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The Penney-Fujiwara Plancherel formula for nilpotent Lie groups

By

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Abstract

We prove the Penney-Fujiwara Plancerel Formula associated to a monomial representation of a nilpotent Lie group. We give also a short proof of a theorem due to Corwin and Greenleaf about the algebra of differential operators on certain nilpotent homogeneous space.

0. Introduction

Let G be a nilpotent connected simply connected Lie group with Lie algebra $\mathfrak{g}$. Let $\mathscr{S}(G)$ denote the Schwartz-space of G, i.e. the space of all complex valued functions φ on G, such that $f\circ\exp$ is a ordinary Schwartz-function on the vector space $\mathfrak{g}$. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. Let $f\in\mathfrak{g}^*$ be such that $<f,[\mathfrak{h},\mathfrak{h}]>=(0)$. We obtain a unitary character χ_f of $H=\exp(\mathfrak{h})$ by letting

$$\chi_f(\exp(Y))=e^{-i<f,Y>},\ Y\in\mathfrak{h}.$$

Let $\mathscr{B}=\{X_1,\cdots,X_r\}$ be a Malcev-basis relative to $\mathfrak{h}$, i.e. $\mathfrak{g}=\sum_{1\le i\le r}^{\oplus}\mathbf{R}X_i\oplus\mathfrak{h}$ and for any $j=1,\cdots,r$, the subspace $\mathfrak{g}_j=\mathrm{span}\{X_j,\cdots,X_r,\mathfrak{h}\}$ is a subalgebra. The mapping $E_{\mathscr{B}}:\mathbf{R}^r\to G/H: E_{\mathscr{B}}(t_1,\cdots,t_r)=\exp(t_1X_1)\cdots\exp(t_rX_r)H$ is then a diffeomorphism. We obtain a G-invariant measure $d\dot{g}=d_{\mathscr{B}}\dot{g}$ on the quotient space G/H by setting

$$\int_{G/H}\xi(g)d_{\mathscr{B}}\dot{g}=\int_{\mathbf{R}^r}\xi(E_{\mathscr{B}}(T))dT,\ \xi\in C_c(G/H),$$

where $C_c(G/H)$ denotes the space of the continuous functions with compact support on G/H.

Let $\mathscr{S}(G/H,f)$ be the space of all C^∞-functions ξ on G, such that $\xi(gh)=\chi_f(h^{-1})\xi(g)$ for all $g\in G, h\in H$ and such that the function $T\mapsto\xi(E_{\mathscr{B}}(T))$ is a Schwartz-function on $\mathbf{R}^r$. Pick a Haar measure dh of H and let for $\varphi\in\mathscr{S}(G)$

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$$P_{H,f}(\varphi)(g)=P(\varphi)(g)=\int_H \varphi(gh)\chi_f(h)dh, g\in G.$$

It is easy to see that $P(\varphi)$ is in $\mathscr{S}(G/H,f)$ and that the mapping P is linear surjective and continuous, if we provide our spaces with the standard Fréchet topologies. Let $S_{H,f}$ be the tempered distribution on G defined by

$$< S_{H,f}, \varphi > = P_{H,f}(\varphi)(e)=\int_H \varphi(h)\chi_f(h)dh, \varphi\in\mathscr{S}(G).$$

We observe that the distribution $S_{H,f}$ is $\overline{\chi_f}$-H invariant, i.e. for any $h\in H$, we have that $\lambda_h(S_{H,f})=\overline{\chi_f(h)}S_{H,f}$, where λ_h denotes left translation by h. Indeed, for $\varphi\in\mathscr{S}(G)$

$$< \lambda_h(S_{H,f}), \varphi > = < S_{H,f}, \lambda_{h^{-1}}\varphi >$$

$$=\int_H \varphi(hh')\chi_f(h')dh'=\overline{\chi_f(h)}\int_H \varphi(h')\chi_f(h')dh'=\overline{\chi_f(h)}< S_{H,f}, \varphi >.$$

