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On steady Stokes and Navier-Stokes problems with zero velocity at infinity in a three-dimensional exterior domain

By

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1. Introduction

In the paper we study the exterior Stokes and Navier-Stokes problems with zero conditions at infinity in weighted function spaces. Let us formulate these problems. Let Ω be an exterior domain in $\mathbf{R}^3$ (i.e. $\Omega=\mathbf{R}^3\setminus\bar{G}$, where G is a bounded domain). Without any loss of generality we can assume that the Cartesian coordinate system in $\mathbf{R}^3$ is chosen so, that the origin lies outside $\bar{\Omega}$ i.e. the point $x=0$ belongs to G. We also assume the boundary $\partial\Omega$ to be a smooth compact manifold. In Ω we consider the Stokes

$$-\nu\Delta\mathbf{v}+\nabla p=\mathbf{f},\quad x\in\Omega,$$

$$\nabla\cdot\mathbf{v}=g,\quad x\in\Omega, \tag{1.1}$$

and Navier-Stokes

$$-\nu\Delta\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{v}+\nabla p=\mathbf{f},\quad x\in\Omega,$$

$$\nabla\cdot\mathbf{v}=0,\quad x\in\Omega, \tag{1.2}$$

systems of equations with the boundary conditions

$$\mathbf{v}=\mathbf{h},\quad x\in\partial\Omega. \tag{1.3}$$

Moreover, we assume the velocity field $\mathbf{v}$ to vanish at infinity

$$\lim_{|x|\to\infty}\mathbf{v}(x)=0. \tag{1.4}$$

In (1.1)-(1.4) $x=(x_1,x_2,x_3)\in\mathbf{R}^3$, $\nabla=(\partial/\partial x_1,\partial/\partial x_2,\partial/\partial x_3)$, $\mathbf{v}=(v_1,v_2,v_3)$ and p are the velocity field and the pressure function in the flow, ν is the coefficient of viscosity, $\mathbf{f}$ and $\mathbf{h}$ are given vector functions in $\mathbf{R}^3$ and g is a given scalar function. By "$\cdot$" we denote the scalar product in $\mathbf{R}^3$.

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The mathematical study of the flow of viscous incompressible fluid around the three-dimensional obstacle (problem (1.2)-(1.4)) and the flow past the obstacle (the case $\lim_{|x|\to\infty} \mathbf{v}(x)=\mathbf{v}_\infty\neq 0$) was the subject of many papers. The existence theory of solutions with a finite Dirichlet integral (D-solutions) for both problems is well known (see [7]). In [7] it is also proved that the solutions approach their limits at infinity pointwise. In 1965 R. Finn [4] introduced so-called PR-solutions (physically reasonable solution), i.e. solutions satisfying the relation

$$|\mathbf{v}(x)|=O(|x|^{-1}), \quad \text{if} \quad \mathbf{v}_\infty=0,$$

$$|\mathbf{v}(x)-\mathbf{v}_\infty|=O(|x|^{-\frac{1}{2}-\varepsilon}), \quad \text{if} \quad \mathbf{v}_\infty\neq 0,$$

where ε may be arbitrary small. In the case of the flow past the obstacle ($\mathbf{v}_\infty\neq 0$) it was proved [1] (see also [5]) that every D-solution is a PR-solution. Moreover, in [1], [5] the asymptotic behaviour of solutions was investigated and the existence of a wake region behind the obstacle was shown. The uniqueness of such solutions was studied under additional smallness assumptions (e.g. [5]).

For the flow around the obstacle ($\mathbf{v}_\infty=0$) PR-solutions were constructed only under certain smallness assumptions on data of the problem [4], [5]. For sufficiently small data the uniqueness of D-solutions satisfying the energy inequality

$$\nu\int_\Omega|\nabla\mathbf{v}|^2dx\leq\int_\Omega\mathbf{f}\cdot\mathbf{v}\ dx$$

is known. Moreover, it was shown that the PR-solution admits the representation

$$\mathbf{v}(x)=\tilde{\mathbf{E}}(x)\cdot\Big(\int_{\partial\Omega}T(\mathbf{v},p)\mathbf{n}\,dS+\int_\Omega\mathbf{f}dx\Big)+\mathbf{w}(x), \tag{1.5}$$

where $\tilde{\mathbf{E}}$ is the velocity part of the fundamental matrix to the Stokes system, $T(\mathbf{v},p)$ is a stress tensor, $\mathbf{n}$ is a normal vector to $\partial\Omega$ and

$$|\mathbf{w}(x)|=O(|x|^{-1}) \quad \text{as} \quad |x|\to\infty \tag{1.6}$$

(e.g. [5]). For the derivatives $\partial\mathbf{v}/\partial x_k$ and for the pressure function p in [17] was derived the relation

$$|\nabla\mathbf{v}(x)|+|p(x)|=O(|x|^{-2}) \quad \text{as} \quad |x|\to\infty. \tag{1.7}$$

We do not mention here the other results obtained in this direction and only notice that the exhaustive list of referencies concerning these problems can be found in the books of G. P. Galdi [5] and in the survey paper of R. Farwig [3].

In (1.5) the chosen main term is of the order $O(|x|^{-1})$ (just as $|\tilde{\mathbf{E}}(x)|$) and the "remainder" $\mathbf{w}(x)$ has the same order of decay. Hence, the formulae (1.5)-(1.7) do not give an asymptotic representation of the solution and should be considered as the decay estimates only. The problem whether it is possible to find the asymptotic representation of the solution with the remainder $\mathbf{w}(x)$ of the order $o(|x|^{-1})$ was open. In this paper we construct (for small data) the solution $(\mathbf{v},p)$ of (1.2)-(1.4) which has the asymptotic representation

$$\mathbf{v}(x)=\frac{1}{r}\mathbf{V}(\theta,\varphi)+O(|x|^{-2+\varepsilon}),\quad \varepsilon>0,$$

$$p(x)=\frac{1}{r^2}P(\theta,\varphi)+O(|x|^{-3+\varepsilon}),\quad \varepsilon>0,$$

where (r,θ,φ) are spherical coordinates in $\mathbf{R}^3$.

The most efficient and convenient way to investigate elliptic problems in unbounded domains is to use function spaces with weighted norms. However, in applying such approach to the nonlinear Navier-Stokes problem (1.2), (1.3), (1.4) there appear certain peculiarities and difficulties which are emphasized and overcome in the paper. The special attention is given to the derivation of the asymptotic formulae the essence of which subtend that the remainder must have a better decay rate at infinity as that of the chosen leading term (compare with (1.5), (1.6)).

For small data we consider the nonlinear problem (1.2), (1.3), (1.4) as a perturbation of the linear one. The results related to the linear Stokes problem (1.1), (1.3), (1.4) (see Section 2.1) are proved by applying the general theory of elliptic problems in domains with conical points [6], [8], [9], [13]. In order to employ these results, we regard Ω as a domain with infinitely remoted conical point, i.e. Ω implies a compact perturbation of a complete cone $\mathbf{K}=\mathbf{R}^3\backslash\{0\}$. The investigations lead us to the conclusion

$$|D^\alpha \mathbf{v}(x)|\sim|x|^{-k-1},\quad |\alpha|=k \quad \text{as} \quad |x|\to\infty. \tag{1.8}$$

Thus, the nonlinear term $(\mathbf{v}\cdot\nabla)\mathbf{v}$ is equivalent to $|x|^{-3}$ as $|x|\to\infty$. The behaviour (1.8) of the solution $\mathbf{v}$ to the linear and nonlinear problems forces us to deal with the weighted spaces where the operator of the Stokes problem looses the Fredholm property. The latter is related to the fact that $\lambda=-1$ is an eigenvalue of the operator pencil associated with the Stokes system. To overcome this difficulty, we narrov the domain of definition of the Stokes operator and study the Stokes problem (1.1), (1.3), (1.4) in "weighted spaces with detached asymptotics". We prove that in such spaces the Stokes operator is Fredholm and that the problem (1.1), (1.3), (1.4) is solvable if the right-hand side $\mathbf{f}$ satisfies certain orthogonality conditions (Section 2.2). The corresponding arguments are closed to those from the paper [10]. Finally, in Section 3 we prove that the orthogonality conditions are always valid for $\mathbf{f}=(\mathbf{v}\cdot\nabla)\mathbf{v}$, provided that $\mathbf{v}$ is solenoidal and belongs to the space mentioned above. This allows us to reduce the nonlinear problem to the operator equation and to prove its solvability for small data applying the Banach contraction principle.

Notice that the analogous problem for "large" data remains open and that for the two-dimensional exterior domain Ω such a result is not known even for small data.

