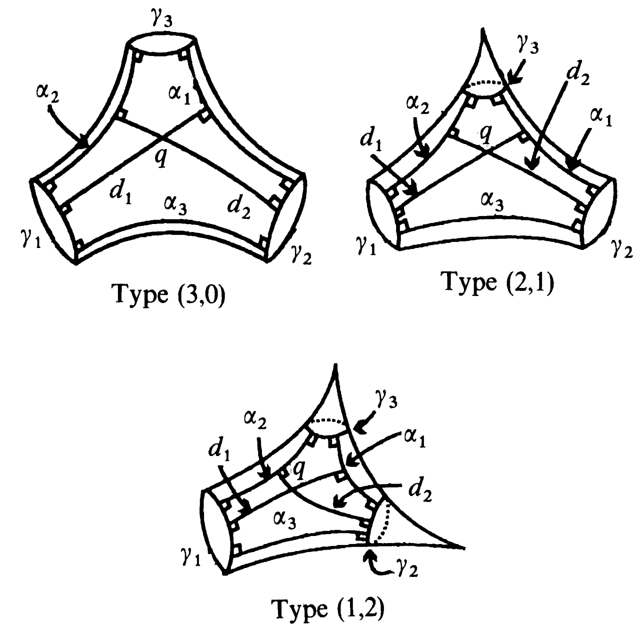

Figure 1. Pairs of Pants

primitive parabolic $g \in G$ so that the curve is the image of the *ϵ-horocycle* of g by the projection. For a rectifiable curve C in $R := \Sigma/G$, we denote by $l_R(C)$ the hyperbolic length of C on R. In this paper, a bordered Riemann surface of finite type $(0, j, 3-j)$ whose borders consisting of geodesics is called the *pair of pants of type* $(3-j, j)$ (cf. Figure 1.)

Let $\epsilon_0 = 2\operatorname{arcsinh} 1$ and $\epsilon < \epsilon_0$. Let P be a pair of pants of type $(3-j, j)$. Then there exist geodesics $\{\gamma_i\}_{i=1}^{3-j}$, $\{\alpha_i\}_{i=1}^{3}$, $\{\mathrm{d}_i\}_{i=1}^{2}$ ϵ-horocycles $\{\gamma_i\}_{j=3-j+1}^{3}$, if $j \neq 0$, and the point q as in Figure 1.

The following lemma can be proved in the argument similar to that of Theorem 2.4.3 and 3.1.8 in [8] and Lemma 4.4 in [20]. Hence we omit the proof. The author would like to thank Professor Toshihiro Nakanishi for teaching about the joint work [20] with Professor Marjatta Näätänen.

Lemma 3. *Let P be a pair of pants of type $(3-j, j)$. Let $\{\gamma_i\}_{i=1}^{3}$, $\{\alpha_i\}_{i=1}^{3}$ and $\{d_i\}_{i=1}^{2}$, and q as in Figure 1. Let d_3 be the shortest geodesic connegting γ_β and α_β. Let l_i and $l(d_i)$ be the lengths of γ_i and d_i respectively. Then*
(a) *d_3 passes through q.*
(b) *Let $L_i = \cosh(l_i/2)$ $(1 \leq j \leq 3-j)$. then*
(b-1) *If P is of type* $(3,0)$, *then*

$$\cosh(l(d_i)) = \frac{(L_1^2 + L_2^2 + L_3^2 + 2L_1L_2L_3 - 1)^{1/2}}{\sinh(l_i/2)}, \text{ for } i = 1, 2, 3.$$

(b-2) *If P is of type* (2,1), *then*

$$\cosh(l(d_i))=\frac{L_1+L_2}{\sinh(l_i/2)},\ \textit{for}\ j=1,2,\ \textit{and}\ e^{l(d_3)}=\frac{L_1+L_2}{\sinh(\epsilon/2)}.$$

(b-3) *If P is of type* (1,2), *then*

$$\cosh(l(d_1))=\frac{L_1+1}{\sinh(l_1/2)},\ \textit{and}\ e^{l(d_i)}=\frac{L_1+1}{\sinh(\epsilon/2)},\ \textit{for}\ i=2,3.$$

2.4. Let Γ be a b-group which contains triangle groups as structure groups. We denote by $\{P_k\}_{k=1}^{s_0}$ the blocks of Γ. Let π is the projection from Δ_Γ to R and f the a conformal mapping from Σ to Δ_Γ.

We may assume that for $1\leq k\leq s$, P_k is a pair of pants of type $(3-j_k, j_k)$. Let $\{\gamma_{k,j}\}_{j=1}^3$ be boundary curves of thick$_\varepsilon(G)/G\cap P$, We assume that for $1\leq j\leq 3-j_k$, $\gamma_{k,j}$ is a geodesic (see Figure 1).

