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Uniqueness in inverse hyperbolic problems
—Carleman estimate for boundary value problems—

By

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1. Introduction

We give sharp Carleman estimates including boundary conditions to show the unique continuation across a lateral boundary for hyperbolic equations, and we show the uniqueness in hyperbolic inverse problems by the use of the above unique continuation.

T. Carleman [4] proposed a weighted inequality to show the uniqueness in Cauchy problems to which Holmgren's theorem are not applicable, and we call this type of weighted estimates the Carleman estimates. The Carleman estimate has been playing an important role to show the uniqueness not only in Cauchy problems but in inverse problems. Especially for inverse hyperbolic problems, the uniqueness is one of the most interesting problems in this field, and many researchers study applications of the Carleman estimate; e.g. Bukhgeĭm [2], Bukhgeĭm and Klibanov [3], Isakov [6], Lavrent'ev, Romanov and Shishat-skiĭ [9], Yamamoto [19] etc.

The author gives a new type of the Carleman estimates in the present paper, and he shows sharp results in the uniqueness. The main interest of this research lies in an inverse problem to identify unknown coefficients of the wave equation from measurement on a lateral boundary. The problem is attractive for many researchers, since it is a mathematical model in geophysics to find properties of geophysical media by observation of wave fields on a part of the surface of the Earth. We wish to know conditions for the uniqueness of solutions, but the uniqueness has not been shown for the case observation is done on a part of a boundary. We show sharp estimates to give conditions for the uniqueness to this case. Proofs of uniqueness theorems of inverse problems are based on the following two points;
(1) the Bukhgeĭm-Klibanov method presented in [3],
(2) Carleman estimates near the boundary for boundary value problems.
We remark the method (1) is an application of the Carleman estimate to inverse problems and effective for various inverse problems to determine coefficients in the equations for which the Carleman estimate holds. Since the Carleman estimate depends essentially on a relation between the type of differential equations and the shape of a domain, and many serious difficulties arise in particular for hyperbolic

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equations; one can see examples for non-uniqueness and counterexamples for the Carleman estimate [1], [8] etc. Our aimed equations are hyperbolic equations with boundary conditions on a part of the lateral surface: Observation data for our inverse problems are given as boundary conditions on a part of the lateral surface which may not be strongly pseudoconvex. We should pay very much attention to show the Carleman estimate. Recently Tataru [16] proposed the Carleman estimate including boundary data in order to show the unique continuation near the boundary, and we develop his idea to introduce the Carleman-Lopatinski condition and obtain delicate uniqueness theorems. We should remark that we use not only the initial conditions but also boundary conditions to give our Carleman estimates, and this idea is essential in our argument.

We shall address our inverse problem precisely. Let u be a solution to the following initial-boundary value problem for a hyperbolic equation:

$$\{\partial_t^2-\triangle_x-a(x)\}u(t,x)=f(x)R(t,x) \quad 0<t<T, \quad x\in\Omega, \tag{1.1}$$

$$u(0,x)=\partial_t u(0,x)=0 \qquad x\in\Omega, \tag{1.2}$$

$$Bu|_{(0,T)\times\partial\Omega}=0, \tag{1.3}$$

where $\Omega\subset\mathbf{R}^{n-1}$ is a bounded domain with a smooth boundary $\partial\Omega$ and B denotes a boundary operator. We assume that $a(x)$ and $R(t,x)$ are given functions, and our inverse problem is identification of $f(x)$. We denote the solution to (1.1)-(1.3) by $u[f](t,x)$ for $f(x)$, and $\Gamma\subset\partial\Omega$ is a part of the boundary which is given a priori. A question of our inverse problem is how to conclude $f_1(x)=f_2(x)$ $x\in\Omega$ under the observation

$$\tilde{B}u[f_1](t,x)=\tilde{B}u[f_2](t,x) \quad 0<t<T, \quad x\in\Gamma, \tag{1.4}$$

where $\tilde{B}$ is a boundary operator associated with the operator B; what conditions should be posed for B, $\tilde{B}$ and Γ in order to identify $f(x)$? When Γ is the hole boundary $\Gamma=\partial\Omega$, and when B and $\tilde{B}$ is the Dirichlet and the Neumann boundary operator respectively, a strong affirmative result is known (See e.g. Bukhgeĭm and Klibanov [3], and Yamamoto [19] for a stability estimate). In the case $\Gamma\neq\partial\Omega$ or the boundary operators B and $\tilde{B}$ are different types from those mentioned above, we have counterexamples for usual Carleman estimates (See [1], [8]) and the condition for unique identification has been an open problem.

We introduce the Carleman-Lopatinski condition which is a new type of the Carleman estimates including boundary conditions. We show uniqueness theorems for identification of the force term f by using our Carleman estimate. The Carleman-Lopatinski condition implies a suitable choice of the boundary operators B and $\tilde{B}$.

This paper consists of four sections. In §2 we state our results, and we give our Carleman estimates near the boundary in §3. We give proofs of the main results in the final section. We remark that the Carleman estimates given in §2 are meaningful for not only the research of inverse problems but the unique continuation of solution to linear differential equations.

2. Notation and Results

We state the results for our inverse problems in this section, and we have two types of uniqueness ; the one is the local uniqueness (Theorem 2.1 and Theorem 2.2) and the other is the global uniqueness (Theorem 2.3 and Theorem 2.4). We remark that the latter results are derived from the former ones for a special domain.

Before stating the results, we should recall the problem (1.1)-(1.3) and the question stated in the previous section ; we must clarify conditions to conclude $f(x)\equiv 0$ when $\tilde{B}u[f](t,x)=0$. The following theorems are answers to the questions. We denote the Dirichlet and Neumann boundary operators by B_D and B_N respectively, and we denote the normal derivative of v on a surface S by $\frac{\partial}{\partial\nu}u|_S$. The Carleman-Lopatinski conditions are key ideas in the present research, and we give the definitions in §3. The strong pseudoconvexity is used in the usual sense and is stated precisely in §3. Our results are as follows.

Theorem 2.1. *Let $S\subset\mathbf{R}^n$ be an oriented C^2 hypersurface transversely intersecting with both a cylinder $(0,T)\times\partial\Omega$ and a plane $\{t=0\}$, and let $(0, x_0)\in\partial\Omega\cap S|_{t=0}$. We assume that S is strongly pseudoconvex w.r.t. d'Alembertian $\square$ at $(0, x_0)$ and that $\{\square, B_D\}$ satisfies the strong Carleman-Lopatinski condition w.r.t. S at $(0, x_0)$. We assume that $R(t,x)\in W^{3,\infty}((0,T)\times\Omega)$ and $R(0,x)\neq 0$ $(x\in\Omega)$. We assume that $(u,f)\pm H^2((0,T)\times\Omega)\times L^2(\Omega)$ satisfies*

$$\{\partial_t^2-\triangle_x-a(x)\}u(t,x)=f(x)R(t,x)\quad 0<t<T,\quad x\in\Omega,$$

$$u(0,x)=\partial_t u(0,x)=0\qquad x\in\Omega,$$

$$u|_{(0,T)\times\partial\Omega}=0,$$

and that $\frac{\partial}{\partial t}u\in H^2((0,T)\times\Omega)$. If $u|_S=\frac{\partial}{\partial\nu}u|_S=0$ in a neighborhood of $(0,x_0)$, then there is a neighborhood $V\subset\mathbf{R}^n$ of $(0,x_0)$ such that $f(x)=0$ in $V\cap(0,T)\times\Omega$.

Theorem 2.2. *Let $S\subset\mathbf{R}^n$ be an oriented C^2 hypersurface satisfying the same hypothesis as in Theorem 2.1. We assume that $\{\square, B_N\}$ satisfies the <u>weak</u> Carleman-Lopatinski condition w.r.t. S at $(0,x_0)\in\partial\Omega\cap S|_{t=0}$ and that $R(t,x)\in W^{3,\infty}((0,T)\times\Omega)$ and $R(0,x)\neq 0$ $(x\in\Omega)$. We assume that $(u,f)\in H^2((0,T)\times\Omega)\times L^2(\Omega)$ satisfies*

$$\{\partial_t^2-\triangle_x-a(x)\}u(t,x)=f(x)R(t,x)\quad 0<t<T,\quad x\in\Omega,$$

$$u(0,x)=\partial_t u(0,x)=0\qquad x\in\Omega,$$

$$\frac{\partial}{\partial \nu} u|_{(0,T)\times\partial\Omega}=0,$$

and that $\frac{\partial}{\partial t}u\in H^2((0,T)\times\Omega)$. *If* $u|_S=\frac{\partial}{\partial \nu}u|_S=0$ *in a neighborhood of* $(0,x_0)$, *then there is a neighborhood* $V\subset\mathbf{R}^n$ *of* $(0,x_0)$ *such that* $f(x)=0$ *in* $V\cap(0,T)\times\Omega$.

Remark 1. *In the case where the boundary operator* B *in* (1.3) *is of the Neumann type, we remark the strong Carleman-Lopatinski condition is not fulfilled, and the situation is similar to the case of the uniformly Lopatinski condition in mixed hyperbolic problems* (*see* [11]). *The boundary operators* B *in* (1.3) *and* $\tilde{B}$ *in* (1.4) *should be chosen so that the unique continuation across a lateral boundary holds for solutions to*

$$\{\partial_t^2-\triangle_x-a(x)\}u(t,x)=0.\quad 0<t<T,\quad x\in\Omega.$$

We should remark the cases $(B=B_D,\tilde{B}=B_N)$ *and* $(B=B_N,\tilde{B}=B_D)$ *are quitely different.*

We show global uniqueness theorems as a simple consequence of above results. Let Ω be a disk in $\mathbf{R}^{n-1}$ and let a subboundary $\Gamma\subset\partial\Omega$ be a part of $\partial\Omega$;

$$\Omega:=\{x\in\mathbf{R}^{n-1}:\ |x|<R\},\tag{2.1}$$

$$\Gamma_\delta\supset\{x\in\partial\Omega:\ x_n<\delta\},\tag{2.2}$$

where R and δ are positive numbers.