Let now $H=\exp(\mathfrak{h})$ and $K=\exp(\mathfrak{k})$ be two closed connected subgroups of G and f be an element in $\mathfrak{g}^*$ sucht that $\mathfrak{h}$ and $\mathfrak{k}$ are subordinated to f. We can construct a $\overline{\chi_f}$-invariant distribution $S^H_{K,f}$ on $\mathscr{S}(G/H,f)$ in the following way. Pick a K-invariant measure $d\dot{k}$ on $K/K\cap H$ and let

$$< S^H_{K,f}, \xi > = \int_{K/H\cap K} \xi(k)\chi_f(k)d\dot{k}.$$

It follows as above that for all $k\in K$

$$< S^H_{K,f}, \lambda_{k^{-1}}\xi > = \overline{\chi_f(k)}\ < S^H_{K,f}, \xi >, \xi\in\mathscr{S}(G/H,f), k\in K.$$

Let $\phi\in\mathfrak{g}^*$ and let $\mathfrak{b}$ be a polarization at ϕ. Let $B=\exp(\mathfrak{b})$ and let χ_ϕ be the character of B associated to ϕ. It is wellknown that the representation $\pi_\phi=\mathrm{Ind}_B^G\chi_\phi$ is irreducible and that the space $\mathscr{S}(G/B,\phi)$ is in fact the space of the C^∞-vectors of π_ϕ (see [1]).

Let now $H=\exp(\mathfrak{h})$ be a closed connected subgroup of G and let $\tau=\mathrm{Ind}_H^G\chi_f$ be the monomial representation induced from χ_f. It has been shown in [1] that there exists a certain affine subspace $\mathscr{V}$ of $\Gamma_f=f+\mathfrak{h}^\perp\subset\mathfrak{g}^*$, such that

$$\tau\simeq\int_{\mathscr{V}}^{\oplus}\pi_\phi d\phi=\tau', \tag{0.1}$$

where $d\phi$ denotes Lebesgue measure on $\mathscr{V}$ and where π_ϕ is the irreducible representation associated to ϕ $(\phi\in\mathscr{V})$.

The general distribution-theoretic Plancherel formula is due to Penney (see [20]). It is associated to a desintegration of an induced representation and it is of the form

$$<\tau(\omega)\alpha_\tau, \alpha_\tau> = \int_{\mathscr{V}} <\pi_\phi(\omega)\beta_\phi, \beta_\phi> d\phi, \omega \in \mathscr{S}(G), \tag{0.2}$$

where α_τ is the canonical cyclic generalized vector for τ and β_ϕ is an (appropriately H-covariant) generalized vector for π_ϕ. In general the determination of appropriate distributions is problematic. In the case when G is nilpotent, (0.2) was obtained by Fujiwara in a different form (see [13]) when the multiplicities occuring in the decomposition (0.1) are finite. Groundbreaking work on extending results of [13] to other classes of homogeneous spaces has been done by Fujiwara and Yamagami [12] and Lipsman [17, 18, 19]. However, beyond the nilpotent case the technical difficulties involved in (0.2) are considerable. Recently, Currey studied a class of completely solvable homogeneous spaces when τ is induced from a "Levi" component. In this situation, he overcomes these problems and he gives an explicit and natural construction for a smooth decomposition.

The first aim of this note is a desintegration of the distribution $S_{H,f}$ into an integral $\int_{\mathscr{V}} S_{B(\phi),\phi} d\phi$ in pure distributions $S_{B(\phi),\phi}$ of positive type associated with the irreducible representations π_ϕ, where $\mathscr{V}$ is a certain affine subspace of $\mathfrak{g}^*$. In other words, we are going to prove (0.2) without taking into account the multiplicities occuring in the decomposition (0.1).

In the second part of the paper we give a short proof of the main result of [7]. Let

$$C^\infty(G,\tau) = \{\xi \in C^\infty(G) : \xi(gh) = \chi(h^{-1})\xi(g), g \in G,\ h \in H\}.$$

Let Diff(G) be the algebra of all C^∞ differential operators taking $C^\infty(G,\tau)$ into itself, and $D_\tau(G/H)$ the algebra of operators $D|C^\infty(G,\tau)$ of $D \in$ Diff(G) commuting with the action of τ on that space. This algebra of differential operators is commutative (see [6]). Commutativity was proven by showing that $D_\tau(G/H)$ is isomorphic to a generating subalgebra of the field $\mathbf{C}(f+\mathfrak{h}^\perp)^H$ of $Ad^*(H)$-invariant rational functions on Γ_f. In [6], Corwin and Greenleaf have formulated the following conjecture:

If $m(\pi) < \infty$ for generic $\pi \in spec(\tau)$, then $D_\tau(G/H) \simeq \mathbf{C}[f+\mathfrak{h}^\perp]^H$, where $\mathbf{C}[f+\mathfrak{h}^\perp]^H$ is the algebra of $Ad^*(H)$-invariant polynomial functions on Γ_f.

