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Moduli of parabolic stable sheaves on a projective sheme

By

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Introduction

The moduli spaces of parabolic vector bundles have been studied especially on algebraic curves ([5]). M. Maruyama and K. Yokogawa have extended the concept of parabolic sheaves to a higher dimensional case and constructed the moduli spaces of parabolic stable sheaves on a higher dimensional smooth projective variety ([3]). Moreover K. Yokogawa has extended it to the moduli space of parabolic semi-stable sheaves on a smooth projective variety and shown that it is projective under some boundedness conditions ([9]).

In this paper we will remove the assumption of smoothness of the base scheme from the result in [3]. Since one of strong tools to study moduli spaces is the variation of moduli spaces of parabolic sheaves in a degeneration of smooth varieties, we do not restrict ourselves to the case where the underlying space is reduced or irreducible. We even allow supports of parabolic sheaves to move around inside the base scheme.

To construct the moduli space of parabolic sheaves, we will use almost the same method as that of [3]. The moduli space is obtained as a quotient space of some subscheme of a product of Quot-schemes by an action of $\mathrm{PGL}(V)$. Our task will be done in the framework of the geometric invariant theory, and hence our problem essentially reduces to the study of the stability of points in the Quot-scheme. Since the base scheme is not necessarily smooth, we can not use the Gieseker space in the calculation of the stability as in [3]. So we will use another method for calculating the stability which is based on the Simpson's idea in the case of stable sheaves ([8]).

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Notation and convention

Let X be a projective scheme over a noetherian scheme $S, \mathcal{O}_X(1)$ an S-very ample invertible sheaf and E a coherent sheaf on X. For an integer m, $E(m)$ denotes $E \otimes \mathcal{O}_X(m)$. If s is a point of S, then X_s denotes the fiber of X over s, $E(s)$ does

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$E\otimes k(s)$, $h^i(E(s))$ does $\dim H^i(X_s, E(s))$ and $\chi(E(s))$ does $\sum_{i\geq 0}(-1)^i h^i(E(s))$. If V is a locally free sheaf on S, $\mathbf{P}(V)$ means Proj $S(V)$ where $S(V)$ is the symmetric algebra of V over $\mathcal{O}_S$. $\mathrm{Grass}_r(V)$ denotes the Grassmannian of rank r quotient bundles of V. For a polynomial H and an integer m, $H[m](x)$ denotes the polynomial $H(x+m)$ in x. For a morphism $g: T\to S$ of schemes, E_T denotes the sheaf $(1_X\times g)^*(E)$ on $X\times_S T$.

1. Definition of parabolic sheaf

Let X be a projective scheme over a field k, $\mathcal{O}_X(1)$ a very ample invertible sheaf on X and D an effective Cartier divisor on X.

Let E be a coherent sheaf on X. The Hilbert polynomial of E can be written in the form

$$\chi(E(m))=\sum_{i=0}^{d} a_i(E)\binom{m+d-i}{d-i}$$

with $a_i(E)$ integers. We use the positive $a_0(E)$ instead of the rank of E.

We fix a positive integer d. E is said to be of pure dimension d if $E\neq 0$ and for any non zero coherent subsheaf E' of E, $\dim\mathrm{Supp}(E')=d$. If E is of pure dimension d and $\dim(D\cap\mathrm{Supp}\,E)<\dim\mathrm{Supp}\,E$, then the canonical homomorphism

$$\imath: E\otimes\mathcal{O}_X(-D)\to E$$

is injective. Indeed the restriction of $\imath$ to $X\backslash D$ is an isomorphism. Hence $\mathrm{Supp}(\ker\imath)\subset D\cap\mathrm{Supp}\,E$ and $\dim\mathrm{Supp}(\ker\imath)<d$. Since $E\otimes\mathcal{O}_X(-D)$ is of pure dimension d, we have $\ker\,\imath=0$.

Definition 1.1. Let X, $\mathcal{O}_X(1)$ and D be as above. Let E be a purely d dimensional coherent sheaf on X such that $\dim(D\cap\mathrm{Supp}\,E)<\dim\mathrm{Supp}\,E$.
Assume that there is a filtration of coherent sheaves

$$E=F_1(E)\supset F_2(E)\supset\cdots\supset F_{l+1}(E)=E(-D)$$

and a sequence of real numbers such that $0\leq\alpha_1\leq\alpha_2\leq\cdots\leq\alpha_l<1$. Then we call such a triple $(E,\{F_i(E)\},\alpha_*)$ a parabolic sheaf on X.

We call l the length of the parabolic sheaf. We put $G_i=F_i(E)/F_{i+1}(E)$, $\alpha_{l+1}=1$ and $\alpha_0=\alpha_l-1$. For a real number α take an integer i such that $\alpha_{i-1}<\alpha-[\alpha]\leq\alpha_i$ where $[\alpha]$ is the largest integer with $\alpha-[\alpha]\geq 0$, and then put $E_\alpha:=F_i(E)(-[\alpha]D)$. We denote the parabolic sheaf (E,F_*,α_*) simply by E_* when it causes no confusion.

Definition 1.2. Let E_* and F_* be parabolic sheaves. An $\mathcal{O}_X$-homomorphism $f: E\to F$ is said to be a parabolic homomorphism if $f(E_\alpha)\subset F_\alpha$ for all real numbers α.

Definition 1.3. A parabolic sheaf E_* is said to be a parabolic subsheaf of F_* if $E\subset F$ and $E_\alpha\subset F_\alpha$ for all real numbers α.

Let F'_* be a parabolic sheaf and E' be a non-zero coherent subsheaf F' such that F'/E' is of pure dimension d. If we put $E'_\alpha := F'_\alpha \cap E'$, then E'_* is a parabolic sheaf. We call it the induced parabolic subsheaf of F'_*.

Definition 1.4. Let $f: E_* \to F_*$ be a parabolic homomorphism. We call F_* a quotient parabolic sheaf of E_* if f is surjective.

Let E'_* be a parabolic sheaf and $f: E' \to G$ be a surjective homomorphism such that G is of pure dimension d. If we put $G_\alpha := f(E'_\alpha)$, then G_* is a parabolic sheaf. We call it the induced parabolic quotient sheaf.

Definition 1.5. Let $(E, F_*(E), \alpha_*)$ be a parabolic sheaf. Then we put

$$\text{par-}\chi(E_*(m)) := \chi(E(-D)(m)) + \sum_{i=1}^{l} \alpha_i \chi(G_i(m)) \quad (m \in \mathbf{Z}).$$

Writing down the Hilbert polynomial of E in the form

$$\chi(E(m)) = \sum_{i=0}^{d} a_i(E) \binom{m+d-i}{d-i},$$

we put

$$\text{par-}P_{E_*}(m) = \text{par-}\chi(E_*(m))/a_0(E).$$

We can easily check the following equation.

$$\text{par-}\chi(E_*(m)) = \int_0^1 \chi(E_\alpha(m))d\alpha.$$

Definition 1.6. Let E_* be a parabolic sheaf on X. E_* is said to be parabolic stable if for every parabolic subsheaf F_* of E_* with $0 < a_0(F) < a_0(E)$,

$$\text{par-}P_{F_*}(m) < \text{par-}P_{E_*}(m)$$

for all sufficiently large integers m.

Remark 1.7. In the above definition we may assume that E/F is of pure dimension d. Indeed let F_* be any parabolic subsheaf of E_* and T be the coherent subsheaf of E containing F such that $\dim \mathrm{Supp}(T/F) < d$ and E/T is of pure dimension d. Let T_* be the induced parabolic sheaf for T. Then for sufficiently large integers m, we have

$$\begin{aligned} \text{par-}P_{F_*}(m) &= \int_0^1 \chi(F_\alpha(m))d\alpha / a_0(F) \\ &\leq \int_0^1 \chi(T_\alpha(m))d\alpha / a_0(T) \\ &= \text{par-}P_{T_*}(m). \end{aligned}$$

Hence we may check the inequality in Definition 1.6 for T_*.