Theorem 2.3. *Suppose that* $T>R$ *and* $a_j(x)\in L^\infty(\Omega)$, $u_j(t,x)\in W^{3,\infty}((0,T)\times\Omega)$ $j=1,2$.
Suppose that each pair of the functions $\{a_j(x),u_j(t,x)\}_{(j=1,2)}$ *satisfy*

$$(\partial_t^2-\triangle_x-a_j(x))u_j(t,x)=0\quad 0<t<T,\quad x\in\Omega,\tag{2.3}$$

$$u_j(0,x)=\alpha(x),\quad \partial_t u_j(0,x)=\beta(x)\quad x\in\Omega,\tag{2.4}$$

$$u_j(t,x)=g(t,x)\quad 0<t<T,\quad x\in\partial\Omega,\tag{2.5}$$

where $\alpha\in H^1(\Omega)$, $\beta\in L^2(\Omega)$, *and* $g\in L^2((0,T)\times\partial\Omega)$, *and we assume that* $|\alpha(x)|\geq\alpha_0>0$ *almost everywhere on* Ω *with a positive number* $\alpha_0>0$. *If*

$$B_N u_1(t,x)=B_N u_2(t,x)\quad 0<t<T,\quad x\in\Gamma_\delta,\tag{2.6}$$

then

$$a_1(x)=a_2(x)\quad x\in\Omega,$$

$$u_1(t,x)=u_2(t,x)\quad 0\leq t\leq T,\quad x\in\Omega.$$

Theorem 2.4. *Suppose that $T>R$ and $a_j \in L^\infty(\Omega)$, $u_j \in W^{3,\infty}((0,T)\times\Omega)$ $j=1$, 2. Suppose that each pair of the functions $\{a_j(x), u_j(t,x)\}_{(j=1,2)}$ satisfy*

$$(\partial_t^2 - \triangle_x - a_j(x))u_j(t,x) = 0 \quad 0<t<T, \quad x\in\Omega, \tag{2.7}$$

$$u_j(0,x) = \alpha(x), \quad \partial_t u_j(0,x) = \beta(x) \quad x\in\Omega, \tag{2.8}$$

$$B_N u_j(t,x) = g(t,x) \quad 0<t<T, \quad x\in\partial\Omega, \tag{2.9}$$

where $\alpha\in H^1(\Omega)$, $\beta\in L^2(\Omega)$, and $g\in L^2((0,T)\times\partial\Omega)$, and we assume that $|\alpha(x)|\geq \alpha_0>0$ almost everywhere on Ω with a positive number $\alpha_0>0$. If

$$u_1(t,x) = u_2(t,x) \quad 0<t<T, \quad x\in\Gamma_\delta, \tag{2.10}$$

then

$$a_1(x) = a_2(x) \quad x\in\Omega,$$

$$u_1(t,x) = u_2(t,x) \quad 0\leq t\leq T, \quad x\in\Omega.$$

Remark 2. *We remark that we have the same results on the global uniqueness if we replace the principal part of the equation* (2.3) *with a strictly hyperbolic operator for which the Carleman estimates in* §3 *holds. Furthermore the results holds for an arbitrary domain Ω as far as boundary operators satisfy the Carleman-Lopatinski conditions.*

3. Carleman estimates near the boundary

We shall give the Carleman estimates near a boundary for solutions to boundary value problems, and the estimates are extensions of those by Tataru[16]. We prove the Carleman estimates with boundary data as well as the energy inequalities for hyperbolic mixed problems (See e.g. [14]).

We use the following notation. Let $P(x,D)$ be a m-th order partial differential operator in a domain $\Sigma\subset\mathbf{R}^n$ with a smooth noncharacteristic boundary $\partial\Sigma$. We assume its principal symbol $p(x,\xi)$ is real and has C^1 coefficients. We decompose $P=P^m+P^{sub}$ where P^m is the principal part of P and P^{sub} is bounded from H^{m-1} into L^2. Let S be an oriented C^2 hypersurface intersecting with $\partial\Sigma$, and let ϕ be a real valued C^2 weight function. We denote the Poisson bracket of two symbols p and q by $\{\cdot,\cdot\}$; i.e.

$$\{p,q\}(x,\xi): =\sum_{j=1}^{n}\left(\frac{\partial p}{\partial\xi_j}\frac{\partial q}{\partial x_j}-\frac{\partial p}{\partial x_j}\frac{\partial q}{\partial\xi_j}\right).$$

Let us first recall the strongly pseudoconvex condition (See e.g. [5]).

Definition 3.1. *We shall say that ϕ is strongly pseudoconvex w.r.t. P at x_0, in case*

$$\{p(x,\xi-i\tau\nabla\phi), p(x,\xi+i\tau\nabla\phi)\}(x_0,\xi)/\tau i>0$$

$$on\ \{(\xi,\tau)\in\mathbf{R}^{n+1}\backslash\{0\}:\ p(x_0,\xi+i\tau\nabla\phi(x_0))=0,\tau\geq 0\}$$

Definition 3.2. *Let S be an oriented C^2 hypersurface which is a level surface of a smooth function $\phi(x)$, and let $x_0\in S$, $\nabla\phi(x_0)\neq 0$ on S. We shall say that S is strongly pseudoconvex w.r.t P at $x_0\in S$, in case*

$$\mathrm{Re}\{p,\{p,\phi\}\}(x_0,\xi)>0$$

$$on\quad \{\xi\in\mathbf{R}^n\backslash\{0\}:\ p(x_0,\xi)=\{p,\phi\}(x_0,\xi)=0\}$$

and

$$\{p(x_0,\xi-i\tau\nabla\phi),p(x_0,\xi+i\tau\nabla\phi)\}(x_0,\xi)/\tau i>0$$

$$on\quad \{(\xi,\tau)\in\mathbf{R}^{n+1}:\ p(x_0,\xi+i\tau\nabla\phi)=\{p(x_0,\xi+i\tau\nabla\phi),\phi\}(x_0,\xi)=0,\tau>0\}.$$

As in 28.3 of [5], one can easily check that the strong pseudoconvexity dose not depend on the choice of the function ϕ.

Let $B:=\{B^k(x,D)\}_{k=1,2,\cdots,\mu}$ be a set of boundary operators on $\partial\Sigma$. We consider a boundary value problem

$$\begin{cases} P(x,D)u=F & \text{in } \Sigma, \\ B^k(x,D)u=g^k & \text{on } \partial\Sigma, \quad k=1,2,\cdots,\mu. \end{cases} \tag{3.1}$$

In order to discuss local property of solutions, we are enough to consider the case of a half space $\Sigma:=\{x=(x_1,\cdots,x_n)\in\mathbf{R}^n:\ x_n>0\}$. Since the boundary $\partial\Sigma$ is noncharacteristic, we assume the coefficients of $D_{x_n}^m$ in P^m should be 1. We denote $x'=(x_1,\cdots,x_{n-1})$ and $\xi'=(\xi_1,\cdots,\xi_{n-1})$ for the tangential space and the corresponding tangential Fourier variable respectively. Thus $x=(x',x_n)$ and $\xi=(\xi',\xi_n)$.

3.1. Weighted norms and pseudodifferential operators with a parameter. We introduce Sobolev spaces $H_\tau^m(\Sigma)$ and $H_\tau^m(\partial\Sigma)$ defined by the following norms respectively. For non-negative integer m and non-negative number $\tau\geq 0$

$$|u|_{m,\tau}^2:=\sum_{j=0}^{m}\tau^{2(m-j)}|u|_{Hj(\Sigma)}^2,$$

$$\langle u\rangle_{m,\tau}^2:=\sum_{j=0}^{m}\tau^{2(m-j)}|u|_{Hj(\partial\Sigma)}^2,$$

where we denote that the usual Sobolev norm by $|\cdot|_{H^j}$, the L^2 inner product in Σ by $(\cdot,\cdot)$ and the L^2 inner product on $\partial\Sigma$ by $\langle\cdot,\cdot\rangle$. Equivalent norms are given by

$$\|u\|_{m,\tau}^2:=|(|D_x|^2+\tau^2)^{m/2}u|_{L^2(\Sigma)}^2,$$

$$\langle\langle u\rangle\rangle_{m,\tau}^2:=|(|D_{x'}|^2+\tau^2)^{M/2}u|_{L^2(\partial\Sigma)}^2.$$

We denote $\lambda:=(1+|\xi'|^2+\tau^2)^{1/2}$ and $\Lambda:=\mathrm{Op}\{(1+|\xi'|^2+\tau^2)^{1/2}\}$. For a real number s, we define the following norms

$$|u|^2_{m,s,\tau}:=|\Lambda^s u|^2_{m,\tau},$$

$$\langle u\rangle^2_{m,s,\tau}:=\sum_{j=0}^{m}\langle D^j_{x_n}\Lambda^s u\rangle^2_{m-j,\tau},$$

and we define a Sobolev space $H^{m,s}_\tau(\Sigma)$ with the above norm $|\cdot|_{m,s,\tau}$.

