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On Absolute continuity of the Gibbs measure under translations

By

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1. Introduction

The question of absolute continuity of measures under transformations of the spaces has been treated in many cases.

One typical example is the case of infinite product measures on $\mathbb{R}^\infty$: Kakutani [10] proved that two infinite product measures are either equivalent or mutually singular. Later, Shepp [23] showed that l^2 is the space of admissible shifts for infinite product measures on $\mathbb{R}^\infty$.

Such properties, dichotomy between equivalence and mutual singularity, and characterization of admissible shifts, were also shown in some different cases: in the case of Gaussian measures [4], [7], [9], and in the case of the Brownian motion measure and the pinned Brownian motion measure over the compact Lie group [1], [14], [22], [24].

In this paper, we treat Gibbs measures of unbounded lattice spin systems. We will show that each element of $l^2(\mathbb{Z}^d)$ is an admissible direction for the Gibbs measure if the second derivative of the potential satisfies a certain integrability condition. We will also show that, in one dimensional case, dichotomy between equivalence and mutual singularity holds for the Gibbs measure if the self-potential is uniformly convex. In this case, $l^2(\mathbb{Z})$ is the space of admissible shifts.

The organization of this paper is as follows. In Section 2, we fix some notation and collect known results about Gibbs measures which we need in the later sections. In Section 3, we show that each element of $l^2(\mathbb{Z}^d)$ is an admissible direction for the Gibbs measure. Lemma 3.2, which is based on (3.5), plays an essential role. We give an expression for the Radon-Nikodym derivative in terms of formal Hamiltonian which is defined reasonably. In Section 4, we treat one dimensional case. We show that, in the case of uniformly convex self-potentials, the Hellinger integral vanishes if the transformation is a shift by $h \notin l^2(\mathbb{Z})$.

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2. Notation and preliminaries

In this section, we fix notation, and recall some known results about Gibbs measures which we need in the later sections.

We set $\Omega := \mathbb{R}^{\mathbb{Z}^d}$. Let $\mathcal{C}$ be the class of all finite subsets of $\mathbb{Z}^d$.

For $\Lambda \subset \mathbb{Z}^d$, we denote by $\mathcal{F}_\Lambda$ the σ-algebra generated by the coordinate map $(x_i)_{i\in\mathbb{Z}^d} \mapsto x_i$ $(i \in \Lambda)$. We use the symbol $\mathcal{F}$ if $\Lambda = \mathbb{Z}^d$.

For $\Lambda \in \mathcal{C}$, we denote by dx_Λ the Lebesgues measure on $\mathbb{R}^\Lambda$. For $i \in \mathbb{Z}^d$, we set $|i| = \max_{1\le l\le d} |i_l|$.

The interaction for our lattice spin systems is given by a family $\Phi = (\Phi_\Lambda)_{\Lambda\in\mathcal{C}}$ of functions on Ω where

$$\Phi_\Lambda(x) = \begin{cases} V(x_i) & \text{if} \quad \Lambda = \{i\}, \\ W_{i,j}(x_i - x_j) & \text{if} \quad \Lambda = \{i,j\} \quad (i \ne j), \\ 0 & \quad \text{otherwise.} \end{cases}$$

Here, $V, W_{i,j} : \mathbb{R} \to \mathbb{R}$, and $W_{i,j} = W_{j,i} = W_{i-j}$ are even functions.

Throughout this paper, we suppose the following conditions on V and $W_{i,j}$:

(V.1) There exist $A > 0, C > 0$ such that

$$V(x) \ge 2Ax^2 - C.$$

(W.1) There exists a positive decreasing function J on $\mathbb{N}$ such that

$$J(r) \le Kr^{-d-\epsilon} \quad \text{for some} \quad K > 0, \epsilon > 0,$$

$$J_1 := \sum_{r\in\mathbb{Z}^d\setminus\{0\}} J(|r|) < A,$$

and

$$|W_{i,j}(x)| \le \frac{1}{4} J(|i-j|)x^2$$

holds.

Remark 2.1. We will use the convention $W_{i,i} = W_0 = 0$ and $J(0) = 0$.

Before introducing Gibbs measures for Φ, we recall the notion of the *regular measure*. For a probability measure μ on $(\Omega, \mathcal{F})$ and for $\Lambda \in \mathcal{C}$, we denote by $g_\Lambda(x|\mu)$ the probability density (with respect to dx_Λ) of the image measure of μ by the projection $\pi_\Lambda : \Omega \to \mathbb{R}^\Lambda$.

Definition 2.1. A probability measure μ on $(\Omega, \mathcal{F})$ is said to be regular if there exist $\bar{A} > 0$ and $\bar{\delta} > 0$ independent of $\Lambda \in \mathcal{C}$ such that

$$g_\Lambda(x|\mu) \le \exp\left(- \sum_{i\in\Lambda} (\bar{A}x_i^2 - \bar{\delta}) \right) \quad \text{for all} \quad \Lambda \in \mathcal{C}.$$

Next, we introduce the notion of the *tempered measure.* Set $\Lambda_l := \{i \in \mathbb{Z}^d; |i| \le l\}$ for $l \in \mathbb{N}$. Let us define $\mathcal{S}$, a subspace of Ω, by

$$\mathcal{S} = \bigcup_{N \in \mathbb{N}} \mathcal{S}_N,$$
$$\mathcal{S}_N = \left\{ x \in \Omega;\ \sup_{l \ge 1} \frac{1}{(2l+1)^d} \sum_{i \in \Lambda_l} x_i^2 \le N^2 \right\}.$$

We say a probability measure μ on $(\Omega, \mathcal{F})$ is *tempered* if $\mu(\mathcal{S}) = 1$. By the same argument in [19, Proposition 5.2], one can show that each regular measure is tempered.