We will often use the following lemma in the sequel whose proof we refer to [9, Proposition 2.2].

Lemma 1.8. *Let $f: X \to S$ be a proper morphism of noetherian schemes and $\varphi: I \to F$ be an $\mathcal{O}_X$-homomorphism of coherent $\mathcal{O}_X$-modules with F flat over S. Then there exists a unique closed subscheme Z of S such that for all morphisms $g: T \to S$, $g^*(\varphi)=0$ if and only if g factors through Z.*

Let $f: X \to S$ be a projective and flat morphism of noetherian schemes, $D \subset X$ be an effective relative Cartier divisor with respect to f and $\mathcal{O}_X(1)$ be an f-very ample invertible sheaf. Let E be a coherent $\mathcal{O}_X$-module, flat over S. Assume that $E(s)$ is of pure dimension d on every geometric fiber X_s of f and that dim $\dim(D_s \cap \operatorname{Supp} E(s)) < \dim \operatorname{Supp} E(s)$. Then the canonical homomorphism $E \otimes_X \mathcal{O}_X(-D) \to E$ is injective (EGA IV, Proposition (11.3.7)).

Definition 1.9. Let X, S, D and $\mathcal{O}_X(1)$ be as above. Let $(\mathcal{L.N.S}ch/S)$ be the category of locally noetherian schemes over S. Let $H, H_1, \cdots, H_l$ be numerical polymials such that $\deg H = d$ and $\deg H_i < d$ for any i. We fix a sequence of rational numbers $\alpha_* = (\alpha_1, \alpha_2, \cdots, \alpha_l)$ such that $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_l < 1$. Let T be a locally noetherian scheme over S. We call $(E, F_*(E), \alpha_*)$ a flat family of parabolic sheaves on X_T over T if E is a T-flat coherent $\mathcal{O}_{X_T}$module such that for every geometric point t of $T, E(t)$ is of pure dimension d, $\dim(D_t \cap \operatorname{Supp} E(t)) < \dim \operatorname{Supp} E(t)$ and $E = F_1(E) \supset \cdots \supset F_{l+1}(E) = E(-D)$ is a filtration by coherent sheaves such that each $E/F_i(E)$ is flat over T.

Definition 1.10. For $T \in (\mathcal{L.N.S}ch/S)$ we set

$$\text{par-}\Sigma^{H_*,\alpha_*}_{D/X/S}(T) := \left\{ (E, F_*(E), \alpha_*) \;\middle|\; \begin{array}{l} (E, F_*(E), \alpha_*) \text{ is a flat family} \\ \text{of parabolic sheaves on } X_T/T \\ \text{with the property (i) below} \end{array} \right\} \Big/ \sim$$

where $\sim$ is the equivalence relation defined by (ii).

(i) for every geometric point t of T, $(E(t), F_*(E)(t), \alpha_*)$ is parabolic stable and $\chi(E(t)(n)) = H(n)$, $\chi((E/F_{i+1}(E))(t)(n)) = H_i(n)$ $(i=1, \cdots, l)$ for all integers n.

(ii) $(E, F_*(E), \alpha_*) \sim (E', F'_*(E'), \alpha_*)$ if there exists an invertible sheaf L on T such that $(E, F_*(E), \alpha_*) \otimes_{\mathcal{O}_T} L \cong (E', F'_*(E'), \alpha_*)$.

If $g: T' \to T$ is a morphism of noetherian schemes, then g induces a canonical map by pull-back:

$$g^*: \text{par-}\Sigma^{H_*,\alpha_*}_{D/X/S}(T) \to \text{par-}\Sigma^{H_*,\alpha_*}_{D/X/S}(T')$$

which makes par-$\Sigma^{H_*,\alpha_*}_{D/X/S}$ a contravariant functor of $(\mathcal{L.N.S}ch/S)$ to $(\mathcal{S}ets)$ where $(\mathcal{S}ets)$ denotes the category of sets.

2. Boundedness and Openness

Let $f: X \to S$ be a projective morphism of noetherian schemes, $\mathcal{O}_X(1)$ an f-very ample invertible sheaf on X and D an effective relative Cartier divisor with respect to f. Let $H, H_1, H_2, \cdots, H_l$ be numerical polynomials such that $\deg H = d$ and $\deg H_i < d$ for all i and $\alpha_* = (\alpha_1, \alpha_2, \cdots, \alpha_l)$ be a sequence of real numbers such that $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_l < 1$. Put $\varepsilon_i := \alpha_{i+1} - \alpha_i$ for $i = 1, \cdots, l$, where $\alpha_{l+1} = 1$.

Let $\mathscr{F}(H_*, \alpha_*)$ be the family of classes of parabolic sheaves on the fibers of X over S such that for a parabolic sheaf E_* on a geometric fiber of X/S, E_* is in $\mathscr{F}(H_*, \alpha_*)$ if and only if E_* is parabolic stable, $\chi(E(m)) = H(m)$ and $\chi(E/E_{\alpha_{i+1}}(m)) = H_i(m)$ for all i.

For any $E_* \in \mathscr{F}(H_*, \alpha_*)$, we have

$$\text{par-}\chi(E_*(m)) = H(m) - \sum_{i=1}^{l} \varepsilon_i H_i(m).$$

Definition 2.1. Let X be a projective scheme over a field k, $\mathcal{O}_X(1)$ a very ample invertible sheaf on X and D an effective Cartier divisor on X.

(1) Let E be a coherent sheaf on X. Take an integer e. E is said to be of c-type e if for general members $D_1, D_2, \cdots, D_{d-1}$ of $|\mathcal{O}_X(1)|$, for $Y = D_1 \cap D_2 \cap \cdots \cap D_{d-1}$, $E|_Y$ is of pure dimension 1 and for every non-zero coherent subsheaf E' of $E|_Y$, $\mu^S(E') \leq \mu^S(E) + e$, where $\mu^S(E) = a_1(E)/a_0(E)$ and $\mu^S(E') = a_1(E')/a_0(E')$.

(2) Let E_* be a parabolic sheaf on X. E_* is said to be parabolic e-stable if E_* is parabolic stable and E is of c-type e.

Let $\mathscr{F}^e(H_*, \alpha_*)$ be the subfamily of $\mathscr{F}(H_*, \alpha_*)$ such that for any member E_* of $\mathscr{F}(H_*, \alpha_*)$, E_* is in $\mathscr{F}^e(H_*, \alpha_*)$ if and only if E_* is parabolic e-stable.

We can easily see the following as in [4, Proposition 3.6].

Proposition 2.2. *The family $\mathscr{F}^e(H_*, \alpha_*)$ is bounded.*

We can see the following proposition by [3] Proposition 2.5.

Proposition 2.3. *There exists an integer m_0 such that for $E_* \in \mathscr{F}^e(H_*, \alpha_*)$, and for every parabolic subsheaf F_* of E_* with $0 < a_0(F) < a_0(E)$,*

$$\int_0^1 h^0(F_\alpha(m))d\alpha / a_0(F) < \int_0^1 h^0(E_\alpha(m))d\alpha / a_0(E) \text{ for every } m \geq m_0.$$

The following proposition can be proven similarly as [3, Proposition 2.8].