We introduce classes of pseudodifferential operators (ϕ.d.o) with a parameter τ. The parameter τ is the weight one in the norms of the Sobolev spaces. On the other hand, the parameter τ in the symbols of ϕ.d.o is useful to the proof of energy estimates for strictly hyperbolic equations as well as in the proof of Carleman estimates.

We introduce the following classes of symbols:

$$S^m:=\{a(x,\xi,\tau):|D^\alpha_x D^\beta_\xi a|<C_{\alpha,\beta}\langle\xi,\tau\rangle^{m-|\beta|},\alpha,\beta\in\mathbf{N}^n\}$$

$$S^{m,s}:=\left\{a(x,\xi,\tau)=\sum_{j=0}^{m}\xi^j_n a_j(x,\xi',\tau):|D^\alpha_x D^{\beta'}_{\xi'}a_j|<C_{\alpha,\beta'}\langle\xi',\tau\rangle^{m+s-j-|\beta'|},\alpha\in\mathbf{N}^n,\right.$$

$$\left.\beta'\in\mathbf{N}^{n-1}\right\}$$

$$C^kS^m:=\{a(x,\xi,\tau):|D^\alpha_x D^\beta_\xi a|<C_{\alpha,\beta}\langle\xi,\tau\rangle^{m-|\beta|},|\alpha|\le k,\beta\in\mathbf{N}^n\}$$

$$C^kS^{m,s}:=\left\{a(x,\xi,\tau)=\sum_{j=0}^{m}\xi^j_n a_j(x,\xi',\tau):|D^\alpha_x D^{\beta'}_{\xi'}a_j|<C_{\alpha,\beta'}\langle\xi',\tau\rangle^{m+s-j-|\beta'|},|\alpha|\le \mathrm{k},\right.$$

$$\left.\beta'\in\mathbf{N}^{n-1}\right\}$$

where we set $\langle\xi,\tau\rangle:=(1+|\xi|^2+|\tau|^2)^{1/2}$.

We define the corresponding spaces with homogenious symbols S^m_{cl} and $S^{m,s}_{cl}$ in the following sense (See Taylor [17]). We will say $a(x,\xi,\tau)\in S^m_{cl}$, provided that $a(x,\xi,\tau)\in S^m$ has a classical expansion

$$a(x,\xi,\tau)\sim\sum_{j\ge 0}a^{(j)}(x,\xi,\tau),$$

where terms $a^{(j)}$ are homogeneous of degree $m-j$ in (ξ,τ), in the sense that the difference between $a(x,\xi,\tau)$ and the sum over $j<N$ belongs to S^{m-N}.

L^2 estimate and Gårding's inequality (the sharp Gårding inequality) hold for the above defined classes of operators with respect to the weighted Sobolev norms.

3.2. Green's formula and modified Gårding inequalities. We denote by $p(x,\xi)$ and $b^k(x,\xi)$ the principal symbols of operators $P(x,D)$ and $B^k(x,D)$ respectively.

and we set $p_\tau(x,\xi,\tau):=p(x,\xi+i\tau\nabla\phi)$ and $b^k_\tau(x,\xi,\tau):=b^k(x,\xi+i\tau\nabla\phi)$, and we define

$$\text{char } P_\tau:=\{(x,\xi,\tau)\in\Sigma\times\mathbf{R}^{n+1}\backslash\{0\}):p_\tau(x,\xi,\tau)=0\}.$$

We set $X:=(x,\xi',\tau)\in\Sigma\times\mathbf{R}^{n-1}_{\xi'}\times\{\tau\geq 0\}$, and for a fixed point $X_0:=(x'_0,0,\xi'_0,\tau_0)\in\Sigma\times\mathbf{R}^{n-1}_{\xi'}\times\{\tau\geq 0\}$ on the boundary $\partial\Sigma$, we consider the symbol $p_\tau(x,\xi,\tau)$ as follows. Let the symbol $p_\tau(x,\xi,\tau)$ be abbreviated to $p_\tau(X,\xi_n)$ to emphasize that the symbol is a polynomial of ξ_n.
Let us factorize $p_\tau(X_0,\xi_n)$ with respect to ξ_n;

$$p_\tau(X_0,\xi_n)=p^+_\tau(X_0,\xi_n)p^-_\tau(X_0,\xi_n)\prod_j(\xi_n-\xi^{(j)}_n)^{m_j}$$

where we assume the imaginary part of the roots of $p^+_\tau(X_0,\xi_n)=0$ and $p^-_\tau(X_0,\xi_n)=0$ are positive and negative respectively, and $\{\xi^{(j)}_n\}$ are all the real roots of $p_\tau(X_0,\xi_n)=0$. Then there exists a suitable small conic neighborhood $U(X_0)$ such that extend a ξ_n-polynomial factorization for $X\in U(X_0)$:

$$p_\tau(X,\xi_n)=p^+_\tau(X,\xi_n)p^-_\tau(X,\xi_n)\prod_j p^{(j)}_\tau(X,\xi_n). \tag{3.2}$$

We note that the imaginary part of all the roots of $p^+_\tau(X,\xi_n)=0$ and $p^-_\tau(X,\xi_n)=0$ positive and negative for any $X\in U(X_0)$ respectively, but we remark that all the roots of $p^{(j)}_\tau(X,\xi_n)=0$ may not be real. For the factorization (3.2) we set

$$m^{(-)}:=\{\text{the degree of } p^-_\tau(X,\xi_n) \text{ w.r.t. } \xi_n\}, \tag{3.3}$$

and we set, in case $m^{(-)}>0$,

$$e^j(X,\xi_n):=\frac{p_\tau(X,\xi_n)}{p^-_\tau(X,\xi_n)}\xi_n^{m^{(-)}-j}, (j=1,\cdots,m^{(-)}). \tag{3.4}$$

We remark that these symbols are not smooth generally.

In order to introduce Green's formula we make some preparations. Let $r(x,\xi,\tau)\in S^{m,0}_{cl}$ and $s(x,\xi,\tau)\in S^{m-1,0}_{cl}$ be

$$r(x,\xi,\tau):=r_0(x,\xi',\tau)\xi^m_n+r_1(x,\xi',\tau)\lambda\xi^{n-1}_n+\cdots+\mathrm{r}_m(x,\xi',\tau)\lambda^m,$$

$$s(x,\xi,\tau):=s_0(x,\xi',\tau)\xi^{m-1}_n+s_1(x,\xi',\tau)\lambda\xi^{m-2}_n+\cdots+s_{m-1}(x,\xi',\tau)\lambda^{m-1},$$

where $\lambda:=(1+|\xi'|^2+\tau^2)^{1/2}$. We define the Bézout form $g_{r,s}$ of two ξ_n-polynomials $\{r(x,\xi,\tau),s(x,\xi,\tau)\}$ by

$$g_{r,s}(\xi_n,\tilde{\xi}_n):=\frac{r(X,\xi_n)s(X,\tilde{\xi}_n)-r(X,\tilde{\xi}_n)s(X,\xi_n)}{\xi_n-\tilde{\xi}_n}$$

$$=[\tilde{\xi}_n^{m-1},\lambda\tilde{\xi}_n^{m-2},\cdots,\lambda^{m-1}]g_{r,s}(x,\xi,\tau)\begin{bmatrix}\xi_n^{m-1}\\ \lambda\xi_n^{m-2}\\ \vdots\\ \lambda^{m-1}\end{bmatrix}, \tag{3.5}$$

and we remark that $g_{r,s}(x,\xi,\tau)$ is a $m\times m$ matrix of which the entries are symbols of order zero w.r.t. (ξ,τ). Furthermore we set a boundary bilinear form

$$G_{r,s}\langle u,v\rangle:=\langle \mathbf{G}_{r,s}(x,D,\tau)\begin{bmatrix}D_{x_n}^{m-1}\\ \Lambda D_{x_n}^{m-2}\\ \vdots\\ \Lambda^{m-1}\end{bmatrix}\mathrm{u},\begin{bmatrix}D_{x_n}^{m-1}\\ \Lambda D_{x_n}^{m-2}\\ \vdots\\ \Lambda^{m-1}\end{bmatrix}\mathrm{v}\rangle,$$

where $\mathbf{G}_{r,s}(x,D,\tau):=\mathrm{Op}\{g_{r,s}(x,\xi,\tau)\}$. The following Green's formula is due to Sakamoto [15]. (See also Lemma 3.6 in [16].)

Lemma 3.3. *We denote pseudo-differential operators with their symbol r and s byR and S respectively. Let u, $v\in H^m(\Sigma)$, then there exists a positive number C such that*

$$|(R(x,D,\tau)u,S^{(*)}(x,D,\tau)v)-(S(x,D,\tau)u,R^{(*)}(x,D,\tau)v)-G_{r,s}\langle u,v\rangle|$$

$$\leq C\{|u|_{m-1,\tau}|v|_{m-1,\tau}+\langle u\rangle_{m-1,-1/2,\tau}\langle v\rangle_{m-1,-1/2,\tau}\},$$

where $R^{(*)}$ and $S^{(*)}$ are the formal adjoint of R and S respectively.

Set a bilinear form by $F_{r,s}(u,u):=(R(x,D,\tau)u,S(x,D,\tau)u)$, and we have the next estimate.