Later, Corwin and Greenleaf proved in [7] this conjecture when there exists a subalgebra which polarizes all generic elements in Γ_f and normalized by $\mathfrak{h}$.

Very recently, we have proved in [2] (and Fujiwara in [14]) this conjecture when there exists a subalgebra which polarizes all generic elements in Γ_f and in particular when H is a normal subgroup of G.

1. The Penney-Fujiwara Plancherel Formula

1.1. Let $H = \exp(\mathfrak{h})$ be a closed connected subgroup of the connected nilpotent

Lie group $G=\exp(\mathfrak{g})$. Let $f\in\mathfrak{g}^*$ such that $<f,[\mathfrak{h},\mathfrak{h}]>=(0)$ and let $\chi_f=\exp(-if_{|\mathfrak{h}})\circ\log$ be its unitary character on H. It has been shown in ([1]) how the representation $\tau=\mathrm{ind}_H^G\chi_f$ can be smoothly disintegrated into irreducibles. There exists a Zariski-open subset $\mathscr{V}_0$ of $\mathscr{V}$ with the following properties. For every $\phi\in\mathscr{V}_0$ there exists a polarization $B(\phi)=\exp(\mathfrak{b}(\phi))$ at ϕ, a Malcev-basis

$$\mathscr{X}(\phi)=\{X_1(\phi),\cdots,X_l(\phi)\}$$

of $\mathfrak{g}$ relative to $\mathfrak{b}(\phi)$, a Malcev-basis

$$\mathscr{Y}(\phi)=\{Y_1(\phi),\cdots,Y_m(\phi)\}$$

of $\mathfrak{b}(\phi)$ relative to $\mathfrak{h}\cap\mathfrak{b}(\phi)$ and a Malcev basis

$$\mathscr{U}(\phi)=\{U_1(\phi),\cdots,U_p(\phi)\}$$

of $\mathfrak{h}$ relative to $\mathfrak{h}\cap\mathfrak{b}(\phi)$, such that the mappings

$$\phi\mapsto X_j(\phi);\phi\mapsto Y_j(\phi);\phi\mapsto U_j(\phi)$$

are rational and continuous on $\mathscr{V}_0$ for all j. The projections

$$T_\phi:\mathscr{S}(G/H,f)\to\mathscr{S}(G/B(\phi),\phi)(\phi\in\mathscr{V}_0)$$

given by

$$T_\phi(\xi)(g)=\int_{B(\phi)/H\cap B(\phi)}\xi(gb)\chi_\phi(b)d_{\mathscr{Y}(\phi)}\dot{b},\ \xi\in\mathscr{S}(G/H,f),g\in G,$$

allow us to define an operator

$$U:S(G/H,f)\to\int_{\mathscr{V}_0}^{\oplus}\mathscr{H}_\phi d\phi=\mathscr{H}_{\tau'},$$

(where $\mathscr{H}_\phi=L^2(G/B(\phi),\phi)$ denotes the Hilbert space of the irreducible representation π_ϕ) by setting

$$U(\xi)(\phi)=T_\phi(\xi)\in\mathscr{H}_\phi,\phi\in\mathscr{V}_0,\xi\in\mathscr{S}(G/H,f).$$