2. Stokes problem

2.1. Weighted function spaces and the solvability of the Stokes problem. Let $C_0^\infty(\overline{\Omega})$ be the set of all infinitely differentiable functions with compact supports in $\overline{\Omega}$ and $L^2(\Omega)$ be the space of measurable square integrable over Ω functions. As usual, for nonnegative integer l and $\beta\in\mathbf{R}$, by $V^l_\beta(\Omega)$ we denote the completion of $C_0^\infty(\overline{\Omega})$ with respect to the weighted norm

$$\|z\,;\,V^l_\beta(\Omega)\|=\Big(\sum_{|\alpha|=0}^{l}\|(1+r)^{\beta-l+|\alpha|}\mathrm{D}^\alpha z\,;\,L^2(\Omega)\|^2\Big)^{1/2},$$

where $\mathrm{D}^\alpha=\partial^{|\alpha|}/\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}$, $|\alpha|=\alpha_1+\alpha_2+\alpha_3$, $\alpha_i\geq 0$. $H^l(\Omega)$ is the Sobolev spaces of functions with the norm

$$\|z\,;\,H^l(\Omega)\|=\Big(\sum_{|\alpha|\leq l}\int_\Omega|\mathrm{D}^\alpha z|^2dx\Big)^{1/2}$$

and $H^{l-1/2}(\partial\Omega)$ is the space of traces on $\partial\Omega$ of functions from $H^l(\Omega)$. $H^{l-1/2}(\partial\Omega)$ is supplied with the natural norm.

We consider the operato~ S^l_β of the Stokes problem (1.1), (1.3), (1.4)

$$\mathscr{D}^l_\beta V(\Omega)\equiv V^{l+1}_\beta(\Omega)^3\times V^l_\beta(\Omega)\ni(\mathbf{v},p)\longmapsto S^l_\beta(\mathbf{v},p)=$$

$$=(\mathbf{f},g,\mathbf{h})\in\mathscr{R}^l_\beta V(\Omega\,;\,\partial\Omega)\equiv V^{l-1}_\beta(\Omega)^3\times V^l_\beta(\Omega)\times H^{l+1/2}(\partial\Omega). \tag{2.1}$$

It is continuous for each $\beta\in\mathbf{R}$ and $l=0,1,2,\cdots$. In order to investigate the properties of S^l_β, we apply general results on formally self-adjoint elliptic problems in domains with conical points, regarding Ω as a domain with infinitely remoted conical point, i.e. for large $|x|$ the domain Ω coincides with a complete cone $\mathbf{K}=\mathbf{R}^3\backslash\{0\}$. Since in this part our reasonings are standard (e.g. [13], Ch. 6), we only underline the scheme of the proofs. First, we consider the Stokes system (1.1) in the complete cone $\mathbf{K}$. We rewrite (1.1) in spherical coordinates (r,ω) with the origin at $x=0$, i.e. $|x|=r$, $\omega=(\varphi,\theta)$, $0\leq\varphi<2\pi$, $0\leq\theta<\pi$. Power solutions of the Stokes problem are the functions of the form

$$\mathbf{v}(x)=r^\lambda\mathbf{V}(\omega),\quad p(x)=r^{\lambda-1}P(\omega),\quad \lambda\in\mathbf{C}, \tag{2.2}$$

which solve the homogeneous Stokes system in $\mathbf{R}^3$:

$$-\nu\Delta\mathbf{v}+\nabla p=0,\quad \nabla\cdot\mathbf{v}=0,\quad x\in\mathbf{R}^3\backslash\{0\}. \tag{2.3}$$

Substituting (2.2) into (2.3) and separating the variables r and ω, we obtain for $(\mathbf{V},\,P)$ the system of partial differential equations on the unit sphere $\Gamma=\{x:|x|=1\}$, depending on the complex parameter λ :

$$\mathbf{S}(\lambda\,;\,\mathrm{D}_\omega)(\mathbf{V},P)=0,\quad \omega\in\Gamma.^2 \tag{2.4}$$

The family of mappings $\lambda\longrightarrow\mathbf{S}(\lambda\,;\,\cdot)$ is called the operator pencil associated to the problem (2.3). The complex numbers λ for which the problem (2.4) has nontrivial solutions are called eigenvalues of (2.4) and the corresponding nontrivial solutions are called eigenvectors. It is evident that the functions (2.2) are power solutions of (2.3) if and only if λ is an eigenvalue of (2.4). The following results are well known (e.g. [13], Ch. 6.4).

Lemma 2.1. *The eigenvalues of the pencil* $\lambda \longrightarrow \mathbf{S}(\lambda\,;\,\cdot)$ *consist of numbers* $\lambda \in \mathbf{Z}$. *If* $\lambda \in \mathbf{Z}$, *there exist nontrivial solutions to* $\mathbf{S}(\lambda\,;\,\mathrm{D}_\omega)(\mathbf{V},P)=0$ *which are smooth on the whole sphere* Γ. *The corresponding power solutions* $(r^\lambda \mathbf{V}(\omega),\ r^{\lambda-1}P(\omega))$ *consist either of homogeneous polynomials (the case* $\lambda \geq 0$), *or can be obtained by differentiating the colomns*

$$\mathbf{E}^{(j)}(x)=\frac{1}{8\pi\nu|x|^3}(\delta_{j1}|x|^2+x_1x_j,\quad \delta_{j2}|x|^2+x_2x_j,$$

$$\delta_{j3}|x|^2+x_3x_j,\ 2\nu x_j)^T,\quad j=1,2,3, \tag{2.5}$$

$$\mathbf{E}^{(4)}(x)=\left(\nabla_x\frac{1}{2\pi\nu|x|},\ 0\right)$$

of the fundamental matrix $\mathbf{E}$ *to the Stokes system* ($\mathbf{E}^{(4)}$ *is the forth column of the fundamental matrix* $\mathbf{E}$ *without the delta-function in its pressure component). In particular, to the eigenvalue* $\lambda=-1$ *correspond three power solutions*

$$(\mathbf{v}^{(j)},p^{(j)})=\mathbf{E}^{(j)},\quad j=1,2,3. \tag{2.6}$$

The power solutions corresponding to $\lambda=0$ *have the form*

$$(\mathbf{v},p)=(\mathbf{c},0),\quad \mathbf{c}\in\mathbf{R}^3. \tag{2.7}$$

Let us consider now the Stokes problem (1.1), (1.3), (1.4) in the exterior domain Ω. Notice that there holds the following Green's formula

$$(-\nu\Delta\mathbf{v}+\nabla p,\mathbf{u})_\Omega+(-\nabla\cdot\mathbf{v},q)_\Omega+(\mathbf{v},\mathbf{n}q-\nu\partial_n\mathbf{u})_{\partial\Omega}=$$

$$=(\mathbf{v},-\nu\Delta\mathbf{u}+\nabla q)_\Omega+(p,-\nabla\cdot\mathbf{u})_\Omega+(\mathbf{n}p-\nu\partial_n\mathbf{v},\mathbf{u})_{\partial\Omega}, \tag{2.8}$$

where $(\cdot,\cdot)_\Xi$ is a scalar product in $L^2(\Xi)$; $(\mathbf{v},p)$, $(\mathbf{u},q)\in C_0^\infty(\bar{\Omega})^4$; $\mathbf{n}$ is a unit normal vector to $\partial\Omega$ and $\partial_n=\nabla\cdot\mathbf{n}$ is a normal derivative.

Because of Green's formula (2.8) the problem (1.1), (1.3) is formally selfadjoint.

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[2]For the spherical components $(V_r,\ V_\theta,\ V_\varphi)$ of the velocity field $\mathbf{V}$ the problem (2.4) takes the form

$$-\nu[(\lambda+1)\lambda-2+\Delta_\Gamma]V_r+2\nu\,\mathrm{div}_\Gamma(V_\theta,\ V_\varphi)+(\lambda-1)P=0,$$

$$-\nu[(\lambda+1)\lambda+\Delta_\Gamma]V_\theta+\frac{\nu V_\theta}{\sin^2\theta}+\frac{2\nu\cos\theta}{\sin^2\theta}\partial_\varphi V_\varphi-2\nu\partial_\theta V_r+\partial_\theta P=0,$$

$$-\nu[(\lambda+1)\lambda+\Delta_\Gamma]V_\varphi-\frac{2\nu}{\sin\theta}\partial_\varphi V_r-\frac{2\nu\cos\theta}{\sin^2\theta}\partial_\varphi V_\theta+\frac{\nu V_\varphi}{\sin^2\theta}+\frac{1}{\sin\theta}\partial_\varphi P=0,$$

$$-(\lambda+2)V_r-\mathrm{div}_\Gamma(V_\theta,\ V_\varphi)=0,$$

where $\Delta_\Gamma=(\sin\theta)^{-1}\partial_\theta(\sin\theta\partial_\theta)+(\sin\theta)^{-2}\partial_\varphi^2$ is the Laplace-Beltrami operator, $\mathrm{div}_\Gamma(V_\theta,\ V_\varphi)=(\sin\theta)^{-1}[\partial_\theta(\sin\theta V_\theta)+\partial_\varphi V_\varphi]$ is the surface divergence of the tangental vector $(V_\theta,\ V_\varphi)$, $\partial_\theta=\partial/\partial\theta$, $\partial_\varphi=\partial/\partial\varphi$.

