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Structure of tensor products of the defining representations of the Lie algebra W_1 of Cartan type

Dedicated to Professor Takeshi Hirai on his sixtieth birthday

By

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Introduction

Lie algebras of Cartan type are infinite-dimensional simple Lie algebras, and are realized as holomorphic vector fields with polynomial coefficients on a vector space over $\mathbb{C}$. Their $\mathbb{Z}$-gradings are naturally given. They were classified by E. Cartan into four series, the general series W_n, the special series S_n, the Hamiltonian series H_n, and the contact series K_n. Irreducible representations of Cartan type Lie algebras were studied by A. Rudakov and I. Kostrikin in 1970's. Rudakov [4] introduced the notion of the height of an irreducible representation. Representations of height greater than or equal to 1 are induced from some representations of the subalgebra which consists of all elements of degree 0. Kostrikin [1] showed that if V is an irreducible $\mathfrak{g}$-module of finite type for a Lie algebra of Cartan type $\mathfrak{g} \not\cong W_1$, then either V or its conjugate V^* is a $\mathfrak{g}$-module of height 1. He also determined all irreducible W_1-modules of finite type whose homogeneous components are equally one-dimensional (see Section 2).

However, since representations of W_n are not necessarily semisimple, it is not sufficient even if we have determined all the irreducible representations.

K. Nishiyama resumed the study on these subjects in the latter half of 1990's. The modules he treated was the m-fold tensor products of the defining representations, denoted as $\mathcal{P}_{(m)}$. He determined the commutant algebra $\mathfrak{C}_m$ of $\mathcal{P}_{(m)}$ in the case $m \leq n$. Then he studied Schur duality and constructed simultaneous decomposition of representations $\mathcal{P}_{(m)}$ as modules of the direct product of W_n and its commutant algebra $\mathfrak{C}_m$. But the case $m > n$ is more complicated. H. Wang extended the study to Cartan type Lie superalgebra $W(n)$, which is a "super-analogue" of Cartan type Lie algebra W_n ([5], [6]).

In this article, we treat the Lie algebra W_1, and give the complete structure of representations $\mathcal{P}_{(m)}$ for $m = 1$ and $m = 2$ of W_1. Our main theorem

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is Theorem 4.5, where we decompose the W_1-module $\mathcal{P}_{(2)}$ into two invariant subspaces and construct Jordan-Hölder composition series of them. The subquotient modules for them can be described in terms of standard irreducible modules $V(k)$ which are introduced in Definition 2.5.

Let us describe the contents of each section briefly. In the first section, we review basic notation and preliminary results. In the second section, we survey elementary properties of the modules $\mathcal{P}_{(m)} = \mathbb{C}[x_1, x_2, \ldots, x_m]$ $(m = 1, 2, \ldots)$ of the Lie algebra W_1. In Definition 2.5, we introduce a series of standard representations $V(k)$ $(k \in \mathbb{N} \cup \{0\})$ of W_1 on the spaces $\mathrm{Span}_{\mathbb{C}}\{v_j \mid j \geq k\}$ spanned by weight vectors v_j of weight j. Proposition 2.8 shows that, for any m, the tensor product representation $\mathcal{P}_{(m)}$ is composed of those modules $V(k)$. In the third and the fourth sections, we give structures of representations $\mathcal{P}_{(1)}$ and $\mathcal{P}_{(2)}$, respectively. The module $\mathcal{P}_{(1)}$ is isomorphic to $V(0)$. In Theorem 4.5 we give the complete structure of the 2-fold tensor product $\mathcal{P}_{(2)}$. The module $\mathcal{P}_{(2)}$ has a submodule $\mathcal{P}' = \mathcal{P}'_s \oplus \mathcal{P}'_a \cong V(0) \oplus V(1)$, which is generated by $\{x_1, x_2\}$. The quotient module is decomposed as $\mathcal{P}/\mathcal{P}' = \mathcal{P}_s^+ \oplus \mathcal{P}_a^+$ into two invariant subspaces, each of which has composition series of lowest weight modules respectively as

$$\mathcal{P}_s^{\pm}= \mathcal{P}_s^+[2] \supset \mathcal{P}_s^+[4] \supset \mathcal{P}_s^+[6] \supset \cdots \supset \mathcal{P}_s^+[2k] \supset \mathcal{P}_s^+[2k+2] \supset \cdots,$$
$$\mathcal{P}_a^{\pm}= \mathcal{P}_a^+[5] \supset \mathcal{P}_a^+[5] \supset \mathcal{P}_a^+[7] \supset \cdots \supset \mathcal{P}_a^+[2k+1] \supset \mathcal{P}_a^+[2k+3] \supset \cdots.$$

The quotient modules of composition series are isomorphic to $V(k)$ for some k, namely,

$$\mathcal{P}_s^+[2k]/\mathcal{P}_s^+[2k+2] \cong V(2k), \quad \mathcal{P}_a^+[2k+1]/\mathcal{P}_a^+[2k+3] \cong V(2k+1).$$

In the last section, we consider the action of the commutant algebra $\mathfrak{C}_2$ given by K. Nishiyama [2], on the representation spaces $\mathcal{P}_{(2)}$ and $\mathcal{P}_{(2)}^+$.