This mapping U is in fact an isometry for the L^2-norms and extends to a unitary operator from $\mathscr{H}_\tau=L^2(G/H,f)$ onto $\mathscr{H}_{\tau'}$ (see [1]). This operator diagonalizes the action of $D_\tau(G/H)$ (see [2]), that is for all $D\in D_\tau(G/H)$, there exist a function $\hat{D}$ on Γ_f such that for all $\xi\in S(G/H,f)$, one has

$$U(D\xi)(\phi)=\hat{D}(\phi)U(\xi)(\phi),\phi\in\mathscr{V}_0.$$

Let dh be a Haar measure of H. We choose now for any $\phi\in\mathscr{V}_0$ a Malcev basis

$$\mathscr{Z}(\phi)=\{Z_1(\phi),\cdots,Z_r(\phi)\}$$

of $\mathfrak{b}(\phi)\cap\mathfrak{h}$, such that for the Malcev basis $\mathscr{B}(\phi)=\mathscr{U}(\phi)\cup\mathscr{Z}(\phi)$ of $\mathfrak{h}$ the measure $d_{\mathscr{B}(\phi)}$ is just the given measure dh. Let also $\mathscr{Y}'(\phi)=\mathscr{Z}(\phi)\cup\mathscr{Y}(\phi)$ be the Malcev basis of $\mathfrak{b}(\phi)$.

We shall use this isometry to prove the Penney-Fujiwara Plancherel theorem.

1.2. Theorem. *Let G be a connected, simply connected nilpotent Lie group, H a connected Lie subgroup, and $\chi=\chi_f$ a unitary character on H associated with some $f\in\mathfrak{g}^*$ such that $f_{|\mathfrak{h}}$ is a Lie homomorphism. Let $\mathscr{V}$ (resp. $\mathscr{V}_0$) the affine subspace of Γ_f (resp. the open dense subset of $\mathscr{V}$) as in (1.1). Let $\mathscr{X}(\phi)$, $\mathscr{Y}(\phi)$, $\mathscr{U}(\phi)$, $\mathscr{Z}(\phi)$, $\mathscr{B}(\phi)$, $\mathscr{Y}'(\phi)$ also as in (1.1) for $\phi\in\mathscr{V}_0$. With the normalizations of the measures given by these bases one has for any $\varphi\in\mathscr{S}(G)$*

$$<S_{H,f},\varphi>=\int_{\mathscr{V}_0}<S_\phi,\varphi>d\phi,$$

where S_ϕ denotes the tempered distribution on $S(G)$ defined by

$$\begin{aligned}<S_\phi,\varphi>&=\int_{H/B(\phi)\cap H}T_\phi(P_{H,f}(\varphi))(h)\chi_f(h)d_{\mathscr{U}(\phi)}\dot{h}\\ &=<S_{H,f}^{B(\phi)},T_\phi(P_{H,f}(\varphi))>,\ \varphi\in\mathscr{S}(G),\ \phi\in\mathscr{V}_0.\end{aligned}$$

Proof. Let $\phi,\psi\in\mathscr{S}(G)$. We shall show that

$$<S_{H,f},\varphi^* * \psi>=\int_{\mathscr{V}_0}<S_\phi,\varphi^* * \psi>d\phi. \tag{1.2.1}$$

Since the factorization theorem of Dixmier-Malliavin says that every Schwartz-function ρ is of the form $\rho=\varphi^* * \psi$ for some elements φ,ψ in $\mathscr{S}(G)$ (see [9]), the theorem follows from (1.2.1). A standard computation tells us that

$$\begin{aligned}<S_{H,f},\varphi^* * \psi>&=\int_{G/H}\Big(\int_H\overline{\varphi(gh')\chi_f(h')}dh'\Big)\Big(\int_H\psi(gh)\chi_f(h)dh\Big)d\dot{g}\\ &=<P_{H,f}(\psi),P_{H,f}(\varphi)>_{L^2(G/H,f)},\end{aligned}$$

where $d\dot{g}$ is the G-invariant measure on G/H which is choosen such that $dg=d\dot{g}dh$.

Let now $\xi=P_{H,f}(\varphi),\eta=P_{H,f}(\psi)\in\mathscr{S}(G/H,f)$. The fact that the map U is an isometry tells us that

$$\int_{\mathscr{V}_0}<T_\phi(\eta),T_\phi(\xi)>_{\mathscr{H}_\phi}d\phi=<U(\eta),U(\xi)>_{\mathscr{H}_{\tau'}}=<\eta,\xi>_{L^2(G/H,f)}.$$

Hence in order to prove the theorem, it suffices to show that for every $\phi\in\mathscr{V}_0$ we have that

$$< T_\phi(\eta), T_\phi(\xi) >_{\mathscr{H}_\phi} = < S_\phi, \varphi^* * \psi > . \tag{1.2.2}$$

