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The theory of second-order differential equations based on Finsler geometry

By

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The first author has felt a great interest in the geometrical approach to the theory of nonlinear dynamical systems on the basis of Finsler metrics ([4], [5]) and gave a lecture with the title "Finsler's metrics of phase space of nonlinear dynamical systems and their applications" at the International Conference on Differential Geometry and its Applications, Bucharest, 1992 (Tensor, N. S., 53, x vi, ↓ 3). The second author participated in the Conference as a member of the Committee of Organization. Since then, the intimate contacts between the present authors have continued through the applications of Finsler geometry in the nonlinear dynamical systems.

Finsler geometry has founded by P. Finsler (1918) as the differential-geometrical development of variation calculus. Consequently the theory of geodesics should be one of the essential fields of the differential geometry of Finsler spaces.

Closely related to the behavior of geodesics, the conception of Finsler space with rectilinear extremals has been proposed and studied by P. Funk and L. Berwald. In particular, Berwald (1941) and the second author [7] have established the necessary and sufficient conditions for a Finsler space to be with rectilinear extremals, and the second author [8] has proved that the transformation of rectilinear coordinate systems is projective.

Directly motivated by the behavior of geodesic equations of two-dimensional Riemannian space and Berwald space, the second author with S. Bácsó [3] have found the notion of Finsler space of Douglas type and investigated the differential-geometrical aspects of the special Finsler spaces. In the two-dimensional case those spaces are characterized by the remarkable fact that the right-hand side of the geodesic equation $y'' = f(x, y, y')$ is a polynomial in y' of degree at most three.

In the recent paper [12] the second author showed that a projectively equivalent class of two-dimensional Finsler spaces is associated to a given second-order differential equation $y'' = f(x, y, y')$ of the normal form such that it is a differential equation of geodesics of every space of this class. Consequently the behavior of $y'' = f(x, y, y')$ is directly connected with that of the spaces.

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The first three sections of the present paper are devoted to the detailed description of Finsler spaces with rectilinear extremals and of Douglas type in the viewpoint of the theory of second-order differential equations. The last section gives a precise proof of the characterization of two-dimensional Douglas space, shown by the first author ([5], [6]). The main results are stated as Theorems 1, 2, 3 and 4, while Theorems A $\sim$ F are those which have been shown already.

1. Preliminaries

1.1. Finsler spaces with the Berwald connection

We consider an n-dimensional Finsler space $F^n = (M^n, L(x,y))$ on a smooth n-manifold M^n ([1], [9]). Its fundamental function L(x, y), a real-valued function on the tangent bundle TM^n, is usually supposed to satisfy certain conditions from the geometrical standpoint, but only the homogeneity and the regularity are supposed here for our following considerations:

(1) $L(x,y)$ be positively homogeneous in $y = (y^i)$ of degree one: $L(x,py) = pL(x,y)$ for $\forall p > 0$.

(2) $L(x,y)$ be regular: $g_{ij} = \dot{\partial}_i\dot{\partial}_j F$ has non-zero $g = \det(g_{ij})$, where $\dot{\partial}_i = \partial/\partial y^i$ and $F = L^2/2$.

Let (g^{ij}) be the inverse matrix of (g_{ij}) and construct

$$2\gamma_j{}^i{}_k(x,y) = g^{ir}(\partial_k g_{rj} + \partial_j g_{rk} - \partial_r g_{jk}),$$
$$2G^i(x,y) = g^{ij}\{(\dot{\partial}_j\partial_r F)y^r - \partial_j F\},$$

where $\partial_j = \partial/\partial x^j$. Then we have $\gamma_j{}^i{}_k(x,y)y^j y^k = 2G^i(x,y)$.