For $x \in \mathcal{S}$ and $\Lambda \in \mathcal{C}$, the following sum

$$\begin{aligned} H_\Lambda(x) : &= \sum_{\Lambda'; \Lambda' \cap \Lambda \neq \emptyset} \Phi_{\Lambda'}(x) \\ &= \sum_{i \in \Lambda} V(x_i) + \frac{1}{2} \sum_{i,j \in \Lambda} W_{i,j}(x_i - x_j) + \sum_{\substack{i \in \Lambda \\ j \in \Lambda^c}} W_{i,j}(x_i - x_j) \end{aligned}$$

is absolutely convergent from (W.1), and then the partition function

$$Z_\Lambda(\omega) = \int e^{-H_\Lambda(x|\omega)} dx_\Lambda$$

is well-defined for all $\omega \in \mathcal{S}$. (We often use such notation as $H_\Lambda(x|\omega)$ if we consider H_Λ as a function on $\mathbb{R}^\Lambda \times \mathbb{R}^{\Lambda^c}$.)

H_Λ is naturally decomposed as $H_\Lambda = U_\Lambda + W_\Lambda$, where

$$U_\Lambda(x) := \sum_{i \in \Lambda} V(x_i) + \frac{1}{2} \sum_{i,j \in \Lambda} W_{i,j}(x_i - x_j)$$

and

$$W_\Lambda(x) := \sum_{\substack{i \in \Lambda \\ j \in \Lambda^c}} W_{i,j}(x_i - x_j).$$

Remark 2.2. From (V.1) and (W.1), U_Λ satisfies

$$U_\Lambda(x) \ge \sum_{i \in \Lambda} \left(\frac{3}{2} A x_i^2 - C \right) \tag{2.1}$$

which we call superstability. (See [19, Section 1].)

Moreover, for $\Lambda, \Delta \in \mathcal{C}$ with $\Lambda \cap \Delta = \emptyset$, by setting

$$W(x_\Lambda | x_\Delta) = \sum_{\substack{i \in \Lambda \\ j \in \Delta}} W_{i,j}(x_i - x_j),$$

we have

$$U_\Lambda(x) + W(x_\Lambda|x_\Delta) + \frac{3}{2}A\sum_{i\in\Delta} x_i^2 \geq \sum_{i\in\Lambda\cup\Delta}(Ax_i^2 - C) \tag{2.2}$$

which is just the same condition as [13, Hypothesis 4.1].

Now, we define a family $\gamma = (\gamma_\Lambda(\cdot|\omega))_{\Lambda\in\mathcal{C},\omega\in\Omega}$ of measures on Ω by

$$\gamma_\Lambda(A|\omega) = \begin{cases} \frac{1}{Z_\Lambda(\omega)}\int e^{-H_\Lambda(x|\omega)}1_A(x|\omega)dx_\Lambda & \text{if} \quad \omega\in\mathcal{S}, \\ 0 & \text{if} \quad \omega\in\Omega\setminus\mathcal{S}, \end{cases}$$

where $A\in\mathcal{F}$ and 1_A denotes the indicator function of A.

We now give a definition of Gibbs measures on $(\Omega,\mathcal{F})$.

Definition 2.2. A Gibbs measure μ for the potential Φ is a probability measure on $(\Omega,\mathcal{F})$ satisfying

$$\mu(A) = \int \gamma_\Lambda(A|\omega)\mu(d\omega) \quad \text{for all} \quad \Lambda\in\mathcal{C}. \tag{2.3}$$

(2.3) is usually called DLR equation.

Remark 2.3. By a similar reason which is described in the paragraph of [8, Definition (3.18)], the Gibbs measure in Definition 2.2 is tempered.

Remark 2.4. In Definition 2.2 above, we use the term ‘Gibbs measure’ as ‘tempered Gibbs measure’ in the sense of Ruelle [19].

The following theorem was shown in [12, Section 4 and appendix]. See also [6] and [17].

Theorem 2.1. *Let $\mathcal{G}_t(\Phi)$ denote the set of Gibbs measures for the potential Φ. If the condition* (V.1) *and* (W.1) *is satisfied, then $\mathcal{G}_t(\Phi)$ is not empty. And each $\mu\in\mathcal{G}_t(\Phi)$ is regular: more precisely, for each $\epsilon>0$, there exists $\tilde{\delta}>0$ such that*

$$g_\Lambda(x|\mu) \leq \exp\left\{-\sum_{i\in\Lambda}((A-\epsilon)x_i^2 - \tilde{\delta})\right\} \quad \textit{for all} \;\; \Lambda\in\mathcal{C}. \tag{2.4}$$

Remark 2.5. As for (2.4), we refer to [20, Theorem 2.2] and [12, Theorems 1.1 and 4.4].

The proof of Theorem 2.1 relies on Ruelle type estimate, which was proved in [19], [20] and [12]. See also [17].

3. Quasi-invariance

In this section, we will show that the Gibbs measure is quasi-invariant under the translation by square summable functions on $\mathbb{Z}^d$.

We keep the notation in the previous section. We denote by $l^2(\mathbb{Z}^d)$ the space of square summable functions on $\mathbb{Z}^d$:

$$l^2(\mathbb{Z}^d) = \Big\{ h = (h_i) \in \Omega ; |h|_2^2 := \sum_{i \in \mathbb{Z}^d} h_i^2 < \infty \Big\}.$$

For convenience, we set

$$\begin{aligned} U_n(x) &= U_{\Lambda_n}(x), \\ W_n(x|\omega) &= W_{\Lambda_n}(x|\omega), \\ Z_n(\omega) &= Z_{\Lambda_n}(\omega), \\ d\lambda_n &= dx_{\Lambda_n}, \\ \mathcal{F}_n &= \mathcal{F}_{\Lambda_n}. \end{aligned}$$

For a measure ν on $(\Omega, \mathcal{F})$ and $h \in \Omega$, we denote by ν^h the image measure of ν by the map $S_h : \Omega \to \Omega$ which is defined by $S_h(x) = x + h$. For a general measurable map $f : \Omega \to \Omega$, we use $f_* \mu$ to denote the image measure.