Lemma 4. *Fix $0<\epsilon<\epsilon_0$ so that $\infty\in$ thick$_\varepsilon(G)$. Then, for $k=1,\cdots,s$, there exist a structure group Γ_k corresponding to P_k and generators $\{C_{k,i}\}_{i=1}^3$ of $H_k:=f^{-1}\Gamma_k f$ such that*

(i) *For $i=1,2,3$, if $C_{k,i}$ is hyperbolic* (*resp. parabolic*), *the the axis* (*resp. ϵ-horocycle*) *of f^{-1} is mapped to $\gamma_{k,i}$ by $\pi\circ f$.*

(ii) $C_{k,3}C_{k,2}C_{k,1}=id$

(iii) $d(\infty, L_{C_{k,i}})$ (*resp.* $d(\infty, L_{C_{k,i\varepsilon}})$) $\leq$ diam(thick$_\varepsilon(G)/G)+\delta(k,i)$, *where $\delta(k,i)$ is $l(d_i)$ as in Lemma* 3 *with respect to curves $\gamma_i:=\gamma_{k,i}$, $i=1,2,3$ and $P:=P_k$, and* diam(E) *is the hyperbolically diameter of $E\subset R$.*

Proof. Fix $k\in\{1,\cdots,s\}$. We only show the case where P_k is of type (3,0). Another cases are proved in the similar manner.

On P_k, let $\{\gamma_{k,i}\}_{i=1}^3$, $\{\alpha_{k,i}\}_{i=1}^3$, and $\{d_{k,i}\}_{i=1}^3$ be geodesics as in Lemma 3. Let q_k be a intersection point of $d_{k,1}$ and $d_{k,2}$. By Lemma 3, $l_R(d_{k,i})=\delta(k,i)$. Take a geodesic β_k in R connecting $c_0:=\pi\circ f(\infty)$ and q_k such that $l_R(\beta_k)\leq$ diam(thick$_\varepsilon(G)/G$). We define the curve $\beta_{k,i}\subset\beta_k\cup d_{k,i}$ connecting q_0 and $\gamma_{k,i}$ so that $l_R(\beta_{k,i})$ $\leq$ diam(thick$_\varepsilon(G)/G)+\delta(k,i)$. We construct a loop $c'_{k,i}:=\beta_{k,i}\gamma_{k,i}\beta_{k,i}^{-1}$ with an initial point q_0. We give an orientation for $c'_{k,i}$ such that $[c'_{k,1}][c'_{k,2}][c'_{k,3}]=1$, where $[-]$ is an equivalence class in $\pi_1(R,q_0)$, the fundamental group of R with the base point q_0.

We take $C_{k,i}\in f\Gamma f^{-1}$ corresponding to $[c'_{k,i}]$ by the canonical isomorphism between $\pi_1(R,q_0)$ and $f\Gamma f^{-1}$. Let $z_k\in\Sigma$ be the end point of the lift of β_k whose initial point is the point at ∞. Let Γ_k be a structure group which stabilizes the component of $\Delta_\Gamma\backslash\cup_{g:\text{A.P.T.}}A_g$ containing $f(z_k)$. Then, by definition, the system $\{C_{k,i}\}_{i=1}^3$ generates H_k and satisfies the assertion of lemma.

2.5. We prove the following theorem.

Theorem 2. *Let Γ be a b-group such that $\infty \in \Delta_\Gamma$ and that the logarithmic capacity of the limit set of Γ is equal to 1. Take $0<\epsilon<\epsilon_1$ such that $\infty \in \mathrm{thick}_\epsilon(\Gamma)$. Let $\{P_k\}_{k=1}^{so}$ be blocks of Γ each of which is a pair of pants. Then there exists $A(\Delta_\Gamma/\Gamma, \{P_k\}_{k=1}^{so}, \epsilon)>0$ such that the Euclidian area $\mathrm{Area}(\boldsymbol{C}\backslash\Delta_\Gamma)$ of $\boldsymbol{C}\backslash\Delta_\Gamma$ is more than $A(\Delta_\Gamma/\Gamma, \{P_k\}_{k=1}^{so}, \epsilon)$. Furhermore, for $M>0$ and some k, if lengths of all closed geodesics in the boundary of P_k are less than M, then there exists $A>0$. depend only on Δ_Γ/Γ, M, and ϵ, so that $\mathrm{Area}(\boldsymbol{C}\backslash\Delta_\Gamma)\geq A$.*

Proof. The direct calculations shows that if $A \in \mathrm{M\ddot{o}b}(\Sigma)$ is hyperbolic,

$$|A'(0)|=4/(\mathrm{tr}^2(A)-4\tanh^2(d(\infty, L_A)))\cosh^2(d(\infty, L_A)), \tag{4}$$

and if A is parabolic and $\infty \in \mathrm{thick}_\epsilon(\langle A\rangle)$,

$$|A'(0)|=1/1+\sinh^2(\epsilon/2)e^{2d(\infty, C_{A,\epsilon})}. \tag{5}$$

Let f be the conformal mapping from Σ to Δ_Γ such that $f(z)=z+O(1)$ near $z=\infty$ (cf. [21, p.207, Corollary 9.9]). For $k\in\{1,\cdots,s\}$. Let H_k and $\{C_{k,i}\}_{i=1}^{3}$ be as in Lemma 4. Since $\infty \in \mathrm{thick}_\epsilon(\langle g\rangle)$ by Lemma 2, Lemma 4, and (4) and (5), $A(\Delta_\Gamma/\Gamma, \{P_k\}_{k=1}^{so}, \epsilon)=64\pi\Sigma_k A(\Delta_\Gamma/\Gamma, P_k, \epsilon)$ where

$$AA(\Delta_1/\Gamma, P_k, \epsilon):=\left\{\sum_{i=1}^{3} S_{k,i}(\sinh^2(l_{k,i}/2), \mathrm{diam}(\mathrm{thick}_\epsilon(G)/G)+\delta(k,i))\right\}^{-1}$$

and

$$S_{k,i}(x,d):=\begin{cases}(x-4)\cosh^2 d((x-4)\cosh^2 d+4)^2/x, & \text{if } 1\leq i\leq 3-j_k\\ \\ 16\mathrm{cinh}^2(\epsilon/2)e^{2d}(1+\sinh^2(\epsilon/2)e^{2d})^2, & \text{otherwise,}\end{cases}$$