The length s of an arc $C : x^i = x^i(t)$, $a \leqq t \leqq b$, on M^n is defined in F^n by the integral $s = \int_a^b L(x,\dot{x})dt$, $\dot{x}^i = dx^i/dt$. An extremal curve of the integral, called a geodesic, is given by the Euler differential equations $d(\dot{\partial}_i L)/dt - \partial_i L = 0$, which can be written in the form

$$\dot{x}^i\{\ddot{x}^j + 2G^j(x,\dot{x})\} - \dot{x}^j\{\ddot{x}^i + 2G^i(x,\dot{x})\} = 0. \tag{1.1}$$

In order to introduce various geometrical quantities in F^n, we are concerned with a Finsler connection $F\Gamma = (F_j{}^i{}_k(x,y), N^i{}_j(x,y), V_j{}^i{}_k(x,y))$ of F^n. For a Finslerian tensor field, $F\Gamma$ gives rise to the h and v-covariant derivatives: Let us treat of a tensor field $X^i(x,y)$ of $(1,0)$ -type, for brevity. Then we get two tensor fields $\nabla^h X$ and $\nabla^v X$ as follows:

$$\nabla^h{}_j X^i = \delta_j X^i + X^r F_r{}^i{}_j, \qquad \nabla^v{}_j X^i = \dot{\partial}_j X^i + X^r V_r{}^i{}_j,$$

where $\delta_j = \partial_j - N^r{}_j(x,y)\dot{\partial}_r$. The h and v-covariant derivatives $\nabla^h X$ and $\nabla^h X$ are tensor fields of $(1,1)$-type.

In the following we need the Berwald connection $B\Gamma = ({G_j}^i{}_k(x,y), G^i{}_j(x, y), 0)$, defined by $G^i{}_j = \dot{\partial}_j G^i$ and ${G_j}^i{}_k = \dot{\partial}_k G^i{}_j$. "${V_j}^i{}_k = 0$"for $B\Gamma$ means ${\nabla_i}^v = \dot{\partial}_i$. We have the commutation formulae, called the Ricci identities:

$$\begin{aligned} {\nabla_k}^h({\nabla_j}^h X^i) - {\nabla_j}^h({\nabla_k}^h X^i) &= X^r {H_r}^i{}_{jk} - (\dot{\partial}_r X^i) R^r{}_{jk}, \\ \dot{\partial}_k({\nabla_j}^h X^i) - {\nabla_j}^h(\dot{\partial}_k X^i) &= X^r {G_r}^i{}_{jk}. \end{aligned}$$

Here three tensor fields H, R and G appear. $R^i{}_{jk}$ is called the (v) h-torsion tensor, defined by

$$R^i{}_{jk} = \partial_k G^i{}_j - {G_j}^i{}_r G^r{}_k - [j,k],$$

where $[j,k]$ denotes the interchange of indices j,k of the preceding terms. ${H_h}^i{}_{jk}$ and ${G_h}^i{}_{jk}$ are called the h and v-curvature tensors respectively, defined by

$${H_h}^i{}_{jk} = \dot{\partial}_h R^i{}_{jk}, \qquad {G_h}^i{}_{jk} = \dot{\partial}_h {G_j}^i{}_k.$$

It is noted that ${G_h}^i{}_{jk}$ is symmetric in the subscripts. The contracted tensor ${G_r}^r{}_{jk} = G_{jk}$ is called the hv-Ricci tensor.

For the later use we shall make special mention of the three classes of special Finsler spaces as follows:

(1) Riemannian spaces, characterized by $g_{ij} = g_{ij}(x)$, that is, the C-tensor $C_{ijk} = (\dot{\partial}_k g_{ij})\ /2$ vanishes.

(2) Locally Minskowski spaces, characterized by the existence of an adapted coordiante system (x^i) around each point such that $L = L(y)$. We have ${G_j}^i{}_k = 0$ in (x^i). The tensorial characterization is $R^i{}_{jk} = 0$ and ${G_h}^i{}_{jk} = 0$.

(3) Berwald spaces, characterized by ${G_j}^i{}_k = {G_j}^i{}_k(x)$, that is, ${G_h}^i{}_{jk} = 0$. Then $2G^i(x,y) = {G_j}^i{}_k(x) y^j y^k$.

The classes (1) and (2) are contained in the class (3).