1. Notation and Preliminaries

1.1. Cartan type Lie algebra W_n

A $\mathbb{Z}$-graded Lie algebra is a Lie algebra $\mathfrak{g}$ endowed with a decomposition $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ with a property $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for every $i, j \in \mathbb{Z}$. A $\mathbb{Z}$-graded $\mathfrak{g}$-module is a $\mathfrak{g}$-module $\mathcal{V} = \oplus\mathcal{V}_j$ such that $\mathfrak{g}_i \cdot \mathcal{V}_j \subset \mathcal{V}_{i+j}$.

The general algebra $\mathfrak{g} = W_n$ of Cartan type consists of all $\mathbb{C}$-derivations of the polynomial algebra $P(n) = \mathbb{C}[z_1, z_2, \ldots, z_n]$. The algebra $P(n)$ has a natural grading as

$$P(n) = \bigoplus_{j=0}^{\infty} P_j,$$
$$P_j = P(n)_j = \{\, f(z) \in P(n) \mid \text{homogeneous polynomials of degree } j\,\},$$

where $z = (z_1, \ldots, z_n)$. Any element in $\mathfrak{g}$ has the form $v = \sum_{i=1}^{n} f_i(z)\partial/\partial z_i$, with $f_i(z) \in P(n)$. A $\mathbb{Z}$-grading of $\mathfrak{g}$ is given as

$$\mathfrak{g} = \bigoplus_{j=-1}^{\infty} \mathfrak{g}_j, \quad \mathfrak{g}_j = \left\{ \sum_{i=1}^{n} f_i(z) \frac{\partial}{\partial z_i} \,\middle|\, f_i(z) \in P(n)_{j+1} \right\}.$$

The algebra $\mathfrak{g}$ acts naturally on the polynomial algebra $P(n)$ with $\mathfrak{g}_i P_j \subset P_{i+j}$. It is a $\mathbb{Z}$-graded $\mathfrak{g}$-module with the natural grading of polynomials, and is called *the defining representation* of $\mathfrak{g}$.

Let us consider the m-fold tensor product $\mathcal{P}_{(m)} := \otimes^m P(n)$. The space $\mathcal{P}_{(m)}{}^+ := \otimes^m \big(P(n)/\mathbb{C}\big)$ is called the top level of $\mathcal{P}_{(m)}$, where $\mathbb{C}$ denotes the space of constant functions in $P(n)$.

The enveloping algebra of $\mathfrak{g} = W_n$ is denoted by $\mathfrak{U}(\mathfrak{g})$. In this article, the grading on a monomial $u = w_{d_1}^{j_1} \cdot w_{d_2}^{j_2} \cdot \cdots \cdot w_{d_k}^{j_k} \in \mathfrak{U}(\mathfrak{g})$, with $w_{d_l} \in \mathfrak{g}_{d_l}$ is defind as $\mathrm{d}(u) = j_1 d_1 + j_2 d_2 + \cdots + j_k d_k$. We can extend the definition of the grading to the linear span of monomials, and we call an element $u \in \mathfrak{U}(\mathfrak{g})_l$ a *homogeneous element* of degree l in $\mathfrak{U}(\mathfrak{g})$. A homogeneous element $u \in \mathfrak{U}(\mathfrak{g})_l$ acts homogeneously, that is, u maps a homogeneous component $\mathcal{V}_k$ into $\mathcal{V}_{k+l}$ of $\mathcal{V} = \oplus_j \mathcal{V}_j$.

Remark. The filtration usually defined on elements in $\mathfrak{U}(\mathfrak{g})$ for a Lie algebra $\mathfrak{g}$, which we touch on in Lemma 4.1, is different from that of the above definition of grading.

1.2. The symmetric group $\mathfrak{S}_m$ and a semigroup $\mathfrak{M}_m$

Let $\mathfrak{S}_m$ be the symmetric group of degree m. Denote the set of Young diagrams of size m as $\mathcal{Y}_m$. Define a Young tableau $B = B(D)$ on a given Young diagram $D \in \mathcal{Y}_m$ to be a numbering of the boxes by integers $1, 2, \ldots, m$. For such B, we define two subgroups of $\mathfrak{S}_m$ as

$$\begin{aligned} P_B &:= \{ g \in \mathfrak{S}_m \mid g \text{ preserves numbers in each row} \}, \\ Q_B &:= \{ g \in \mathfrak{S}_m \mid g \text{ preserves numbers in each column} \}, \end{aligned}$$

and an element in the group algebra $\mathbb{C}[\mathfrak{S}_m]$ as

$$c_B := \left(\sum_{g \in P_B} g \right) \left(\sum_{g \in Q_B} (\operatorname{sgn} g) g \right), \tag{1.1}$$

which is called a Young symmetrizer of $B(D)$. It is well-known that $\mathbb{C}[\mathfrak{S}_m] \cdot c_B$ is an irreducible $\mathfrak{S}_m$-module, and $\mathbb{C}[\mathfrak{S}_m] \cdot c_{B(D)} \cong \mathbb{C}[\mathfrak{S}_m] \cdot c_{B'(D')}$ as $\mathfrak{S}_m$-modules if and only if associated Young diagrams coincide, that is, $D = D'$. We denote the equivalence class of those modules by σ_D.