$(k=1,\cdots,s_0)$, satisfies the assertion.

Corollary 1. *For a b-group Γ, let f be a conformal mapping from Σ to Δ_1. Then $\|\{f,-\}\|\leq\dfrac{3}{2}\{1-A(\Delta_\Gamma/\Gamma, \{P_k\}_{k=1}^{so}, \epsilon_0)/\pi\}^{1/2}$, where each P_k is a block of Γ which is a pair of pants.*

Proof. Let $G:=f^{-1}\Gamma f$. Since G is torsion free, for $\varphi\in B(G)$, it holds $\|\varphi\|=\sup\{(|z|^2-1)^2|\varphi(z)|/4| \ z\in\mathrm{thick}_{\epsilon_0}(G)\}$ (cf. [23, Lemma 1] and [4, p.198,Exercise 8.2]). Hence by an argument similar to that of Lemma 6.7 in [9, p.151](the Nehari-Kraus theorem), we conclude the assertion.

3. Sequences of terminal regular b-groups

In this section, by using Theorem 2, we study a behavior of a sequence corresponding to terminal regular b-groups contained in a Bers boundary. Theorem 1 is proved in Section 3.2.

3.1. Let G be a finitely generated Fuchsian group of the first kind acting on Σ. For $\varphi \in \partial T(G)$, we denote by C_φ the partition with respect to G_φ. We show the following lemma.

Lemma 5. *Let $\{\varphi_m\}_{m=1}^{\infty} \subset \partial T(G)$ be a sequence corresponding to terminal regular b-groups. Then there exist a subsequence $\{\varphi_{m_j}\}_{j=1}^{\infty}$, a maximal partition $\{C_k\}_{k=1}^{3p-3+n}$ on $R := \Sigma/G$, a number $k_0 \in \{0, 1, \cdots, 3p-3+n\}$, and homeomorphisms $\{f_j\}_{j=1}^{\infty}$ of R onto itself such that*

(1) *For $j \geq 1$, $C_{\varphi_{mj}} = \{f_j(C_k)\}_{k=1}^{3p-3+n}$,*

(2) *If $k_0 > 0$, then $l_R(f_j(C_k)) \to \infty$ as $j \to \infty$ for $1 \leq k \leq k_0$, and*

(3) *If $k_0 < 3p-3+n$, then $f_j(C_k) = C_k$ for $k > k_0$*

If, in addition, $\mathrm{Area}(\mathbf{C} \backslash \Delta_{\varphi_{mj}}) \to 0$ *as $j \to \infty$, then it also holds that*

(4) *No component of $R \backslash \cup_{k > k_0} C_k$ is a pair of pants, and hence $k_0 > 0$.*

Proof. Since the number of graphs induced by the maximal partition on R is finite (cf.[2],[11]), we may assume that all graphs induced from $\{C_{\varphi_m}\}_{m=1}^{\infty}$ are the same. Let us denote $C_{\varphi_1} = \{C'_k\}_{k=1}^{3p-3+n}$. Then, there exist homemorphisms $\{h_m\}_{m=1}^{\infty}$ of R onto itslf such that $C_{\varphi_m} = \{h_m(C'_k)\}_{k=1}^{3p-3+n}$ (cf.[8, Appendix]). By taking the subsequence of $\{h_m\}_{m=1}^{\infty}$ and renumbering the curves $\{C'_k\}_{k=1}^{3p-3+n}$ if necessary, we may suppose that there exist $k_0 \in \{0, 1, \cdots, 3p-3+n\}$ and $M > 0$ such that if $k_0 > 0$, then $l_R(h_m(C'_k)) \to \infty$ as $m \to \infty$ for $1 \leq k \leq k_0$, and that if $k_0 < 3p-3+n$, then $l_R(h_m(C'_k)) < M$ for $k_0 < k \leq 3p-3+n$ and $m \geq 1$.

Since the number of closed geodesics in R whose hyperbolic length are less than M is finite (cf.[2]), there exists a subsequence $\{\varphi_{m_j}\}_{j=1}^{\infty}$ such that $h_{m_j}(C'_k) = h_{m_l}(C'_k)$ for $j, l \geq 1$ and $k_0 < k \leq 3p-3+n$.