1.2. Projective change of Finsler spaces

We consider a change of Finsler metrics: $F^n = (M^n, L(x,y)) \to \overline{F}^n = (M^n, \overline{L}(x,y))$. If any geodesic of F^n coincides with a geodesic of $\overline{F}^n$ as a set of points and vice versa, then the change is called projective and F^n is said to be projectively related to $\overline{F}^n$.

F^n is projectively related to $\overline{F}^n$, if and only if there exists locally a scalar field $P(x,y)$, positively homogeneous in y^i of degree one, satisfying

$$\overline{G}^i(x,y) = G^i(x,y) + P(x,y) y^i.$$

Let us put $\dot{\partial}_i P = P_i$. Then we get

$$\overline{G}^i{}_j = G^i{}_j + P_j y^i + P\delta^i{}_j.$$

From these equations we obtain an invariant of projective change as follows:

$$Q^h = G^h - \frac{1}{n+1} G^r{}_r y^h.$$

Consequently we are led to the series of projective invariants by means of successive differentiations of Q^h with respect to y^i. In particular, we get

$$Q^h{}_i = \dot\partial_i Q^h = G^h{}_i - \frac{1}{n+1}(G_r{}^r{}_i y^h + G^r{}_r \delta^h{}_i),$$

$$Q_i{}^h{}_j = \dot\partial_j Q^h{}_i = G_i{}^h{}_j - \frac{1}{n+1}(G_{ij} y^h + G_r{}^r{}_i \delta^h{}_j + G_r{}^r{}_j \delta^h{}_i),$$

where G_{ij} is the hv-Ricci tensor. Next we get the invariant tensor field

$$D_i{}^h{}_{jk} = \dot\partial_k Q_i{}^h{}_j = G_i{}^h{}_{jk} - \frac{1}{n+1}\{(\dot\partial_k G_{ij}) y^h + G_{ij}\delta^h{}_k + G_{jk}\delta^h{}_i + G_{ki}\delta^h{}_j\},$$

called the *Douglas tensor*, where $G_i{}^h{}_{jk}$ is the hv-curvature tensor.

1.3. Associated Finsler spaces of dimension two

We are concerned with a Finsler space F^2 of dimension two. Let us denote a coordinate system (x^1, x^2) by (x, y) and (y^1, y^2) by (p, q). Then the equation (1.1) of a geodesic of F^2 has the form

$$p(\dot q + 2G^2) - q(\dot p + 2G^1) = 0. \tag{1.2}$$

In this case we have $2G^i = G_1{}^i{}_1 p^2 + 2G_1{}^i{}_2 pq + G_2{}^i{}_2 q^2, i = 1, 2$. Hence, in terms of $y' = dy/dx$ and $y'' = d^2y/dx^2$, (1.2) can be written as

$$\begin{aligned} y'' &= f(x, y, y') = f_3(y')^3 + f_2(y')^2 + f_1 y' + f_0, \\ f_3 &= G_2{}^1{}_2,\ f_2 = 2G_1{}^1{}_2 - G_2{}^2{}_2, \\ f_1 &= G_1{}^1{}_1 - 2G_1{}^2{}_2,\ f_0 = -G_1{}^2{}_1, \end{aligned} \tag{1.3}$$

where $G_j{}^i{}_k = G_j{}^i{}_k(x, y; 1, y')$, because $G_j{}^i{}_k(x, y; p, q)$ are positively homogeneous in (p, q) of degree zero [11].

Recently the second author has shown [12]: In the two-dimensional case any path space is projectively related to a Finsler space. In other words, to a given second-order differential equation $y'' = \Phi(x, y, y')$ of the normal form, there exists a class of projectively related Finsler spaces the geodesics of which are nothing but $y'' = \Phi(x, y, y')$. Therefore $y'' = \Phi(x, y, y')$ can be regarded as the differential equation of geodesics of a two-dimensional Finsler space F^2. We shall say that F^2 is *associated to* $y'' = \Phi(x, y, y')$.