For two measures ν_1, ν_2 on a measurable space $(X, \mathcal{B})$, we denote by $H(\nu_2|\nu_1)$ the relative entropy of ν_2 with respect to ν_1:

$$H(\nu_2|\nu_1) = \begin{cases} \nu_{2,\omega} \Big(\log\big(\frac{d\nu_2}{d\nu_1}(\omega)\big)\Big) & \text{if} \quad \nu_2 \ll \nu_1, \\ \infty & \text{otherwise.} \end{cases}$$

Here, we use such notation as $\nu_{2,\omega}$ to denote the integration by the measure ν_2 in ω.

We now start the study of the quasi-invariance of a Gibbs measure. Let $\mu \in \mathcal{G}_t(\Phi)$. As was mentioned in Theorem 2.1, μ is regular. Moreover, from (2.4), it is easy to deduce that for each $a < A$, there exist $0 < M_a < \infty$ such that

$$\mu_x \bigg(\exp\Big(a \sum_{i \in \Lambda} x_i^2 \Big) \bigg) \le M_a^{|\Lambda|} \quad \text{for all} \quad \Lambda \in \mathcal{C}. \tag{3.1}$$

Below, we fix such M_a so that M_a is increasing in $a < A$.

Throughout this section, besides (V.1) and (W.1), we suppose on V and $W_{i,j}$ the following condition.

(V.2) V is C^2-class and

$$|V''(x)| \le K_1 e^{ax^2} \quad \text{for some} \quad K_1 > 0,\ 0 < a < \frac{1}{2}A.$$

(W.2) W is C^2-class and

$$|W''_{i,j}(x)| \le \frac{1}{2} J(|i-j|).$$

We begin by the case of 'finite dimensional translation'.

Let Ω_0 denote the following subspace of Ω:

$$\Omega_0 := \{\omega \in \Omega; \omega_i = 0 \quad \text{for all but finitely many} \quad i \in \mathbb{Z}^d\}.$$

Lemma 3.1. *For $h \in \Omega_0$, set $n := \max\{|i|; h(i) \neq 0\}$. Then, μ^h is absolutely continuous relative to μ. Moreover, the Radon-Nikodym derivative is given by*

$$\frac{d\mu^h}{d\mu}(x) = \exp(H_n(x) - H_n(x-h)). \tag{3.2}$$

Proof. (3.2) is an easy consequence of DLR equation (2.3) and well-known (see, e.g., [2, Theorem 4.4]). But we give a proof for convenience.

For each bounded measurable function f on Ω, we have

$$\begin{aligned}
\mu^h(f) &= \mu(f(x+h)) \\
&= \mu_\omega\biggl(\frac{1}{Z_n(\omega)}\int f(x+h|\omega)e^{-H_n(x|\omega)}d\lambda_n(x)\biggr) \\
&= \mu_\omega\biggl(\frac{1}{Z_n(\omega)}\int f(x|\omega)e^{-H_n(x-h|\omega)}d\lambda_n(x)\biggr) \\
&= \mu_\omega\biggl(\frac{1}{Z_n(\omega)}\int f(x|\omega)e^{\left(H_n(x|\omega)-H_n(x-h|\omega)\right)}e^{-H_n(x|\omega)}d\lambda_n(x)\biggr) \\
&= \mu\bigl(f(x)\exp(H_n(x) - H_n(x-h))\bigr).
\end{aligned}$$

The proof is completed. □

Next, we study the convergence of $H_n(x) - H_n(x-h)$. First, we note that, from (W.2),

$$\partial_i H_{\{i\}}(x_i|\omega) := \lim_{\epsilon \to 0} \frac{H_{\{i\}}(x_i + \epsilon|\omega) - H_{\{i\}}(x_i|\omega)}{\epsilon}$$

exists for each $\omega \in \mathcal{S}$.

The following lemma plays a crucial role in this section.

Lemma 3.2. *Define the linear map T from Ω_0 to $L^2(\Omega, \mu)$ by*

$$T(h) = \sum_{i \in \mathbb{Z}^d} h_i \partial_i H_{\{i\}}(x).$$

Then, there exists a constant $R > 0$ independent of $h \in \Omega_0$ such that

$$\|T(h)\|_2 \leq R|h|_2 \tag{3.3}$$

where $\|\cdot\|_2$ denotes the norm of $L^2(\Omega, \mu)$.

Proof. First, we note that

$$R' := \sup_{i\in\mathbb{Z}^d} \mu(|\partial_i H_{\{i\}}|^2) < \infty. \tag{3.4}$$

To show (3.4), we deduce from (V.2) and (W.2) that $|V'(x)| \le Ke^{ax^2}$ for some $K > 0$ and $|W'_{i,j}(x)| \le 1/2J(|i-j|)|x|$ respectively. By using these two inequality, we have (3.4) as follows:

$$\begin{aligned}
\mu(|\partial_i H_{\{i\}}|^2) &\le 2\mu(|V'(x_i)|^2) + 2\mu\left(\left|\sum_{j\neq i} W'_{i,j}(x_i - x_j)\right|^2\right)\\
&\le 2K^2\mu(e^{2ax_i^2}) + \frac{1}{2}\mu\left(\left(\sum_{j\neq i} J(|i-j|)(|x_i| + |x_j|)\right)^2\right)\\
&\le 2K^2 M_{2a} + \frac{J_1}{2}\mu\left(\sum_{j\neq i} J(|i-j|)(|x_i| + |x_j|)^2\right)\\
&= 2K^2 M_{2a} + J_1\mu\left(\sum_{j\neq i} J(|i-j|)\left(|x_i|^2 + |x_j|^2\right)\right)\\
&\le 2K^2 M_{2a} + 2J_1^2 \sup_i \mu\left(x_i^2\right).
\end{aligned}$$

Here, we have applied Schwarz inequality to $\left(\sum_{j\neq i} J(|i-j|)(|x_i| + |x_j|)\right)^2$ in the second line.