Lemma 3.4. *Suppose that the symbols $r\in C^1S_{cl}^{m,0}$ and $s\in C^1S_{cl}^{m-1,0}$ are real, and that $u\in H^m(\Sigma)$, then there exists a positive number C such that*

$$|2\mathrm{Re}\ F_{r,is}(u,u)-G_{r,s}\langle u,u\rangle|\leq C\{|u|^2_{m,-1,\tau}+\langle u\rangle^2_{m-1,-1/2,\tau}\}.$$

We remark $F_{r,is}(u,u):=(R(x,D,\tau)u,iS(x,D,\tau)u)$.

We futher introduce the principal and subprincipal symbols for the bilinear form $F_{r,s}(u,u)$ as follows. We define the principal symbol $f_{r,s}(x,\xi,\tau)$ by

$$f_{r,s}(x,\xi,\tau):=r(x,\xi,\tau)\overline{s(x,\xi,\tau)}$$

and its subprincipal symbol $f_{r,s}^{sub}(x,\xi,\tau)$ by the formal subprincipal symbol of the operator $S^{(*)}R$;

$$f_{r,s}^{sub}(x,\xi,\tau)=\frac{i}{2}\left(\{r,\bar{s}\}+\sum_j(r_{x_j,\xi_j}\bar{s}-r\bar{s}_{x_j,\xi_j})\right).$$

Hence we have the next estimate.

Lemma 3.5. *Suppose that the symbols* $r \in C^1 S_{cl}^{m,0}$ *and* $s \in C^1 S_{cl}^{m-1,0}$ *are real and that* $u \in H^m(\Sigma)$, *then there exists a positive number* C *such that*

$$|\mathrm{Re}(2F_{r,is}(u,u) - 2F_{r,is}^{sub}(u,u) - G_{r,s}\langle u,u\rangle)|$$

$$\leq C\{|u|_{m,-3/2,\tau}^2 + \langle u\rangle_{m-1,-1/2,\tau}^2\},$$

where $F_{r,is}^{sub}(u,u)$ *is a bilinear form and the real part of its symbol is* $\mathrm{Re}\, f_{r,s}^{sub}(x,\xi,\tau)$.

For $z=(z_0,\cdots,z_{m-1}) \in \mathbf{C}^m$, we set bilinear forms

$$G_{r,s}[z,\bar{z}] := [\bar{z}_{m-1},\bar{z}_{m-2},\cdots,\bar{z}_0]\boldsymbol{g}_{r,s}\begin{bmatrix} z_{m-1} \\ z_{m-2} \\ \vdots \\ z_0 \end{bmatrix},$$

$$(\mathrm{Im}\, G_{r,s})[z,\bar{z}] := [\bar{z}_{m-1},\bar{z}_{m-2},\cdots,\bar{z}_0]\left\{\frac{\boldsymbol{g}_{r,s}-\boldsymbol{g}_{r,s}^*}{2}\right\}\begin{bmatrix} z_{m-1} \\ z_{m-2} \\ \vdots \\ z_0 \end{bmatrix},$$

where $\boldsymbol{g}_{r,s}=\boldsymbol{g}_{r,s}(x,\xi,\tau)$ is the matrix derived from the Bézout form (3.5) and $\boldsymbol{g}_{r,s}^* := \overline{(\boldsymbol{g}_{r,s})}^T$. For the symbols $\{b_\tau^k\}$ crresponding to the boundary operators, let $\beta^k := \{$the degree of $b_\tau^k(X,\xi_n)$ w.r.t. $\xi_n\}$, and

$$b_\tau^k(X,\xi_n) = \sum_{l=0}^{\beta^k} b_{\tau,l}^k(X)\Lambda^l \xi_n^{\beta^k-l}.$$

Further for $z=(z_0,\cdots,z_{m-1}) \in \mathbf{C}^m$, we set

$$b_\tau^k[z] := \sum_{l=0}^{\beta^k} b_{\tau,l}^k(X) z_{\beta^k-l}.$$

We also define $e^j[z]$ for the symbol $e^j(X,\xi_n)$ in the same manner. From the definition (3.4) of $e^j(X,\xi_n)$, the degree of $e^j(X,\xi_n)$ with respect to ξ_n is $m-j$ and that

$$e^j(X,\xi_n) = \sum_{l=0}^{m-j} e_l^j(X)\Lambda^l \xi_n^{m-j-l},$$

and, for $z=(z_0,\cdots,z_{m-1}) \in \mathbf{C}^m$, we set

$$e^j[z] := \sum_{l=0}^{m-j} e_l^j(X) z_{m-j-l}.$$

Under the above preparations we state Gårding inequalities.

Lemma 3.6 (Gårding's inequality). *If* $r \in C^1 S_{cl}^{m,0}$ *and* $s \in C^1 S_{cl}^{m-1,0}$, *and if the bilinear form* $(\mathrm{Re}\, G_{r,s})[z,\bar{z}]$ *is positive, i.e.*

$$(\mathrm{Re}\ G_{r,s})[z,\bar{z}] \geq c|z|^2,$$

then we have

$$\mathrm{Re}(G_{r,s}\langle u,u\rangle) \geq \frac{c}{2}\langle u\rangle^2_{m-1,0,\tau}.$$

We also state the following sharp Gårding inequality for $m\times m$ systems.

Lemma 3.7 (sharp Gårding inequality). *If* $r\in C^1S^{m,0}_{cl}$ *and* $s\in C^1S^{m-1,0}_{cl}$, *and if the bilinear form* $(\mathrm{Re}\ G_{r,s})[z,\bar{z}]$ *is non-negative, i.e.*

$$(\mathrm{Re}\ G_{r,s})[z,\bar{z}] \geq 0,$$

then there exists $c>0$ *such that*

$$\mathrm{Re}(G_{r,s}\langle u,u\rangle) \geq -c\langle u\rangle^2_{m-1,-1/2,\tau}.$$

A proof of the above Gårding inequality can be seen in [17] and [18]. Since the imaginary part of the zeros of $p^-_\tau(X,\xi_n)=0$ as a polynomial of ξ_n is negative, we obtain the following proposition.

Proposition 3.8. *There exists a positive number* $c>0$ *such that*

$$|u|^2_{m^{(-)}-1,1/2,\tau}+\langle u\rangle^2_{m^{(-)}-1,0,\tau} \leq c(|P^-_\tau u|^2_{0,-1/2,\tau}+|u|^2_{m^{(-)}-1,-1/2,\tau}).$$

Proof. We decompose $p^-_\tau(X,\xi_n)$ into

$$p^-_\tau(X,\xi_n)=\xi_n^{m^{(-)}}+a^r_1(X)\lambda\xi_n^{m^{(-)}-1}+\cdots+a^r_{m^{(-)}}(X)\lambda^{m^{(-)}}$$

$$+i(a^i_1(X)\lambda\xi_n^{m^{(-)}-1}+\cdots+a^i_{m^{(-)}}(X)\lambda^{m^{(-)}})$$

$$=\mathrm{Re}\ p^-_\tau(X,\xi_n)+i\ \mathrm{Im}\ p^-_\tau(X,\xi_n),$$

where $\mathrm{Re}\ p^-_\tau(X,\xi_n)$ and $\mathrm{Im}\ p^-_\tau(X,\xi_n)$ are real symbols whose coefficients $a^r_j(X)$ and $a^i_j(X)$ are homogeneous symbols of degree zero with respect to (ξ',τ). Since all the roots of $p^-_\tau(X,\xi_n)=0$ lie in the lower half plane $\{\mathrm{Im}\ \xi_n<0\}$, we see that $a^i_1(X)>0$ and all the roots of $\{\mathrm{Re}\ p^-_\tau\}(X,\xi_n)=0$ and $\{\mathrm{Im}\ p^-_\tau\}(X,\xi_n)=0$ are real and distinct. We remark that $m^{(-)}-1$ roots of $\{\mathrm{Im}\ p^-_\tau\}(X,\xi_n)=0$ separate $m^{(-)}$ roots of $\{\mathrm{Re}\ p^-_\tau\}(X,\xi_n)=0$ and that the coefficient of the leading term of $\mathrm{Im}\ p^-_\tau(X,\xi_n)$ is positive (see Hermite theorem in [11]), i.e.

$$a^i_1(X)>0. \tag{3.6}$$

We begin with

$$|\Lambda^{-1/2}P^-_\tau(x,D,\tau)u|^2_{0,\tau}=|\Lambda^{-1/2}(\mathrm{Re}\ P^-_\tau(x,D,\tau)+i\ \mathrm{Im}\ P^-_\tau(x,D,\tau))u|^2_{0,\tau}$$

$$=\{|\Lambda^{-1/2}\mathrm{Re}\ P^-_\tau u|^2_{0,\tau}+|\Lambda^{-1/2}\mathrm{Im}\ P^-_\tau u|^2_{0,\tau}\} \tag{3.7}$$

$$-i\{\mathrm{Re}\, P_\tau^- u,\Lambda^{-1}\mathrm{Im}\, P_\tau^- u)-(\Lambda^{-1}\mathrm{Im}\, P_\tau^- u,\mathrm{Re}\, P_\tau^- u)\}. \tag{3.8}$$