2. Projectively flat Finsler spaces of dimension two

A Finsler space is called *projectively flat*, if it has a covering by coordinate neighborhoods in which it is projectively related to a locally Minkowski space.

Next a Finsler space is said to be *with rectilinear extremals*, if it is covered by coordinate neighborhoods in which any geodesic is represented by linear equations $x^i = x^i_o + ta^i$ in a parameter t, where x^i_0 and a^i are constants. Such a coordinate system (x^i) is called rectilinear [8].

It is well-known [1] that a Finsler space is projectively flat, if and only if it is with rectilinear extremals. The tensorial characterization of projective flatness has been established by L. Berwald and rigorously by the second author [7] as follows:

Theorem A. *A Finsler space of dimension n is projectively flat, if and only if*

(1) $n \geqq 3 : D_h{}^i{}_{jk} = 0$ *and* $W^i_{jk} = 0$,

(2) $n = 2 : D_h{}^i{}_{jk} = 0$ *and* $K_{ij} = 0$.

Here $D_h{}^i{}_{jk}$ is the Douglas tensor. The Weyl torsion tensor $W^i{}_{jk} = 0$ is equivalent to the fact that the $(v)h$-torsion tensor $R^i{}_{jk}$ is of the form

$$R^i{}_{jk} y^j = L^2 K(\delta^i{}_k - l^i l_k),$$

where K is called the scalar curvature. The space F^n, $n \geqq 3$, with $W^i{}_{jk} = 0$ is said to be of scalar curvature K. But $W^i{}_{jk}$ vanishes always in any Finsler space of dimension two. The tensor K_{ij} is defined from the h-curvature tensor $H_i{}^h{}_{jk}$ as follows:

$$H_{ij} = H_i{}^r{}_{jr}, \qquad H_i = \frac{1}{n-1}(nH_{ri} + H_{ir})y^r,$$

$$K_{ij} = (n-1)(\nabla_j{}^h H_i - \nabla_i{}^h H_j).$$

Throughout the present paper we are mainly concerned with the case $n = 2$. To understand the necessary and sufficient condition as above in $n = 2$, we need two essential scalars I and R of a two-dimensional Finsler space F^2. With respect to the orthonormal Berwald frame (l, m) of F^2 ([1], [9]), the C-tensor C_{ijk} and the $(v)h$-torsion tensor $R^i{}_{jk}$ are written in the form

$$LC_{ijk} = Im_i m_j m_k, \qquad R^i{}_{jk} = \varepsilon LRm^i(l_j m_k - l_k m_j),$$

where $\varepsilon = \pm 1$ is the signature of F^2. $I(x, y)$ is called the *main scalar* and $R(x, y)$ the h-scalar curvature or the *Gauss carvature.*

For a scalar field $S(x, y)$ the h-scalar derivatives $(S_{,1}, S_{,2})$ and the v-scalar derivatives $(S_{;1}, S_{;2})$ are defined as follows:

$$\nabla_j{}^h S = S_{,1} l_i + S_{,2} m_i, \qquad L(\dot{\partial}_i S) = S_{;1} l_i + S_{;2} m_i.$$

If $S(x,y)$ is positively homogeneous in y^i of degree r, then $S_{;1} = rS$. Since both I and R are of degree zero, we get $I_{;1} = R_{;1} = 0$.

Then the Douglas tensor $D_h{}^i{}_{jk}$ and the tensor K_{ij} of F^2 are written in the form

$$3LD_h{}^i{}_{jk} = -(6I_{,1} + \varepsilon I_{2;2} + 2II_2)m_h l^i m_j m_k,$$

$$K_{ij} = L(3R_{,2} - R_{;2,1})(l_i m_j - l_j m_i),$$

where $I_2 = I_{,1;2} + I_{,2}$. Therefore we have

Theorem B. *A two-dimensional Finsler space is projectively flat, if and only if the main scalar I and the Gauss curvature R saisfy the differential equations*

$$6I_{,1} + \varepsilon I_{2;2} + 2II_2 = 0, \qquad 3R_{,2} - R_{;2,1} = 0,$$

where $I_2 = I_{,1;2} + I_{,2}$.