Let $f_j = h_{n_j} \circ (h_{n_1})^{-1}$ and $C_k = h_{n_1}(C'_k)$ for $j \geq 1$ and $1 \leq k \leq 3p-3+n$. Then, by definition, the subsequence $\{\varphi_{m_j}\}_{j=1}^{\infty}$ the partition $\{C_k\}_{k=1}^{3p-3+n}$ on R, the number k_0 and homemorphisms $\{f_j\}_{j=1}^{\infty}$ satisfy (1)-(3) in this lemma.

From now on, we assume that $\mathrm{Area}(\mathbf{C} \backslash \Delta_{\varphi_{mj}}) \to 0$ as $j \to \infty$. Suppose that there exists a component P_i of $R \backslash \cup_{k > k_0} C_k$ such that P_i is a pair of pants. Since $\{C_k\}_{k > k_0} \subset C_{\varphi_{mj}}$ for each $j \geq 1$ and P_i does not contain the simple closed geodesic which is not homotopic to boundary components of P_i, P_i is a block of $G_{\varphi_{mj}}$ for every $j \geq 1$. Hence each $G_{\varphi_{mj}}$ contains a triangle group as a structure group corresponding to P_i.

Take $\epsilon > 0$ so that $\infty \in \mathrm{thick}_\epsilon(G)$. Since the lengths of all closed geodesics in the boundary of P_i are less than M, by Theorem 2, there exists $A > 0$, depend only on R, M, and ϵ such that $\mathrm{Area}(\mathbf{C} \backslash \Delta_{\varphi_{mj}}) \leq A$ for $j \geq 1$. This contradicts the assumption.

3.2. To prove Theorem 1, it suffices to show the following proposition.

Proposition 6. *Let $\{\varphi_m\}_{m=1}^{\infty}$ be a sequence corresponding to terminal regular b-groups in $\partial T(G)$ satisfying* (1)–(3) *in Lemma* 5 *with respect to a partition $\{C_k\}_{k=1}^{3p-3+n}$, a number k_0, and homeomorphisms $\{f_m\}_{m=1}^{\infty}$ of $R:=\Sigma/G$ onto itself. Suppose that the sequence satisfies* (a) *in Theorem* 1 *and converges to $\varphi_0 \in \partial T(G)$. Then G_{φ_0} is a b-group with no moduli such that $C_{\varphi_0}=\{C_k\}_{k>k_0}$. Especially, if no component of $R \backslash \cup_{k>k_0} C_k$ is a pair of pants, then G_{φ_0} is a totally degenerate group.*

Proof. We prove the case $k_0<3p-3+n$. The case where $k_0=3p-3+n$ is proved by the similar manner.

Let $g_k \in G$ be primitive hyperbolic elements corresponding to C_k for $1 \leq k \leq 3p-3+n$. We denote by $\{P_i\}_{i=1}^{s_0}$ the components of $R \backslash \cup_{k>k_0} C_k$ each of which is not a pair of pants. Let $\{P_i\}_{i=s+1}^{s_0}$ be components of $R \backslash (\cup_{k>k_0} C_k \cup \cup_{i=1}^{s_0} P_i)$. Fix a stabilizer group H_i corresponding to P_i in G. Let $G_{i,m}=\chi_{\varphi_m}(H_i)$ for $m \geq 0$. For $m \geq 1$, since G_{φ_m} is a terminal regular b-group and, P_i is not a pair of pants for $1 \leq i \leq s_0$. $G_{i,m}$ is also a terminal regural b-group such that $\Delta_{G_{i,m}}/G_{i,m}$ is homeomorphic to P_i if $1 \leq i \leq s_0$ (cf.[11]). By definition, for $i>s_0$ is a triangle group.

We first show that $\{G_{i,0}\}_{i=1}^{s_1}$ is a basis of the structure groups of G_{φ_0}. It is clear that for $k>k_0$, $\chi_{\varphi_0}(g_k)$ is an A.P.T. in G_{φ_0}. Since $G_{i,0}=\chi_{\varphi_0}(H_i)$, it suffices to show that $\chi_{\varphi_0}(g)$ is loxodromic for any hyperbolic element $g \in G$ which is not conjugate to a power of g_k for any k.

If the geodesic corresponding to g meets C_k for some $k>k_0$ then $\chi_{\varphi_0}(g)$ is loxodromic. Hence we can take $\epsilon(g)$, $N(g)$ satisfying (a) for g in this theorem. Thus we assume that the geodesic corresponding to g is contained in some P_k. By (2) in Lemma 5, there exists $N(g)>0$ so that $\chi_{\varphi_0}(g)$ is loxodromic for $m \geq N(g)$. By assumption (a), there exist $\epsilon(g)>0$ such that for $m \geq N(g)$, inequalities

$$|\mathrm{tr}^2(\chi_{\varphi_m}(g))-4| \geq \epsilon(g)$$

hold. Since $\chi_{\varphi_m}(g) \rightarrow \chi_{\varphi_0}(g)$ as $m \rightarrow \infty$, we have that $\mathrm{tr}^2(\chi_{\varphi_0}(g)) \neq 4$. Since χ_{φ_0} is an isomophism and G is torsion free, $\chi_{\varphi_0}(g)$ is loxodromic.