Let us return to the W_n-module $\mathcal{P}_{(m)}$. The group algebra $\mathbb{C}[\mathfrak{S}_m]$ acts naturally on $\mathcal{P}_{(m)}$ from the right. Since $\mathbb{C}[\mathfrak{S}_m]$ is semisimple and commutes with W_n, we have the following lemma.

Lemma 1.1. *Let $\mathcal{V}$ be a submodule of $\mathcal{P}_{(m)}$. Then $\mathcal{V}$ is decomposed into a sum of $W_n \times \mathfrak{S}_m$-invariant subspaces as*

$$\mathcal{V} = \sum_{D \in \mathcal{Y}_m}^{\oplus} \mathrm{Hom}_{\mathfrak{S}_m}(\sigma_D, \mathcal{V}) \otimes \sigma_D \tag{1.2}$$

with and graded W_n-modules $\mathrm{Hom}_{\mathfrak{S}_m}(\sigma_D, \mathcal{V})$.

Put $\mathfrak{M}_m := \{\varphi : [m] \longrightarrow [m]\}$, the semigroup of all maps from the set $[m] := \{1, 2, \dots, m\}$ into itself. Denote the semigroup ring of $\mathfrak{M}_m$ by $\mathbb{C}[\mathfrak{M}_m]$. The set of all invertible elements in $\mathfrak{M}_m$ coincides with the symmetric group $\mathfrak{S}_m$.

The semigroup ring $\mathbb{C}[\mathfrak{M}_m]$ acts on $\mathcal{P}_{(m)}$ faithfully and is contained in the commutant algebra $\mathfrak{C}_m$ of W_n-module $\mathcal{P}_{(m)}$. Moreover, if $m \leq n$, $\mathbb{C}[\mathfrak{M}_m]$ coincides with $\mathfrak{C}_m$. (See [2]).

Modules $\mathcal{V}$ should be decomposed as $W_n \times \mathfrak{M}_m$-modules. However, as $\mathfrak{M}_m$ is not necessarily semisimple, this decomposition is more complicated than the decomposition as $W_n \times \mathfrak{S}_m$-modules.

2. General structure of W_1-modules

Hereafter we always treat the algebra $\mathfrak{g} = W_1$:

$$\mathfrak{g} = \mathrm{Span}_{\mathbb{C}}\left\{ w_j = z^{j+1}\frac{\partial}{\partial z} \;\middle|\; j = -1,\, 0,\, 1,\, 2,\, 3,\, \cdots \right\},$$

as a vector space, where $\mathrm{Span}_{\mathbb{C}}\{I\}$ denotes the $\mathbb{C}$-linear space spanned by a set I of vectors. Bracket relations of w_j's are given by

$$[w_i, w_j] = (j - i)w_{i+j} \quad (i, j \in \mathbb{Z}_{\geq -1}).$$

Representation space treated here is $\mathcal{P}_{(m)}$, and it is identified with $\mathbb{C}[x_1, \dots, x_m]$ by the correspondence $z^{i_1} \otimes z^{i_2} \otimes \cdots \otimes z^{i_m} \mapsto x_1^{i_1} x_2^{i_2} \cdot \cdots \cdot x_m^{i_m}$. The action of w_j on $\mathcal{P}_{(m)}$ is

$$w_j \cdot f(x) = \sum_{i=1}^{m} x_i^{j+1} \frac{\partial}{\partial x_i} f(x).$$

Let

$$\mathfrak{N}(\mathfrak{g}) := \mathfrak{U}(\mathfrak{g}_+) \quad \text{with} \quad \mathfrak{g}_+ := \sum_{j=0}^{\infty} \mathfrak{g}_j.$$

A $\mathbb{Z}$-grading on homogeneous elements in $\mathfrak{N}(\mathfrak{g})$ are similarly defined as for $\mathfrak{U}(\mathfrak{g})$:

$$\mathfrak{N}(\mathfrak{g})_l = \mathfrak{N}(\mathfrak{g}) \cap \mathfrak{U}(\mathfrak{g})_l.$$

The $\mathfrak{N}(\mathfrak{g})$-module $\mathcal{P}_{(m)}$ is finitely generated (see [2], Lemma 3.5).

Lemma 2.1. *Let $\mathcal{V}$ be a $\mathbb{Z}$-graded $\mathfrak{g}$-module. Every $\mathfrak{N}(\mathfrak{g})$-submodule of $\mathcal{V}$ finitely generated by a set of homogeneous elements annihilated by w_{-1} is a $\mathfrak{U}(\mathfrak{g})$-module.*

Proof. Let $\mathcal{V}'$ be an $\mathfrak{N}(\mathfrak{g})$-submodule generated by $\{v_1, v_2, \dots, v_k\} \subset \operatorname{Ker} w_{-1}$. Every element $v' \in \mathcal{V}'$ can be written as $v' = \sum_i u_i v_i$ with $u_i \in \mathfrak{N}(\mathfrak{g})$. We know that $\mathfrak{U}(\mathfrak{g}) = \mathfrak{N}(\mathfrak{g}) + \mathfrak{U}(\mathfrak{g})w_{-1}$ by The PBW theorem. Then for u in $\mathfrak{U}(\mathfrak{g})$, there are elements $u_i' \in \mathfrak{N}(\mathfrak{g}), u_i'' \in \mathfrak{U}(\mathfrak{g})$ such that $uu_i = u_i' + u_i'' w_{-1}$ and we have

$$uv' = u\sum_i u_i v_i = \sum_i (u_i' + u_i'' w_{-1})v_i = \sum_i u_i' v \in \mathcal{V}'.$$

So $\mathcal{V}'$ is invariant under the actions of $\mathfrak{U}(\mathfrak{g})$. □

We call elements in $\operatorname{Ker} w_{-1}$ lowest weight vectors. A $\mathfrak{U}(\mathfrak{g})$-submodule generated by a lowest weight vector is called a lowest weight module.