Now we treat, in particular, of a locally Minkowski space of dimension two. Then $G_j{}^i{}_k = 0$ in an adapted coordinate system (x, y), and hence (1.3) is reduced to $y'' = 0$. Since a Finsler space which is projectively related to a locally Minkowski space is called projectively flat, we have

Theorem 1. *A differential equation $y'' = f(x,y,y')$ is reduced to $d^2\overline{y}/d\overline{x}^2 = 0$ by a change of variables $(x,y) \to (\overline{x},\overline{y})$ if and only if the associated Finsler space F^2 is projectively flat. Then $(\overline{x},\overline{y})$ is a rectilinear coordinate system of F^2.*

Remark. Such a Finsler space is characterized by Theorem B. Further the fundamental function $L(x,y;p,q)$ has been given ([10], [11]) as follows:

Theorem C. *The fundamental functions of all projectively flat Finsler spaces of dimension two are given, in a rectilinear coordinate system (x,y), by*

$$L(x,y;p,q) = p\int_0^z (z-t)H(t,y-tx)dt + E_x p + E_y q,$$

where $(p,q) = (\dot{x},\dot{y}), z = q/p$ and H, $E(x,y)$ are arbitrary functions.

Remark. See p. 55 of the book [2], published originaly in 1937, where the condition for $d^2y/dx^2 = \phi(x,y,dy/dx)$ to be reduced to the form $d^2y/dx^2 = 0$ is given in terms of the notion of dual equation.

3. Douglas spaces of dimension two

For a Berwald space the connection coefficients $G_j{}^i{}_k$ are functions of position (x^i) alone in any local coordinate system. Hence, in a two-dimensional

Berwald space the right-hand side $f(x, y, y')$ of (1.3) is a polynomial in y' of degree at most three. What is the necessary and sufficient condition for a two-dimensional Finsler space such that $f(x, y, y')$ be of the form above?

Suppose that f be of the form:

$$f(x, y, y') = g_3(y')^3 + g_2(y')^2 + g_1 y' + g_0,$$

with $g_k(x, y), k = 0, 1, 2, 3$. Then (1.2) can be written as

$$g_0 p^3 + g_1 p^2 q + g_2 p q^2 + g_3 q^3 = 2(G^1 \dot{y} - G^2 \dot{x}).$$

Consequently $G^1 q - G^2 p$ is a homogeneous polynomial in (p, q) of degree three. Thus we are led to the

Definition ([3]). A Finsler space of dimension n is said to *be of the Douglas type*, or called *a Douglas space*, if $D^{ij} = G^i y^j - G^j y^i$ are homogeneous polynomial in y^i of degree three.

Now we treat of the condition $\dot{\partial}_k \dot{\partial}_j \dot{\partial}_i \dot{\partial}_h D^{lm} = D^{lm}{}_{hijk} = 0$ for a Douglas space. It is observed that the contracted tensor $D^{lr}{}_{hijr}$ is written as

$$D^{ir}{}_{hijr} = (n+1) D_h{}^l{}_{ij},$$

where $D_h{}^l{}_{ij}$ is the Douglas tensor. Conversely $D^{lm}{}_{hijk}$ is written in terms of the Douglas tensor as

$$D^{lm}{}_{hijk} = (\dot{\partial}_k D_h{}^l{}_{ij}) y^m + \{D_i{}^l{}_{jk} \delta^m{}_h + (h, i, j, k)\} - [l, m],$$

where (h, i, j, k) denotes the cyclic permutation of these subscripts of the preceding terms. As a consequence the tensor $D^{lm}{}_{hijk}$ vanishes if and only if the Douglas tensor vanishes. Therefore we obtain the fundamental theorem [3] of the theory of Douglas spaces as follows:

Theorem D. *A Finsler space is a Douglas space, if and only if the Douglas tensor vanishes identically.*

In the two-dimensional case Theorem D may be stated in the words of the theory of ordinary differential equations as follows:

Theorem 2. *The right-hand side of a differential equation $y'' = f(x, y, y')$ is a polynomial in y' of degree at most three, if and only if the associated Finsler space of dimension two is a Douglas space.*

Remark. Since the notion of Douglas space is characterized by the tensor equation in virtue of Theorem D, the property of $f(x, y, y')$ in Theorem 2 is independent of the choice of variables (x, y).