Second, we claim that, for $i \neq j$,

$$|\mu(\partial_i H_{\{i\}}\partial_j H_{\{j\}})| \le LJ(|i-j|) \tag{3.5}$$

where we set $L := \sup_{i,j} \mu(|x_i \partial_j H_{\{j\}}|)$. To show (3.5), according to the following decomposition of $\partial_i H_{\{i\}}$

$$\partial_i H_{\{i\}}(x) = \left(V'(x_i) + \sum_{k\neq i,j} W'_{i,k}(x_i - x_k)\right) + W'_{i,j}(x_i - x_j),$$

we write

$$\begin{aligned}
&\mu(\partial_i H_{\{i\}}\partial_j H_{\{j\}})\\
&\quad = \mu\left(\left(V'(x_i) + \sum_{k\neq i,j} W'_{i,k}(x_i - x_k)\right)\partial_j H_{\{j\}}\right) + \mu\left(W'_{i,j}(x_i - x_j)\partial_j H_{\{j\}}\right).
\end{aligned}$$

Since $V'(x_i) + \sum_{k \neq i,j} W'_{i,k}(x_i - x_k)$ does not depend on x_j, we can show that the first term vanishes. Indeed, by noting that

$$\begin{aligned}\int \partial_j H_{\{j\}}(x_j|\omega) e^{-H_{\{j\}}(x_j|\omega)} dx_j &= -\int \partial_j \left(e^{-H_{\{j\}}(x_j|\omega)} \right) dx_j \\ &= 0,\end{aligned}$$

we have

$$\begin{aligned}&\mu\left(\left(V'(x_i) + \sum_{k \neq i,j} W'_{i,k}(x_i - x_k)\right) \partial_j H_{\{j\}}\right) \\ &= \mu_\omega\left(\frac{1}{Z_n(\omega)} \int \left(V'(\omega_i) + \sum_{k \neq i,j} W'_{i,k}(\omega_i - \omega_k)\right) \partial_j H_{\{j\}}(x_j|\omega) e^{-H_{\{j\}}(x_j|\omega)} dx_j\right) \\ &= \mu_\omega\left(\frac{\left(V'(\omega_i) + \sum_{k \neq i,j} W'_{i,k}(\omega_i - \omega_k)\right)}{Z_n(\omega)} \int \partial_j H_{\{j\}}(x_j|\omega) e^{-H_{\{j\}}(x_j|\omega)} dx_j\right) \\ &= 0.\end{aligned}$$

As for the second term,

$$\begin{aligned}&|\mu(W'_{i,j}(x_i - x_j)\partial_j H_{\{j\}})| \\ &\quad \leq \frac{1}{2} J(|i-j|) \mu\left(|x_i - x_j||\partial_j H_{\{j\}}|\right) \\ &\quad \leq \frac{1}{2} J(|i-j|) \mu\left((|x_i| + |x_j|)|\partial_j H_{\{j\}}|\right) \\ &\quad \leq L J(|i-j|).\end{aligned}$$

We have obtained (3.5).

By combining (3.4) and (3.5), we have (3.3) as follows:

$$\begin{aligned}\mu\left(|T(h)|^2\right) &= \mu\left(\sum_{i,j \in \mathbb{Z}^d} h_i h_j \partial_i H_{\{i\}} \partial_j H_{\{j\}}\right) \\ &\leq \sum_{i,j \in \mathbb{Z}^d} \left|h_i h_j \mu\left(\partial_i H_{\{i\}} \partial_j H_{\{j\}}\right)\right| \\ &\leq \left(R' \sum_{i \in \mathbb{Z}^d} h_i^2 + L \sum_{\substack{i,j \in \mathbb{Z}^d \\ i \neq j}} |h_i h_j| J(|i-j|)\right) \\ &\leq R' \sum_{i \in \mathbb{Z}^d} h_i^2 + L \left(\sum_{\substack{i,j \in \mathbb{Z}^d \\ i \neq j}} h_i^2 J(|i-j|)\right)^{1/2} \left(\sum_{\substack{i,j \in \mathbb{Z}^d \\ i \neq j}} h_j^2 J(|i-j|)\right)^{1/2}\end{aligned}$$

$$\le (R' + LJ_1) \sum_{i \in \mathbb{Z}^d} h_i^2.$$

The proof is completed. □

For $h \in \Omega$ and $n \in \mathbb{N}$, we define $h^n \in \Omega_0$ by

$$h_j^n = \begin{cases} h_j & \text{if } |j| \le n, \\ 0 & \text{if } |j| > n. \end{cases}$$

Lemma 3.3. *For $h \in l^2(\mathbb{Z}^d)$, $H_n(x) - H_n(x - h^n)$ converges in $L^1(\Omega, \mu)$.*

Proof. By Taylor expansion up to the second order and noting that (W.2) ensures the absolute convergence for each $x \in \mathcal{S}$, we have

$$\begin{aligned}
&H_n(x) - H_n(x - h^n) \qquad (3.6)\\
&= \sum_{i \in \Lambda_n} h_i V'(x_i) + \frac{1}{2} \sum_{i,j \in \Lambda_n} (h_i - h_j) W'_{i,j}(x_i - x_j) \\
&\quad + \sum_{\substack{|i| \le n \\ |j| > n}} h_i W'_{i,j}(x_i - x_j) \\
&\quad - \sum_{i \in \Lambda_n} h_i^2 \int_0^1 \int_0^1 V''(x_i + tsh_i) tdtds \\
&\quad - \frac{1}{2} \sum_{i,j \in \Lambda_n} (h_i - h_j)^2 \int_0^1 \int_0^1 W''_{i,j}(x_i - x_j + ts(h_i - h_j)) tdtds \\
&\quad - \sum_{\substack{|i| \le n \\ |j| > n}} h_i^2 \int_0^1 \int_0^1 W''_{i,j}(x_i - x_j + tsh_i) tdtds \\
&= \sum_{i \in \Lambda_n} h_i \partial_i H_{\{i\}}(x) \\
&\quad - \sum_{i \in \Lambda_n} h_i^2 \int_0^1 \int_0^1 V''(x_i + tsh_i) tdtds \\
&\quad - \frac{1}{2} \sum_{i,j \in \Lambda_n} (h_i - h_j)^2 \int_0^1 \int_0^1 W''_{i,j}(x_i - x_j + ts(h_i - h_j)) tdtds \\
&\quad - \sum_{\substack{|i| \le n \\ |j| > n}} h_i^2 \int_0^1 \int_0^1 W''_{i,j}(x_i - x_j + tsh_i) tdtds.
\end{aligned}$$