Lemma 2.2. *Let $u \in \mathfrak{N}(\mathfrak{g})$ be a monomial. Then, the operator u is injective on $\mathcal{P}^+_{(m)}$.*

Proof. We prove this lemma only for the case $m = 2$. It is sufficient to prove the lemma when u is an element in $\mathfrak{g}$. Put an element in $\mathcal{P}_{(2)}$ of homogeneous degree q as $v = \sum_{0\le p\le q} a_p {x_1}^p {x_2}^{q-p}$ with coefficients $a_p \in \mathbb{C}$, then,

$$w_j v = \sum_{0\le p\le q+j} \{(p-j)a_{p-j} + (q-p)a_p\} {x_1}^p {x_2}^{q-p+j},$$

where $a_p = 0$ for $p < 0$ and $p > q$ by definition. Now we assume that $w_j v = 0$, that is,

$$(p-j)a_{p-j} + (q-p)a_p = 0 \quad \text{for} \quad 0 \le p \le q+j. \tag{2.1}$$

If we solve these equation successively, we get $a_p = 0$ for $0 \le p \le q$.

Thus we have $v = 0$ if $uv = 0$, and the operator u is proved to be an injection.

Proof for the case $m \ge 3$ is similar. □

Lemma 2.3. *Let $\mathcal{V}$ be a lowest weight $\mathfrak{g}$-module. If the lowest weight is not 0, then an operator w_{-1} on $\mathcal{V}$ is surjective.*

Proof. We set

$$\mathfrak{N}(\mathfrak{g})^{(r)} = \operatorname{Span}_{\mathbb{C}}\left\{ u = w_i^{r_i} w_{i-1}^{r_{i-1}} \cdots w_1^{r_1} \;\middle|\; r_1 + \cdots + r_i \le r \right\}.$$

Then the enveloping algebra $\mathfrak{N}(\mathfrak{g})$ has a natural filtration

$$\mathbb{C} = \mathfrak{N}(\mathfrak{g})^{(0)} \subset \mathfrak{N}(\mathfrak{g})^{(1)} \subset \mathfrak{N}(\mathfrak{g})^{(2)} \subset \cdots \subset \mathfrak{N}(\mathfrak{g}).$$

We will prove that

$$uv \in w_{-1}\mathfrak{N}(\mathfrak{g})v \quad \text{for } u \in \mathfrak{N}(\mathfrak{g})^{(l)} \text{ and } v \in \operatorname{Ker} w_{-1} \tag{2.2}$$

by induction on the degree l of this filtration.

When $l = 1$, the assertion is true because

$$(j+2)w_j v = (w_{-1}w_{j+1} - w_{j+1}w_{-1})v = w_{-1}w_{j+1}v \in w_{-1}\mathfrak{U}(\mathfrak{g})v,$$

where j is an integer greater than or equal to -1.

We assume that (2.2) is true for some l, that is,

$$\text{if } u \in \mathfrak{N}(\mathfrak{g})^{(l)} \text{ and } v \in \operatorname{Ker} w_{-1},\ uv = w_{-1}u'v, \quad \exists u' \in \mathfrak{N}(\mathfrak{g})^{(l)},$$

and let us prove that

$$w_i uv \in w_{-1}\mathfrak{N}(\mathfrak{g})^{(l+1)}v \ \text{ for } \ u \in \mathfrak{N}(\mathfrak{g})^{(l)} \tag{2.3}$$

by induction on $i \geq 0$. When $i = 0$, $w_0 w_j uv = c w_j uv$ for some $c \in \mathbb{C}$ because w_0 acts as a scalar. As the hypothesis of induction, we assume $w_{i-1}u'v \in w_{-1}\mathfrak{N}(\mathfrak{g})^{(l+1)}v$ for $u' \in \mathfrak{N}(\mathfrak{g})^{(l)}$, then we have

$$w_i uv = w_i w_{-1}u'v = (w_{-1}w_i - (i+1)w_{i-1})u'v \in w_{-1}\mathfrak{N}(\mathfrak{g})^{(l+1)}v.$$

Thus the assertion (2.3) for any i, and then the assertion (2.2) for any l are proved. □

Immediately, we have the following lemma.