Therefore, applying the general results (see Ch. 6.1, 6.4 in [13]) we get

Theorem 2.1. *(i) The mapping S^l_β (see (2.1)) is Fredholm if and only if $\beta-l+1/2\notin\mathbf{Z}$. In the case where $\beta-l+1/2\in\mathbf{Z}$ the range of S^l_β is not closed.*
(ii) The operators S^l_β and $S^l_{2l-\beta}$ are Fredholm simultaneously and

$$\operatorname{coker} S^l_{2l-\beta}=\{(\mathbf{u},q,(\mathbf{n}q-\nu\partial_n\mathbf{u})|_{\partial\Omega}):(\mathbf{u},q)\in\ker S^l_\beta\}. \tag{2.9}$$

(iii) The mapping S^l_β is an isomorphism, if $\beta\in(l-1/2,l+1/2)$. For $\beta>l+1/2$, S^l_β is a monomorphism and for $\beta<l-1/2$, S^l_β is an epimorphism.

Proof. The part (i) of the theorem follows from Theorem 3.4 in [6] (see also Theorem 4.1.2 and Remark 4.1.5 in [13]) and Lemma 2.1 (the statement is true if the line $\{\lambda\in\mathbf{C}:\operatorname{Re}\lambda=\beta-l+1/2\}$ is free of eigenvalues of the problem (2.4)).

The part (ii) is a sequence of general results on self-adjoint elliptic problems (see [8] and Ch. 6.1 in [13]).

In order ot prove (iii), we mention that by (i) the mapping S^l_l is a Fredholm operator ($l-l+1/2\notin\mathbf{Z}$). Since $S^l_\beta=S^l_{2l-\beta}$ for $\beta=l$, we have

$$\operatorname{Ind}S^l_l\equiv\dim\ker S^l_l-\dim\operatorname{coker}S^l_l=0.$$

For each $(\mathbf{v},p)\in\ker S^l_l$ there holds the relation

$$0=(-\nu\Delta\mathbf{v}+\nabla p,\mathbf{v})_\Omega+(\nu\partial_n\mathbf{v}-\mathbf{n}p,\mathbf{v})_{\partial\Omega}=\sum_{j=1}^{3}(\nabla v_j,\nabla v_j)_\Omega,$$

i.e. $v_i=c_i$. By virtue of homogeneous boundary conditions $c_i=0$. Further, from the Stokes equations it follows $\nabla p=0$ and hence, $p=c_0$. Since $c_0\notin V^l_l(\Omega)$, we conclude $c_0=0$. Thus,

$$\ker S^l_l=\{0\}$$

and the mapping S^l_l is an isomorphism. Since the strip $\{\lambda\in\mathbf{C}: 0<\operatorname{Re}\lambda=\beta-\lambda+1/2<1\}$ is free of eigenvalues of the problem (2.4), it follows from Theorem 3.3 [6] and Theorems 4.2.1, 4.2.4 [13] that S^l_l is an isomorphism for each $\beta\in(l-1/2, l+1/2)$.

Finally, an increase of β narrows the space $\mathcal{D}^l_\beta V(\Omega)$. It is not difficult to verify that in the case $\beta>l+1/2$ the cokernel of S^l_β is not trivial (we will see it later in Theorem 2.2). In particular, this means that in the case $\beta<l-1/2$ the kernel of S^l_β is not trivial (see the formula (2.9)). Hence, S^l_β is a monomorphism for $\beta>l+1/2$ and S^l_β is an epimorphism for $\beta<l-1/2$.

We conclude the investigation of the Stokes problem (1.1), (1.3), (1.4) in spaces $\mathcal{D}^l_\beta V(\Omega)$ with an assertion concerning the asymptotics of the solution.

Theorem 2.2. *Let $(\mathbf{f},g,\mathbf{h})\in\mathcal{R}^l_\gamma V(\Omega;\partial\Omega)$, $l\geq1$, $\gamma\in(l+1/2,l+3/2)$. Then the solution $(\mathbf{v},p)\in\mathcal{D}^l_\beta V(\Omega)$, $\beta\in(l-1/2,l+1/2)$ of the problem (1.1), (1.3), (1.4)[3] admits the asymptotic representation*

$$\binom{\mathbf{v}}{p}=\sum_{k=1}^{3} c_k \mathbf{E}^{(k)}+\binom{\tilde{\mathbf{v}}}{\tilde{p}},\ \binom{\tilde{\mathbf{v}}}{\tilde{p}}\in \mathcal{D}_\gamma^l V(\Omega) \tag{2.10}$$

where $c_k=c_k(\mathbf{f},g,\mathbf{h})$ *are constants. Moreover, there holds the estimate*

$$\sum_{k=1}^{3}|c_k|+\|(\mathbf{v},\tilde{p})\,;\,\mathcal{D}_\gamma^l V(\Omega)\|\leq c\|(\mathbf{f},g,\mathbf{h})\,;\,\mathcal{R}_\gamma^l V(\Omega\,;\,\partial\Omega)\|. \tag{2.11}$$

The theorem follows immediately from Theorem 1.2 [6] (see also Theorems 4.2.1, 6.4.3 [13]) after we take into account the obtained information on eigenvalues of the pencil $\mathbf{S}(\lambda;\cdot)$ (Lemma 2.1) and Theorem 2.1.

Remark 2.1. The analogous results are also true in weighted function spaces generated by L^q-norms ($q>1$). However, we ignore these generalizations, since they are not necessary for our purposes (e.g. [9] and Ch. 3.6 and Remark 4.1.6 in [13]).

Let us consider the problem (1.1), (1.3), (1.4) in weighted Hölder spaces. For $l\geq 0$ being an integer, $\delta\in(0,1)$ and $\beta\in\mathbf{R}$, the weighted Hölder space $\Lambda_\beta^{l,\delta}(\Omega)$ is defined as the completion of $C_0^\infty(\overline{\Omega})$ in the norm

$$\|\varphi\,;\,\Lambda_\beta^{l,\delta}(\Omega)\|=\sum_{|\alpha|\leq l}\sup_{x\in\Omega}(|x|^{\beta-l-\delta+|\alpha|}|D^\alpha\varphi(x)|)+$$

$$+\sum_{|\alpha|=[l]}\sup_{x\in\Omega}\Big(|x|^\beta\sup_{y\in\Omega}\Big(\frac{|D^\alpha\varphi(x)-D^\alpha\varphi(y)|}{|x-y|^\delta}\Big)\Big).$$

Further, by $C^{l,\delta}(\partial\Omega)$ we denote the space of traces on $\partial\Omega$ of functions in $C^{l,\delta}(\Omega)$, i.e.

$$C^{l,\delta}(\partial\Omega)=\{\varphi|_{\partial\Omega}:\varphi\in C^{l,\delta}(\Omega)\},$$

where $C^{l,\delta}(\Omega)$ is the usual Hölder space.

The operator $\mathfrak{S}_\beta^{l,\delta}$ of the Stokes problem (1.1), (1.3), (1.4) realizes the continuous mapping:

$$\mathcal{D}_\beta^{l,\delta}\Lambda(\Omega)\equiv\Lambda_\beta^{l+1,\delta}(\Omega)^3\times\Lambda_\beta^{l,\delta}(\Omega)\ni(\mathbf{v},p)\longmapsto\mathfrak{S}_\beta^{l,\delta}(\mathbf{v},p)=$$

$$=(\mathbf{f},g,\mathbf{h})\in\mathcal{R}_\beta^{l,\delta}\Lambda(\Omega\,;\,\partial\Omega)\equiv\Lambda_\beta^{l-1,\delta}(\Omega)^3\times\Lambda_\beta^{l,\delta}(\Omega)\times C^{l+1,\delta}(\partial\Omega)^3.$$

According to general results on elliptic boundary value problems (e.g. [13]) $\mathfrak{S}_\beta^{l,\delta}$ inherits (after the obvious recalculation of indeces) the properties of the operator S_β^l and, therefore, there holds the following assertion.

Theorem 2.3. (*i*) *The mapping* $\mathfrak{S}_\beta^{l,\delta}$ *is Fredholm if and only if* $\beta-l-\delta-1\notin\mathbf{Z}$. *In the case* $\beta-l-\delta-1\in\mathbf{Z}$ *the range of* $\mathfrak{S}_\beta^{l,\delta}$ *is not closed.*
(*ii*) *If* $\beta\in(l+\delta+1,l+\delta+2)$ *the mapping* $\mathfrak{S}_\beta^{l,\delta}$ *is an isomorphism, if* $\beta>l+\delta+2$,

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[3]The existence of such a solution follows from Theorem 2.1 (iii).