By Lemma 3.2, the first term converges in $L^1(\Omega, \mu)$ as n goes to infinity. Since we have from (V.2)

$$\begin{aligned}
&\sum_{i\in\Lambda_n} h_i^2\mu\left(\left|\int_0^1\int_0^1 V^{''}(x_i+tsh_i)tdtds\right|\right)\\
&\quad\le \sum_{i\in\Lambda_n} h_i^2\mu\left(\int_0^1\int_0^1 e^{a(x_i+tsh_i)^2}tdtds\right)\\
&\quad\le \frac{1}{2}\sum_{i\in\Lambda_n} h_i^2\mu\left(e^{2ax_i^2+2ah_i^2}\right)\\
&\quad\le \frac{1}{2}\sum_{i\in\Lambda_n} h_i^2 e^{2ah_i^2}M_{2a}\\
&\quad\le \frac{1}{2}|h|_2^2 e^{2a|h|_2^2}M_{2a},
\end{aligned}$$

the second term also converges in $L^1(\Omega,\mu)$. As for the third term, we can show uniform absolute convergence as follows:

$$\begin{aligned}
&\frac{1}{2}\sum_{i,j\in\Lambda_n}(h_i-h_j)^2\left|\int_0^1\int_0^1 W_{i,j}^{''}(x_i-x_j+ts(h_i-h_j))\right|tdtds\\
&\quad\le \frac{1}{2}\sum_{i,j\in\Lambda_n}(h_i-h_j)^2\int_0^1\int_0^1\frac{1}{2}J(|i-j|)tdtds\\
&\quad\le \sum_{i,j\in\Lambda_n}\left(h_i^2+h_j^2\right)\frac{1}{4}J(|i-j|)\\
&\quad\le \frac{J_1}{2}|h|_2^2.
\end{aligned}$$

The fourth term converges to 0 uniformly as $n\to\infty$. To see this, note that

$$\begin{aligned}
&\sum_{i\ne j} h_i^2\left|\int_0^1\int_0^1 W_{i,j}^{''}(x_i-x_j+tsh_i)tdtds\right|\\
&\quad= \sum_{\substack{|i|\le n\\ |j|>n}} h_i^2\int_0^1\int_0^1\frac{1}{2}J(|i-j|)tdtds\\
&\quad\le \sum_{\substack{|i|\le n\\ |j|>n}} h_i^2\frac{1}{4}J(|i-j|)\\
&\quad\le \frac{J_1}{4}|h|_2^2<\infty.
\end{aligned}$$

The proof is completed. □

We denote by $H(x)-H(x-h)$ the above limit of $H_n(x)-H_n(x-h^n)$ in $L^1(\Omega,\mu)$.

Now, we close this section by the following theorem.

Theorem 3.1. *We suppose that the condition* (V.1), (V.2), (W.1) *and* (W.2) *are satisfied. Let $\mu\in\mathcal{G}_t(\Phi)$. Then, for each $h\in l^2(\mathbb{Z}^d)$, μ^h and μ are* *equivalent and its Radon-Nikodym derivative is given by*

$$\frac{d\mu^h}{d\mu}(x) = \exp(H(x) - H(x-h)). \tag{3.7}$$

Proof. We first show that $\{(d\mu^{h^n}/d\mu)(x)\}$ is a uniformly integrable sequence. Indeed, by Lemma 3.1, we may give a following expression for $H(\mu^{h^n}|\mu)$:

$$\begin{aligned} H\left(\mu^{h^n}|\mu\right) &= \mu^{h^n}(H_n(x) - H_n(x-h^n)) \\ &= \mu(H_n(x+h^n) - H_n(x)). \end{aligned}$$

By the same method in the proof of Lemma 3.3, we can show that $H_n(x + h^n) - H_n(x)$ converges in $L^1(\Omega, \mu)$. In particular

$$\sup_n H\left(\mu^{h^n}|\mu\right) < \infty. \tag{3.8}$$

(3.8) shows that $\{(d\mu^{h^n}/\,d\mu)(x)\}$ is a uniformly integrable sequence. Therefore, by combining this with Lemma 3.3, we obtain that $(d\mu^{h^n}/d\mu)(x)$ converges to $\exp(H(x)-H(x-h))$ in $L^1(\Omega, \mu)$. (See, e.g., [11, Theorem 1.2.8].) In particular, μ^h is absolutely continuous relative to μ and its Radon-Nikodym derivative is given by (3.7).

Conversely, by considering μ^h and μ as $(S_h)_*\mu$ and $(S_h)_*\mu^{-h}$ respectively, we have $\mu \ll \mu^h$. The proof is completed. □

4. Equivalence-singularity dichotomy

In this section, we treat one dimensional case. We will show that dichotomy between equivalence and singularity holds for Gibbs measures if the self-potential is uniformly convex.

We keep the notation of the previous sections. And besides (V.1), (V.2), (W.1) and (W.2), we suppose the following conditions:

(V.3) $$V''(x) \geq 2m > 0 \quad \text{for all} \quad x \in \mathbb{R}.$$

(W.3) $$J_1 < \frac{4}{3}m.$$

(W.4) $$J_2 := \sum_{n=1}^{\infty} nJ(n) < \infty.$$

Remark 4.1. Under the conditions above, the uniqueness of (tempered) Gibbs measure holds. For the uniqueness problem in one dimension, see [16].