Lemma 2.4. *Let $\mathcal{V}$ be a lowest weight $\mathfrak{g}$-module. then,*

$$\mathcal{V}_i = w_{-1}(\mathcal{V}_{i+1}), \quad \textit{and} \quad \dim \mathcal{V}_i \leq \dim \mathcal{V}_{i+1}. \tag{2.4}$$

Definition 2.5 (W_1-module $V(k)$). Let $k \in \mathbb{N} \cup \{0\}$ and $V = \operatorname{Span}_{\mathbb{C}}\{v_j \mid j \geq k,\ v_j \in V_j\}$ be a graded vector space with $V_j = \mathbb{C}v_j$. The relations

$$w_i v_j = (j + ik)v_{i+j} \quad (j \geq k,\ i \geq -1) \tag{2.5}$$

give a $\mathbb{Z}$-graded $\mathfrak{g}$-module structure on the space V, which we denote by $V(k)$.

Lemma 2.6. (1) *Let U be some $\mathbb{Z}$-graded $\mathfrak{g}$-module and $U = \oplus_{j=k}^{\infty} U_j$ as a vector space with $U_j = \mathbb{C}v_j$. If the conditions $w_0 u_j = j u_j$ for all $j \geq k$ and $w_{-1}u_k = 0$ hold, then U is isomorphic to $V(k)$ as a $\mathfrak{g}$-module.*

(2) *$\mathbb{Z}$-graded $\mathfrak{g}$-modules $V(k)$ and $V(l)$ are isomorphic if and only if $k = l$.*

Proof. Let V be a $\mathbb{Z}$-graded $\mathfrak{g}$-module with a basis $\{v'_j \in V_j \mid j \geq k\}$, where the elements w_0 and w_{-1} in $\mathfrak{g}$ act as $w_i v'_k = (i+1)k v'_{i+k}, w_0 v'_j = j v'_j$.

To show that $V \cong V(k)$, let us show

$$w_i v'_j = (j + ik)v'_{i+j} \quad \text{for } \ j \geq k, i \geq -1. \tag{2.6}$$

The action of

$$[w_{-1}, w_{j-k}] = (j-k+1)w_{j-k-1} = w_{-1}w_{j-k} - w_{j-k}w_{-1}$$

on v'_k determines the action of w_{-1} as $w_{-1}v'_j = (j-k)v'_{j-1}$ for $j > k$. If we put $w_i v'_j = c_{i,j}v'_{i+j}$ with $c_{i,j} \in \mathbb{C}$, then, the action of

$$[w_{-1}, w_i] = (i+1)w_{i-1} = w_{-1}w_i - w_i w_{-1}$$

on v'_j forces the following condition on $c_{i,j}$'s

$$(i+1)c_{i-1,j}\, v'_{i+j-1} = \{(j+i-k)c_{i,j} + (j-k)c_{i,j-1}\}\, v'_{i+j-1}.$$

If we solve this equation successively, and we get $c_{i,j} = j + ik$, and it concludes the equations (2.6). The isomorphisms $V(k) \cong V$ and $V \cong U$ are clear. □

Modules $V(k)$ are irreducible when $k \neq 0$. The module $V(0)$ has an invariant subspace $\mathbb{C}\cdot v_0$ and

$$V(0)/\mathbb{C}\cdot v_0 \cong V(1).$$

Remark. Kostrikin [1] introduced $\mathbb{Z}$-graded irreducible $\mathfrak{g}$-modules $V(\alpha, \beta)$ with equally one-dimensional homogeneous components, that is, $\dim V(\alpha,\beta)_j = 1$ for $-\infty < j < \infty$, where $\alpha, \beta \in \mathbb{C}$. Our module $V(k)$ $(k \geq 1)$ coincides with the maximal submodule of $V(0,k)$. Here $(\cdot)_k$ denotes the homogeneous subspace of degree k.

Lemma 2.7. *Let $\mathcal{V} \subset \mathcal{P}_{(m)}$ be a $\mathbb{Z}$-graded $\mathfrak{g}$-submodule generated by an element v_k in $\left(\mathcal{P}_{(m)}\right)_k \cap \operatorname{Ker} w_{-1}$. Then $\mathcal{V}$ has $V(k)$ as its quotient module.*

Proof. Let $\mathcal{V}'$ be the proper maximal submodule of $\mathcal{V}$. Applying Lemma 2.4 for a $\mathfrak{g}$-module $\mathcal{V}/\mathcal{V}'$, we have $\dim(\mathcal{V}/\mathcal{V}')_i \geq 1$ for $i \geq k$. Now, assume that there is an integer $i \geq k$ such that $\dim(\mathcal{V}/\mathcal{V}')_i = 1$ and $\dim(\mathcal{V}/\mathcal{V}')_{i+1} > 1$. Then, the fact $w_{-1}(\mathcal{V}/\mathcal{V}')_{i+1} \subset (\mathcal{V}/\mathcal{V}')_i$ means that $\mathcal{V}$ contains a lowest weight vector v', which is not in the submodule $\mathcal{V}'$. This vector v' generates another submodule $\mathcal{V}''$ of $\mathcal{V}$ and $\mathcal{V}'' + \mathcal{V}' \supsetneqq \mathcal{V}'$, $\mathcal{V}'' + \mathcal{V}' \not\ni v_k$. This contradicts the maximality of $\mathcal{V}'$, and we have $\dim\left(\mathcal{V}/\mathcal{V}'\right)_i = 1$ for all $i \geq k$. Therefore, Lemma 2.6 shows that $\mathcal{V}/\mathcal{V}' \cong V(k)$. □