$\mathfrak{S}_\beta^{l,\delta}$ is a monomorphism and if $\beta<l+\delta+1$, $\mathfrak{S}_\beta^{l,\delta}$ is an epimorphism.
(iii) Let $(\mathbf{f},g,\mathbf{h})\in\mathcal{R}_\gamma^{l,\delta}\Lambda(\Omega;\partial\Omega)$, $\gamma\in(l+\delta+2,l+\delta+3)$. Then the solution $(\mathbf{v},p)\in\mathcal{D}_\beta^{l,\delta}\Lambda(\Omega)$, $\beta\in(l+\delta+1,l+\delta+2)$, admits the asymptotic represention (2.10) *with $(\tilde{\mathbf{v}},\tilde{p})\in\mathcal{D}_\gamma^{l,\delta}\Lambda(\Omega)$ and there holds the estimate*

$$\sum_{k=1}^{3}|c_k|+\|(\tilde{\mathbf{v}},\tilde{p});\mathcal{D}_\gamma^{l,\delta}\Lambda(\Omega)\|\le c\|(\mathbf{f},g,\mathbf{h});\mathcal{R}_\gamma^{l,\delta}\Lambda(\Omega;\partial\Omega)\|. \tag{2.12}$$

2.2. Stokes problem in weighted function spaces with detached asymptotics. We start this section by explaining the motivation of the presented below investigations on the linear Stokes problem (1.1), (1.3), (1.4). In Section 3 we are going to study (for small data) the nonlinear Navier-Stokes problem (1.2), (1.3), (1.4) as a perturbation of the linear problem (1.1), (1.3), (1.4). Let $(\mathbf{v},p)$ be the solution of (1.2), (1.3), (1.4) admiting the asymptotic representation (2.10), i.e.

$$|\mathbf{v}(x)|\sim r^{-1} \quad \text{as} \quad r\to\infty.$$

The nonlinear term $(\mathbf{v}\cdot\nabla)\mathbf{v}$ in (1.2) is then equivalent to r^{-3} as $r\to\infty$. Considering $(\mathbf{v}\cdot\nabla)\mathbf{v}$ as a right-hand side of the linear problem, we get $(\mathbf{v}\cdot\nabla)\mathbf{v}\in\Lambda_\beta^{l-1,\delta}(\Omega)^3$ with $\beta-l-\delta-1=1\in\mathbf{Z}$. As it follows from Theorem 2.3, in such the case $\mathfrak{S}_\beta^{l,\delta}$ gives off the Fredholm property (the same is true for the operator S_β^l). On the other hand, for $\beta>l+\delta+2$ the operator $\mathfrak{S}_\beta^{l,\delta}$ is a monomorphism with a nontrivial cokernel and the solvability of (1.1), (1.3), (1.4) requires additional compatibility conditions. Thus, the nonlinear problem (1.2), (1.3), (1.4) cannot be treated as a perturbation of the linear one regarded in the classical Kondratjev and Hölder weighted spaces. In order to overcome this difficulty, we narrow the domain of definition of the Stokes operator introducing the weighted function spaces with detached asymptotics, which reflect more precisely the behaviour of the solutions at infinity. Such kind of spaces were first introduced in [14], [15] (see also [13] Ch. 6.2 and Ch. 12.2) in connection with the investigation of boundary value problems in domains with edges on the boundary. The asymptotics was separated only in the solution itself and the right-hand sides were specified to decay sufficiently fast near the edge. For the Stokes and Navier-Stokes problems in domains with cylindrical outlets to infinity this technique has been applied in [12]. The complete separating of the asymptotics (both in the solution and in the right-hand side) become relevant in [10], [11], [16]. The considerations below are similar to those in [10].

Let us fix a natural number l and a weight index $\gamma\in(l+1/2,l+3/2)$ and introduce the space $\mathfrak{D}_\gamma^l V(\Omega)$ of vector functions $(\mathbf{v},p)$ which can be represented in the form

$$\mathbf{v}(x)=\frac{1}{r}\mathbf{V}(\omega)+\tilde{\mathbf{v}}(x), \quad p(x)=\frac{1}{r^2}P(\omega)+\tilde{p}(x), \tag{2.13}$$

where $(\mathbf{V},P)\in H^{l+1}(\Gamma)^3\times H^l(\Gamma)$ and $(\tilde{\mathbf{v}},\tilde{p})\in\mathcal{D}_\gamma^l(\Omega)$. The norm in the space $\mathfrak{D}_\gamma^l V(\Omega)$ is defined by the formula

$$\|(\mathbf{v},p)\,;\mathfrak{D}^l_\gamma V(\Omega)\|=(\|(\mathbf{V},P)\,;H^{l+1}(\Gamma)^3\times H^l(\Gamma)\|^2+$$

$$+\|(\tilde{\mathbf{v}},\tilde{p})\,;\mathcal{D}^l_\gamma V(\Omega)\|^2)^{1/2}. \tag{2.14}$$

The Stokes operator (1.1), (1.3), (1.4) acts continuously from $\mathfrak{D}^l_\gamma V(\Omega)$ into the space $\mathfrak{R}^l_\gamma V(\Omega,\partial\Omega)$ which contains triples of functions $(\mathbf{f},g,\mathbf{h})$ admiting the representation

$$\mathbf{f}(x)=\frac{1}{r^3}\mathbf{F}(\omega)+\tilde{\mathbf{f}}(x),\quad g(x)=\frac{1}{r^2}G(\omega)+\tilde{g}(x),\quad \mathbf{h}(x)=\tilde{\mathbf{h}}(x), \tag{2.15}$$

where $(\mathbf{F},G)\in H^{l-1}(\Gamma)^3\times H^l(\Gamma)$, $(\tilde{\mathbf{f}},\tilde{g},\tilde{\mathbf{h}})\in\mathcal{R}^l_\gamma V(\Omega,\partial\Omega)$. The norm in $\mathfrak{R}^l_\gamma V(\Omega,\partial\Omega)$ is defined by

$$\|(\mathbf{f},g,\mathbf{h})\,;\mathfrak{R}^l_\gamma V(\Omega,\partial\Omega)\|=(\|(\mathbf{F},G)\,;H^{l-1}(\Gamma)^3\times H^l(\Gamma)\|^2+$$

$$+\|(\tilde{\mathbf{f}},\tilde{g},\tilde{\mathbf{h}})\,;\mathcal{R}^l_\gamma V(\Omega\,;\partial\Omega)\|^2)^{1/2}. \tag{2.16}$$

Remark 2.2. Let $(\mathbf{v},p)\in\mathcal{D}^l_\beta(\Omega)$ be a solution of the problem (1.1), (1.3), (1.4) with the right-hand side $(\mathbf{f},g,\mathbf{h})\in\mathcal{R}^l_\gamma(\Omega)$. If $\beta\in(l-1/2,l+1/2)$ and $\gamma\in(l+1/2,l+3/2)$, the solution admits the asymptotic representation (2.10) (Theorem 2.2). Hence, $(\mathbf{v},p)$ can be represented in the form (2.13) where

$$\left(\frac{1}{r}\mathbf{V}(\omega),\frac{1}{r^2}P(\omega)\right)=\sum_{k=1}^{3}c_k\mathbf{E}^{(k)}.$$

(see formulae (2.5) for the definition of the functions $\mathbf{E}^{(k)}$). This was the reason why we have used the combination of words "spaces with detached asymptotics" in the title of the section.

Lemma 2.2. *Let* $\mathbf{S}(\lambda\,;\cdot)$ *be the operator pencil associated to the problem* (2.3). *The problem*

$$\mathbf{S}(-1\,;D_\omega)(\mathbf{V},P)=(\mathbf{F},G),\quad \omega\in\Gamma, \tag{2.17}$$

with $(\mathbf{F},G)\in H^{l-l}(\Gamma)^3\times H^l(\Gamma)$ *has a solution* $(\mathbf{V},P)\in H^{l+1}(\Gamma)^3\times H^l(\Gamma)$ *if and only if there holds the compatibility condition*

$$\int_\Gamma\mathbf{F}\cdot\mathbf{c}\,d\Gamma_\omega=0,\quad \forall\mathbf{c}\in\mathbf{R}^3. \tag{2.18}$$

The solution $(\mathbf{V},P)$ *is not unique; the homogeneous problem* (2.17) *has three linearly independent solutions*

$$\xi^{(1)}(\omega)=\frac{1}{8\pi\nu}(1+\sin^2\theta\cos^2\varphi,\ \sin^2\theta\sin\varphi\cos\varphi,\ \sin\theta\cos\theta\cos\varphi,\ 2\nu\sin\theta\cos\varphi),$$

$$\xi^{(2)}(\omega)=\frac{1}{8\pi\nu}(\sin^2\theta\cos\varphi\sin\varphi,\ 1+\sin^2\theta\sin^2\varphi,\ \sin\theta\cos\theta\sin\varphi,\ 2\nu\sin\theta\sin\varphi),$$

$$\xi^{(3)}(\omega)=\frac{1}{8\pi\nu}(\sin\theta\cos\theta\cos\varphi,\ \sin\theta\cos\theta\sin\varphi,\ 1+\cos^2\theta,\ 2\nu\cos\theta). \tag{2.19}$$

Notice that $\xi^{(j)}(\omega)$ *are the traces of the fundamental colomns* $\mathbf{E}^{(j)}$, $j=1,2,3$, *on the the sphere* Γ (*see* (2.5)).