Proposition 2.4. *Assume that X is flat over S and $H^i(X_t, \mathcal{O}_X(1) \otimes k(t)) = 0$ for all $i > 0$ and all $t \in S$. Let (E_*, α_*) be a flat family of parabolic sheaves on X/S. Then there exist open sets S^s and S^e of S such that for all algebraically closed fields k, we have*

$$S^s(k)=\{t\in S(k)|E_*\otimes k(t) \text{ is parabolic stable}\}.$$

$$S^e(k)=\{t\in S(k)|E_*\otimes k(t) \text{ is parabolic } e\text{-stable}\}.$$

3. Construction of moduli spaces of parabolic stable sheaves

Let S be a noetherian scheme and $f: X\to S$ be a projective and flat morphism. Let $D\subset X$ be an effective relative Cartier divisor with respect to f and $\mathcal{O}_X(1)$ be an f-very ample invertible sheaf such that $H^i(X_s, \mathcal{O}_{X_s}(1))=0$ for all $s\in S$ and all $i>0$. Let $H, H_1, \cdots, H_l$ be numerical polynomials such that $\deg H=d$, $\deg H_i<d$ and the leading coefficients of all these polynomials are positive. Let $\alpha_*=(\alpha_1, \cdots, \alpha_l)$ be a sequence of *rational numbers* such that $0\leq\alpha_1<\cdots<\alpha_l<1$. Put $\varepsilon_i:=\alpha_{i+1}-\alpha_i$ for $i=1, \cdots, l$. We fix a positive integer e. Since $\mathcal{F}^e(H_*, \alpha_*)$ is bounded, there exists an integer m_0 such that for every $E_*\in\mathcal{F}^e(H_*, \alpha_*)$ we have the following properties (Proposition 2.3):

(a) $E_\alpha(m)$ and $(E/E_\alpha)(m)$ are generated by global sections for all $0\leq\alpha\leq 1$ and for all $m\geq m_0$.
(b) $H^i(E_\alpha(m))=0$ and $H^i(E/E_\alpha(m))=0$ for all $i>0$, $m\geq m_0$ and $0\leq\alpha\leq 1$.
(c) For any $m\geq m_0$ and for any parabolic subsheaf F_* of E_* with $0<a_0(F)<a_0(E)$, we have

$$\int_0^1 h^0(F_\alpha(m))d\alpha/a_0(F)<\int_0^1 h^0(E_\alpha(m))d\alpha/a_0(E).$$

We fix an integer $m\geq m_0$. Let V_m be a free $\mathcal{O}_S$-module of rank $H(m)$. Let $Q:=\mathrm{Quot}^{H[m]}_{V_m\otimes\mathcal{O}_X/X/S}$ be the Quot-scheme and $\varphi: V_m\otimes\mathcal{O}_{X_Q}\to\tilde{E}$ be the universal quotient sheaf. Put $Q_i:=\mathrm{Quot}^{H_i[m]}_{\tilde{E}/X_Q/Q}$ for $i=1, \cdots, l$ and let $\varphi_i: \tilde{E}\otimes\mathcal{O}_{X_{Q_i}}\to\tilde{E}_i$ be the universal quotient sheaves.

We define a sequence of morphisms of schemes

$$R_1\to R_2\to\cdots\to R_l\to Q$$

and surjections of coherent sheaves $\bar{E}_{i+1}\otimes_{X_{R_{i+1}}}\mathcal{O}_{X_{R_i}}\to\bar{E}_i$ as follows: First put $R_l:=Q_l=\mathrm{Quot}^{H_l[m]}_{\tilde{E}/X_Q/Q}$ and $\bar{E}_l:=\tilde{E}_l$. By descending induction on i, we put $R_i:=\mathrm{Quot}^{H_i[m]}_{\bar{E}_{i+1}/X_{R_{i+1}}/R_{i+1}}$ and let $\bar{E}_{i+1}\otimes_{X_{R_{i+1}}}\mathcal{O}_{X_{R_i}}\to\bar{E}_i$ be the universal quotient sheaf. Then, we have the following sequence of surjections of $\mathcal{O}_{X_{R_1}}$-modules, which are flat over R_1

$$V_m\otimes\mathcal{O}_{X_{R_1}}\xrightarrow{\phi}\tilde{E}\otimes_{X_Q}\mathcal{O}_{X_{R_1}}\xrightarrow{\phi_l}\bar{E}_l\otimes_{X_{R_l}}\mathcal{O}_{X_{R_1}}\xrightarrow{\phi_{l-1}}\cdots$$

$$\xrightarrow{\phi_2}\bar{E}_2\otimes_{X_{R_2}}\mathcal{O}_{X_{R_1}}\xrightarrow{\phi_1}\bar{E}_1\to 0.$$

R_1 represents the functor

$$T \mapsto \left\{ (g_1, \cdots, g_l) \in \prod_{i=1}^{l} Q_i(T) \,\middle|\, g_1^*(\varphi_1|_{\ker\varphi_2}) = 0, \cdots, g_{l-1}^*(\varphi_{l-1}|_{\ker\varphi_l}) = 0 \right\}$$

and R_1 is a closed subscheme of $\prod_{i=1}^{l} Q_i$ (Lemma 1.8).

Put $\phi_l' : \tilde{E}(-D) \otimes_{X_Q} \mathcal{O}_{X_{R_1}} \to \tilde{E}_{R_1} \xrightarrow{\phi_l} \tilde{E}_l \otimes \mathcal{O}_{X_{R_1}}$. Then there exists a closed subscheme $\bar{\Gamma}$ of R_1 such that for any morphism $g: T \to R_1$, g factors through $\bar{\Gamma}$ if and only if $g^*(\phi_l') = 0$ (Lemma 1.8). Let $\bar{F}_{i+1}(\tilde{E})$ be the kernel of the composition

$$\phi_i \circ \phi_{i+1} \circ \cdots \circ \phi_l : \tilde{E} \otimes_{X_Q} \mathcal{O}_{X_{\bar{\Gamma}}} \to \tilde{E}_i \otimes_{X_{R_1}} \mathcal{O}_{X_{\bar{\Gamma}}}$$

for $i = 1, \cdots, l$ and put $\bar{F}_1(\tilde{E}) := \tilde{E} \otimes_{X_Q} \mathcal{O}_{X_{\bar{\Gamma}}}$.

By the openness of pure dimensionality, Chevalley's theorem and the upper semi-continuity of cohomologies, there exists an open subscheme Q^0 of Q such that for any algebraically closed field K,

$$Q^0(K) = \left\{ x \in Q(K) \,\middle|\, \begin{array}{l} \tilde{E}(x) \text{ is of pure dimension } d, \\ \dim(D_x \cap \operatorname{Supp} \tilde{E}(x)) < d, \\ V_m \otimes k(x) \xrightarrow{\sim} H^0(\tilde{E}(x)) \text{ and} \\ H^j(\tilde{E}(x)) = 0 \text{ for } j > 0 \end{array} \right\}.$$

The homomorphism $\tilde{E} \otimes \mathcal{O}_{X_{Q^0}}(-D) \xrightarrow{\iota} \tilde{E} \otimes \mathcal{O}_{X_{Q^0}}$ is injective and its cokernel is flat over Q^0, since $\dim(D_x \cap \operatorname{Supp} \tilde{E}(x)) < d$ for all $x \in Q^0$ (EGA IV, Proposition (11.3.7)). Put

$$U_i := \left\{ x \in Q_i \times_Q Q^0 \,\middle|\, \begin{array}{l} H^j(\ker \varphi_i(x)) = 0,\ H^j(\tilde{E}_i(x)) = 0 \\ \text{for } j \geq 1,\ \ker \varphi_i(x) \text{ is generated} \\ \text{by global sections and} \\ V_m \otimes k(x) \to H^0(\tilde{E}_i(x)) \text{ is surjective.} \end{array} \right\}.$$

Then U_i is an open subset of Q_i and $(f_{U_i})_*(\tilde{E}_i \otimes_{X_{R_i}} \mathcal{O}_{X_{U_i}})$ is a locally free O_{U_i}-module of rank $H_i(m)$.

Let Γ be the open subscheme of $\bar{\Gamma} \times_{\prod_{j=1}^{l} Q_j} (\prod_{j=1}^{l} U_j)$ such that for a point s of $\bar{\Gamma} \times_{\prod_{j=1}^{l} Q_j} (\prod_{j=1}^{l} U_j)$, s is in Γ if and only if $\chi((\operatorname{coker} \iota)(s)(n)) = H_l[m](n)$.