Proposition 2.8. *Let $\mathcal{V}$ be a $\mathbb{Z}$-graded submodule of $\mathcal{P}_{(m)}$ or $\mathcal{P}^+_{(m)}$. Then $\mathcal{V}$ contains $V(k)$ as a subquotient module with multiplicity* $\dim \mathcal{V}_k - \dim \mathcal{V}_{k-1}$ *for any $k \in \mathbb{N}$.*

Proof. Apply the proof of the previous lemma to submodules of $\mathcal{V}$. □

3. The module $\mathcal{P}_{(1)}$ (the case $m = 1$)

First, we treat the case $m = 1$. Here, we have

$$\mathcal{P} = \mathcal{P}_{(1)} = \operatorname{Span}_{\mathbb{C}}\left\{v_j = z^j \mid j = 0, 1, \dots\right\},$$

and $\mathfrak{g}$ acts as $w_i v_j = j v_{i+j}$ $(j \geq 0)$. This space $\mathcal{P}_{(1)}$ of polynomials is nothing but $V(0)$ as a $\mathfrak{g}$-module.

The module $\mathcal{P}_{(1)}$ has an invariant subspace $\mathbb{C}\cdot v_0$ and the space of top level is given as

$$\mathcal{P}^+ = \mathcal{P}^+_{(1)} = \mathcal{P}_{(1)}/\mathbb{C}\cdot v_0 \simeq \mathrm{Span}_{\mathbb{C}}\{\tilde{v}_j = v_j + \mathbb{C}\cdot v_0 \,|\, j = 1, 2, \dots\},$$

and $\mathcal{P}^+_{(1)} \simeq V(1)$ as $\mathfrak{g}$-modules.

4. The module $\mathcal{P}_{(2)}$ (the case $m = 2$)

In this section, we treat the case $m = 2$. Here, $\mathcal{P} = \mathcal{P}_{(2)} = \otimes^2 P(1) \cong \mathbb{C}[x_1, x_2]$. Let $\mathcal{P}_s$ and $\mathcal{P}_a$ be the subspaces consisting of symmetric elements and anti-symmetric ones, respectively. By Lemma 1.1, we have the following isomorphism:

$$\mathcal{P}_s \cong \mathrm{Hom}_{\mathfrak{S}_2}(\sigma_{\square\!\square}, \mathcal{P}) \otimes \sigma_{\square\!\square}, \qquad \mathcal{P}_a \cong \mathrm{Hom}_{\mathfrak{S}_2}(\sigma_{\substack{\square\\\square}}, \mathcal{P}) \otimes \sigma_{\substack{\square\\\square}}$$

as $\mathfrak{g} \times \mathfrak{S}_2$-modules. The $\mathfrak{g}$-module $\mathcal{P}$ is decomposed as $\mathcal{P} = \mathcal{P}_s \oplus \mathcal{P}_a$, into two $\mathfrak{g}$-modules. The top level $\mathcal{P}^+ = \mathcal{P}^+_{(2)}$ is also decomposed as $\mathcal{P}^+ = \mathcal{P}_s^+ \oplus {\mathcal{P}_a}^+$. More precisely, put

$$\mathcal{P}'_s = \mathrm{Span}_{\mathbb{C}}\left\{x_1^i + x_2^i \,|\, i = 0, 1, \dots\right\}, \quad \mathcal{P}'_a = \mathrm{Span}_{\mathbb{C}}\left\{x_1^i - x_2^i \,|\, i = 1, 2, \dots\right\},$$

then, $\mathcal{P}_s^+ = \mathcal{P}_s/\mathcal{P}'_s$, ${\mathcal{P}_a}^+ = \mathcal{P}_a/\mathcal{P}'_a$ and $\mathcal{P}'_s \cong V(0)$, $\mathcal{P}'_a \cong V(1)$. Dimensions of weight subspaces of these modules are as follows:

weight	0	1	2	3	4	5	6	$\cdots$
$\mathcal{P}$	1	2	3	4	5	6	7	$\cdots$
$\mathcal{P}^+$	0	0	1	2	3	4	5	$\cdots$
$\mathcal{P}_s$	1	1	2	2	3	3	4	$\cdots$
$\mathcal{P}_a$	0	1	1	2	2	3	3	$\cdots$
$\mathcal{P}_s^+$	0	0	1	1	2	2	3	$\cdots$
$\mathcal{P}_a^+$	0	0	0	1	1	2	2	$\cdots$

4.1. A key lemma and its consequences

The following lemma is a key to our main theorem.

Lemma 4.1. *Let $k \geq 2$ be an integer and $v = \sum_{p=0}^{k} a_p {x_1}^p x_2^{k-p} \in \mathcal{P}$ $(a_p \in \mathbb{C})$ be an element of homogeneous degree k in $\mathrm{Ker}\, w_{-1}$. Put $a^l_{i,p} := a_{p-l+i} + a_{p-i}$ with $a_p = 0$ for $p < 0$ or $p > k$. Then,*

$$\mathfrak{N}(\mathfrak{g})_l v = \mathrm{Span}_{\mathbb{C}}\left\{ \sum_{p=0}^{k+l} a^l_{i,p} x_1^p x_2^{k+l-p} \;\middle|\; i = 0, 1, \dots, \left[\frac{l}{2}\right] \right\}. \tag{4.1}$$

Proof. Denote by $\mathcal{D}_{k,l}$ the space in the right hand side of (4.1). Then $\mathcal{D}_{k,l} = \mathrm{Span}_{\mathbb{C}}\left\{v_0^l, v_1^l, \ldots, v_{[l/2]}^l\right\}$ with $v_i^l = \sum_{p=0}^{k+l} a_{i,p}^l x_1^p x_2^{k+l-p}$, where $a_{i,p}^l = a_{p-l+i} + a_{p-i}$.