We write $P = P_{H,f}$, $B = B(\phi)$, $\mathscr{X} = \mathscr{X}(\phi)$, $\mathscr{Y} = \mathscr{Y}(\phi)$, $\mathscr{U} = \mathscr{U}(\phi)$, $\mathscr{B} = \mathscr{B}(\phi)$, $\mathscr{Y}' = \mathscr{Y}'(\phi)$, $\mathscr{Z} = \mathscr{Z}(\phi)$. We see that

$$\begin{aligned} < T_\phi(\eta), T_\phi(\xi) >_{\mathscr{H}_\phi} &= \int_{G/B} [(\int_{B/B\cap H} \eta(gb)\chi_\phi(b)d_{\mathscr{Y}}\dot{b})\overline{(\int_{B/B\cap H} \xi(gb)\chi_\phi(b)d_{\mathscr{Y}}\dot{b})}]d_{\mathscr{X}}\dot{g} \\ &= \int_{G/B} [\left(\int_{B/B\cap H} (\int_H \psi(gbh)\chi_f(h)d_{\mathscr{B}}h)\chi_\phi(b)d_{\mathscr{Y}}\dot{b}\right) . \\ &\quad \overline{\left(\int_{B/B\cap H} (\int_H \varphi(gbh')\chi_f(h')d_{\mathscr{B}}h')\chi_\phi(b)d_{\mathscr{Y}}\dot{b}\right)}]d_{\mathscr{X}}\dot{g}. \end{aligned} \tag{1.2.3}$$

On the other hand

$$\begin{aligned} T_\phi(P(\varphi^* * \psi))(h) &= \int_{B/B\cap H} (\int_H (\varphi^* * \psi)(hbh')\chi_f(h')d_{\mathscr{B}}h'\chi_\phi(b)d_{\mathscr{Y}}\dot{b} \\ &= \int_{B/B\cap H} (\int_H \int_G \varphi^*(g)\psi(g^{-1}hbh')dg\chi_f(h')d_{\mathscr{B}}h'\chi_\phi(b)d_{\mathscr{Y}}\dot{b} \\ &= \int_{B/B\cap H} (\int_H \int_G \overline{\varphi(gh^{-1})}\psi(gbh')dg\chi_f(h')d_{\mathscr{B}}h'\chi_\phi(b)d_{\mathscr{Y}}\dot{b} \\ &= \int_{B/B\cap H} (\int_G \overline{\varphi(gh^{-1})}\eta(gb)dg\chi_\phi(b)d_{\mathscr{Y}}\dot{b} \\ &= \int_G T_\phi(\eta)(g)\overline{\varphi(gh^{-1})}dg \\ &= \int_{G/B}\int_B \overline{\varphi(gbh^{-1})}T_\phi(\eta)(gb)d_{\mathscr{Y}'}bd_{\mathscr{X}}\dot{g} \\ &= \int_{G/B}\int_B \overline{\varphi(gbh^{-1})}T_\phi(\eta)(g)\chi_\phi(b^{-1})d_{\mathscr{Y}'}bd_{\mathscr{X}}\dot{g}. \end{aligned}$$

It follows that

$$\begin{aligned} < S_\phi, \varphi^* * \psi > &= \int_{H/B\cap H} T_\phi(P(\varphi^* * \psi))(h)\chi_f(h)d_{\mathscr{U}}\dot{h} \\ &= \int_{H/B\cap H} [\int_{G/B} (\int_B \overline{\varphi(gbh^{-1})}T_\phi(\eta)(g)\chi_\phi(b^{-1})d_{\mathscr{Y}'}b)d_{\mathscr{X}}\dot{g}]\chi_f(h)d_{\mathscr{U}}\dot{h} \end{aligned}$$

$$= \int_{H/B\cap H} [\int_{G/B} (\int_B \overline{\varphi(gbh^{-1})\chi_\phi(b)} T_\phi(\eta)(g) d_{\mathscr{Y}'} b) d_{\mathscr{X}} \dot{g}] \chi_f(h) d_{\mathscr{U}} \dot{h}$$