Consider the injection

$$\tilde{E} \otimes_{X_\Gamma} \mathcal{O}_{X_\Gamma}(-D) \hookrightarrow \bar{F}_{l+1}(\tilde{E}).$$

Since the Hilbert polynomials on each fiber of the above sheaves are the same to each other, we have $\tilde{E} \otimes_{X_\Gamma} \mathcal{O}_{X_\Gamma}(-D) = \bar{F}_{l+1}(\tilde{E})$.

Let Γ^s be the open subscheme of Γ such that a geometric point x of Γ is in Γ^s if and only if $(\tilde{E}(-m) \otimes_{X_Q} \mathcal{O}_{X_\Gamma}, \bar{F}_*(\tilde{E})(-m), \alpha_*) \otimes k(x)$ is parabolic e-stable.

Fix an integer i such that $0 \leq i \leq l$. For a noetherian scheme T over S and a T-valued point $\tilde{E} \otimes \mathcal{O}_T \to E_i$ of U_i, $(f_T)_*(E_i)$ is a locally free sheaf of rank $H_i(m)$ and $(f_T)_*(\tilde{E} \otimes \mathcal{O}_T) \to (f_T)_*(E_i)$ is surjective. So $(f_T)_*(E_i)$ defines a T-valued point of

$Z_i := \mathrm{Grass}_{H_i(m)}((f_{Q^0})_*(\tilde{E}_{Q^0}))$. Thus we have a Q^0-morphism

$$U_i \to \mathrm{Grass}_{H_i(m)}((f_{Q^0})_*(\tilde{E}_{Q^0})).$$

Proposition 3.1. *$U_i \to \mathrm{Grass}_{H_i(m)}((f_{Q^0})_*(\tilde{E}_{Q^0}))$ is an immersion.*

Proof. Let $(f_{Q^0})_*(\tilde{E}_{Q^0})_{Z_i} \to \tilde{N}$ be the universal quotient bundle on $Z_i = \mathrm{Grass}_{H_i(m)}((f_{Q^0})_*(\tilde{E}_{Q^0}))$ and $\tilde{J}_i$ be its kernel. We have the following exact sequence:

$$0 \to \tilde{J}_i \to (f_{Q^0})_*(\tilde{E}_{Q^0}))_{Z_i} \to \tilde{N} \to 0,$$

which leads us another commutative diagram:

$$\begin{array}{ccccccc} 0 \to & (f_{Z_i})^*(\tilde{J}_i) & \to & (f_{Z_i})^*((f_{Q^0})_*(\tilde{E}_{Q^0})_{Z_i}) & \to & (f_{Z_i})^*(\tilde{N}) & \to 0 \\ & & \searrow^{h} & \downarrow & & \downarrow & \\ & & & \tilde{E}_{Z_i} & \longrightarrow & \mathrm{coker} h. & \end{array}$$

Let W_i be the stratum with Hilbert polynomial $H_i[m]$ of the flattening stratification of coker h. Let F be the kernel of $\tilde{E}_{Z_i} \to \mathrm{coker} h$. Let W'_i be the open subscheme of W_i such that

$$W'_i(k) = \left\{ x \in W_i(k) \;\middle|\; \begin{array}{l} H^j(X_x, F(x)) = 0, H^j(X_x, \mathrm{coker} h(x)) = 0 \text{ for} \\ j \geq 1, F(x) \text{ is generated by global sections} \\ \text{and } V_m \otimes k(x) \to \mathrm{coker} h(x) \text{ is surjective} \end{array} \right\}$$

for all algebraically closed fields k.

Then we have a factorization $U_i \xrightarrow{\theta_1} W'_i \hookrightarrow \mathrm{Grass}_{H_i(m)}(f_Q{}^0)_*(\tilde{E}_Q{}^0))$. We will show that θ_1 is an isomorphism. Since $\mathrm{coker}(h_{W'_i})$ is flat over W'_i, we have a morphism $\theta_2 : W'_i \to \mathrm{Quot}^{H_i[m]}_{\tilde{E}/X_Q/Q}$ such that $(\theta_2)^*(\tilde{E}_i) \cong \mathrm{coker}(h)_{W'_i}$ as quotients of $\tilde{E}_{W'_i}$. By the definition of U_i and W'_i, this morphism factors through U_i:

$$W'_i \xrightarrow{\theta_1} U_i \hookrightarrow \mathrm{Quot}^{H_i[m]}_{\tilde{E}/X_Q/Q}.$$

Take a T-valued point g of U_i. $\theta_1(g)$ corresponds to the quotient bundle

$$(f_T)_*(\tilde{E}) \to (f_T)_*((1 \times g)^*(\tilde{E}_i)).$$

We have the following exact commutative diagram.

$$\begin{array}{ccccccccc} 0 & \to & (f_T)^*(f_T)_*(\ker(\varphi_i)_T) & \to & (f_T)^*(f_T)_*(\tilde{E}_T) & \to & (f_T)^*(f_T)_*((\tilde{E}_i)_T) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & (\ker \varphi_i)_T & \to & \tilde{E}_T & \to & (\tilde{E}_i)_T & \to & 0. \end{array}$$

Since the leftmost down-arrow is surjective, we have $(\tilde{E}_i)_T \cong \mathrm{coker}(h)_T$ as quotients of $\tilde{E}_T$. Therefore we have $\theta_2 \circ \theta_1 = id$.

Conversely take a T-valued point g of W_i'. $\theta_2(g)$ corresponds to the quotient sheaf $\mathrm{coker}(h)_T$ of $\tilde{E}_T$. We have the following commutative diagram:

$$\begin{array}{ccccccc} 0 \to (f_T)^*((\tilde{J}_i)_T) & \to & (f_T)^*(f_T)_*(\tilde{E}_T) & \to & (f_T)^*(\tilde{N}_T) & \to 0 \\ & \searrow^{h_T} & \downarrow & & \downarrow & \\ & & \tilde{E}_T & \longrightarrow & \mathrm{coker}(h)_T. & \end{array}$$

From this we obtain the following commutative diagram.

$$\begin{array}{ccc} (f_T)_*(\tilde{E}_T) & \to & \tilde{N}_T \\ {\scriptstyle id}\downarrow & & \downarrow \\ (f_T)_*(\tilde{E}_T) & \to & (f_T)_*(\mathrm{coker}(h)_T). \end{array}$$

Since the homomorphism $(f_T)_*(\tilde{E}_T) \to (f_T)_*(\mathrm{coker}(h)_T)$ is surjective and both $(f_T)_*(\mathrm{coker}(h)_T)$ and N_T are locally free and have the same rank $H_i(m)$, we have $\tilde{N}_T \cong (f_T)_*(\mathrm{coker}(h)_T)$ as quotients of $\tilde{E}_T$. Hence we have $\theta_1 \circ \theta_2 = id$ from which we conclude that $\theta_1 : U_i \xrightarrow{\sim} W_i'$ is an isomorphism. Hence $U_i \xrightarrow{\theta_1} W_i' \hookrightarrow Z_i$ is an immersion.