The purpose of this section is to show the following theorem.

Theorem 4.1. *Let $d = 1$ and suppose* (V.1), (V.2), (V.3) *and* (W.1), (W.2), (W.3), (W.4). *Let μ be the unique Gibbs measure for the potential*

Φ. *Then, dichotomy between equivalence and singularity holds for* μ *in the following form*:

(1) *For each* $h \in l^2(\mathbb{Z})$, μ^h *and* μ *are equivalent.*

(2) *For each* $h \in \Omega \setminus l^2(\mathbb{Z})$, μ^h *and* μ *are mutually singular.*

For the proof of Theorem 4.1, we use the Hellinger integral, which we denote by ρ:

$$\rho(\nu^1, \nu^2) = \int \sqrt{\frac{d\nu^1}{d\nu^3}}\sqrt{\frac{d\nu^2}{d\nu^3}} d\nu^3$$

where, ν^1, ν^2, ν^3 are probability measures on Ω in relation that $\nu^1, \nu^2 \ll \nu^3$. It is well-known that this definition is independent of the choice of such ν^3. See, e.g. [25, Section 1.4].

We prepare some properties of ρ. For a probability measure ν on $(\Omega, \mathcal{F})$, we denote by ν_n the restriction of ν to $\mathcal{F}_n$, and by $\nu(X|\mathcal{F}_n)$ the conditional expectation of X with respect to $\mathcal{F}_n$.

Proposition 4.1. *The following properties holds for* ρ.

$$\rho(\nu^1, \nu^2) = \lim_{n\to\infty} \rho(\nu_n^1, \nu_n^2). \tag{4.1}$$

$$\rho(\nu^1, \nu^2) = 0 \quad \textit{is equivalent to} \quad \nu^1 \perp \nu^2. \tag{4.2}$$

Proof. Here we set

$$\alpha_n^p(x) := \frac{d\nu_n^p}{d\nu_n^3}(x), \quad \alpha^p(x) := \frac{d\nu^p}{d\nu^3}(x) \quad (p = 1, 2)$$

for short. First, as for (4.1), we note that

$$\begin{aligned} \rho(\nu_n^1, \nu_n^2) &= \int \sqrt{\alpha_n^1(x)}\sqrt{\alpha_n^2(x)} d\nu_n^3 \\ &= \int \sqrt{\alpha_n^1(x)}\sqrt{\alpha_n^2(x)} d\nu^3, \end{aligned}$$

where we regard $\alpha_n^p(x)$ as a function on $(\Omega, \mathcal{F}_n)$ in the first line and as on $(\Omega, \mathcal{F})$ in the second line. Since $\alpha_n^p = \nu^3(\alpha^p|\mathcal{F}_n)$, α_n^p converges to α^p in $L^1(\Omega, \nu^3)$. (See, e.g., [11, Proposition 2.2.4 and Theorem 2.6.6].) We have obtained (4.1).

As for (4.2), we refer to [25, Lemma 1.4.1]. □

To show Theorem 4.1, we need the following lemma.

Lemma 4.1. *Suppose the same condition in Theorem* 4.1, *and let* μ *be a Gibbs measure. Then,*

$$M := \sup_n \mu\left(\exp\left(\frac{1}{4}\sum_{\substack{|i|\le n\\ |j|>n}} J(|i-j|)\left(x_i^2 + x_j^2\right)\right)\right) < \infty. \tag{4.3}$$

Proof. The idea of the proof is the same as in [18, Lemma 7.1].
First, we note that

$$\mu\left(\exp\left(\frac{1}{4}\sum_{\substack{|i|\le n\\|j|>n}} J(|i-j|)\left(x_i^2+x_j^2\right)\right)\right)$$
$$\le \mu\left(\exp\left(\frac{1}{2}\sum_{\substack{|i|\le n\\|j|>n}} J(|i-j|)x_i^2\right)\right)^{1/2}\mu\left(\exp\left(\frac{1}{2}\sum_{\substack{|i|\le n\\|j|>n}} J(|i-j|)x_j^2\right)\right)^{1/2}.$$

We will show that

$$\sup_n \mu\left(\exp\left(\frac{1}{2}\sum_{\substack{|i|\le n\\|j|>n}} J(|i-j|)x_j^2\right)\right) < \infty. \tag{4.4}$$

As for the other term, we can show the same conclusion by the same method below. To show (4.4), we note that

$$\sum_{|i|\le n,|j|>n+N} J(|i-j|) \le 2\sum_{k=N+1}^{\infty} kJ(k) \tag{4.5}$$

holds for $n, N \in \mathbb{N}$. Indeed, we first rewrite the left hand side as

$$\begin{aligned}
\sum_{|i|\le n,|j|>n+N} J(|i-j|) &= \sum_{i=-n}^{n}\sum_{j>n+N} J(|i-j|) + \sum_{i=-n}^{n}\sum_{j<-(n+N)} J(|i-j|)\\
&= \sum_{i=-n}^{n}\sum_{j>n+N} J(|i+j|) + \sum_{i=-n}^{n}\sum_{j>n+N} J(|i+j|)\\
&= 2\sum_{i=0}^{2n}\sum_{j>n+N} J(i+j-n)\\
&= 2\sum_{i=0}^{2n}\sum_{j>N} J(i+j).
\end{aligned}$$

Then, we set $J^N(i) = J(i+N)$ and obtain (4.5) as follows:

$$\begin{aligned}\sum_{|i|\leq n,|j|>n+N} J(|i-j|) &= 2\sum_{i=0}^{2n}\sum_{j=1}^{\infty} J^N(i+j) \\ &\leq 2\sum_{i=0}^{\infty}\sum_{j=1}^{\infty} J^N(i+j) \\ &= 2\sum_{k=1}^{\infty}\sum_{i+j=k} J^N(i+j) \\ &= 2\sum_{k=1}^{\infty} kJ^N(k) \\ &= 2\sum_{k=1}^{\infty} kJ(k+N) \\ &= 2\sum_{k=N+1}^{\infty} (k-N)J(k) \\ &\leq 2\sum_{k=N+1}^{\infty} kJ(k).\end{aligned} \tag{4.6}$$