Proof. The elementary calculations show that there holds the following Green's formula

$$\langle \mathbf{S}(\lambda\,;\mathrm{D}_\omega)(\mathbf{V},P),\ (\mathbf{U},Q)\rangle=\langle(\mathbf{V},P),\ \mathbf{S}(1+\bar\lambda\,;\mathrm{D}_\omega)(\bar{\mathbf{U}},\bar{Q})\rangle, \tag{2.20}$$

where $\lambda\in\mathbf{C}$ and $\langle\cdot,\cdot\rangle$ stays for the scalar product in $L^2(\Gamma)$. Hence, the pencils $\mathbf{S}(\lambda\,;\mathrm{D}_\omega)$ and $\mathbf{S}(1+\bar\lambda\,;\mathrm{D}_\omega)$ are formally adjoint. Thus, the formally adjoint operator to $\mathbf{S}(-1\,;\mathrm{D}_\omega)$ is $\mathbf{S}(0\,;\mathrm{D}_\omega)$. Since (2.17) is an elliptic problem, it has the Fredholm property. As usual, the solvability conditions for (2.17) can be obtained from (2.20), taking in it $\mathbf{S}(-1\,;\mathrm{D}_\omega)(\mathbf{V},P)=(\mathbf{F},G)$ and substituting instead of $(\mathbf{U},Q)$ the eigenvectors of the adjoint problem

$$\mathbf{S}(0\,;\mathrm{D}_\omega)(\mathbf{U},Q)=(\mathbf{0},0). \tag{2.21}$$

From Lemma 2.1 we know that the eigenvectors of (2.21) have the form

$$(\mathbf{U},Q)=(\mathbf{c},0),\quad \mathbf{c}\in\mathbf{R}^3$$

(see (2.7)) and we immediately get the solvability condition (2.18). From Lemma 2.1 it also follows the formulae (2.19) (see (2.5)) for linearly independent solutions $\xi^{(k)}$, $k=1,2,3$, of the homogeneous problem (2.17).

Let us denote by $\mathfrak{R}_\perp$ the subspace of $\mathfrak{R}^l_\gamma V(\Omega\,;\partial\Omega)$ consisting from elements $(\mathbf{f},g,\mathbf{h})\in\mathfrak{R}^l_\gamma V(\Omega\,;\partial\Omega)$ which satisfy the orthogonality condition (2.18), i.e.

$$\mathfrak{R}_\perp=\left\{(\mathbf{f},g,\mathbf{h})\in\mathfrak{R}^l_\gamma V(\Omega\,;\partial\Omega):\ \int_\Gamma \mathbf{F}(\omega)\cdot\mathbf{c}\,d\Gamma_\omega=0,\quad \forall\mathbf{c}\in\mathbf{R}^3\right\}$$

(see the representation formula (2.15) for $\mathbf{f}$).

Theorem 2.4. *Let* $(\mathbf{f},g,\mathbf{h})\in\mathfrak{R}_\perp$. *Then the Stokes problem* (1.1), (1.3), (1.4) *has a unique solution* $(\mathbf{v},p)\in\mathfrak{D}^l_\gamma V(\Omega)$ *and there holds the estimate*

$$\|(\mathbf{v},p)\,;\mathfrak{D}^l_\gamma V(\Omega)\|\le c\|(\mathbf{f},g,\mathbf{h})\,;\mathfrak{R}^l_\gamma V(\Omega\,;\partial\Omega)\| \tag{2.22}$$

Proof. By Lemma 2.2 the problem (2.17) is solvable in $H^{l+1}(\Gamma)^3\times H^l(\Gamma)$ for every right-hand side $(\mathbf{F},G)\in H^{l-1}(\Gamma)^3\times H^l(\Gamma)$ satisfying the compatibility condition (2.18). We denote the operator of this problem by $\mathbf{A}$. Let us fix anyhow a linear inverse operator $\mathbf{B}$ to the epimorphism

$$\mathbf{A}: H^{l+1}(\Gamma)^3\times H^l(\Gamma)\longmapsto\mathcal{R}(\mathbf{A}),$$

where $\mathcal{R}(\mathbf{A})$ is an image of $\mathbf{A}$. We put $(\mathbf{U}^0,Q^0)=\mathbf{B}(F,G)$. Then

$$\|(\mathbf{U}^0,Q^0)\,;\,H^{l+1}(\Gamma)^3\times H^l(\Gamma)\|\le c\|(\mathbf{F},G)\,;\,H^{l-1}(\Gamma)^3\times H^l(\Gamma)\|\le$$

$$\le c\|(\mathbf{f},g,\mathbf{h})\,;\,\mathfrak{R}^l_\gamma V(\Omega\,;\,\partial\Omega)\| \tag{2.23}$$

(see the definition (2.16) of the norm in $\mathfrak{R}^l_\gamma V(\Omega\,;\,\partial\Omega)$). Substituting the sums

$$\mathbf{V}(x)=\frac{1}{r}\mathbf{U}^0(\omega)+\mathbf{u}(x),\quad p(x)=\frac{1}{r^2}Q^0(\omega)+q(x).$$

into the Stokes problem (1.1), (1.3), (1.4), we derive for $(\mathbf{u},q)$ the same problem with the new right-hand side

$$\mathbf{f}^0(x)=\tilde{\mathbf{f}}(x),\quad g^0(x)=\tilde{g}(x),\quad \mathbf{h}^0(x)=\tilde{\mathbf{h}}(x)-\Big(\frac{1}{r}\mathbf{U}^0(\omega)\Big)\Big|_{\partial\Omega}. \tag{2.24}$$

Thus,

$$(\mathbf{f}^0,g^0,\mathbf{h}^0)\in\mathcal{R}^l_\gamma V(\Omega\,;\,\partial\Omega),\quad \gamma\in(l+1/2,l+3/2).$$

If $\beta\in(l-1/2,l+3/2)$, then

$$\mathcal{R}^l_\gamma V(\Omega\,;\,\partial\Omega)\subset\mathcal{R}^l_\beta V(\Omega\,;\,\partial\Omega)$$

and according to Theorem 2.1 (iii) and Theorem 2.2 there exists a unique solution $(\mathbf{u},q)\in\mathcal{D}^l_\beta V(\Omega)$ of this problem admiting the asymptotic representation (2.10) with $(\tilde{\mathbf{u}},\tilde{q})\in\mathcal{D}^l_\gamma V(\Omega)$ and the estimate (2.11) holds true. The sum $\sum_{k=1}^{3}c_k\mathbf{E}^{(k)}$ can be represented in the form $(r^{-1}\mathbf{V}^0,r^{-2}P^0)$ and because of (2.11)

$$\|(\mathbf{V}^0,P^0)\,;\,H^{l+1}(\Gamma)^3\times H^l(\Gamma)\|\le c\|(\mathbf{f}^0,g^0,\mathbf{h}^0)\,;\,\mathcal{R}^l_\gamma V(\Omega\,;\,\partial\Omega)\|. \tag{2.25}$$

Thus,

$$(\mathbf{v},p)=\Big(\frac{1}{r}(\mathbf{V}^0+\mathbf{U}^0),\frac{1}{r^2}(P^0+Q^0)\Big)+(\tilde{\mathbf{u}},\tilde{q}),\quad (\tilde{\mathbf{u}},\tilde{q})\in\mathfrak{D}^l_\gamma V(\Omega).$$

In virtue of (2.23), (2.11), (2.25) we have

$$\|(\mathbf{v},p)\,;\,\mathfrak{D}^l_\gamma V(\Omega)\|\le c(\|(\mathbf{f},g,\mathbf{h})\,;\,\mathfrak{R}^l_\gamma V(\Omega\,;\,\partial\Omega)\|+\|(\mathbf{f}^0,g^0,\mathbf{h}^0)\,;\,\mathcal{R}^l_\gamma V(\Omega\,;\,\partial\Omega)\|)$$

$$\le c\|(\mathbf{f},g,\mathbf{h})\,;\,\mathfrak{R}^l_\gamma V(\Omega\,;\,\partial\Omega)\|.$$

The uniqueness of the solution $(\mathbf{v},p)\in\mathfrak{D}^l_\gamma V(\Omega)$, $\gamma\in(l+1/2,l+3/2)$, follows from the inclusion $\mathfrak{D}^l_\gamma V(\Omega)\subset\mathcal{D}^l_\beta V(\Omega)$ with $\beta\in(l-1/2,l+1/2)$ and from the uniqueness of the solution to (1.1), (1.3), (1.4) in the space $\mathcal{D}^l_\beta V(\Omega)$ (see Theorem 2.1 (iii)).

Remark 2.3. Since the solution is unique, $(\mathbf{v},p)$ does not depend on the choise of the operator $\mathbf{B}$ i.e. it is not important how we fix a nonunique solution of the problem (2.17) on the sphere Γ.

The analogous results are also valid in Hölder spaces with detached asymptotics.