Thus, if $1 \leq i \leq s_0$, $G_{i,0}$ is either a quasi-Fuchsian group or a totally degenerate group without A.P.T.s (cf.[15], [17, p.225, Theorem D.21], and [17, p.268, Theorem C.25]). We assume that $G_{i,0}$ is a quasi-Fuchsian group for some i. By the arguments above, for $m \geq 1$, the isomorphism $\chi_{\varphi_m} \circ \chi_{\varphi_0}^{-1}$ from $G_{i,0}$ onto $G_{i,m}$ is allowable in the sense of Bers (cf. [5, p.574]). Since $\chi_{\varphi_m} \circ \chi_{\varphi_0}^{-1}$ converges to the identity on $G_{i,0}$ by the quasiconformally stability for quasi-Fuchsian groups (cf. [5, Proposition 6]), $G_{i,m}$ is quasi-Fuchsian for sufficiently large m. This is contradiction. Thus, $G_{i,0}$ is a totally degenerate group without A.P.T.s for $i=1,\cdots,s_0$. Thus, G_{φ_0} is a b-group with no moduli such that $C_{\varphi_0}=\{C_k\}_{k>k_0}$.

3.3. Remark. (1) *Any* sequence corresponding to terminal regular b-groups which conveges to $\varphi_0 \in \partial T(G)$ corresponding to totally degenerate group without

A.P.T.s satisfies (a) and (b) in Theorem 1.

(2) For any $\varphi\in\partial T(G)$, there exists $\{\varphi_m\}_{m=1}^{\infty}$ in $\partial T(G)$ corresponding to terminal regular b-groups such that (2-i) Area$(C\backslash\Delta_{\varphi_m})$ tends to zero, and (2-ii) $\{\varphi_m\}_{m=1}^{\infty}$ converges to φ_0.

(3) *Any* totally degenerate group G_{φ_0} *with* A.P.T.s has $\{\varphi_m\}_{m=1}^{\infty}$ in $\partial T(G)$ corresponding to terminal regular b-groups which converges to φ_0 such that (3-i) Area$(C\backslash\Delta_{\varphi_m})$ tends to zero, and (3-ii) it does not satisfy (a) in Theorem 1.

(4) If dim $T(G)>1$, there exists $\{\varphi_m\}_{m=1}^{\infty}$ corresponding to terminal regular b-groups in $\partial T(G)$ satisfying (a) in Theorem 1 such that $\{G_m\}_{m=1}^{\infty}$ converges to a b-group but not a totally degenerate group.

Proof. Before proving (1)-(4) above, we note that $\{\varphi_m\}_{m=1}^{\infty}$ corresponding to terminal regular b-groups in $\partial T(G)$ which converges to $\varphi_0\in\partial T(G)$ corresponding to a totally degenerate group without A.P.T.s satisfies that Area$(C\backslash\Delta_{\varphi_m})\to 0$ as $m\to\infty$. Indeed, it follow from the following two facts: (1) The measure of $C\backslash\Delta_{\varphi_0}=\Lambda(G_{\varphi_0})$ is zero by Thurston's theorem (cf.[18], and (2) $\{\Delta_{\varphi_m}\}_{m=1}^{\infty}$ converges to Δ_{φ_0} in the sense of kernel convergence with respect to ∞ (cf. [21, Theorem 1.8]).

Let us prove Remark (1)-(4).

(1) By the argument above, the sequence satisfies (a) in Theorem 1. Since G_{φ_0} has no A.P.T.s, that also satisfies (a) in Theorem 1.

(2) Since the set of differentials corresponding to terminal regular b-groups and the set of those corresponding to totally degenerate groups with-out A.P.T.s are dense in $\partial T(G)$ (cf.[19] and [5, Theorem 14]), by the standard arguments and Remark (1), we find a sequence satisfying (2-i) and (2-ii). This remark was pointed out to the author by Professor Hiroshige Shiga.

(3) Let $\{g_k\}_{k=1}^{s}$ be hyperbolic elements in G so that $\{\chi_{\varphi_0}(g_k)\}_{k=1}^{s}$ is a basis of A.P.T.s of G_{φ_0}. Take $L_0>0$ so that $2\cosh(L_0/2):=\max_{\Gamma\leq k\leq s}|\mathrm{tr}(g_k)|$. By applying the argument in Lemma 5 for $\{\psi_m\}_{m=1}^{\infty}$ corresponding to totally degenerate groups without A.P.T.s in $\partial T(G)$ which converges to φ_0, there exists a terminal regular b-group G_{φ_m} such that $l_{\Sigma/G}(C)>mL_0$ for each $C\in C_{\varphi_m}$, $\|\varphi_m-\psi_m\|<1/m$, and that Area$(C\backslash\Delta_{\varphi_m})<1/m$ for $m\geq 1$. By the definition of L_0, $\chi_{\varphi_m}(g_k)$ is loxodromic for each $m\geq 1$ and $k=1,\cdots,s$. Since $\chi_{\varphi_0}(g_k)$ is parabolic, $\{\varphi_m\}_{m=1}^{\infty}$ satisfies (3-i) and (3-ii).