Theorem A shows that the condition $D_h{}^i{}_{jk} = 0$ for a Douglas space is contained as a part in the condition for a Finsler space to be projectively flat. Therefore we have

Theorem E. *If a Finsler space F^n is projectively flat, then F^n is a Douglas space.*

Thus Theorem 1 gives rise to the following expression of Theorem E in the viewpoint of the theory of ordinary differential equations:

Theorem 3. *If a differential equation $y'' = f(x, y, y')$ can be reduced to $d^2\overline{y}/d\overline{x}^2 = 0$ by a change of variables $(x, y) \to (\overline{x}, \overline{y})$, then $f(x, y, y')$ must be a polynomial in y' of degree at most three.*

Remark. Theorem 3 coincides with Theorem 1 (p. 51) of Arnold's book [2]. It should be emphasized that Theorem 3 is only an analytical version of the geometrical Theorem E.

The remainder of the present section is devoted to studying Douglas spaces of dimension two. Let $F^2 = (\pi(x, y), L(x, y; p, q))$ be a two-dimensional Finsler space which is defined in a domain $\pi(x, y)$ of the (x, y)-plane and has the fundamental function $L(x, y; p, q)$. Since L is positively homogeneous in (p, q) of degree one, we can introduce

$$W(x, y; p, q) = L_{pp}/q^2 = -L_{pq}/pq = L_{qq}/p^2,$$

which is called the Weierstrass invariant, positively homogeneous in (p, q) of degree -3. Then the Euler equation $d(\dot{\partial}_i L)/dt - \partial_i L = 0$ of a geodesic can be written in the single equation

$$p\dot{q} - \dot{p}q + (L_{xq} - L_{yp})/W = O. \tag{3.1}$$

Hence (1, 1) shows $2(G^2 p - G^1 q) = (L_{xq} - L_{yp})/W$. Therefore we have

Theorem F. *A two-dimensional Finsler space is a Douglas space, if and only if $(L_{xq} - L_{yp})/W$ is a homogeneous polynomial in (p, q) of degree three.*

Example 1. Let us consider a two-dimensional Finsler space F^2 with the metric

$$L = (q - p)\log|z - 1| - (q + p)\log|z + 1| - 2xq, \qquad z = q/p.$$

we have $(L_{xq} - L_{yp})/W = p(p^2 - q^2)$. Thus F^2 is a Douglas space. The geodesics equation is $y'' = (y')^2 - 1$.

4. Another characterization of two-dimensional Douglas space

For $F^2 = (\pi(x, y), L(x, y; p, q))$ we introduce the *associated fundamental function* $A(x, y, z)$ of three variables by $A(x, y, z) = L(x, y; 1, z)$. Then we have the relation between L and A as follows: $L(x, y; p, q) = pA(x, y, q/p)$.

Putting $A' = \partial A/\partial z$, we get $L_{xq} - L_{yp} = A'_y z + A'_x - A_y$ and $W = A''/p^3$. Consequently (3.1) leads to the equation of geodesics in terms of $A(x, y, z)$ as follows:

$$A''y'' + A'_y y' + A'_x - A_y = 0, \qquad z = y'. \tag{4.1}$$

Now we consider the equation (1.3) of geodesics. From (4.1) it follows that

$$A''f + A'_y z + A'_x - A_y = 0 \tag{4.2.1}$$

must be identically satisfied by (x, y, z), where $f = f(x, y, z)$ and $A = A(x, y, z)$. Differentiate (4.2.1) successively by z. Putting $S = \log |A''|$ and $P = S_x + S_y z$, we obtain