Let us fix the numbers $l\geq 1$, $\delta\in(0,1)$ and $\gamma\in(l+\delta+2,l+\delta+3)$ and introduce the space $\mathfrak{D}_\gamma^{l,\delta}\Lambda(\Omega)$ of functions $(\mathbf{v},p)$ which are represented in the form (2.13) with $(\mathbf{V},P)\in C^{l+1,\delta}(\Gamma)^3\times C^{l,\delta}(\Gamma)$, $(\tilde{\mathbf{v}},\tilde{p})\in\mathcal{D}_\gamma^{l,\delta}\Lambda(\Omega)$. The norm in $\mathfrak{D}_\gamma^{l,\delta}\Lambda(\Omega)$ is defined by the formula

$$\|(\mathbf{v},p)\,;\mathfrak{D}_\gamma^{l,\delta}\Lambda(\Omega)\|=\|(\mathbf{V},P)\,;\,C^{l+1,\delta}(\Gamma)^3\times C^{l,\delta}(\Gamma)\|+\|(\tilde{\mathbf{v}},\tilde{p})\,;\,\mathcal{D}_\gamma^{l,\delta}\Lambda(\Omega)\|.$$

Let, further, $\mathfrak{R}_\gamma^{l,\delta}\Lambda(\Omega\,;\,\partial\Omega)$ be the space of triples $(\mathbf{f},g,\mathbf{h})$ admiting the representation (2.15) with $(\mathbf{F},G)\in C^{l-1,\delta}(\Gamma)^3\times C^{l,\delta}(\Gamma)$, $(\tilde{f},\tilde{g},\tilde{h})\in\mathcal{R}_\gamma^{l,\delta}\Lambda(\Omega\,;\,\partial\Omega)$ and

$$\|(\mathbf{f},g,\mathbf{h})\,;\mathfrak{R}_\gamma^{l,\delta}\Lambda(\Omega\,;\,\partial\Omega)\|=\|(\mathbf{F},G)\,;\;C^{l-1,\delta}(\Gamma)^3\times C^{l,\delta}(\Gamma)\|+$$

$$+\|(\tilde{\mathbf{f}},\tilde{g},\tilde{\mathbf{h}})\,;\,\mathcal{R}_\gamma^{l,\delta}\Lambda(\Omega\,;\,\partial\Omega)\|.$$

Theorem 2.5. *Let $(\mathbf{f},g,\mathbf{h})\in\mathfrak{R}_\gamma^{l,\delta}\Lambda(\Omega\,;\,\partial\Omega)$ and let there holds the orthogonality condition* (2.18). *Then the Stokes problem* (1.1), (1.3), (1.4) *has a unique solution $(\mathbf{v},p)\in\mathfrak{D}_\gamma^{l,\delta}\Lambda(\Omega)$ satisfying the estimate*

$$\|(\mathbf{v},p)\,;\mathfrak{D}_\gamma^{l,\delta}\Lambda(\Omega)\|\leq c\|(\mathbf{f},g,\mathbf{h})\,;\mathfrak{R}_\gamma^{l,\delta}\Lambda(\Omega\,;\,\partial\Omega)\|. \tag{2.26}$$

3. Navier-Stokes problem

In this section we prove in weighted spaces with detached asymptotics the solvability (for sufficiently small data) of the nonlinear stationary Navier-Stokes system (1.2), (1.3), (1.4). We start with some auxiliary results.

Lemma 3.1. ([2]). (i) *Let $u\in V_\beta^l(\Omega)$, $l\leq 3/2$, $2\leq q\leq 6/(3-2l)$. Then $u\in L^q_{\beta-l-3/q+3/2}(\Omega)$ and*

$$\|u\,;\,L^q_{\beta-l-3/q+3/2}(\Omega)\|\leq c\|u\,;\,V_\beta^l(\Omega)\|. \tag{3.1}$$

(ii) *If $l>3/2$, $m+\delta\leq l-3/2$, $\delta\in(0,1)$, then $u\in\Lambda^{m,\delta}_{m+\delta+\beta-(l-3/2)}(\Omega)$ and*

$$\|u\,;\,\Lambda^{m,\delta}_{m+\delta+\beta-(l-3/2)}(\Omega)\leq c\|u\,;\,V_\beta^l(\Omega)\|. \tag{3.2}$$

Lemma 3.2. *The mapping $\mathfrak{M}$*

$$\mathfrak{D}_\gamma^l V(\Omega)\ni(\mathbf{v},0)\longmapsto\mathfrak{M}(\mathbf{v},0)=(-(\mathbf{v}\cdot\nabla)\mathbf{v},0,0)\in$$

$$\in\mathfrak{R}_\gamma^l V(\Omega\,;\,\partial\Omega),\quad \gamma\in(l+1/2,l+3/2), \tag{3.3}$$

is continuous and there holds the estimate

$$\|\mathfrak{M}(\mathbf{v},0)\,;\mathfrak{R}_\gamma^l V(\Omega\,;\,\partial\Omega)\|\leq c_*\|(\mathbf{v},0)\,;\mathfrak{D}_\gamma^l V(\Omega)\|^2. \tag{3.4}$$

Proof. From the representation (2.13) for $\mathbf{v}$ we have

$$\hat{\mathbf{f}}\equiv-(\mathbf{v}\cdot\nabla)\mathbf{v}=-\Big(\frac{1}{r}\mathbf{V}(\omega)\cdot\nabla\Big)\Big(\frac{1}{r}\mathbf{V}(\omega)\Big)-$$

$$-\Big(\frac{1}{r}\mathbf{V}(\omega)\cdot\nabla\Big)\tilde{\mathbf{v}}-(\tilde{\mathbf{v}}\cdot\nabla)\Big(\frac{1}{r}\mathbf{V}(\omega)\Big)-(\tilde{\mathbf{v}}\cdot\nabla)\tilde{\mathbf{v}}=\mathbf{f}_0+\mathbf{f}_1+\mathbf{f}_2+\mathbf{f}_3. \tag{3.5}$$

Let us write $(\mathbf{v}\cdot\nabla)\mathbf{v}$ in the form (2.15). One can easily verify that

$$\mathbf{f}_0(x)=\frac{1}{r}\mathbf{F}(\omega) \tag{3.6}$$

and due to embedding theorems $\mathbf{F}(\omega)\in H^{l-1}(\Gamma)$ and

$$\|\mathbf{F}\,;\,H^{l-1}(\Omega)\|\leq c\|\mathbf{V}\,;\,H^{l+1}(\Gamma)\|^2 \tag{3.7}$$

From the definition of the norm in $V^l_\gamma(\Omega)$ it follows that

$$\mathrm{f}_1=\Big(\frac{1}{r}\mathrm{V}\cdot\nabla\Big)\tilde{\mathrm{v}}\in V^{l-1}_\gamma(\Omega),\quad \mathrm{f}_2=(\tilde{\mathrm{v}}\cdot\nabla)\Big(\frac{1}{r}\mathbf{V}\Big)\in V^{l-1}_\gamma(\Omega),$$

$$\|\mathbf{f}_1\,;\,V^{l-1}_\gamma(\Omega)\|+\|\mathbf{f}_2\,;\,V^{l-1}_\gamma(\Omega)\|\leq c\|\mathbf{V}\,;\,H^{l+1}(\Gamma)\|\|\tilde{\mathbf{v}}\,;\,V^{l+1}_\gamma(\Omega)\|\leq$$

$$\leq c\|\mathbf{v}\,;\,\mathfrak{D}^l_\gamma V(\Omega)\|^2. \tag{3.8}$$

The first inclusion in (3.8) is evident, since $\mathrm{V}\in H^{l+1}(\Gamma)\subset C^{l-1}(\Gamma)$ and $\nabla\tilde{\mathrm{v}}\in V^l_\gamma(\Omega)\subset V^{l-1}_{\gamma-1}(\Omega)$. In order to prove the second inclusion one can employ the approach due to V.A. Kondratijev [6]: to divide-the cone $\mathbf{R}^3\backslash\{0\}$ into the annula $\omega_k=\{x:2^k<|x|<2^{k+1}\}$, $k=1,2,\cdots$, and, after the change of variables $\omega_k\ni x\longmapsto 2^{-k}x\in\omega_1$, to apply the Sobolev embedding theorems in ω_1. Making the inverse change of variables and summing the obtained inequalities we derive (3.8).
In order to estimate the last term $\mathbf{f}_3=(\tilde{\mathbf{v}}\cdot\nabla)\tilde{\mathbf{v}}$ in (3.5), we consider the sums

$$\mathrm{D}^\alpha(\tilde{v}_k\partial\tilde{v}_i/\partial x_k)=\sum_{\substack{|\nu|+|\mu|=|\alpha|\\ 0\leq|\nu|\leq|\alpha|}}\mathrm{D}^\nu\tilde{v}_k\mathrm{D}^\mu(\partial\tilde{v}_i/\partial x_k),$$

$$i,k=1,2,3,\quad |\alpha|=0,1,\cdots,l-1. \tag{3.9}$$

Lemma 3.1 shows that

$$\mathrm{D}^\nu\tilde{v}_k\in V^{l+1-|\nu|}_\gamma(\Omega)\subset\Lambda^{l-|\nu|-1,1/2}_\gamma(\Omega),\quad |\nu|\leq|\alpha|\leq l-1.$$