For RHS of (3.7), we have

$$|\Lambda^{-1/2}\mathrm{Re}\, P_\tau^- u|_{0,\tau}^2+|\Lambda^{-1/2}\mathrm{Im}\, P_\tau^- u|_{0,\tau}^2 \geq c|\Lambda^{-1/2}u|_{m^{(-)},\tau}^2-d|u|_{m^{(-)}-1,\tau}^2, \tag{3.9}$$

and we can check it as follows; since an equality

$$(\mathrm{Re}\, P_\tau^-)(x,D,\tau)=\mathrm{D}_{x_n}^{m^{(-)}}+\sum_{j=1}^{m^{(-)}} a_j^r(x,D',\tau)\Lambda^j D_{x_n}^{m^{(-)}-j}$$

leads us to

$$|\Lambda^{-1/2}D_{x_n}^{m^{(-)}}u|_{0,\tau}^2 \leq |\Lambda^{-1/2}(\mathrm{Re}\, P_\tau^-)(x,D,\tau)u|_{0,\tau}^2+d|\Lambda^{1/2}u|_{m^{(-)}-1,\tau}^2, \tag{3.10}$$

we have, by virtue of $a_1^i(X)>0$,

$$|\Lambda^{1/2}u|_{m^{(-)}-1,\tau}^2 \leq |\Lambda^{-1/2}(\mathrm{Im}\, P_\tau^-)(x,D,\tau))u|_{0,\tau}^2+d|u|_{m^{(-)}-1,\tau}^2, \tag{3.11}$$

Thus (3.10) and (3.11) yield the estimate (3.9). For the two terms in (3.8), we use Green's formula, Lemma 3.4 and conclude

$$|((\mathrm{Re}\, P_\tau^-)u,\Lambda^{-1}(\mathrm{Im}\, P_\tau^-)u)-(\Lambda^{-1}(\mathrm{Im}\, P_\tau^-)u,(\mathrm{Re}\, P_\tau^-)u)+G_{\mathrm{Re}\, p_\tau^-,\lambda^{-1}\mathrm{Im}\, p_\tau^-}\langle u,u\rangle|$$

$$\leq d(|u|_{m^{(-)}-1,\tau}^2+\langle u\rangle_{m^{(-)}-1,-1/2,\tau}^2).$$

By the Hermite theorem in [11], since the Bézout form for Re p_τ^- and λ^{-1} Im p_τ^- is positive definite, we can apply Gårding's inequality of Lemma 3.6 to show

$$G_{Re\, p_\tau^-,\lambda^{-1}\mathrm{Im}\, p_\tau^-}\langle u,u\rangle \geq \langle u\rangle_{m^{(-)}-1,\tau}^2.$$

Hence we have

$$|u|_{m^{(-)}-1,1/2,\tau}^2+\langle u\rangle_{m^{(-)}-1,0,\tau}^2 \leq c(|P_\tau^- u|_{0,-1/2,\tau}^2+|u|_{m^{(-)}-1,\tau}^2).$$

By the interpolation inequality for the second term in the RHS, we conclude

$$|u|_{m^{(-)}-1,\tau}^2 \leq \varepsilon|u|_{m^{(-)}-1,1/2,\tau}^2+1/\varepsilon|u|_{m^{(-)}-1,-1/2,\tau}^2$$

and this completes a proof of Proposition 3.8.

By Gårding's inequality, we get the same kind of proposition as in [16].

Proposition 3.9. *Let* $r\in S_{cl}^{m,0}$ *and* $s\in S_{cl}^{m-1,0}$. *If there exists a suitable small conic neighborhood* $U(X_0)$ *such that*

$$(\mathrm{Re}\, G_{r,s})[z,\bar{z}]\geq c|z|^2 \quad \text{on} \quad U(X_0)\cap\{e^j[z]=b_\tau^k[z]=0, j=1,\cdots,m^{(-)},k=1,\cdots,\mu\}$$

for any $X_0\in\Sigma\times\mathbf{R}_\xi^{n-1}\times\{\tau\geq 0\}$ *on the boundary* $\partial\Sigma$, *then there exist* $c>0$ *and* $d>0$ *such that the following estimate holds*

$$\mathrm{Re}\Big(G_{r,s}\langle u,u\rangle+d\sum_{k=1}^{\mu}\langle B_\tau^k u\rangle_{m-\beta^k-1,\tau}^2\Big)\geq c\langle u\rangle_{m-1,0,\tau}^2-d(|u|_{m-1,-1/2,\tau}^2+|P_\tau u|_{0,-1/2,\tau}^2)$$

for $u \in H^m(\Sigma)$.

Proof. By the use of a partition of unity, we are enough to give a proof for a localized problem in a neighborhood of X_0. The assumption leads us to have

$$(\mathrm{Re}\ G_{r,s})[z,\bar{z}]+d\left(\sum_{j=1}^{m^{(-)}} e^j[z]\overline{e^j}[\bar{z}]+\sum_{k=1}^{\mu} b_\tau^k[z]\overline{b_\tau^k}[\bar{z}]\right)\geq c|z|^2$$

for large enough $d>0$, and an application of Gårding's inequality gives

$$\mathrm{Re}(G_{r,s}\langle u,u\rangle)+d\left(\sum_{j=1}^{m^{(-)}}\langle E^j u\rangle_{j-1,\tau}^2+\sum_{k=1}^{\mu}\langle B_\tau^k u\rangle_{m-\beta^k-1,\tau}^2\right)\geq c\langle u\rangle_{m-1,0,\tau}^2$$

(See Lemma 3.12 of [16]). We apply Proposition 3.8 to the function $E^{m^{(-)}}u$ to show

$$\langle E^{m^{(-)}}u\rangle_{m^{(-)}-1,0,\tau}^2\leq c(|P_\tau u|_{0,-1/2,\tau}^2+|u|_{m-1,-1/2,\tau}^2),$$

and we reach

$$\sum_{j=1}^{m^{(-)}}\langle E^j u\rangle_{j-1,\tau}^2\leq c(|P_\tau u|_{0,-1/2,\tau}^2+|u|_{m-1,-1/2,\tau}^2).$$

This completes a proof of Proposition 3.9.

According to the modified sharp Gårding inequality (Lemma 3.7), we can extend Proposition 3.9 to the case that the bilinear form $(\mathrm{Re}\ G_{r,s})[z,\bar{z}]\geq 0$ on $U(X_0)\cap\{e^j[z]=b_\tau^k[z]=0,\ j=1,\cdots,m^{(-)},k=1,\cdots,\mu\}$:

Corollary 3.10. *Let* $r\in S_{cl}^{m,0}$ *and* $s\in S_{cl}^{m-1,0}$. *If there exists a suitable small conic neighborhood* $U(X_0)$ such that

$$(\mathrm{Re}\ G_{r,s})[z,\bar{z}]\geq 0 \quad on \quad U(X_0)\cap\{e^j[z]=b_\tau^k[z]=0, j=1,\cdots,m^{(-)},k=1,\cdots,\mu\}$$

for any $X_0\in\Sigma\times\mathbf{R}_{\xi'}^{n-1}\times\{\tau\geq 0\}$ *on the boundary* $\partial\Sigma$, *then there exist* $c>0$ *and* $d>0$ *such that the following estimate holds.*

$$\mathrm{Re}\left(G_{r,s}\langle u,u\rangle+d\sum_{k=1}^{\mu}\langle B_\tau^k u\rangle_{m-\beta^k-1,\tau}^2\right)\geq -c\langle u\rangle_{m-1,-1/2,\tau}^2-d(\tau^{-1}|u|_{m-1,\tau}^2+|P_\tau u|_{0,-1/2,\tau}^2)$$

for $u\in H^m(\Sigma)$.

Next proposition is a basic estimate for the Carleman estimate.

Proposition 3.11. *Let* $r\in S_{cl}^{m,0}$ *and* $s\in S_{cl}^{m-1,0}$ *be real symbols, and set*

$$F_{r,is}(u,u):=(R(x,D,\tau)u,iS(x,D,\tau)u).$$

If

$$f^{sub}=f_{r,is}^{sub}(x,\xi,\tau)>c(|\xi|^2+\tau^2)^{m-1} \quad on \quad \mathrm{char}\ P_\tau,$$

then there exist $c>0$ *and* $d>0$ *such that we have*

$$\mathrm{Re}\{F_{r,is}(u,u)-G_{r,s}\langle u,u\rangle\}\geq c|u|^2_{m-1,\tau}-d(|P_\tau u|^2_{0,-1,\tau}+\langle u\rangle^2_{m-1,-1/2,\tau})$$

for large enough $\tau>0$ *and for* $u\in H^m$.