The operator $\pi_\phi(\varphi^*)$ is Hilbert-Schmidt, its kernel is the function

$$I(g,g') = \int_B \overline{\varphi(gbg'^{-1})\chi_\phi(b)} d_{\mathscr{Y}'} b,$$

and the function $(g,h) \mapsto I(g,h)$ is in $\overline{\mathscr{S}(G/B,\phi)} \otimes \mathscr{S}(H/B\cap H, f)$. Hence, using Fubini, we can deduce that

$$< S_\phi, \varphi^* * \psi > = \int_{G/B} \int_{H/B\cap H} \int_B \overline{\varphi(gbh^{-1})} T_\phi(\eta)(g) \overline{\chi_\phi(b)} d_{\mathscr{Y}'} b \chi_f(h) d_{\mathscr{U}} \dot{h} d_{\mathscr{X}} \dot{g}. \qquad (1.2.4)$$

Now for any $q \in C_c(G)$ we have that

$$\int_{B/B\cap H} \int_H q(b'h^{-1}) \chi_f(h) \overline{\chi_\phi(b')} d_{\mathscr{B}} h d_{\mathscr{Y}} \dot{b}' = \int_B \int_{H/H\cap B} q(b'h^{-1}) \chi_\phi(h) \overline{\chi_\phi(b')} d_{\mathscr{U}} \dot{h} d_{\mathscr{Y}'} b'. \qquad (1.2.5)$$

Indeed,

$$\int_{B/B\cap H} \int_H q(b'h^{-1}) \chi_\phi(h) \overline{\chi_\phi(b')} d_{\mathscr{B}} h d_{\mathscr{Y}} \dot{b}' = \int_{\mathbf{R}^m} \int_{\mathbf{R}^{r+p}} q(E_{\mathscr{Y}}(T)(E_{\mathscr{B}}(S))^{-1}) \overline{\chi_\phi(E_{\mathscr{Y}}(T))} \chi_\phi(E_{\mathscr{B}}(S)) dS dT$$

$$= \int_{\mathbf{R}^m} \int_{\mathbf{R}^p} \int_{\mathbf{R}^r} q(E_{\mathscr{Y}}(T)(E_{\mathscr{U}}(S) E_{\mathscr{Z}}(R))^{-1}) \overline{\chi_\phi(E_{\mathscr{Y}}(T))} \chi_\phi(E_{\mathscr{U}}(S) E_{\mathscr{Z}}(R) dR dS dT$$

$$= \int_{\mathbf{R}^m} \int_{\mathbf{R}^r} \int_{\mathbf{R}^p} q(E_{\mathscr{Y}}(T)(E_{\mathscr{Z}}(R)^{-1}(E_{\mathscr{U}}(S))^{-1}) \overline{\chi_\phi(E_{\mathscr{Y}}(T))} \chi_\phi(E_{\mathscr{U}}(S) E_{\mathscr{Z}}(R) dS dR dT$$

$$= \int_{\mathbf{R}^{m+r}} \int_{\mathbf{R}^p} q(E_{\mathscr{Y}'}(T)(E_{\mathscr{U}}(S))^{-1} \overline{\chi_\phi(E_{\mathscr{Y}'}(T))} \chi_\phi(E_{\mathscr{U}}(S) dS dT$$

$$= \int_{H/H\cap B} \int_B q(b'h^{-1}) \chi_\phi(h) \overline{\chi_\phi(b')} d_{\mathscr{U}} \dot{h} d_{\mathscr{Y}'} b'.$$

Hence by (1.2.4) and (1.2.5), we have that

$$<S_\phi, \varphi^* * \psi> = \int_{G/B}\left(\int_{H/B\cap H}\int_B \overline{\varphi(gbh^{-1})}T_\phi(\eta)(g)\overline{\chi_\phi(b)}d_{\mathscr{B}}b\chi_f(h)d_{\mathscr{U}}\dot{h}\right)d_{\mathscr{X}}\dot{g}$$

$$= \int_{G/B} T_\phi(\eta)(g)\left(\int_{B/B\cap H}\int_H \overline{\varphi(gbh^{-1})\chi_f(h)d_{\mathscr{B}}h\overline{\chi_\phi(b)}}d_{\mathscr{Y}}\dot{b}\right)d_{\mathscr{X}}\dot{g}$$