Since $\gamma\in(l+1/2,l+3/2)$, we have

$$\|\mathrm{D}^\alpha(\tilde{v}_k\partial\tilde{v}_i/\partial x_k)\,;\,L_{\gamma-l+1+|\alpha|}(\Omega)\|^2\leq$$

$$\leq c\sum_{\substack{|\nu|+|\mu|=|\alpha|\\ 0\leq|\nu|\leq|\alpha|}}\int_\Omega|\mathrm{D}^\nu\tilde{\mathbf{v}}|^2|\mathrm{D}^\mu\nabla\tilde{\mathbf{v}}|^2(1+r^2)^{\gamma-l+1+|\alpha|}\,dx$$

$$=c\sum_{\substack{|\nu|+|\mu|=|\alpha| \\ 0\leq|\nu|\leq|\alpha|}}\int_\Omega |D^\nu\tilde{\mathbf{v}}|^2(1+r^2)^{\gamma-l+|\nu|+1/2}|D^\mu\nabla\tilde{\mathbf{v}}|^2(1+r^2)^{\gamma-l+|\mu|}(1+r^2)^{-\gamma+l+1/2}dx$$

$$\leq c\sum_{\substack{|\nu|+|\mu|=|\alpha| \\ 0\leq|\nu|\leq|\alpha|}}\sup_{x\in\Omega}(|D^\nu\tilde{\mathbf{v}}|^2(1+r^2)^{\gamma-l+|\nu|+1/2})\int_\Omega |D^\mu\nabla\tilde{\mathbf{v}}|^2(1+r^2)^{\gamma-l+|\mu|}dx$$

$$\leq c\sum_{\substack{|\nu|+|\mu|=|\alpha| \\ 0\leq|\nu|\leq|\alpha|}}\|D^\nu\tilde{\mathbf{v}}\,;\,\Lambda_\gamma^{l-\nu-1,1/2}(\Omega)\|^2\|D^\mu\nabla\tilde{\mathbf{v}}\,;\,L^2_{\gamma-l+|\mu|}(\Omega)\|^2$$

$$\leq c\|\tilde{\mathbf{v}}\,;\,V_\gamma^{l+1}(\Omega)\|^4. \tag{3.10}$$

Summing the inequalities (3.10) over $|\alpha|=0,\cdots,l-1$, we deduce $(\tilde{\mathbf{v}}\cdot\nabla)\tilde{\mathbf{v}}\in V_\gamma^{l-1}(\Omega)$ and

$$\|(\tilde{\mathbf{v}}\cdot\nabla)\tilde{\mathbf{v}}\,;\,V_\gamma^{l-1}(\Omega)\|\leq c\|(\tilde{\mathbf{v}}\,;\,V_\gamma^{l-1}(\Omega)\|^2. \tag{3.11}$$

From (3.5)-(3.11) we conclude that $\hat{\mathbf{f}}$ is represented in the form

$$\hat{\mathbf{f}}(x)=\frac{1}{r^3}\mathbf{F}(\omega)+\tilde{\mathbf{f}}(x) \tag{3.12}$$

with $-\tilde{\mathbf{f}}=\mathbf{f}_1+\mathbf{f}_2+\mathbf{f}_3\in V_\gamma^{l-1}(\Omega)$ and

$$\|(\tilde{\mathbf{f}},0,0)\,;\,\mathfrak{R}_\gamma^l V(\Omega\,;\,\partial\Omega)\|\leq c\|(\mathbf{v},0)\,;\,\mathfrak{D}_\gamma^l V(\Omega)\|^2.$$

Lemma 3.3. *Assume that the function* $\mathbf{H}(x)=r^{-3}\mathbf{F}(\omega)$ *can be represented in the form*

$$\mathbf{H}(x)=\Big(\frac{1}{r}\mathbf{V}(\omega)\cdot\nabla\Big)\Big(\frac{1}{r}\mathbf{V}(\omega)\Big), \tag{3.13}$$

where

$$\operatorname{div}\Big(\frac{1}{r}\mathbf{V}(\omega)\Big)=0. \tag{3.14}$$

Then there holds the relation

$$\int_\Gamma \mathbf{F}(\omega)\cdot\mathbf{c}\,d\Gamma_\omega=0,\quad \forall\mathbf{c}\in\mathbf{R}^3. \tag{3.15}$$

Proof. Let us consider the integral

$$I=\int_R^{2R}\int_\Gamma r^{-3}\mathbf{F}(\omega)\cdot\mathbf{c}r^2\ dr\ d\Gamma_\omega=\ln 2\int_\Gamma \mathbf{F}(\omega)\cdot\mathbf{c}\,d\Gamma_\omega, \tag{3.16}$$

where the number R is arbitrary. From (3.13), (3.14) it follows that

$$I=\int_{\{x\in\mathbf{R}^3:\,R<r<2R\}}\Big(\frac{1}{r}\mathbf{V}(\omega)\cdot\nabla\Big)\Big(\frac{1}{r}\mathbf{V}(\omega)\Big)\cdot\mathbf{c}\,dx=$$

$$=\int_{\partial\{x\in\mathbf{R}^3:\,R<r<2R\}}\frac{1}{r^2}(\mathbf{V}(\omega)\cdot\mathbf{n})(\mathbf{V}(\omega)\cdot\mathbf{c})dS=$$

$$=\int_{\{x\in\mathbf{R}^3:\,r=2R\}}\frac{1}{r^2}(\mathbf{V}(\omega)\cdot\mathbf{n})(\mathbf{V}(\omega)\cdot\mathbf{c})dS-$$

$$-\int_{\{x\in\mathbf{R}^3:\,r=R\}}\frac{1}{r^2}(\mathbf{V}(\omega)\cdot\mathbf{n})(\mathbf{V}(\omega)\cdot\mathbf{c})dS, \qquad (3.17)$$

where $\mathbf{n}$ is a unit outward normal to $\partial\{x\in\mathbf{R}^3:\,R<r<2R\}$. Since the integral

$$\int_{\{x\in\mathbf{R}^3:\,r=t\}}r^{-2}(\mathbf{V}(\omega)\cdot\mathbf{n})(\mathbf{V}(\omega)\cdot\mathbf{c})dS=\int_\Gamma(\mathbf{V}(\omega)\cdot\mathbf{n})(\mathbf{V}\cdot\mathbf{c})d\Gamma_\omega$$

does not depend on t, from (3.17), (3.16) we conclude

$$0=I=\ln 2\int_\Gamma\mathbf{F}(\omega)\cdot\mathbf{c}\,d\Gamma_\omega$$

and, hence,

$$\int_\Gamma\mathbf{F}(\omega)\cdot\mathbf{c}\;d\Gamma_\omega=0,\quad\forall\mathbf{c}\in\mathbf{R}^3.$$

Now we are able to prove the main result of the paper.

Theorem 3.1. *Suppose that* $(\mathbf{f},0,\mathbf{h})\in\mathfrak{R}^l_\gamma V(\Omega\,;\,\partial\Omega)$ *with* $\gamma\in(l+1/2,l+3/2)$. *Assume, in addition, that f satisfies the orthogonality condition* (2.18). *There exists a positive constant* æ_0 *such that if*

$$\|(\mathbf{f},0,\mathbf{h})\,;\,\mathfrak{R}^l_\gamma V(\Omega\,;\,\partial\Omega)\|<\text{æ}_0, \qquad (3.18)$$

then the problem (1.2), (1.3), (1.4) *has a unique solution* $(\mathbf{v},p)\in\mathfrak{D}^l_\gamma V(\Omega)$ *and there holds the estimate*

$$\|(\mathbf{v},p)\,;\,\mathfrak{D}^l_\gamma V(\Omega)\|\le c\|(\mathbf{f},0,\mathbf{h})\,;\,\mathfrak{R}^l_\gamma V(\Omega\,;\,\partial\Omega)\|. \qquad (3.19)$$

Proof. Let $\mathcal{S}$ be the operator of the linear Stokes problem (1.1), (1.3), (1.4) acting from $\mathfrak{D}^l_\gamma V(\Omega)$ into $\mathfrak{R}_\perp=\{(\mathbf{f},g,\mathbf{h})\in\mathfrak{R}^l_\gamma V(\Omega\,;\,\partial\Omega):\int_\Gamma\mathbf{F}(\omega)\cdot\mathbf{c}\;d\Gamma_\omega=0,\ \forall\mathbf{c}\in\mathbf{R}^3\}$. By Theorem 2.4 there exists the bounded inverse operator $\mathcal{S}^{-1}$. If $(\mathbf{v},p)\in\mathfrak{D}^l_\gamma V(\Omega)$, from Lemmas 3.2 and 3.3 we know that $\mathfrak{M}(\mathbf{v},0)\in\mathfrak{R}_\perp$. Hence, the problem (1.2), (1.3), (1.4) is equivalent to the operator equation in the space $\mathfrak{D}^l_\gamma V(\Omega)$:

$$(\mathbf{v},p)=\mathcal{S}^{-1}(\mathbf{f},0,\mathbf{h})+\mathcal{S}^{-1}\mathfrak{M}(\mathbf{v},0)\equiv\mathfrak{A}(\mathbf{v},0).$$

The estimate (3.4) shows that the operator $\mathfrak{A}$ maps the ball

$$\mathcal{B}_{æ_1}=\{(\mathbf{v},0):\|(\mathbf{v},0)\,;\mathfrak{D}_\gamma^l V(\Omega)\|\leq æ_1\}$$

into itself, provided that

$$\|\mathcal{S}^{-1}\|æ_0<æ_1/2,\quad \|\mathcal{S}^{-1}\|c_*æ_1<1/2$$

(c_* is the constant from the inequality (3.4)). Moreover, using the same arguments as in the proof of Lemma 3.2, we show for $(\mathbf{v},0)$, $(\mathbf{u},0)\in\mathcal{B}_{æ_1}$ the estimate

$$\|\mathfrak{A}(\mathbf{v},0)-\mathfrak{A}(\mathbf{u},0)\,;\mathfrak{D}_\gamma^l V(\Omega)\|\leq c_{**}(\|(\mathbf{v},0)\,;\mathfrak{D}_\gamma^l V(\Omega)\|+$$

$$+\|(\mathbf{u},0)\,;\mathfrak{D}_\gamma^l V(\Omega)\|)\|(\mathbf{v}-\mathbf{u},0)\,;\mathfrak{D}_\gamma^l V(\Omega)\|\leq 2c_{**}æ_1\|(\mathbf{v}-\mathbf{u},0)\,;\mathfrak{D}_\gamma^l V(\Omega)\|.$$

Thus, if

$$æ_1<1/2c_{**},$$

the operator $\mathfrak{A}$ is a contraction and the statement of the theorem follows from the Banach contraction principle.