We set $G(n) := \mathrm{Grass}_{H[m](n)}(f_*(V_m \otimes_S \mathcal{O}_X(n)))$ for sufficiently large n and $G_i := \mathrm{Grass}_{H_i(m)}(V_m)$. Then we have an isomorphism

$$\mathrm{Grass}_{H_i(m)}((f_{Q^0})_*(\tilde{E})) \xrightarrow{\sim} G_i \times_S Q^0.$$

Hence we have an immersion

$$\Gamma \hookrightarrow \prod_{i=1}^{l}{}_{Q^0} U_i \hookrightarrow \prod_{i=1}^{l}{}_{Q^0} Z_i \hookrightarrow G(n) \times_S \prod_{i=1}^{l} G_i.$$

$G(n)$ and $G_i (i = 1, \cdots, l)$ have the very ample invertible sheaves $\mathcal{O}_{G(n)}(1)$, $\mathcal{O}_{G_i}(1)$ respectively determined by Plücker embeddings. For a positive rational numbers $\beta_0, \beta_1, \cdots, \beta_l$, take the $\mathbf{Q}$-invertible sheaf

$$L := \mathcal{O}_{G(n)}(\beta_0) \otimes \bigotimes_{i=1}^{l} \mathcal{O}_{G_i}(\beta_i).$$

For a T-valued point $(g, (F, F_1, \cdots, F_l))$ of $\mathrm{GL}(V_m) \times (G(n) \times \prod_{i=1}^{l} G_i)$, we have surjections

$$V_m \otimes f_*(\mathcal{O}_X(n)) \otimes \mathcal{O}_T \xrightarrow{g} V_m \otimes f_*(\mathcal{O}_X(n)) \otimes \mathcal{O}_T \to F,$$

$$V_m \otimes \mathcal{O}_T \xrightarrow{g} V_m \otimes \mathcal{O}_T \to F_i$$

that define a T-valued point of $G(n)\times\prod_{i=1}^{l}G_i$. We denote it by $(g^*F, g^*F_1, \cdots, g^*F_l)$. The functorial morphism

$$\mathrm{GL}(V_m)(T)\times(G(n)\times\prod_{i=1}^{l}G_i)(T)\to G(n)(T)\times\prod_{i=1}^{l}G_i(T)$$

$$(g,(F,F_1,\cdots,F_l))\quad\mapsto\quad(g^*F,g^*F_1,\cdots,g^*F_l)$$

defines an action of $\mathrm{GL}(V_m)$ on $G(n)\times\prod_{i=1}^{l}G_i$. By definition, Γ and Γ^s are $\mathrm{GL}(V_m)$-stable subschemes of $G(n)\times\prod_{i=1}^{l}G_i$.

We have the canonical $\mathrm{GL}(V_m)$-linearization on $\mathcal{O}_{G(n)}(1)|_{G(n)}$ induced by that of $\mathcal{O}_{\mathbf{P}(\Lambda(f_Q)_*(V_m\otimes\mathcal{O}_{X_Q}(n)))}(1)$. Similarly we have a $\mathrm{GL}(V_m)$- linearization on $\mathcal{O}_{G_i}(1)|_{G_i}$ induced by $\mathcal{O}_{\mathbf{P}(\Lambda V_m)}(1)$. Therefore for a suitable large integer a, we have a $\mathrm{PGL}(V_m)$ linearization on $L^{\otimes a}$.

Proposition 3.2. *Let K be an algebraically closed field. Let $V, W_0, \cdots, W_l$ be K-vector spaces with $\dim V=n$, $\dim W_i=m_i$. Let $r_0,\cdots,r_l,\beta_0,\cdots,\beta_l$ be positive integers. Set*

$$X:=G_0\times_K G_1\times_K\cdots\times_K G_{r_l}$$

where $G_i:=\mathrm{Grass}_{r_i}(V\otimes_K W_i)$ for $i=0,\cdots,l$. We consider the canonical $\mathrm{PGL}(V)$-linearization on some power of $L=\mathcal{O}_{G_0}(\beta_0)\otimes\cdots\otimes\mathcal{O}_{G_l}(\beta_l)$. Then for a K-valued point $x=(E_0,E_1,\cdots,E_l)$ of X, x is a properly stable point (resp. *semi-stable point*) *of X with respect to L if and only if for all proper non-zero vector subspaces V' of V,*

$$\dim V\left(\sum_{j=0}^{l}\beta_j\dim_K F_j\right)\quad>\quad\dim V'\left(\sum_{j=0}^{l}\beta_j\dim_K E_j\right)$$

$$(\text{resp. }\geq)$$

where F_j is the image of the composition $V'\otimes_K W_j\hookrightarrow V\otimes_K W_j\to E_j$.

Proof. Since there is a canonical isogeny $\mathrm{SL}(V)\to\mathrm{PGL}(V)$, we may prove the proposition for the stability with respect to the $\mathrm{SL}(V)$-linearization instead of the $\mathrm{PGL}(V)$-linearization.

Take any one parameter subgroup λ of $\mathrm{SL}(V)$. For a suitable basis $e_1,\cdots,e_n$ of V, the dual action of λ is represented by $e_i\mapsto t^{u_i}e_i$, where $u_1\leq\cdots\leq u_n$ and $\sum_{i=0}^{n}u_i=0$.

Take a basis $f_1^{(j)},\cdots,f_{m_j}^{(j)}$ of W_j. For $p=m_j(i-1)+k$, put $h_p^{(j)}:=e_i\otimes f_k^{(j)}$ for $i=1,\cdots,n$ and $k=1,\cdots,m_j$. The dual action of λ on $V\otimes W_j$ is represented by

$$(h_1^{(j)},\cdots,h_{m_jn}^{(j)})\mapsto(t^{u_1}h_1^{(j)},\cdots,t^{u_1}h_{m_j}^{(j)},t^{u_2}h_{m_j+1}^{(j)},\cdots,t^{u_n}h_{m_jn}^{(j)}).$$

We define integers $s_1^{(j)}\leq\cdots\leq s_{m_jn}^{(j)}$ by putting $s_p^{(j)}:=u_i$ for $p=(i-1)m_j+k$ with $1\leq k\leq m_j$.

Let $\alpha_j\colon V\otimes_K W_j\to E_j$ be the given surjections for $j=0,\cdots,l$. Let $U_p^{(j)}$ be the vector subspace of $V\otimes_K W_j$ generated by $h_1^{(j)},\cdots,h_p^{(j)}$. We put $U_0^{(j)}=0$. For each

$1 \leq i \leq r_j$, let $\mu_i^{(j)}$ be the unique integer such that $\dim\alpha_j(U^{(j)}_{\mu_i^{(j)}}) = i$ and $\dim\alpha_j(U^{(j)}_{\mu_i^{(j)}-1}) = i-1$. Then we have $0 < \mu_1^{(j)} < \mu_2^{(j)} < \cdots < \mu_{r_j}^{(j)} \leq m_j n$ and $\{\alpha_j(h^{(j)}_{\mu_1^{(j)}}), \cdots, \alpha_j(h^{(j)}_{\mu_{r_j}^{(j)}})\}$ is a basis of E_j. We consider the Plücker embedding

$$\prod_{j=0}^{l} G_j \hookrightarrow \prod_{j=0}^{l} \mathbf{P}(\bigwedge^{r_j}(V \otimes W_j)).$$

Put $P^{(j)}_{i_1^{(j)}, \cdots, i_{r_j}^{(j)}} := \alpha_j(h^{(j)}_{i_1^{(j)}}) \wedge \cdots \wedge \alpha_j(h^{(j)}_{i_{r_j}^{(j)}})$ for $1 \leq i_1^{(j)} < \cdots < i_{r_j}^{(j)} \leq m_j n$. We have the following immersion defined by L

$$\prod_{j=0}^{l} \mathbf{P} \bigwedge^{r_j}(V \otimes W_j) \hookrightarrow \mathbf{P}\left(\bigotimes_{j=0}^{l} (S^{(\beta_j)}(\bigwedge^{r_j}(V \otimes W_j))) \right),$$

where $S^{(\beta_j)}(\bigwedge^{r_j}(V \otimes W_j))$ is the β_j-th symmetric product of $\bigwedge^{r_j}(V \otimes W_j)$ over K. Let $\{M^{(j)}_{w_j}(P^{(j)})\}_{w_j}$ be the set of monomials in $\{P^{(j)}_{i_1^{(j)}, \cdots, i_{r_j}^{(j)}}\}$ of degree β_j. Then the homogeneous coordinates of

$$\mathbf{P}\left((S^{(\beta_0)}(\bigwedge^{r_0}(V \otimes W_0)) \right) \otimes \cdots \otimes (S^{(\beta_l)}(\bigwedge^{r_l}(V \otimes W_l)) \Big)$$

can be represented by $(\prod_{j=0}^{l} M^{(j)}_{w_j}(P^{(j)}))_{w_j}$. Write the action of λ as

$$\lambda(t) \cdot M^{(j)}_{w_j}(P^{(j)}) = t^{\lambda(M^{(j)}_{w_j}(P^{(j)}))} M^{(j)}_{w_j}(P^{(j)}).$$