First we prove

$$(4.2) \qquad uv \in \mathrm{Span}_{\mathbb{C}}\left\{ \sum_{p=0}^{k+l} a_{i,p}^l x_1^p x_2^{k+l-p} \;\middle|\; i = 0, 1, \ldots, \left[\frac{l}{2}\right] \right\} = \mathcal{D}_{k,l}$$

for $u \in \mathfrak{N}(\mathfrak{g})_l \cap \mathfrak{N}(\mathfrak{g})^{(n)}$ by induction on n, the degree of filtration defined in the proof of Lemma 2.3.

When $u \in \mathfrak{N}(\mathfrak{g})^{(0)}$, i.e., $u = c \in \mathbb{C}$, the assertion holds clearly.

Assume that the assertion is true for some $n \in \mathbb{N} \cup \{0\}$, that is, if $u \in \mathfrak{N}(\mathfrak{g})_l \cap \mathfrak{N}(\mathfrak{g})^{(n)}$, there is a sequence $\{c_i \in \mathbb{C}\}$ for $0 \le i \le [l/2]$ such that uv is written as $uv = \sum_{i=0}^{[l/2]} c_i v_i^l$, with $v_i^l = \sum_{p=0}^{k+l} a_{i,p}^l x_1^p x_2^{k+l-p}$, $0 \le i \le [l/2]$, where $a_{i,p}^l = a_{p-l+i} + a_{p-i}$. Here v_i^l can be also defined for $[l/2] < i \le l$, and we have $v_i^l = v_{l-i}^l$ because $a_{i,p}^l = a_{l-i,p}^l$. The operator w_j acts on v_i^l as $x_1^{j+1}\partial/\partial x_1 + x_2^{j+1}\partial/\partial x_2$, so we have

$$(4.3) \qquad w_j v_i^l = \sum_p \{(p-j)a_{i,p-j}^l + (k-p+l)a_{i,p}^l\} x_1^p x_2^{k+l+j-p}.$$

Since $v \in \mathrm{Ker}\, w_{-1}$, we have

$$(4.4) \qquad (p+1)a_{p+1} + (k-p)a_p = 0 \quad \text{for } 0 \le p \le k-1.$$

This equality is valid for $p \in \mathbb{Z}$ because $a_p = 0$ for $p < 0$ and for $p > k$. Put $b_p := pa_p$, then it can be rewritten as

$$b_p - b_{p+1} = ka_p \quad (p \in \mathbb{Z}).$$

Consider the coefficient of $x_1^p x_2^{k+l+j-p}$ in (4.3):

$$\begin{aligned}
&(p-j)(a_{p-l+i-j} + a_{p-i-j}) + (k-p+l)(a_{p-l+i} + a_{p-i}) \\
&\quad = \{(p-l+i-j) + (l-i)\}a_{p-l+i-j} + \{(p-i-j) + i\}a_{p-i-j} \\
&\qquad + \{-(p-l+i) + (k+i)\}a_{p-l+i} + \{-(p-i) + (k+l-i)\}a_{p-i} \\
&\quad = (b_{p-l+i-j} - b_{p-l+i}) + ka_{p-l+i} + (b_{p-i-j} - b_{p-i}) + ka_{p-i} \\
&\qquad + (l-i)(a_{p-l+i-j} + a_{p-i}) + i(a_{p-i-j} + a_{p-l+i}) \\
&\quad = k(a_{p-l+i-j} + \cdots + a_{p-l+i-1}) + ka_{p-l+i} \\
&\qquad + k(a_{p-i-j} + \cdots + a_{p-i-1}) + ka_{p-i} \\
&\qquad + (l-i)a_{i,p}^{l+j} + ia_{i+j,p}^{l+j} \\
&\quad = k(a_{p-l+i-j} + \cdots + a_{p-l+i}) + k(a_{p-i} + \cdots + a_{p-i-j}) \\
&\qquad + (l-i)a_{i,p}^{l+j} + ia_{i+j,p}^{l+j} \\
&\quad = k(a_{i,p}^{l+j} + \cdots + a_{i+j,p}^{l+j}) + (l-i)a_{i,p}^{l+j} + ia_{i+j,p}^{l+j},
\end{aligned}$$

where $a^{l+j}_{i+p} := a_{p-l-j+i} + a_{p-i}$ for $0 \le i \le l+j$. Thus

$$w_j v^l_i = (k+l-i)v^{l+j}_i + k\left(v^{l+j}_{i+1} + \cdots + v^{l+j}_{i+j-1}\right) + (k+i)v^{l+j}_{i+j}, \tag{4.5}$$

and the assertion (4.2) for $u \in \mathfrak{N}(\mathfrak{g})^{(n+1)}$ is proved, and then, we get

$$\mathfrak{N}(\mathfrak{g})_l v \subset \mathcal{D}_{k,l}.$$

To get the converse inclusion, we prove

$$\dim \mathfrak{N}(\mathfrak{g})_l v \ge \left[\frac{l}{2}\right] + 1 \tag{4.6}$$

by induction on l. Of course it is true for $l = 0$. We assume that the assertion is true for numbers less than some l. From the equation (4.5) we have

$$w_1 v^l_i = (k+l-i)v^{l+1}_i + (k+i)v^{l+1}_{i+1} \quad \left(0 \le i \le \left[\frac{l}{2}\right]\right). \tag{4.7}$$