Remark 3.1. A decisive point in the proof of Theorem 3.1 was the fact that for each solenoidal vector function $\mathbf{v}\in\mathfrak{D}_\gamma^l V(\Omega)$ the nonlinear term $(\mathbf{v}\cdot\nabla\mathbf{v})$ satisfies the orthogonality condition (2.18). This is true (see Lemma 3.3) only for the three-dimensional exterior domain Ω and that is the main reason why the same method cannot be applied in the case of two-dimensional exterior domains.

Remark 3.2. The compatibility condition (2.18) does not contain the integral characteristic of the boundary function $\mathbf{h}$, i.e. $\int_{\partial\Omega}\mathbf{h}\cdot\mathbf{n}\,ds$, which is usually called "the flux" of $\mathbf{h}$. To explain this fact, we notice that the potential vector field

$$\mathbf{w}(x)=\nabla\psi(x),\quad \psi(x)=1/4\pi|x|$$

is not only solenoidal and harmonic, but also satisfies the system of homogeneous Navier-Stokes equations

$$-\nu\Delta\mathbf{w}+(\mathbf{w}\cdot\nabla)\mathbf{w}+\nabla q=0,$$

where $q=\frac{1}{2}|\nabla w|^2$. This can be seen from the identity $(\mathbf{v}\cdot\nabla)\mathbf{v}=-\mathbf{v}\times\mathrm{rot}\,\mathbf{v}+\frac{1}{2}\nabla|\mathbf{v}|^2$. Thus, taking

$$\mathbf{v}=a_*\nabla\psi+\mathbf{u},\quad p=a_*q+s,$$

one gets for $(\mathbf{u},s)$ the system of equations which differs from the Navier-Stokes system only by first order terms $a_*(\mathbf{w}\cdot\nabla)\mathbf{u}+a_*(\mathbf{u}\cdot\nabla)\mathbf{w}$. The new boundary function $h-a_*\nabla\psi|_{\partial\Omega}$ have zero flux, provided we have chosen the constant a_* in an appropriate way. The flux of **h** is related to the next eigenvalue $\lambda=-2$ of the pencil $\mathbf{S}(\lambda\,;\,\cdot)$.

Analogously to Theorem 3.1 can be proved

Theorem 3.2. *Let* $(\mathbf{f},0,\mathbf{h})\in\mathfrak{R}_\gamma^{l,\delta}\Lambda(\Omega\,;\,\partial\Omega)$, $l\geq 1$, $\delta\in(0,1)$, $\gamma\in(l+\delta+2,l+\delta+3)$ and let **f** *satisfies the orthogonality condition* (2.18). *There exists a positive number* $æ_0$ *such that if*

$$\|(\mathbf{f},0,\mathbf{h})\,;\,\mathfrak{R}_\gamma^{l,\delta}\Lambda(\Omega\,;\,\partial\Omega)\|<æ_0,$$

then the problem (1.2), (1.3), (1.4) *has a unique solution* $(\mathbf{v},p)\in\mathfrak{D}_\gamma^{l,\delta}\Lambda(\Omega)$ *and there holds the estimate*

$$\|(\mathbf{v},p)\,;\,\mathfrak{D}_\gamma^{l,\delta}\Lambda(\Omega)\|\leq c\|(\mathbf{f},0,\mathbf{h})\,;\,\mathfrak{R}_\gamma^{l,\delta}(\Omega\,;\,\partial\Omega)\|. \tag{3.20}$$

Remark 3.3. From Theorem 3.2 it follows, in particular (see the definitions of the norm in $\mathfrak{D}_\gamma^{l,\delta}\Lambda(\Omega)$), that in the case where **f** has a compact support the solution $(\mathbf{v},p)\in\mathfrak{D}_\gamma^{l,\delta}\Lambda(\Omega)$, $l\geq 1$, $\delta\in(0,1)$, $\gamma\in(l+\delta+2,l+\delta+3)$, admits the asymptotic representation

$$\mathbf{v}(x)=\frac{1}{r}\mathbf{V}(\omega)+\tilde{\mathbf{v}}(x),\quad p(x)=\frac{1}{r^2}P(\omega)+\tilde{p}(x),$$

with

$$|\mathrm{D}^\alpha\tilde{\mathbf{v}}(x)|=O(r^{-2-|\alpha|+\varepsilon}),\quad |\alpha|=0,1,\cdots,l+1,\quad \varepsilon>0,$$

$$|\mathrm{D}^\alpha\tilde{p}(x)|=O(r^{-3-|\alpha|+\varepsilon}),\quad |\alpha|=0,1,\cdots,l,\quad \varepsilon>0.$$

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References

[1] K.I. Babenko, On stationary solutions of the problem of flow past a body of viscous incompressible fluid, Mat. Sbornik, **91** (133), 3-27 (1973). English Transl.: Math. USSR, Sbornik, **20** (1973), 1-25.
[2] W. Borchers and K. Pileckas, Existence, uniqueness and asymptotics of steady jets, Arch. Rat. Mech. Anal., **120** (1992), 1-49.
[3] R. Farwig, The stationary Navier-Stokes equations in 3D-exterior domain, to appear.
[4] R. Finn, On the exterior stationary problem for the Navier-Stokes equations, and associated perturbation problems, Arch. Rat. Mech. Anal., **19** (1965), 363-406.

[5] G.P. Galdi, An introduction to the mathematical theory of Navier-Stokes equations. I. Linearized steady problems, II. Nonlinear steady problems. Springer Tracts in Natural Philosophy, **38, 39**, Springer-Verlag, New York-Berlin-Heidelberg,1994.

[6] V.A. Kondratjev, Boundary value problems for elliptic equations in domains with conical or corner points, Trudy Moskov. Mat. Obshch., **16** (1967), 209-292. English Transl.: Trans. Moscow Math. Soc., **16** (1967).

[7] O.A. Ladyzhenskaya, The mathematical theory of viscous incompressible flow. Gordon and Breach, New York, London, Paris, 1969.

[8] V.G. Maz'ya and B.A. Plamenevskii, On the coefficients in the asymptotics of solutions of elliptic boundary value problems in domains with conical points, Math. Nachr., **76** (1977), 29-60. English Transl.: Amer. Math. Soc. Transl., **123**-2 (1984), 57-88.

[9] V.G. Maz'ya and B.A. Plamenevskii, Estimates in L_p and Hölder classes and the Miranda-Agmon maximum principle for solutions of elliptic boundary value problems in domains with singular points on the boundary, Math. Nachr., **81** (1978), 25-82. English Transl.: Amer. Math. Soc. Transl., **123**-2 (1984), 1-56.

[10] S.A. Nazarov, On the two-dimensional aperture problem for Navier-Stokes equations, C.R. Acad. Sci. Paris, **323** (1996) Ser.1, 699-703.

[11] S.A. Nazarov, The Fredholm property of the operator of the boundary value problem with a condition of Chaplygin-Youkovskii-Kutta type on an edge of the boundary, Funk. analis i ego prilozch., **31**-3 (1997), 44-56.

[12] S.A. Nazarov and K. Pileckas, Asymptotic conditions at infinity for the Stokes and Navier-Stokes problems in domains with cylindrical outlets to infinity. Quaderni di Matematica, Advances in Fluid Dynamics, **4** (1999), 141-243.

[13] S.A. Nazarov and B.A. Plamenevskii, Elliptic boundary value problems in domains with piecewise smooth boundaries. Walter de Gruyter and Co, Berlin, 1994.

[14] S.A. Nazarov and B.A. Plamenevskii, Elliptic problems with the radiation condition on edges of the boundary, Mat. Sbornik, **183**-10 (1992), 13-44. English Transl.: Math. Sbornik, **77**-1 (1994), 149-176.

[15] S.A. Nazarov and B.A. Plamenevskii, Self-adjoint elliptic problems with the radiation conditions on edges of the boundary, Algebra i Analiz, **4**-3 (1992), 196-225. English Transl.: St. Petersburg Math. J., **4**-3 (1992), 569-594.

[16] S.A. Nazarov, M. Specovius-Neugebauer and G. Thäter, Quiet flows for Stokes and Navier-Stokes problems in domains with cylindrical outlets to infinity (to appear).

[17] A. Novotny and M. Padula, Note on decay of solutions of steady Navier-Stokes equations in 3-D exterior domains, Differential Integral Equations, **8** (1995), 1833-1844.