Proof. By Lemma 3.4 in [16], there exists a symbol $q\in S^{m-1,-1}_{cl}$ such that

$$\mathrm{Re}(f^{sub}(x,\xi,\tau)+p_\tau(x,\xi,\tau)q(x,\xi,\tau))>c(|\xi|^2+\tau^2)^{m-1}.$$

Furthermore we can decompose the above symbol into the form

$$\mathrm{Re}(f^{sub}(x,\xi,\tau)+\mathrm{p}_\tau(x,\xi,\tau)q(x,\xi,\tau))=c/2(|\xi|^2+\tau^2)^{m-1}+\sum_{j=1,2}\{a_j(x,\xi,\tau)\}^2,$$

where $a_j\in C^1S^{m-1,0}(j=1,2)$ are real symbols respectively. (See Lemma 3.5 in [16].) We firstly show our aimed estimate for symbols with smooth coefficients. By Proposition 3.4 or Lemma 3.7 in [16], we have

$$\left|\mathrm{Re}\ F^{sub}(u,u)+(P_\tau(x,D,\tau)u,Q(x,D,\tau)u)-c/2|u|^2_{m-1,\tau}-\sum_{j=1,2}|A_j(x,D,\tau)u|^2_{0,\tau}\right|$$

$$\leq d(|u|^2_{m,-3/2,\tau}+\langle u\rangle^2_{m-1,-1/2,\tau}).$$

Since the boundary is noncharacteristic, one can estimate the m-th order derivative of u along the normal direction by $P_\tau(x,D,\tau)u$ and $|u|_{m-1,\tau}$;

$$|u|^2_{m,-1,\tau}\leq c(|u|^2_{m-1,\tau}+|P_\tau(x,D,\tau)u|^2_{0,-1,\tau}).$$

Here we use an estimate

$$|u|^2_{m,-3/2,\tau}\leq\tau^{-1}|u|^2_{m,-1/2,\tau}$$

and we have

$$\mathrm{Re}F^{sub}(u,u)\leq c|u|^2_{m-1,\tau}-d(|P_\tau(x,D,\tau)u|^2_{0,-1,\tau}+\langle u\rangle^2_{m-1,-1/2,\tau})$$

for large enough τ. Using Lemma 3.5, we have

$$|\mathrm{Re}(F(u,u)-F^{sub}(u,u)-G_{r,s}\langle u,u\rangle)|$$

$$\leq d(|u|^2_{m,-3/2,\tau}+\langle u\rangle^2_{m-1,-1/2,\tau})$$

$$\leq d'(|P_\tau(x,D,\tau)u|^2_{0,-1,\tau}+\tau^{-1}|u|^2_{m-1,\tau}+\langle u\rangle^2_{m-1,-1/2,\tau}).$$

This implies that the desired estimate holds for smooth coefficients case. In order to obtain the result for symbols r and s with C^1 coefficients, we have only to check that the estimate is stable with respect to small perturbations of r and s in $C^1S^{m,0}_{cl}$ and C^1 $S^{m-1,0}_{cl}$ respectively. Indeed, replace the symbols r and s with smooth ε-approximations, and, from the estimate in Lemma 3.7 of [16], we have

$$|\mathrm{Re}(F_{r,s}(u,u)-G_{r,s}\langle u,u\rangle)-\mathrm{Re}(F_{r_\varepsilon,s_\varepsilon}(u,u)-G_{r_\varepsilon,s_\varepsilon}\langle u,u\rangle)|$$

$$\leq c_1\varepsilon(|u|^2_{m,-1,\tau}+\langle u\rangle^2_{m-1,-1/2,\tau})$$

$$\leq c_2\varepsilon(|P_\tau(x,D,\tau)u|^2_{0,-1,\tau}+|u|^2_{m-1,\tau}+\langle u\rangle^2_{m-1,-1/2,\tau}).$$

This implies the desired estimate for C^1 coefficients.

3.3. Proof for the Carleman estimates near the boundary. Sakamoto [14] introduces a basic concept which is a refinement of the uniform Lopatinski condition. It is essential to show the energy inequality for hyperbolic mixed problems, and it is also important for our case. We shall firstly define Carleman-Lopatinski (C-L) conditions near the boundary, and we prepare

$$p^r_\tau(x,\xi,\tau):\ =\mathrm{Re}\,p_\tau(x,\xi,\tau),\ p^i_\tau(x,\xi,\tau):\ =\mathrm{Im}\,p_\tau(x,\xi,\tau).$$

Definition 3.12 (strong Carleman-Lopatinski condition). *Let $S=\{\phi=0\}$ be a C^2 hypersurface, $x_0'\in S\cap\partial\Sigma$ and suppose $(\xi_0',\tau_0)\in\mathbf{R}^{n-1}_{\xi'}\times\{\tau\geq 0\}$. We shall say that $\{P,B\}$ satisfies the strong Carleman-Lopatinski condition w.r.t. S at $X_0=(x_0',0,\xi_0',\tau_0)$, in case*
there exists a suitable small conic neighborhood $U(X_0)$ such that

$$G_{p^r_\tau,\frac{1}{\tau}p^i_\tau}[z,\bar z]\leq c|z|^2 \quad on \quad U(X_0)\cap\{e^j[z]=b^k_\tau[z]=0, j=1,\cdots,m^{(-)},k=1,\cdots,\mu\},$$

where $\{P,B\}$ comes from the boundary value problem (3.1).

Definition 3.13 (weak Carleman-Lopatinski condition). *Let $S=\{\phi=0\}$ be a C^2 hypersurface, $x_0'\in S\cap\partial\Sigma$ and suppose $(\xi_0',\tau_0)\in\mathbf{R}^{n-1}_{\xi'}\times\{\tau\geq 0\}$. We shall say that $\{P,B\}$ satisfies the weak Carleman-Lopatinski condition w.r.t. S at $X_0=(x_0',0,\xi_0',\tau_0)$, in case*
there exists a suitable small conic neighborhood $U(X_0)$ such that the following (A) and (B) hold :
(A) *There exist $q\in C^1S^{m-1,0}_{cl}$ and real symbols $q^{(1)}$, $q^{(2)}\in C^1S^{m-1,1-m}_{cl}$ such that*
(1) $q\equiv(q^{(1)}+iq^{(2)})\tau^{-1}\mathrm{Im}\,p_\tau$ (mod $\mathrm{Re}\,p_\tau$) *and* $q^{(2)}>0$ *on* char $\mathrm{Re}\,P_\tau$
(2) $\mathrm{Im}\ G_{p_\tau,\bar q}[z,\bar z]\leq 0$ *on* $U(X_0)\cap\{e^j[z]=b^k_\tau[z]=0,\ j=1,\cdots,m^{(-)},\ k=1,\cdots,\mu\}$
(3) $\mathrm{Im}\{p_\tau,\bar q\}\leq -c(|\xi|^2+\tau^2)^{m-1}$ *on* char P_τ.
(B) *There exists a real symbol $r\in C^1S^{m-1,0}_{cl}$ such that*
$G_{\frac{1}{\tau}p^i_\tau,r}[z,\bar z]\geq c|z|^2$ *on* $U(X_0)\cap\{e^j[z]=b^k_\tau[z]=0,\ j=1,\cdots,m^{(-)},k=1,\cdots,\mu\}$.

The main goal of this section is to show the following two theorems.

Theorem 3.14. *Suppose that ϕ is a strongly pseudoconvex function w.r.t. P and that $\{P,B\}$ satisfies the strong Carleman-Lopatinski condition w.r.t. the level set of ϕ. Then for large enough τ we have*

$$|e^{\tau\phi}u|^2_{m-1,\tau}+\langle e^{\tau\phi}u\rangle^2_{m-1,0,\tau}\leq d\left(\frac{1}{\tau}|e^{\tau\phi}Pu|^2_{0,\tau}+\sum_{k=1}^{\mu}\langle e^{\tau\phi}B^k u\rangle^2_{m-\beta^k-1,\tau}\right) \tag{3.12}$$

whenever $u\in H^{m-1}(\Sigma)$ is supported in a fixed compact set and the RHS is finite.

Theorem 3.15. *Suppose $\{P,B\}$ satisfies the weak Carleman-Lopatinski condi-*

tion w.r.t. the level set of ϕ. Then for large enough τ we have

$$|e^{\tau\phi}u|^2_{m-1,\tau}+\tau^{1/2}\langle e^{\tau\phi}u\rangle^2_{m-1,-1/2,\tau}\leq d\Big(\frac{1}{\tau}|e^{\tau\phi}Pu|^2_{0,\tau}+\sum_{k=1}^{\mu}\langle e^{\tau\phi}B^k u\rangle^2_{m-\beta^k-1,\tau}\Big) \quad (3.13)$$

whenever $u\in H^{m-1}(\Sigma)$ is supported in a fixed compack set and the RHS is finite.

Proof of Theorem 3.14. We define a bilinear form $F(u,u)$ by

$$F(u,u):=F_{p^r_\tau,\frac{i}{\tau}p^i_\tau}(u,u)=2((\mathrm{Re}\,P_\tau)(x,D,\tau)u,\ \frac{i}{\tau}(\mathrm{Im}\,P_\tau)(x,D,\tau)u).$$

From the pseudoconvex condition, we have

$$f^{sub}_{p^r_\tau,\frac{i}{\tau}p^i_\tau}(x,\xi,\tau)=\frac{2}{\tau}\{\mathrm{Re}\,p_\tau,\mathrm{Im}p_\tau\}>0 \quad \text{on} \quad \text{char}\ P_\tau.$$

Proposition 3.11 leads us to

$$\mathrm{Re}(F(u,u)-G_{p^r_\tau,\frac{i}{\tau}p^i_\tau}\langle u,u\rangle)\geq c|u|^2_{m-1,\tau}-d(|P_\tau u|^2_{0,-1,\tau}+\langle u\rangle^2_{m-1,-1/2,\tau}), \quad (3.14)$$

and the strong Carleman-Lopatinski condition and Proposition 3.9 make us to deduce

$$\mathrm{Re}\Big(G_{p^r_\tau,\frac{i}{\tau}p^i_\tau}\langle u,u\rangle+\sum_{k=1}^{\mu}\langle B^k_\tau u\rangle^2_{m-\beta^k-1,\tau}\Big)$$

$$\geq c\langle u\rangle^2_{m-1,0,\tau}-d(|P_\tau u|^2_{0,-1/2,\tau}+|u|^2_{m-1,-1/2,\tau}). \quad (3.15)$$

We combine (3.14) with (3.15) to see, for large enough τ.

$$\mathrm{Re}\ F(u,u)+\sum_{k=1}^{\mu}\langle B^k_\tau u\rangle^2_{m-\beta^k-1,\tau}\geq c(|u|^2_{m-1,\tau}+\langle u\rangle^2_{m-1,0,\tau})-d|P_\tau u|^2_{0,-1/2,\tau}. \quad (3.16)$$