From (4.5) and (W.4), we can fix $N \in \mathbb{N}$ so that

$$J_3 := \sup_n J_3^{(n)} < \frac{A}{2} \quad \text{for all} \quad n \in \mathbb{N}, \tag{4.7}$$

where $J_3^{(n)} := \sum_{|i|\leq n,|j|>n+N} J(|i-j|)$. We devide $\sum_{\substack{|i|\leq n\\|j|>n}} J(|i-j|)x_j^2$ as

$$\sum_{\substack{|i|\leq n\\|j|>n}} J(|i-j|)x_j^2 = \sum_{\substack{|i|\leq n\\n<|j|\leq n+N}} J(|i-j|)x_j^2 + \sum_{\substack{|i|\leq n\\|j|>n+N}} J(|i-j|)x_j^2,$$

and then by using Schwarz inequality, we have

$$\begin{aligned}&\mu\left(\exp\left(\frac{1}{2}\sum_{\substack{|i|\leq n\\|j|>n}} J(|i-j|)x_j^2\right)\right) \\ &\leq \mu\left(\exp\left(\sum_{\substack{|i|\leq n\\n<|j|\leq n+N}} J(|i-j|)x_j^2\right)\right)^{1/2} \mu\left(\exp\left(\sum_{\substack{|i|\leq n\\|j|>n+N}} J(|i-j|)x_j^2\right)\right)^{1/2}.\end{aligned}$$

As for the first integral, from (3.1) and (W.1), we obtain

$$\mu\left(\exp\left(\sum_{\substack{|i|\le n\\ n<|j|\le n+N}} J(|i-j|)x_j^2\right)\right) \le \mu\left(\exp\left(J_1\sum_{n<|j|\le n+N} x_j^2\right)\right) \le M_{J_1}^{2N} < \infty. \tag{4.8}$$

For the second term, we can show that

$$\mu\left(\exp\left(\sum_{\substack{|i|\le n\\ j>n+N}} J(|i-j|)x_j^2\right)\right) \le M_{2J_3} < \infty.$$

To see this, set $a_j^{(n)} = \sum_{|i|\le n} J(|i-j|)$ and $b_j^{(n)} = a_j^{(n)}/J_3^{(n)}$. Then $b_j^{(n)}$ satisfies $\sum_{|j|>n+N} b_j^{(n)} = 1$. By using Hölder's inequality, we have

$$\begin{aligned}
&\mu\left(\exp\left(\sum_{|i|\le n,|j|>n+N} J(|i-j|)x_j^2\right)\right) \\
&= \mu\left(\exp\left(\sum_{|j|>n+N} b_j^{(n)} 2J_3^{(n)} x_j^2\right)\right) \\
&\le \prod_{|j|>n+N}\left(\int e^{2J_3^{(n)}x_j^2}\,d\mu(\omega)\right)^{b_j^{(n)}} \\
&\le \prod_{|j|>n+N} M_{2J_3^{(n)}}^{b_j^{(n)}} \le M_{2J_3}.
\end{aligned} \tag{4.9}$$

By combining (4.8) and (4.9), we obtain

$$\mu\left(\exp\left(\sum_{\substack{|i|\le n\\ |j|>n}} J(|i-j|)x_j^2\right)\right) \le M_{2J_3}^{1/2} M_{J_1}^{N}\ .$$

The proof is completed. □

We now give the proof of Theorem 4.1.

Proof of Theorem 4.1. It is sufficient to give a proof for the case of $h \in \Omega \setminus l^2(\mathbb{Z})$. And since $\mu^{2h} \perp \mu$ if and only if $\mu^h \perp \mu^{-h}$, we only have to show that

$$\lim_{n\to\infty} \rho(\mu_n^h, \mu_n^{-h}) = 0 \quad \text{for} \quad h \in \Omega \setminus l^2(\mathbb{Z}^d).$$

First, we note that $(\mu^h)_n = (\mu^{h^n})_n$ and use μ_n^h to denote this measure. And then, we deduce from (3.1) that μ_n^h is absolutely continuous relative to μ_n and the Radon-Nikodym derivative is

$$\begin{aligned}\frac{d\mu_n^h}{d\mu_n}(x) &= \mu\left(e^{(H_n(x)-H_n(x-h^n))}|\mathcal{F}_n\right) \\ &= e^{(U_n(x)-U_n(x-h))}\mu\left(e^{(W_n(x)-W_n(x-h^n))}|\mathcal{F}_n\right).\end{aligned} \tag{4.10}$$

From (W.2), we can show that

$$\begin{aligned}&\mu\left(e^{(W_n(x)-W_n(x-h^n))}|\mathcal{F}_n\right) \\ &\quad\le \exp\left(\frac{3}{4}\sum_{\substack{|i|\le n\\|j|>n}} J(|i-j|)h_i^2\right)\mu\left(\exp\left(\frac{1}{4}\sum_{\substack{|i|\le n\\|j|>n}} J(|i-j|)(x_i^2+x_j^2)\right)\middle|\mathcal{F}_n\right).\end{aligned} \tag{4.11}$$