Precisely writing for

$$M^{(j)}_{w_j}(P^{(j)}) = \prod_{\Sigma \nu(i_1^{(j)}, \cdots, i_{r_j}^{(j)}) = \beta_j} (P^{(j)}_{i_1^{(j)}, \cdots, i_{r_j}^{(j)}})^{\nu(i_1^{(j)}, \cdots, i_{r_j}^{(j)})},$$

we have

$$\lambda(M^{(j)}_{w_j}(P^{(j)})) = \sum_{\Sigma \nu(i_1^{(j)}, \cdots, i_{r_j}^{(j)}) = \beta_j} \nu(i_1^{(j)}, \cdots, i_{r_j}^{(j)}) \sum_{p=1}^{r_j} s^{(j)}_{i_p^{(j)}}.$$

We use the one parameter criterion for the stability of x ([6]):

$\mu(x, \lambda) > 0$ (resp. $\mu(x, \lambda) \geq 0$) for all λ
$\Leftrightarrow x$ is a properly stable point (resp. semi-stable point).

For the definition of $\mu^L(x, \lambda)$, see ([6]). From [6] Proposition 2.3, we have

$$\mu(x, \lambda) = -\min\left\{ \prod_{j=0}^{l} \lambda(M^{(j)}_{w_j}(P^{(j)}) \,\middle|\, \prod_{j=0}^{l} M^{(j)}_{w_j}(P^{(j)})(x) \neq 0 \right\}.$$

If $i_k<\mu_k^{(j)}$ for some k, we have $P^{(j)}_{i_1^{(j)},\dots,i_{r_j}^{(j)}}=0$. Therefore

$$\begin{aligned}\mu^L(x,\lambda)&=-\sum_{j=0}^{l}\sum_{p=1}^{r_j}\beta_j s^{(j)}_{\mu_p^{(j)}}\\&=-\sum_{j=0}^{l}\beta_j\sum_{q=1}^{m_jn}s_q^{(j)}(\dim\alpha_j(U_q^{(j)})-\dim\alpha_j(U_{q-1}^{(j)}))\\&=\sum_{j=0}^{l}\beta_j(-r_js^{(j)}_{m_jn}+\sum_{q=1}^{m_jn-1}(s^{(j)}_{q+1}-s_q^{(j)})\dim\alpha_j(U_q^{(j)}))\\&=\sum_{j=0}^{l}\beta_j(-r_ju_n+\sum_{i=1}^{n-1}(u_{i+1}-u_i)\dim\alpha_j(U^{(j)}_{im_j})).\end{aligned}$$

For $u_1,\cdots,u_n$, there exist non-negative rational numbers $\delta_1,\cdots,\delta_{n-1}$ such that

$$u_i=\sum_{q=1}^{n-1}(-q\delta_q)+\sum_{q=n-i+1}^{n-1}(n-q)\delta_q\qquad\text{for } i=1,\cdots,n.$$

Indeed we may put $\delta_{n-i}=(u_{i+1}-u_i)/n$ for $\mathrm{i}=1,\cdots,n-1$. Hence we have

$$\begin{aligned}\mu^L(x,\lambda)&=\sum_{j=0}^{l}\beta_j(-r_j\sum_{q=1}^{n-1}q\delta_{n-q}+\sum_{q=1}^{n-1}n\delta_{n-q}\dim\alpha_j(U^{(j)}_{qm_j}))\\&=\sum_{q=1}^{n-1}\sum_{j=0}^{l}\beta_j(-r_jq+n\dim\alpha_j(U^{(j)}_{qm_j}))\delta_{n-q}.\end{aligned}$$

If $\sum_{j=0}^{l}\beta_j(-r_ji+n\dim\alpha_j(U^{(j)}_{im_j}))>0$ (resp. ≥ 0) for all i, then we have $\mu^L(x,\lambda)>0$ (*resp.* ≥ 0).

Conversely suppose that $\mu^L(x,\lambda)>0$ (*resp.* $\mu^L(x,\lambda)\geq 0$) for any one-parameter subgroup λ of SL(V). If we put $u_1=\cdots=u_i=i-n$, $u_{i+1}=\cdots=u_n=i$ with $1\leq i\leq n-1$, then we have $\delta_{n-i}=1$ and $\delta_q=0$ for $q\neq n-1$, and hence we have

$$\sum_{j=0}^{l}\beta_j(-r_ji+n\dim\alpha_j(U^{(j)}_{im_j}))=\mu^L(x,\lambda)>0\ (\text{resp. }\geq 0).$$

If we put $V':=Ke_1+\cdots+Ke_i\subseteq V$, then $U^{(j)}_{im_j}=V'\otimes_K W_j$. By the above arguments we have

$\mu(x,\lambda)>0$ (*resp.* ≥ 0) for all one parameter subgroups λ

$$\Leftrightarrow\sum_{j=0}^{l}\beta_j(-r_j\dim V'+n\dim\alpha_j(V'\otimes_K W_j))>0\ (\textit{resp.}\geq 0)$$

for all proper non-zero subspaces V' *of* V

$$\Leftrightarrow\sum_{j=0}^{l}\beta_j\dim\alpha_j(V'\otimes_K W_j)-\frac{\dim V'}{\dim V}\sum_{j=0}^{l}\beta_j\dim E_j>0\ (\text{resp. }\geq 0)$$

for all proper non-zero subspaces V' of V.

Recall the immersion

$$\Gamma^s \subsetneq \text{Grass}\,(V_m \otimes f^*(\mathcal{O}_X(n))) \times {}_S\prod \text{Grass}_{H^i(m)}(V_m).$$

We have set $G_i := \text{Grass}_{H_i(m)}(V_m)$, $G(n) := \text{Grass}_{H[m](n)}(V_m \otimes f^*(\mathcal{O}_X(n)))$ for $i = 1, \cdots, l$.

Proposition 3.3. *Putting* $\beta_0 := (H(m) - \sum_{i=1}^l \varepsilon_i H_i(m))/H[m](n)$, $\beta_i := \varepsilon_i$ *for* $i = 1, \cdots, l$, *we take* $L = \mathcal{O}_{G(n)}(\beta_0) \otimes \otimes_{i=1}^l \mathcal{O}_{G_i}(\beta_i)$. *Then there exists an integer* n_0 *such that for any* $n \geq n_0$ *and for any k-valued geometric point* $(E, E_1, \cdots, E_l)$ *of* Γ^s, *the corresponding k-valued point* y *of* $\Gamma^s \subset G(n) \times \prod_{i=1}^l G_i$ *is a properly stable point.*

Proof. Let $(\tilde{E}, \tilde{E}_1, \cdots, \tilde{E})$ be the universal family on $X \times_S \Gamma^s$. Let $\tilde{E}_*(-m)$ be the corresponding parabolic sheaf. Let y be a geometric point of Γ^s over a geometirc point s of S. Let V' be a vector subspace of $V_m \otimes k(y)$. Let $E(V')$ be the coherent subsheaf of $\tilde{E}(y)$ generated by the image of

$$V' \subsetneq V_m \otimes k(y) \xrightarrow{\sim} H^0(X_y, \tilde{E}(y)).$$

Let $\mathscr{F}$ be the family of classes of coherent sheaves on the fibers of X over S such that

$$\mathscr{F} := \left\{ E(V') \middle| \begin{array}{l} y \text{ is a geometric point of } \Gamma^s \text{ and } V' \text{ is a non} \\ \text{zero proper vector subspace of } V_m \otimes k(y) \end{array} \right\}.$$

Since the set of all V' are parameterized by Grassmannians, $\mathscr{F}$ is bounded. Hence there exists an integer n_1 such that for any $n \geq n_1$ and for any $E' \in \mathscr{F}$ with E' on a geometric fiber X_y,

$$H^0(X_y, V' \otimes \mathcal{O}_X(n)(y)) \to H^0(X_y, E'(n))$$

is surjective and $\dim H^0(X_s, E'(n)) = \chi(E'(n))$.