Here we use simple facts

$$\langle u\rangle^2_{m-1,-1/2,\tau}\leq\frac{1}{\tau}\langle u\rangle^2_{m-1,0,\tau},\ |u|^2_{m-1,-1/2,\tau}\leq\frac{1}{\tau}|u|^2_{m-1,\tau}.$$

It is obvious that

$$\tau\ \mathrm{Re}\ F(u,u)=|P_\tau u|^2_{0,\tau}-|\mathrm{Re}\ P_\tau)u|^2_{0,\tau}-|\mathrm{Im}\ P_\tau)u|^2_{0,\tau}\leq|P_\tau u|^2_{0,\tau},$$

and combination of this inequality with (3.16) leads us to

$$|u|^2_{m-1,\tau}+\langle u\rangle^2_{m-1,0,\tau}\leq d\Big(\frac{1}{\tau}|P_\tau u|^2_{0,\tau}+\sum_{k=1}^{\mu}\langle B^k_\tau u\rangle^2_{m-\beta^k-1,\tau}\Big). \quad (3.17)$$

Since the difference between $P(x,D+i\tau\nabla\phi)$ and $P_\tau(x,D,\tau)$ is an operator of order $m-1$, we set $v:=e^{\tau\phi}u$ and get

$$|P_\tau v|^2_{0,\tau}\leq c(|P(x,D+i\tau\nabla\phi)v|^2_{0,\tau}+|v|^2_{m-1,\tau}).$$

In the same way as above, we have

$$\langle B_\tau^k v\rangle^2_{m-\beta^k-1,\tau}\leq c\Big(\sum_{k=1}^{\mu}\langle B(x,D+i\tau\nabla\phi)v\rangle^2_{m-\beta^k-1,\tau}+\langle v\rangle^2_{m-1,-1,\tau}\Big).$$

Hence we apply (3.17) to v, and we obtain the inequality (3.12) for large enough τ.

We have shown the inequality (3.12) for $u\in H^m$, and by virtue of the following lemma, we have the same result for $u\in H^{m-1}(\Sigma)$, $P(x,D)u\in L^2(\Sigma)$ and $B^k u\in H^{m-\beta^k-1}(\partial\Sigma)$.

Lemma 3.16. *Suppose that $u\in H^{m-1}(\Sigma)$ satisfies $P(x,D)u\in L^2(\Sigma)$ and $B^k u\in H^{m-\beta^k-1}(\partial\Sigma)$. Then there exists a sequence $\{u_j\}\in H^m(\Sigma)$ such that*

$$u_j\longrightarrow u \text{ in } H^{m-1}(\Sigma)$$

$$P(x,D)u_j\longrightarrow P(x,D)u \text{ in } L^2(\Sigma)$$

$$B^k u_j\longrightarrow B^k u \text{ in } H^{m-\beta^k-1}(\partial\Sigma).$$

Proof. It is enough to show the results for localized problem. There are two cases ; the support of u is included in Σ, and it contains the boundary $\partial\Sigma$. Since the first case is treated in [5], we consider the second one. Denote $u_j: =\Lambda_j u$, $\Lambda_j: =(1+1/j|D'|)^{-1}$, and we see $u_j\in H^{m-1,1}$ and $u_j\longrightarrow u$ in $H^{m-1}(\Sigma)$. Furthermore, $P(x,D)u_j=\Lambda_j P(x,D)u+[P,\Lambda_j]u$. Since the coefficients of the principal part of P P are Lipschitz continuous, it follows that the operator $[P,\Lambda_j]$ is equibounded from H^{m-1} to L^2 (see e.g. Lemma 17.1.5 in[5]).
Since $[P,\Lambda_j]u\longrightarrow 0$ in $L^2(\Sigma)$ for $u\in H^m(\Sigma)$, it follows that the same result holds for each $u\in H^{m-1}(\Sigma)$ by density argument. Thus one can see, from the commutation properties of pseudodifferential operators, $P(x,D)u_j\longrightarrow P(x,D)u$ in $L^2(\Sigma)$. In particular, $P(x,D)u_j\in \mathrm{L}^2$, $u_j\in H^{m-1,1}$ implies that $D_\nu^m u_j\in L^2$ and $u_j\in H^m(\Sigma)$. Finally, to complete the proof of the theorem we have to estimate $B^k u_j=\Lambda_j B^k u+[B^k,\Lambda_j]u$. The first RHS term converges to $B^k u$ in $H^{m-\beta^k-1}(\partial\Sigma)$ while the operator $[B^k,\Lambda_j]|_{\partial\Sigma}$ is equibounded by the trace regularity and the same commutator estimates as above, and we see pointwise convergent to 0 from $H^{m-1}(\Sigma)$ into $H^{m-\beta^k-1}(\partial\Sigma)$. We obtain the desired result.

In order to start a proof of Theorem 3.15, we prepare a lemma :

Lemma 3.17. *Suppose $\{P,B\}$ satisfies teh hypothesis (A) in the weak Carleman-Lopatinski condition w.r.t. the level set of ϕ. Then for $u\in H^m(\Sigma)$ and large enough $\tau>0$, we have*

$$\mathrm{Re}(P_\tau(x,D,\tau)u,Q(x,D,\tau)u)\geq c\Big(|u|^2_{m-1,\tau}+\tau\sum_{j=0}^{m-1}|Q_j(x,D,\tau)u|^2_{0,\tau}\Big)$$

$$-d\Big(|P_\tau u|^2_{0,-1/2,\tau}+\sum_{k=1}^{\mu}\langle B_\tau^k u\rangle^2_{m-\beta^k-1,\tau}+\langle u\rangle^2_{m-1,-1/2,\tau}\Big). \tag{3.18}$$

where a symbol of q for the operator Q plays a role seen in the weak Carleman-Lopatinski condition, and the operators Q_j have symbol q_j defined by

$$q_j := \tau^{-1}(\operatorname{Im} p_\tau)\xi_n^j\lambda^{-j} \quad (\operatorname{mod} \operatorname{Re} p_\tau) \quad (j=0,1,\cdots,m-1).$$

Proof. From the hypothesis $(A)(1)$, there exist symbols $w_k \in C^1 S_{cl}^{m-1,0}$ $(k=1,2)$ such that

$$\operatorname{Re}(p_\tau \bar{q}) = \tau\left(c\sum_{j=0}^{m-1}|q_j|^2 + \sum_{k=1,2}|w_k|^2\right) \tag{3.19}$$

(See Lemma 4.3 in [16]). Let

$$F(u,u) := (P_\tau u, Qu) - \tau\left(c\sum_{j=0}^{m-1}|Q_j u|_{0,\tau}^2 + \sum_{k=1,2}|W_k u|_{0,\tau}^2\right),$$

and, by the equality (3.19), the real part of the principal symbol of $F(u,u)$ is equal to0. By direct calculation, we have

$$\operatorname{Re} f^{sub}(x,\xi,\tau) = \{\operatorname{Re} p_\tau, \operatorname{Im} q\} - \{\operatorname{Im} p_\tau, \operatorname{Re} q\} > 0 \quad \textit{on} \quad \operatorname{char} P_\tau.$$

Applying Proposition 3.11, we obtain

$$\operatorname{Re}(F(u,u) - G_F\langle u,u\rangle) \geq c|u|_{m-1,\tau}^2 - d(|P_\tau u|_{0,-1,\tau}^2 + \langle u\rangle_{m-1,-1/2,\tau}), \tag{3.20}$$

where $G_F\langle u,u\rangle := G_{p_\tau^i, i \operatorname{Im} q}\langle u,u\rangle + G_{ip_\tau^i, \operatorname{Re} q}\langle u,u\rangle$. Since the hypothesis $(A)(2)$ holds, we have the estimate from Corollary 3.10

$$G_{p_\tau^i, i \operatorname{Im} q}\langle u,u\rangle + G_{ip_\tau^i, \operatorname{Re} q}\langle u,u\rangle + \sum_{k=1}^{\mu}\langle B_\tau^k u\rangle_{m-\beta^k-1,\tau}^2$$

$$\geq -c\langle u\rangle_{m-1,-1/2,\tau}^2 - d(|P_\tau u|_{0,-1/2,\tau}^2 + \tau^{-1}|u|_{m-1,\tau}^2). \tag{3.21}$$

Combination of (3.21) with (3.20) yields (3.18).