To see this, it is sufficient to bound $W_{i,j}(x_i - x_j) - W_{i,j}(x_i - x_j - h_i)$ as follows:

$$\begin{aligned}&|W_{i,j}(x_i-x_j) - W_{i,j}(x_i-x_j-h_i)| \\ &\quad= \left|h_iW'_{i,j}(x_i-x_j) - h_i^2\int_0^1\int_0^1 W''_{i,j}(x_i-x_j-sth_i)tdtds\right| \\ &\quad\le \frac{1}{2}|h_i|J(|i-j|)(|x_i|+|x_j|) + \frac{1}{4}J(|i-j|)h_i^2 \\ &\quad= \frac{1}{2}J(|i-j|)|h_ix_i| + \frac{1}{2}J(|i-j|)|h_ix_j| + \frac{1}{4}J(|i-j|)h_i^2 \\ &\quad\le \frac{1}{4}J(|i-j|)(h_i^2+x_i^2) + \frac{1}{4}J(|i-j|)(h_i^2+x_j^2) + \frac{1}{4}J(|i-j|)h_i^2 \\ &\quad= \frac{3}{4}J(|i-j|)h_i^2 + \frac{1}{4}J(|i-j|)x_i^2 + \frac{1}{4}J(|i-j|)x_j^2.\end{aligned}$$

By combining (4.10) and (4.11), we have

$$\begin{aligned}&\left(\frac{d\mu_n^h}{d\mu_n}(x)\frac{d\mu_n^{-h}}{d\mu_n}(x)\right)^{1/2} \\ &\quad\le \exp\left\{-\frac{1}{2}\big(U_n(x+h)+U_n(x-h)-2U_n(x)\big)\right\}\exp\left(\frac{3}{4}\sum_{\substack{|i|\le n\\|j|>n}} J(|i-j|)h_i^2\right) \\ &\qquad\times\mu\left(\exp\left(\frac{1}{4}\sum_{\substack{|i|\le n\\|j|>n}} J(|i-j|)(x_i^2+x_j^2)\right)\middle|\mathcal{F}_n\right).\end{aligned}$$

And then, we obtain

$$
\begin{aligned}
&\rho(\mu_n^h, \mu_n^{-h}) \qquad (4.12)\\
&= \mu\left(\sqrt{\frac{d\mu_n^h}{d\mu_n}}\sqrt{\frac{d\mu_n^{-h}}{d\mu_n}}\right)\\
&\leq \exp\left(\frac{3}{4}\sum_{\substack{|i|\leq n\\|j|>n}} J(|i-j|)h_i^2\right)\mu\left(\exp\left\{-\frac{1}{2}(U_n(x+h)+U_n(x-h)-2U_n(x))\right\}\right.\\
&\qquad\left.\times\,\mu\left(\exp\left(\frac{1}{4}\sum_{\substack{|i|\leq n\\|j|>n}} J(|i-j|)(x_i^2+x_j^2)\right)\middle|\,\mathcal{F}_n\right)\right)\\
&= \exp\left(\frac{3}{4}\sum_{\substack{|i|\leq n\\|j|>n}} J(|i-j|)h_i^2\right)\mu\left(\exp\left\{-\frac{1}{2}(U_n(x+h)+U_n(x-h)-2U_n(x))\right\}\right.\\
&\qquad\left.\times\exp\left(\frac{1}{4}\sum_{\substack{|i|\leq n\\|j|>n}} J(|i-j|)(x_i^2+x_j^2)\right)\right).
\end{aligned}
$$

On the other hand, by using Taylor expansion and then from (V.3) and (W.2), we obtain

$$
\begin{aligned}
&U_n(x+h)+U_n(x-h)-2U_n(x) \qquad (4.13)\\
&\quad= \sum_{i\in\Lambda_n} h_i^2\int_0^1\int_0^1 \big(V^{''}(x_i+tsh_i)+V^{''}(x_i-tsh_i)\big)tdtds\\
&\qquad+\frac{1}{2}\sum_{i,j\in\Lambda_n}(h_i-h_j)^2\int_0^1\int_0^1 W_{i,j}^{''}(x_i-x_j+ts(h_i-h_j))tdtds\\
&\qquad+\frac{1}{2}\sum_{i,j\in\Lambda_n}(h_i-h_j)^2\int_0^1\int_0^1 W_{i,j}^{''}(x_i-x_j-ts(h_i-h_j)))tdtds\\
&\quad\geq \sum_{i\in\Lambda_n} h_i^2\int_0^1\int_0^1 4mtdtds-\frac{1}{2}\sum_{i,j\in\Lambda_n}(h_i-h_j)^2\int_0^1\int_0^1 J(|i-j|)tdtds\\
&\quad\geq 2m\sum_{i\in\Lambda_n} h_i^2-\frac{1}{2}\sum_{i,j\in\Lambda_n} J(|i-j|)(h_i^2+h_j^2).
\end{aligned}
$$

By combining (4.12) and (4.13), and then (4.3), we obtain

$$
\begin{aligned}
&\rho(\mu_n^h, \mu_n^{-h})\\
&\leq \exp\left(\frac{3}{4}\sum_{\substack{|i|\leq n\\|j|>n}} J(|i-j|)h_i^2\right)\exp\left(-m\sum_{i\in\Lambda_n} h_i^2\right)\exp\left(\frac{1}{2}\sum_{i,j\in\Lambda_n} J(|i-j|)h_i^2\right)
\end{aligned}
$$

$$\times \mu\left(\exp\left(\frac{1}{4}\sum_{\substack{|i|\le n\\|j|>n}} J(|i-j|)(x_i^2+x_j^2)\right)\right)$$
$$\le M\exp\left(\frac{3}{4}\sum_{\substack{|i|\le n\\j\in\mathbb{Z}^d}} J(|i-j|)h_i^2\right)\exp\left(-m\sum_{i\in\Lambda_n} h_i^2\right)$$
$$\le M\exp\left(\frac{3}{4}J_1\sum_{i\in\Lambda_n} h_i^2\right)\exp\left(-m\sum_{i\in\Lambda_n} h_i^2\right).$$

Since $m-3/4J_1>0$ from (W.3),

$$\begin{aligned}\rho(\mu^h,\mu^{-h}) &= \lim_{n\to\infty}\rho(\mu_n^h,\mu_n^{-h})\\ &= 0.\end{aligned}$$

The proof is completed. □

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