$$= \int_{G/B} T_\phi(\eta)(g)\overline{T_\phi(\xi)(g)}d_{\mathscr{X}(\phi)}\dot{g}$$

$$= < T_\phi(\eta), T_\phi(\xi)>_{\mathscr{H}_\phi}.$$

1.3. Corollary. *We keep the same hypotheses and notations as above. Let $\phi\in\mathscr{V}_0$ and $\psi\in\mathscr{S}(G/B(\phi),\phi)$. Let*

$$\beta_\phi(\psi) = \overline{<S_{H,f}^{B(\phi)}, \psi>} = \int_{H/B(\phi)\cap H} \overline{\psi(h)\chi_f(h)}d_{\mathscr{U}}\dot{h}, \tag{1.3.1}$$

then we have that for all $\omega\in\mathscr{D}(G)$ that

$$<S_{H,f}, \omega> = <\tau(\omega)\alpha_\tau, \alpha_\tau> = \int_{\mathscr{V}} <\pi_\phi(\omega)\beta_\phi, \beta_\phi> d\phi,$$

where α_τ is the canonical cyclic genaralized vector for τ i.e $\alpha_\tau(\xi)=\overline{\xi(e)}$, $\xi\in\mathscr{S}(G/H,f)$.

Indeed, it's not difficult to see that $<S_{H,f}, \omega> = <\tau(\omega)\alpha_\tau, \alpha_\tau>$ (see [12, 13]). On the other hand the following computation in ([12], page 177) tells us that for $\phi\in\mathscr{V}_0$ we have

$$<\pi_\phi(\omega)\beta_\phi, \beta_\phi> = \int_{H/B(\phi)\cap H} T_\phi(P_{H,f}(\omega))(h)\chi_f(h)d_{\mathscr{U}(\phi)}\dot{h} = <S_{H,f}^{B(\phi)}, T_\phi(P_{H,f}(\omega))>$$

for all $\omega\in\mathscr{D}(G)$ and theorem (1.2) permits us to conclude.

2. Invariant differential operators

Let G, H, f e.c.t. be as in the introduction. Let

$$C^\infty(G,\tau) = \{\xi\in C^\infty(G) : \xi(gh) = \chi(h^{-1})\xi(g), g\in G, h\in H\}.$$

Let Diff(G) be the algebra of all C^∞ differential operators taking $C^\infty(G,\tau)$ into itself, and $D_\tau(G/H)$ the algebra of operators $D|C^\infty(G,\tau)$ of $D\in\mathrm{Diff}(G)$ commuting with the action of τ on that space. Let $\Gamma_f = \mathrm{f}+\mathfrak{h}^\perp$. It is wellknown that the finite multiplicity condition for τ is equivalent to the condition that for one and hence for almost all $\phi\in\Gamma_f$, we have that

$$2\dim(Ad^*(H)\phi) = \dim(Ad^*(G)\phi).$$

(see [5]).

The aim of this section is to give a short proof of the following theorem proved by Corwin and Greenleaf in [7].

2.1. Theorem. *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra. Let* $f \in \mathfrak{g}^*$, *and* $\mathfrak{h}, \mathfrak{b}$ *two subalgebras of* $\mathfrak{g}$. *Suppose that* $\mathfrak{h}$ *is subordinate to* f, *i.e* $\langle f, [\mathfrak{h}, \mathfrak{h}] \rangle = (0)$ *and that* $\mathfrak{b}$ *is a polarization in* ϕ *for all* $\phi \in \Gamma_f = f + \mathfrak{h}^{\perp}$ *in general position and that* $\mathfrak{b}$ *is normalized by* $\mathfrak{h}$. *Let* $G = \exp \mathfrak{g}$, $H = \exp \mathfrak{h}$, $B = \exp \mathfrak{b}$. *Suppose in addition that the representation* $\tau = \mathrm{Ind}_H^G \chi_f$ *of* G *is decomposed on* $\hat{G}$ *with finite multiplicities. Then the conjecture* (0.3) *hold.*