(4) Let $R=\Sigma/G$ and $C=\{C_k\}_{k=1}^{d}$ a maximal partition on R. Let $\{P_s\}_{s=1}^{s_1}$ be the components of $R\backslash\cup_{k\neq 1}C_k$ such that $C_1\subset P_1$. Since $d=\dim T(G)>1$, we may suppose that $s_1>1$ and that P_2 is a pair of pants. Let R_1 be the infinite Nielsen extension of P_1 (cf.[6]), Γ_1 the Fuchsian group of R_1 and $(\Gamma_1)_{\psi_0}$ a totally degenerate group without A.P.T.s such that $\psi_0\in\partial T(\Gamma_1)$. We define $\{\varphi_m\}_{m=1}^{\infty}$ of a sequence corresponding to terminal gerular b-groups and $\varphi_0\in\partial T(G)$ satisfying the conditions (a), (b), (c), and (d) in Theorem 3 in Section 3.4 for the partition C, $s_0=1$, and $F_1=(\Gamma_1)_{\psi_0}$. Then $\{\varphi_m\}_{m=1}^{\infty}$ satisfies the assertion.

If $\dim T(G)=1$, then G_φ has an A.P.T. if and only if G_φ is a terminal regular b-group. By Remark (1) and the proof of Proposition 6, we have

Corollary 2. *Suppose that* $\dim T(G)=1$. *For a sequence* Φ *corresponding to terminal regular b-groups in* $\partial T(G)$ *which converges to* $\varphi_0 \in \partial T(G)$, *the following three conditions are equivalent:*

(1) G_{φ_0} *is a totally degenerate group.*

(2) Φ *contains a subsequence with* (a) *and* (b) *in Theorem* 1.

(3) Φ *contains a subsequence consisting of mutually distinct elements and satisfies* (a) *in Theorem* 1.

3.4. To complete the proof of Remark (4) in the previous subsection, we will show the following theorem.

Theorem 3. *Let G be a finitely generated Fuchsian group of the first kind acting on* Σ *and* $R=\Sigma/G$. *Let* $C=\{C_k\}_{k=1}^{k_0}$ *be a partition on R and* $\{P_s\}_{s=1}^{s_0}$ *the components of* $R\setminus\cup_{k=1}^{k_0}C_k$ *each of which is not a pair of pants. For* $i=1,\cdots,s_0$, *let* F_i *be a boundary group such that* Δ_{F_i}/F_i *is homeomorphic to* P_i. *Then there exist* $\varphi_0 \in \partial T(G)$ *and corresponding to terminal regular b-groups such that*

(a) $\varphi_m \to \varphi_0$ *as* $m \to \infty$,

(b) $C \subset C_{\varphi_m}$ *for* $m \geq 0$, *and*

(c) *A covering group of* P_i *in* G_{φ_0} *is quasiconformally conjugate to* F_i, *If, in addition, each* F_i *is a totally degenerate group without A.P.T.s, then*

(d) Φ *satisfies* (a) *in Theorem* 1.

This theorem is proved in Section 3.6.

3.5. The following lemma is well-known. However, the author has never seen what is stated in this form.

Lemma 7. *Let R and S be a hyperbolic Riemann surface of type* (p,n). *Let P be a domain in R such that P is homeomorphic to R and that the inclusion mapping i from P to R is homotopic to a homeomorphism of P onto R. Then, for* $K \geq 1$, *there exists* $K_0=K_0\ (K,P,p,n)>1$ *such that if a K-quasiconformal(q.c.) mapping h from P into S which is homotopic to a homeomorphism from P onto S exists, there exists a* K_0*-q.c.mapping g from R to S so that* $g\circ i$ *is homotopic to h.*

Proof. Let $T(R)$ be a Teichmüller space of R (cf. [9, p.120]). Let M be a Riemann surface of type (p,n). If there exists a K-q.c.mapping h_M from P to M homotopic to a homeomorphism of P onto M, then there exists a q.c.mapping $f_{(P,K,M)}$ from R onto S such that $f_{(P,K,M)}\circ i$ is homotopic to h_M. We denote by $X(P,K)$ the closure of the set of such $[M, f_{(P,K,M)}]$ in $T(R)$. Let $\hat{i}$ be a homeomorphism form P to R homotopic to i. Let $\{\gamma_i\}_{i=1}^{N}$ be a system of simple closed geodesics fill up R (cf. [10, p.249]). By the decreasing property for the hyperbolic metric and Wolpert's Theorem (cf. [8, p.153]), $\Sigma_{i=1}^{N} l_S(f_{(P,K,M)}(\gamma_i)) \leq \Sigma_{i=1}^{N} k l_P(\hat{i}^{-1}(\gamma_i))$. Hence

$X(P,K)$ is compact (cf. [10, Lemma 3.1]). Let d_0 be the diameter of $X(P,K)$ with respect to the Teichmüller distance of $T(R)$ (cf. [9, p.125]). Then, $K_0:=e^{d_0}$ satisfies the assertion.