For $E' \in \mathscr{F}$, put $E'_{\alpha_i} := E' \cap \tilde{F}_i(\tilde{E})(y)$. Let F' be the coherent subsheaf of $\tilde{E}(y)$ containing E' such that $\tilde{E}(y)/F'$ is of pure dimension d and dim Supp$(F'/E') < d$. Then by the choice of m ((c)), we have

$$h^0(E') - \sum_{i=1}^{l} \varepsilon_i (h^0(E') - h^0(E'_{\alpha_{i+1}})) \leq h^0(F') - \sum_{i=1}^{l} \varepsilon_i (h^0(F') - h^0(F'_{\alpha_{i+1}}))$$

$$< \frac{a_0(E')}{a_0(H)} (H(m) - \sum_{i=1}^{l} \varepsilon_i H_i(m))$$

where F'_* has the induced parabolic structure and $a^0(H)$ is the leading coefficient of $H(m)/d!$.

Since $\mathscr{F}$ is bounded, there exists a positive real number $\varepsilon<1$ such that for any $E'\in\mathscr{F}$ with $E'\neq\tilde{E}(y)$, we have

$$h^0(E')-\sum_{i=1}^{l}\varepsilon_i(h^0(E')-h^0(E'_{\alpha_{i+1}}))<\frac{a_0(E')}{a_0(H)}(H(m)-\sum_{i=1}^{l}\varepsilon_i H_i(m))(1-\varepsilon).$$

Moreover there exists an integer n_2 such that for any $n\geq n_2$ and for any $E'\in\mathscr{F}$, we have

$$\frac{\chi(E'(n))}{H[m](n)}>\frac{a_0(E')}{a_0(H)}(1-\varepsilon).$$

There exists an integer n_3 such that for any $n\geq n_3$ we have a canonical closed immersion $Q\hookrightarrow \mathrm{Grass}_{H[m](n)}(f_*(V_m\otimes\mathcal{O}_X(n)))$.

We put $n_0:=\max\{n_1, n_2, n_3\}$ and take an integer $n\geq n_0$. Take a non-zero vector subspace V' of $V_m\otimes k(y)$ such that $E'\neq\tilde{E}(y)$, where E' is the subsheaf of $\tilde{E}(y)$ generated by V'. We put $U:=H^0(X_y, \tilde{E}(n)(y))$ and $W:=f_*(\mathcal{O}_X(n))\otimes k(y)$. Let U' be the image of $V'\otimes W$ by the linear map $\alpha: V_m\otimes W\to U$. By the choice of n_1, we have $U'=H^0(X_y, E'(n))$. Let W_i be the image of V' by the linear map

$$\alpha_i: V_m\otimes k(y)=H^0(X_y, \tilde{E}(y))\to H^0(X_y, (\tilde{E}/\tilde{F}_{i+1}(E))(y))$$

and V_i be the kernel of $V'\to W_i$. Then, we have

$$\begin{aligned}
&\dim V_m\left(\beta_0\dim\alpha(V'\otimes W)+\sum_{i=1}^{l}\beta_i\dim\alpha_i(V')\right)\\
&\quad-\dim V'\left(\beta_0\dim h^0(\tilde{E}(y)(n))+\sum_{i=1}^{l}\beta_i h^0(((\tilde{E}/(\tilde{F}_{i+1}((\tilde{E}))(y))\right)\\
&=H(m)\left(\frac{H(m)-\sum_{i=1}^{l}\varepsilon_i H_i(m)}{H[m](n)}\dim U'+\sum_{i=1}^{l}\varepsilon_i\dim W_i\right)\\
&\quad-\dim V'\left(\frac{H(m)-\sum_{i=1}^{l}\varepsilon_i H_i(m)}{H[m](n)}H[m](n)+\sum_{i=1}^{l}\varepsilon_i H_i(m)\right)\\
&=H(m)\left(\frac{H(m)-\sum_{i=1}^{l}\varepsilon_i H_i(m)}{H[m](n)}h^0(E'(n))\right.\\
&\qquad\left.+\sum_{i=1}^{l}\varepsilon_i(\dim V'-\dim V_i)-\dim V'\right)\\
&\geq H(m)\left(\frac{H(m)-\sum_{i=1}^{l}\varepsilon_i H_i(m)}{H[m](n)}h^0(E'(n))\right.
\end{aligned}$$

$$+\sum_{i=1}^{l}\varepsilon_i(h^0(E')-h^0(E'_{\alpha_{i+1}}))-h^0(E')\Bigg)$$

$$>H(m)\left(\frac{H(m)-\sum_{i=1}^{l}\varepsilon_i H_i(m)}{a_0(H)}a^0(E')(1-\varepsilon)\right.$$

$$\left.+\sum_{i=1}^{l}\varepsilon_i(h^0(E')-h^0(E'_{\alpha_{i+1}}))-h^0(E')\right)$$

$$>H(m)\left(h^0(E')-\sum_{i=1}^{l}\varepsilon_i(h^0(E')-h^0(E'_{\alpha_{i+1}}))\right.$$

$$\left.+\sum_{i=1}^{l}\varepsilon_i(h^0(E')-h^0(E'_{\alpha_{i+1}}))-h^0(E')\right)$$

$$=0$$

Hence by Proposition 3.2, we see that every geometric point y of Γ^s is a properly stable point.

By Proposition 3.3, there exists a geometric quotient

$$\xi:\Gamma^s\to M^e$$

of Γ^s by the action of $\mathrm{PGL}(V_m)$ ([7]). For a positive integer e, put

$$\text{par-}\Sigma_{D/X/S}^{H_*,\alpha_*,e}(T):=\{E_*\in\text{par-}\Sigma_{D/X/S}^{H_*,\alpha_*}(T)|E_*(t) \text{ is parabolic } e\text{-stable}\}$$

for any locally noetherian scheme T over S. Then par-$\Sigma_{D/X/S}^{H_*,\alpha_*,e}$ is a subfunctor of par-$\Sigma_{D/X/S}^{H_*,\alpha_*}$.

Theorem 3.4. *M^e is a coarse moduli scheme of* par-$\Sigma_{D/X/S}^{H_*,\alpha_*,e}$ *Namely,*

(i) *there exists a morphism of functors Ψ^e:par-$\Sigma_{D/X/S}^{H_*,\alpha_*,e}\to M^e$ such that for every geometric point s of S,*

$$\Psi^e(\mathrm{Spec}k(s)):\text{par-}\Sigma_{D/X/S}^{H_*,\alpha_*}(\mathrm{Spec}k(s))\tilde{\to}M^e(k(s))$$

is bijective and

(ii) *if M' is a locally noetherian scheme over S and if Ψ':par-$\Sigma_{D/X/S}^{H_*,\alpha_*}\to M'$ is a morphism of functors, then there exists a unique S-morphism $h:M^e\to M'$ such that $h\circ\Psi^e=\Psi'$.*