$$S'f + f' + P = 0, \tag{4.2.2}$$

$$S''f + S'f' + f'' + P' = 0, \tag{4.2.3}$$

$$S'''f + 2S''f' + S'f'' + f''' + P'' = 0, \tag{4.2.4}$$

and finally

$$S^{iv}f + 3S'''f' + 3S''f'' + S'f''' + f^{iv} + P''' = 0. \tag{4.3}$$

Supose that F^2 under consideration be a Douglas space. Then $f^{iv} = 0$ from Theorem 2, and hence (4.3) is reduced to

$$S^{iv}f + 3S'''f' + 3S''f'' + S'f''' + P''' = 0. \tag{4.2.5}$$

Then the coefficients of $(f, f', f'', f''', 1)$ in these five equations (4.2) must satisfy

$$\triangle(A) = \begin{vmatrix} A'' & 0 & 0 & 0 & \delta(A) \\ S' & 1 & 0 & 0 & P \\ S'' & S' & 1 & 0 & P' \\ S''' & 2S'' & S' & 1 & P'' \\ S^{iv} & 3S''' & 3S'' & S' & P''' \end{vmatrix} = 0,$$

where $\delta(A) = A'_y z + A'_x - A_y$.

Theorem 4. *A two-dimensional Finsler space $(\pi(x,y), A(x,y,z))$ with the associated fundamental function $A(x,y,z)$ is a Douglas space, if and only if A satisfies $\triangle(A) = 0$, where $S = \log|A''|$, $P = S_x + S_y z$ and $\delta(A) = A'_y z + A'_x - A_y$.*

Proof. Only the sufficiency must be shown. From $\triangle(A) = 0$ it follows that the five equations

(1) $A''x_1 + (A'_y z + A'_x - A_y)x_5 = 0,$

(2) $S'x_1 + x_2 + Px_5 = 0,$

(3) $S''x_1 + S'x_2 + x_3 + P'x_5 = 0,$

(4) $S'''x_1 + 2S''x_2 + S'x_3 + x_4 + P''x_5 = 0,$

(5) $S^{iv}x_1 + 3S'''x_2 + 3S''x_3 + S'x_4 + P'''x_5 = 0,$

has a non-trivial solution $(x_1, \ldots, x_5)$. Suppose that $x_5 = 0$. Then (1) gives $x_1 = 0$ because of $A'' = pW \neq 0$. Hence (2) leads to $x_2 = 0$, (3) to $x_3 = 0$ and (4) to $x_4 = 0$, which is a contradiction. Thus we have non-zero x_5. Hence (1) and (4.2.1) lead to $f = x_1/x_5$. Then (2), comparing with (4.2.2), gives $f' = x_2/x_5$. Similarly we obtain $f'' = x_3/x_5$ and $f''' = x_4/x_5$. Consequently (5) gives (4.2.5), and, comparing with (4.3), $f^{iv} = 0$ is concluded. Therefore $f(x,y,z)$ is a polynomial in z of degree three. □

The condition $\Delta(A) = 0$ should coincide with the vanishing of the Douglas tensor in the two-dimensional case. In fact, $\triangle(A)$ is constructed from $A(x,y,z)$ by the differentiations one time with respect to (x,y) and six times with respect to z. On the other hand, the Douglas tensor is the set of components $D_i{}^h{}_{jk}$, constructed from $L(x^i, y^i)$ in the same way, that is, by the differentiations one time with respect to x^i and six times with respect to y^i.

Example 2. We treat of F^2 with

$$L(x,y;p,q) = 2p\log|q/p| + qu(x,y),$$

where $u(x,y)$ is a function of (x,y). We have $A(x,y,z) = 2\log|z| + zu$ and $S = \log 2 - 2\log|z|$, $P = 0$. Then $\Delta(A) = 0$ holds for any $u(x,y)$. Thus F^2 is a Douglas space. The geodesic equation is given by $2y'' = u_x(y')^2$.

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