3.6. Let us prove Theorem 3. We only show the case where $s_0=1$. Another cases are proved by the similar manner.

Let $\hat{R}_1$ be the infinite Nielsen extension of P_1. Since $R_1:=\Delta_{F_1}/F_1$ is homeomorphic to P_1, there exists a K_1-q.c.mapping h_0 from $\hat{R}_1$ onto R_1. Let $Q:=h_0(P_1)$, i an inclusion mapping from Q to R_1. Then, by definition, i is homotopic to a homemorfphism from Q onto R_1. Let f be a conformal mapping from Σ to and Δ_{F_1} and $\Gamma_1=f^{-1}F_1f$. We take $\{\psi_m\}_{m=1}^{\infty}$ in $\partial T(\Gamma_1)$ corresponding to terminal regular b-groups which conveges to $\psi_0:=\{f,-\}\in\partial T(\Gamma_1)$. Let $C_{\psi m}=\{C'_{k,m}\}_{k=1}^{k_1}$ and $C_{k,m}$ the geodesic in P_1 (and hence in R) such that $i\circ h_0(C_{k,m})$ is homotopic to $C'_{k,m}$ for $k=1,\cdots,k_1$. Then, $C_m:=\{C_{i,m},C_j\}_{i=1,\cdots,k_1,j=1,\cdots,k_0}$ is a maximal partition on R for $m\geq 1$. Take the terminal regular b-group G_{φ_m} so that $C_{\varphi_m}=C_m$ (cf.[1, Theorem 6]). We may suppose that $\Phi:=\{\varphi_m\}_{m=1}^{\infty}$ converges to some $\varphi_0\in\partial T(G)$. By definition, and satisfy (a) and (b) in this Theorem.

We prove that satisfies (c). Let π be the projection from Σ to R and $\tilde{P}_1$ a component of $\pi^{-1}(P_1)$. We may assume that $\infty\in\tilde{P}_1$. Let H_1 be the stabilizer subgroup of $\tilde{P}_1$ in G and $G_m:=\chi_{\varphi_m}(H_1)$ for $m\geq 0$. Then for $m\geq 1$, G_m is a covering group of P_1 in G_{φ_m} and is a terminal regular b-group (cf.[11]).

Let $S_m=\Delta_{G_m}/G_m$ and π_m the projection from Δ_{G_m} to S_m. Then there exists the injective holomorphic mapping h_m from P_1 to S_m such that $h_m\circ\pi|_{\tilde{P}_1}=\pi_m\circ W_{\varphi_m}|_{\tilde{P}_1}$. By definition, h_m is homotopic to a homeomorphism from P_1 to S_m (cf.[14]). Hence, by Lemma 7, there exist $K_0=K_0\ (K_1,Q,p,n)>0$, and the K_0-q.c.mapping g_m from R_1 to S_m so that $g_m\circ i$ is homotopic to $h_m\circ(h_0|_{P_1})^{-1}$.

Fix the lift $\tilde{h}_0$ of $i\circ h_0|_{P_1}$ from P_1 into Σ. $\tilde{h}_0$ defines the isomorphism ξ from H_1 to Γ_1 by $\xi(h)\circ\tilde{h}_0=\tilde{h}_0\circ h$ for $h\in H_1$. By definition, h_m induces the isomorphism $\chi_{\varphi_m}|_{H_1}$. Since for $m\geq 1$, $g_m\circ i\circ h_0|_{P_1}$ is homotopic to h_m, there exists the lift $\tilde{g}_m$ of g_m from Σ onto Δ_{G_m} so that the isomorphism $\tilde{\eta}_m$ from Γ_1 to G_m defined by $\tilde{\eta}_m(\gamma)=\tilde{g}_m\gamma\tilde{g}_m^{-1}$ satisfies that $\tilde{\eta}_m\circ\xi=\chi_{\varphi_m}|_{H_1}$.

Let $w_m=\tilde{g}_m\circ W_{\psi_m}^{-1}$. Then w_m is a K_0-q.c.mapping from Δ_{ψ_m} onto Δ_{G_m} and defines the isomorphism η_m from G_{ψ_m} to G_m by $\eta_m(g)=w_mgw_m^{-1}$. Then, η_m satisfies that $\eta_m=\chi_{\varphi_m}|_{H_1}\circ(\chi_{\varphi_m}\circ\xi)^{-1}$. Since $g_m\circ i$ is homotopic to h_m, η_m is type preserving. Since G_{ψ_m} and G_m are terminal regular, by the rigidity of triangle groups, w_m can be extended to the K_0-q.c.mapping on $\hat{C}$ compatible with G_{φ_m}. This extension denoted by the same symbol w_m for short.

To prove (c) in Theorem 3, it suffices to show that the family $\{w_m\}_{m=1}^{\infty}$ contains a subsequence which converges to a K_0-q.c.mapping w_0 on $\hat{C}$. Indeed, since $w_mG_{\psi_m}w_m^{-1}=\eta_m(G_{\psi_m})=G_m$ for $m\geq 1$, $w_0G_{\psi_0}w_0^{-1}=G_0$.