Proof. Let T be a locally noetherian sheme over S and take a T-valued point $x=(E(m), E_1(m),\cdots,E_l(m))$ of Γ^s. Let E_* be the corresponding parabolic sheaf. Then we have a morphism of functors

$$\theta : h_{\Gamma^s} \to \text{par-}\sum_{D/X/S}^{H_*,\alpha_*,e}$$

by sending x to E_*. Take a T-valued point $g \in \mathrm{PGL}(V_m)(T)$. Giving g is equivalent to giving an invertible sheaf $\mathscr{M}$ on T and an isomorphism $\sigma: V_m \otimes \mathcal{O}_T \xrightarrow{\sim} V_m \otimes \mathscr{M}$. We obtain surjections

$$V_m \otimes_S \mathcal{O}_{X_T} \xrightarrow{\sigma} V_m \otimes_S \mathcal{O}_{X_T} \otimes_T \mathscr{M} \to E(m) \otimes_T \mathscr{M}$$

$$V_m \otimes_S \mathcal{O}_{X_T} \xrightarrow{\sigma} V_m \otimes_S \mathcal{O}_{X_T} \otimes_T \mathscr{M} \to E(m) \otimes_T \mathscr{M} \to E_i(m) \otimes_T \mathscr{M}.$$

This corresponds to the T-valued point gx of Γ^s. Therefore we have $\theta(gx) = \theta(x)$. Let $h_{\Gamma^s}/h_{\mathrm{PGL}(V_m)}$ be the functor $T \mapsto \Gamma^s(T)/PGL(V_m)(T)$. θ induces a morphism of functors

$$\bar{\theta} : h_{\Gamma^s}/h_{\mathrm{PGL}(V_m)} \to \text{par-}\sum_{D/X/S}^{H_*,\alpha_*,e}.$$

If $\theta(E_*^{(1)}) = \theta(E_*^{(2)})$ for $E_*^{(1)}, E_*^{(2)} \in \Gamma^s(T)$, then there exists an invertible sheaf $\mathscr{M}$ on T such that $E_*^{(1)} \cong E_*^{(2)} \otimes \mathscr{M}$. Composing the following isomorphisms

$$\begin{array}{l} V_m \otimes \mathcal{O}_T \xrightarrow{\sim} f_*(E^{(1)}(m)) \\ \qquad\qquad \wr\downarrow \\ V_m \otimes \mathscr{M} \xrightarrow{\sim} f_*(E^{(2)}(m) \otimes \mathscr{M}) \cong f_*(E^{(2)}(m)) \otimes \mathscr{M}, \end{array}$$

we obtain an isomorphism $V_m \otimes \mathcal{O}_T \xrightarrow{\sim} V_m \otimes \mathscr{M}$ and this gives a T-valued point $g \in \mathrm{PGL}(V_m)(T)$. By definition $g \cdot E_*^{(1)} = E_*^{(2)}$. Hence $\bar{\theta}$ is injective.

Take any T-valued point E_* of $\text{par-}\sum_{D/X/S}^{H_*,\alpha_*,e}(T)$. There exists an open covering $T = \cup U_i$ and surjections $V_m \otimes_S \mathcal{O}_{X_{U_i}} \to E_{U_i}(m)$. With respect to this surjection we can consider $(E(m)_{U_i}, (E/E_{\alpha_2})(m)_{U_i}, \cdots, (E/E_{\alpha_{l+1}})(m)_{U_i})$ as an U_i-valued point of Γ^s. Sending it by θ, we obtain $(E_{U_i})_*$. Hence the sheaves associated to the presheaves $h_{\Gamma^s}/h_{\mathrm{PGL}(V_m)}$ and $\text{par-}\sum_{D/X/S}^{H_*,\alpha_*,e}$ for Zariski topology equal to each other. Since ξ is a geometric quotient, we have a morphism of functors $h_{\Gamma^s}/h_{\mathrm{PGL}(V_m)} \to h_{M^e}$. Thus we have a morphism

$$\Psi^e : \text{par-}\sum_{D/X/S}^{H_*,\alpha_*,e} \to h_{M^e}.$$

For every geometric point s of S,

$$\Gamma^s(k(s))/\mathrm{PGL}(V_m)(k(s)) \xrightarrow{\theta(k(s))} \text{par-}\sum_{D/X/S}^{H_*,\alpha_*,e}(k(s))$$

is bijective and since ξ is a geometric quotient,

$$\Gamma^s(k(s))/\mathrm{PGL}(V_m)(k(s)) \to M^e(k(s))$$

is also bijective. Hence

$$\Psi^e(k(s)):\text{par-}\sum_{D/X/S}^{H_*,\alpha_*,e}(k(s)))\to M^e(k(s))$$

is bijective.

For any locally noetherian scheme M' over S and for any morphism of functors $\Psi':\text{par-}\sum_{D/X/S}^{H_*,\alpha_*,e}\to M'$

$$\Gamma^s\to\text{par}\sum_{D/X/S}^{H_*,\alpha_*,e}\to M'$$

is $PGL(V_m)$-equivariant, where the action of $PGL(V_m)$ on M' is trivial. Since M^e is a categorical quotient, there exists a unique morphism $h:M^e\to M'$ such that $h\circ\xi=\Psi'$ on $h_{\Gamma^s}/h_{PGL(V_m)}$. Since the sheaves associated to $h_{\Gamma^s}/h_{PGL(V_m)}$ and $\text{par-}\sum_{D/X/S}^{H_*,\alpha_*,e}$ are the same to each other, we have $h\circ\Psi^e=\Psi'$. The uniquess of h is obvious.

Theorem 3.5. *There exists a coarse moduli scheme of* $\text{par-}\sum_{D/X/S}^{H_*,\alpha_*}$

Proof. For positive integers e,e' with $e<e'$, we have a canonical morphism of functors

$$\text{par-}\sum_{D/X/S}^{H_*,\alpha_*,e}\hookrightarrow\text{par-}\sum_{D/X/S}^{H_*,\alpha_*,e'}\xrightarrow{\Psi^e}h_{M^{e'}}.$$

By the universality of M^e, there exists a unique morphism $j^{e,e'}:M^e\to M^{e'}$ such that $j^{e,e'}\circ\Psi^e=\Psi^{e'}|_{\text{par}-\Sigma_{D/X/S}^{H_*\alpha_*e}}$, and $j^{e,e'}$ is an open immersion. Set

$$M:=\varinjlim_{e>0}M^e.$$

For any locally noetherian sheme T over S and for any $E\in\text{par-}\sum_{D/X/S}^{H_*,\alpha_*}(T)$, there exists an open covering $\{U_i\}$ of T such that each U_i is noetherian. For each $(E_{U_i})_*$, there exists an integer e_i such that for any geometric point t of U_i, $E_{U_i}(t)_*$ is parabolic e_i-stable. We have $(E_{U_i})_*\in\text{par-}\sum_{D/X/S}^{H_*,\alpha_*,e_i}(U_i)$. We obtain a morphism $\Psi^{e_i}((E_{U_i})_*):U_i\to M^{e_i}\hookrightarrow M$. By the functoriality we have $\Psi^{e_i}((E_{U_i})_*)|_{U_i\cap U_j}=\Psi^{e_j}((E_{U_j})_*)|_{U_i\cap U_j}$. By gluing these morphisms we obtain a morphism $\Psi(E_*):T\to M$ such that $\Psi(E_*)|_{U_i}=\Psi^{e_i}((E_{U_i})_*)$ for all i. Then we obtain a morphism of functors

$$\Psi:\text{par-}\sum_{D/X/S}^{H_*,\alpha_*}\to h_M.$$

For every geometric point s of S,

$$\text{par-}\sum_{D/X/S}^{H_*,\alpha_*}(k(s))\xrightarrow{\Psi(k(s))}M(k(s))$$

is bijective because each $\Psi^e(k(s))$ is bijective.

For any locally noetherian scheme M' over S and for any morphism of functors Ψ': par-$\sum_{D/X/S}^{H_*,\alpha_*}\to M'$, there exists a morphism $h_e: M^e\to M'$ such that $h_e\circ\Psi^e=\Psi'$ on par-$\sum_{D/X/S}^{H_*,\alpha_*,e}$. By the functoriality and the universality of M^e, $h_e=h_{e'}\circ j^{e,e'}$ for $e<e'$. Then we obtain a morphism $h: M\to M'$ such that $h|_{M^e}=h_e$. By the construction we have $\Psi'=h\circ\Psi$. The uniqueness of such h is obvious.

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