Proof. First of all, let us remark that $\mathfrak{c} = \mathfrak{h} + \mathfrak{b}$ is a subalgebra of $\mathfrak{g}$, as $\mathfrak{h}$ normalizes $\mathfrak{b}$. Let $C = \exp(\mathfrak{c})$. Then $\tau = \mathrm{Ind}_C^G \tau_0$ where $\tau_0 = \mathrm{Ind}_H^G \chi_f$ and so by [6, (35)] the algebra $D_\tau(G/H)$ is isomorphic to the algebra $D_{\tau_0}(C/H)$. On the other hand, by the finite multiplicity condition, we know that $ad^*(\mathfrak{h})(f) \supset \mathfrak{c}^{\perp}$ and so $f + \mathfrak{c}^{\perp}$ is contained in the H-orbit of f. Hence the restriction map defines an H-covariant isomorphism between the algebra of H-invariant polynomial functions defined on Γ_f and the algebra of H-invariant polynomial functions defined on $f_{|\mathfrak{c}} + \mathfrak{h}^{\perp, \mathfrak{c}} \subset \mathfrak{c}^*$. Hence, we can suppose that $G = C$. In particular $\mathfrak{b}$ is now a normal subgroup of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{h} + \mathfrak{b}$.

The Fourier transform denoted here by U maps the space $L^2(G/H, f)$ onto the Hilbert space $L^2(\Gamma_f)$. The transformation U is defined for $\xi \in \mathscr{S}(G/H, f)$ by

$$U(\xi)(\phi) = \int_{B/B \cap H} \xi(b) \chi_\phi(b) d\dot{b}, \ \phi \in \Gamma_f.$$

Let us take a Malcev-basis $\mathscr{X} = \{X_1, \cdots, X_r\}$ of $\mathfrak{g}$ relative to $\mathfrak{h}$. Since $\mathfrak{g} = \mathfrak{h} + \mathfrak{b}$, we can assume that $\mathscr{X} \subset \mathfrak{b}$. But then for any $\phi \in \Gamma_f$, the set $\mathscr{X}$ is also a Malcev-basis of $\mathfrak{b}$ relative to $\mathfrak{h} \cap \mathfrak{b} = \mathfrak{h} \cap \mathfrak{g}(\phi)$. We can write then U in the following form:

$$U(\xi)(\phi) = \int_{\mathbf{R}^r} \xi(\exp(t_1 X_1) \cdots \exp(t_r X_r)) e^{-i(\sum_{k=1}^r t_k \phi(X_k))} dt_1 \cdots dt_r.$$

Hence in these coordinates U is just the ordinary Fourier transform on $\mathbf{R}^r$. We can transfer the representation τ of G on $L^2(G/H, f)$ to $L^2(\Gamma_f)$ with this map U and we get a representation of G on $L^2(\Gamma_f)$. In particular,

$$\rho(h)\eta(\phi) = \chi_f(h) \eta(Ad^*(h^{-1})\phi), \ \rho(b)\eta(\phi) = \chi_\phi(b)\eta(\phi)$$

for $b \in B, h \in H, \eta \in L^2(\Gamma_f)$.

Let now D be an element of $D_\tau(G/H)$. Then D commutes with $\tau(b)$ for all $b \in B$. Furthermore, D is represented by an element of the envelopping universal algebra $\mathfrak{u}(\mathfrak{g}_{\mathbf{C}})$ of $\mathfrak{g}_{\mathbf{C}}$, hence, it can be written on $S(G/H, f)$ as a differential operator with polynomial coefficients. Let $D' = U \circ D \circ U^{-1}$ be the corresponding operator acting on $S(\Gamma_f)$ the Schwartz space of Γ_f. Then, since U is the ordinary Fourier transform,

D' is also a differential operator with polynomial coefficients and D' commutes with the multiplication with the functions $e^{i(\cdot)(X)}$, $X \in \mathfrak{b}$, and hence D' is itself a multiplication operator with a polynomial function P_D. As D commutes with the action of H, the function P_D must be H-invariant. Then P_D is a H-invariant polynomial on Γ_f. On the other hand, if P is a H-invariant polynomial on Γ_f, then the multiplication with P defines an operator D' on $S(\Gamma_f)$ which commutes with the action of G. Hence $D = U^{-1} \circ D' \circ U$ is an element of $D_\tau(G/H)$. Hence we see that $D_\tau(G/H)$ is isomorphic to the algebra of H-invariant polynomial functions defined on Γ_f.

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