Take primitive hyperbolic $g_1,g_2\in H_1$ so that g_1 is not conjugate to g_2 in H_1 and that $\alpha_{i,0}:=\chi_{\varphi_0}(g_i)$ and $\beta_{i,0}:=\chi_{\psi_0}\circ\xi(g_i)$ are loxodromic. Let $\alpha_{i,m}=\chi_{\varphi_m}(g_i)$ and $\beta_{i,m}:=\chi_{\psi_m}\circ\xi(g_i)$. Then there exists $N_1>0$ such that for $m\geq N_1$, $\alpha_{i,m}$ and $\beta_{i,m}$ are loxodromic. For $m \geq N_1$, let $\{a_{2i-1,m}, a_{2i,m}\}$ and $\{b_{2i-1,m}, b_{2i,m}\}$ be the set of the fixed points of $\alpha_{i,m}$ and $\beta_{i,m}$ respectively. By discreteness, the cardinality of $\{a_{j,m}\}_{i=1}^{4}$ and $\{b_{i,m}\}_{i=1}^{4}$ are equal to 4 for $m \geq N_1$ or $m=0$. Since $\alpha_{i,m} \to \alpha_{i,0}$ and $\beta_{i,m} \to \beta_{i,0}$, we may suppose that there exist $N_0 \geq N_1$ and $d>0$ such that $k(a_{j,m}, a_{j,0})$, $k(b_{j,m}, b_{j,0}) < d$ for $j=1, \cdots, 4$ and $m \geq N_0$ and that $k(a_{i,0}, a_{j,0})$, $k(b_{i,0}, b_{j,0}) > 4d$ for $i \neq j$, where $k(-,-)$ is the spherical distance on $\hat{C}$. Let $B_i = \{z \in \hat{C} | k(z, a_{i,0}) \leq d\}$. Since $w_m(\{a_{i,m}\}_{i=1}^{4}) = \{b_{i,m}\}_{i=1}^{4}$ by applying an argument similar to that of Theorem 4.2 in [13, p.70] for domains $\{\hat{C} \backslash B_i \cup B_j\}_{i \neq j}$, there exists a subsequence $\{w_{m_j}\}_{j=1}^{\infty}$ and a K_0-q.c.mapping w_0 so that converges uniformly to w_0.

It is easy to observe that if each F_i is a totally degenerate group without A.P.T.s, then $\{\varphi_m\}_{m=1}^{\infty}$ satisfies (d).

DEPARTMENT OF MATHEMATICS
OSAKA CITY UNIVERSITY

References

[1] W. Abikoff, On the boundaries of Teichmüller spaces and Kleinian groups III, Acta Math., **134** (1975), 211–236.
[2] W. Abikoff, The real analytic theory of Teichmüller space, Lecture Notes in Math. **820**, Springer-Verlag, 1980.
[3] L.V. Ahlfors, Conformal invariant, Topics in Geometric function theory, McGrow-Hill, 1973.
[4] A.F. Beardon, The geometry for discrete groups, Graduate Texts in Mathematics 91, Springer-Verlag, New York, 1983.
[5] L. Bers, On boundaries of Teichmüller spaces and on Kleinian groups: I, Ann. of Math., **79** (1970), 570–600.
[6] L. Bers, Nielsen extensions of Riemann surface, Ann. Acad. Aci. Fenn. **2** (1976), 17–22.
[7] L. Bers, Finite dimensional Teichmüller spaces and generalizations, Bull. A.M.S. (New Series), **5** (1981), 131–172.
[8] P. Buser, Geometry and spectra of compact Riemann surfaces, Progress in Mathematics 106, 1992, Birkhauser.
[9] Y. Imayoshi and M. Taniguchi, An introduction to Teichmüller spaces, Springer-Verlag, 1991.
[10] S.P. Kerchoff, The Nielsen realization problem, Ann. of Math., **117** (1983), 235–265.
[11] I. Kra, Horocyclic coordinate for Riemann surfaces and Moduli spaces I: Teichmüller and Riemann spaces of Kleinian groups, Jour. of Amer. Math. Soc., **3** (1990), 499–578.
[12] O. Lehto, Univalent functions and Teichmüller spaces, Springer-Verlag, 1986.
[13] O. Lehto and K.I. Virtarnen, Quasiconformal mappings in the plane, 2nd Ed., Springer-Verlag, Berlin and New York, 1973.
[14] B. Maskit, On boundaries of Teichmüller spaces and on Kleinian groups: II, Ann. of Math., **91** (1970), 607–639.
[15] B. Maskit, Decomposition of certain Kleinian groups, Acta. Math., **130** (1973), 243–263.
[16] B. Maskit, On the classification of Kleinian groups:I-Koebe groups, Acta. Math.., **135** (1975), 249–270.
[17] B. Maskit, Kleinian groups, Springer-Verlag, Berlin and New York, (1987).
[18] K. Matsuzaki and M. Taniguchi, Hyperbolic Manifolds and Kleinian Groups, Oxford science Publications, 1998.
[19] C.T. McMullen, Cusps are dence, Ann. of Math., **133** (1991), 217–247.
[20] T. Nakanishi and M. Näätänen, Paramertrization of Teichmüller space by Length parameters, accepted for publication in Analysis and Topology (C.Andreian-Cazacu, O.Lehto and Th. M. Rassias, eds.) WorldScientific, Singapore.
[21] Ch. Prommerenke, Boundary behaviour of conformal maps, Springer-Verlag, 1991.
[22] H. Shiga and H. Tanigawa, Grunsky's inequality and its applications to Teichmüller spaces,

Kodai. Math. J., **16** (1993), 361–378.
[23] H. Yamamoto, Inner radii of Teichmüller spaces of finitely generated Fuchsian groups, Kodai. Math. J., **14** (1991), 350–357.