Proof of Theorem 3.15. From Lemma 3.17 and the hypothesis (A), we have an estimate

$$|u|_{m-1,\tau}^2 + c\tau\sum_{j=0}^{m-1}|Q_j(x,D,\tau)u|_{0,\tau}^2$$

$$\leq d\left(|P_\tau(x,D,\tau)u|_{0-1/2,\tau}^2 + \sum_{k=1}^{\mu}\langle \mathrm{B}_\tau^k u\rangle_{m-\beta^k-1,\tau}^2 + \langle u\rangle_{m-1,-1/2,\tau}^2\right). \tag{3.22}$$

On the other hand, by the hypothesis (B), we obtain

$$\operatorname{Re} G_{p_\tau^i, r}\langle u,u\rangle + \tau\sum_{k=1}^{\mu}\langle B_\tau^k u\rangle_{m-\beta^k-1,-1/2,\tau}^2$$

$$\geq c\tau\langle u\rangle_{m-1,-1/2,\tau}^2 - d(|P_\tau(x,D,\tau)u|_{0-1/2,\tau}^2 + |u|_{m-1,-1/2,\tau}^2). \tag{3.23}$$

Application of Lemma 3.4 yields

$$|\mathrm{Re}(iP_\tau^i u, Ru) - G_{p_{i,r}^i}\langle u,u\rangle| \leq d\tau(|u|^2_{m-1,-1/2,\tau} + \langle u\rangle^2_{m-2,0,\tau}), \tag{3.24}$$

and we obtain, for positive $\alpha>0$,

$$|\mathrm{Re}(iP_\tau^i u, Ru)| \leq 2\alpha\tau^{3/2}|\tau^{-1}P_\tau^i u|^2_{0,\tau} + \frac{2\tau^{1/2}}{\alpha}|u|^2_{m-1,\tau}. \tag{3.25}$$

Therefore by (3.22)-(3.25), we see

$$|u|^2_{m-1,\tau} + \tau^{1/2}\langle u\rangle^2_{m-1,-1/2,\tau} \leq d\left(\frac{1}{\tau}|P_\tau u|^2_{0,\tau} + \sum_{k=1}^{\mu}\langle B_\tau^k u\rangle^2_{m-\beta^k-1,\tau}\right)$$

and it completes a proof of the Theorem 3.15.

3.4. Unique continuation and applications. The Carleman estimates near the boundary in the previous section suggest that the following unique continuation property. We remark that we also use the same notation as above.

Theorem 3.18. *Let $\phi\in C^2(\Sigma)$, $\phi(x_0)\neq 0$, $S:=\{\phi(x)=\phi(x_0)\}$, and assume that the level surface of ϕ is strong pseudoconvex w.r.t. P at x_0 and $\{P, B\}$ satisfies the strong Carleman-Lapatinski condition w.r.t. the level surface of ϕ at x_0. Let $u\in H^{m-1}(\Sigma)$ be a solution to*

$$\begin{cases} P(x,D)u=0 & \text{in } \Sigma \\ B^j(x,D)u=0 & \text{on } \partial\Sigma, j=1,2,\cdots,\mu. \end{cases}$$

If there exists a neighborhood $U(x_0)$ such that $u=0$ in $\{x\in U(x_0)\,;\, \phi(x)>\phi(x_0)\}$ then $u=0$ in a neighborhood of x_0.

Remark 3. *We should note that the strong Carleman-Lopatinski condition is stable w.r.t. small C^1 perturbation for the function ϕ.*

Theorem 3.19. *Let $\phi\in C^2(\Sigma)$, $\phi(x_0)\neq 0$, $S:=\{\phi(x)=\phi(x_0)\}$. Assume that the level surface of ϕ is strong pseudoconvex w.r.t. P at and that $\{P,B\}$ satisfies the weak Carleman-Lopatinski condition w.r.t. the level surface of ϕ in a neighborhood of x_0. Let $u\in H^{m-1}(\Sigma)$ be a solution to*

$$\begin{cases} P(x,D)u=0 & \text{in } \Sigma \\ B^j(x,D)u=0 & \text{on } \partial\Sigma,\ j=1,2,\cdots,\mu. \end{cases}$$

If there exists a neighborhood $U(x_0)$ such that $u=0$ in $\{x\in U(x_0)\,;\, \phi(x)>\phi(x_0)\}$ then $u=0$ in a neighborhood of x_0.

We apply the above results to the wave equation with the Dirichlet (or Nneumann) boundary condition. Note that the strong (or weak) Carleman-Lopatinski condition holds for the cases.

4. Proof of the main results for inverse problems

Extend $u(t,x)$ as an even functions w.r.t. $t\in[-T,T]$, and we get, by the equations (1.1)-(1.4),

$$(\partial_t^2-\triangle_x-a(x))u(t,x)=f(x)R(t,x)-T<t<T,x\in\Omega \tag{4.1}$$

$$u(0,x)=\partial_t u(0,x)=0 \qquad x\in\Omega \tag{4.2}$$

$$u(t,x)=0 \qquad -T<t<T, \quad x\in\partial\Omega \tag{4.3}$$

$$\frac{\partial u}{\partial n}(t,x)=0 \qquad -r_0<t<r_0, \quad x\in\partial\Omega\cap D(\tilde{x}_0,r_0), \tag{4.4}$$

and we define P_a by

$$P_a: =\partial_t^2-\triangle_x-a(x).$$

By the assumption $R(0,x)\neq 0(\forall x\in\Omega)$, there exists a number $\eta\in(0,r_0)$ such that

$$R(t,x)\neq 0, \quad -\eta<t<\eta, \quad x\in\Omega.$$

Therefore from (4.1), it follows that

$$f(x)=\frac{P_a u(t,x)}{R(t,x)}, \quad -\eta<t<\eta, \quad x\in\Omega,$$

and we have

$$0=\frac{\partial f(x)}{\partial t}=\frac{\partial}{\partial t}\left\{\frac{P_a u(t,x)}{R(t,x)}\right\}, \quad -\eta<t<\eta, \quad x\in\Omega.$$

Let

$$h(t,x): =\frac{\partial_t R(t,x)}{R(t,x)}, \quad Q: =\partial_t-h(t,x)$$

then we see

$$(QP_a u)(t,x)=0, \quad -\eta<t<\eta, \quad x\in\Omega.$$

We set $v(t,x): =(Qu)(t,x)$ and have the following system for u and v:

$$P_a v(t,x)=[P_a,Q]u(t,x), \quad -\eta<t<\eta, \quad x\in\Omega, \tag{4.5}$$

$$Qu(t,x)=v(t,x), \quad -\eta<t<\eta, \quad x\in\Omega. \tag{4.6}$$

where $[A,B]=AB-BA$. Since $u(0,x)=0$, we can solve an ordinary differential equation, for fixed $x\in\Omega$,

$$\frac{d}{dt}u(t,x)-h(t,x)u(t,x)=v(t,x), \quad -\eta<t<\eta$$

in the form

$$u(t,x)=\int_0^t H(t,s,x)v(s,x)ds, \tag{4.7}$$

wherre $H(t,s,x)=\frac{h(t,x)}{h(s,x)}$. For simplicity, we denote H by the operator

$$(Hv)(t,x):=\int_0^t H(t,s,x)v(s,x)ds.$$

The following inequality Lemma 4.1 can be proved (e.g. Lemma 3 in Bukhgeim [2]). For $(t^0,x^0)\in(-T,T)\times\Omega$ we set a weight function by

$$\phi=\frac{-(t-t^0)^2+|x-x^0|^2}{2},$$

and for $\varepsilon>0$ we set

$$\Omega^{(\varepsilon)}:=\{(t,x): x\in\Omega, \phi>\epsilon\}.$$

Lemma 4.1 (Bukhgeim). *Suppose* $t\cdot\partial_t\phi\leq 0$ *and* $\Omega^{(\varepsilon)}\subset(-\eta,\eta)\times\Omega$. *Then*

$$\int_{\Omega(\varepsilon)}e^{2\tau\phi}\left|\int_0^t|\rho(s,x)|ds\right|^2 dtdx\leq\eta^2\int_{\Omega(\varepsilon)}e^{2\tau\phi}|\rho(s,x)|^2 dtdx$$

for $\forall\rho\in L^2(\Omega^{(\varepsilon)})$.

From (4.5) and (4.7), we have

$$P_a v(t,x)=[P_a,Q]\int_0^t H(t,s,x)v(s,x)ds. \tag{4.8}$$

We remark that $[P_a,Q]$ is a 1st order differential operator. Let $\chi\in C^\infty([-\eta,\eta]\times\bar{\Omega})$ be $0\leq\chi(t,x)\leq 1$ and,

$$\chi(t,x)=\begin{cases}1 & (t,x)\in\Omega^{(\varepsilon)}\\ 0 & (t,x)\in\Omega^{(0)}\backslash\Omega^{(\varepsilon-\varepsilon/2)},\end{cases}$$

and set

$$w(t,x):=\chi(t,x)v(t,x), \quad |t|<\eta, \quad x\in\Omega.$$

In (4.3) and (4.4), we apply Theorem 3.14 to obtain

$$\tau|e^{\tau\phi}w|^2_{1,\tau}+\langle e^{\tau\phi}w\rangle^2_{1,0,\tau}\leq C|e^{\tau\phi}Pw|^2_{0,\tau},$$

and application of Lemma 4.1 and (4.8) lead us to see

$$|e^{\tau\phi}Pw|^2_{0,\tau}=|e^{\tau\phi}P\chi v|^2_{0,\tau}$$

$$\leq |e^{\tau\phi}[P,\chi]v|_{0,\tau}^2 + |e^{\tau\phi}\chi P v|_{0,\tau}^2$$

$$\leq |e^{\tau\phi}[P,\chi]v|_{0,\tau}^2 + |e^{\tau\phi}[\chi,H]v|_{0,\tau}^2 + e|^{\tau\phi}\chi v|_{1,\tau}^2.$$

Therefore, for large enough τ, we have

$$\tau|e^{\tau\phi}w|_{1,\tau}^2 + \langle e^{\tau\phi}w\rangle_{1,0,\tau}^2$$

$$\leq C(|e^{\tau\phi}[P,\chi]v|_{0,\tau}^2 + |e^{\tau\phi}[\chi,H]v|_{0,\tau}^2).$$

A straightforward consequence of the usual proof for unique continuation is $v=0$. According to the equality (4.7), we get $u=0$. By the assumption $R(0,x)\neq 0$ and the equation (4.1), we obtain $f(x)=0$.

Thus the proof of Theorem 2.1 is complete.

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