In the case where l is an even number, we have

$$w_1 v^l_{l/2} = \left(k + \frac{l}{2}\right) v^{l+1}_{l/2} + \left(k + \frac{l}{2}\right) v^{l+1}_{l/2+1} = (2k+l)v^{l+1}_{l/2}. \tag{4.8}$$

The latter equality of (4.8) is deduced from $v^l_i = v^l_{l-i}$. As $\{v_i \mid 0 \le i \le l/2\}$ are linearly independent by the hypothesis of induction, and $k + l - i \ne 0$ for $0 \le i \le l/2$ and $k > 0$, elements in the set $\{w_1 v^l_i \mid 0 \le i \le l/2\}$ are linearly independent.

In the case where l is an odd number, we have

$$\begin{aligned} w_1 v^l_{(l-1)/2} &= \frac{1}{2}(2k+l+1)v^{l+1}_{(l-1)/2} + \frac{1}{2}(2k+l-1)v^{l+1}_{(l+1)/2}, \\ w_2 v^{l-1}_{(l-1)/2} &= \frac{1}{2}(2k+l-1)v^{l+1}_{(l-1)/2} + kv^{l+1}_{(l+1)/2} + \frac{1}{2}(2k+l-1)v^{l+1}_{(l+3)/2} \\ &= (2k+l-1)v^{l+2}_{(l-1)/2} + kv^{l+2}_{(l+1)/2}. \end{aligned}$$

The condition $k > 0$, $l \ge 1$ leads the determinant of the coefficient matrix

$$-(2k-l-1)^2 + k(2k+l+1) = -k(4k+3l-5) - (l-1)^2 < 0,$$

and it shows that $w_2 v^{l-1}_{(l-1)/2}$ and $w_1 v^l_{(l-1)/2}$ are linearly independent. These facts lead that

$$v^{l+1}_i \in \mathrm{Span}_{\mathbb{C}}\left\{w_1 v^l_0,\, w_1 v^l_1,\, \ldots,\, w_1 v^l_{(l-1)/2},\, w_2 v^{l-1}_{(l-1)/2}\right\}$$

for $0 \le i \le (l+1)/2$. Altogether, the assertion (4.6) is true for $l+1$. Consequently we get the converse inclusion $\mathfrak{N}(\mathfrak{g})_l v \supset \mathcal{D}_{k,l}$. $\square$

Remark. The latter half of the above proof can also be carried out by using Lemma 2.4.

We have the following corollary.

Corollary 4.2. *Let $v \in \mathcal{P}$ be a homogeneous element annihilated by w_{-1}. If v is not constant as a polynomial, then,*

$$\dim \mathfrak{N}(\mathfrak{g})_l \, v = \left[\frac{l}{2}\right] + 1.$$

4.2. Main results for $\mathcal{P}_{(2)}$

Theorem 4.3. *Let $\mathfrak{g} = W_1$ and $\mathcal{P} = \mathcal{P}_{(2)}$.*
(1)

$$\dim (\mathcal{P}_s^+ \cap \operatorname{Ker} w_{-1})_i = \begin{cases} 1 & (i = 2, 4, 6, \cdots) \\ 0 & (i = 3, 5, 7, \cdots) \end{cases}. \tag{4.9}$$

(2) *Denote by v_{2i}^s $(i = 1, 2, \cdots)$ a non-zero element of $(\mathcal{P}_s^+)_{2i} \cap \operatorname{Ker} w_{-1}$ of dimension one. Denote by $\mathcal{P}_s^+[2i]$ the $\mathfrak{g}$-module generated by v_{2i}^s. Then,*

$$\dim \left(\mathcal{P}_s^+[2i]\right)_l = \left[\frac{l}{2} - i + 1\right]. \tag{4.10}$$

Proof. Applying the Corollary 4.2 to $v = v_2^s$, we get (4.10) in the case of $i = 1$. Equality (4.9) is clear in the case of $i = 2$, and we get (4.10) and (4.9) for all i inductively. $\square$

Similarly, we have the following result for $\mathcal{P}_a^+$.

Theorem 4.4. *Let $\mathfrak{g} = W_1$ and $\mathcal{P} = \mathcal{P}_{(2)}$.*
(1)

$$\dim (\mathcal{P}_a{}^+ \cap \operatorname{Ker} w_{-1})_i = \begin{cases} 1 & (i = 3, 5, 7, \cdots) \\ 0 & (i = 4, 6, 8, \cdots) \end{cases}. \tag{4.11}$$

(2) *Denote by $v_{2i+1}^a (i = 1, 2, \cdots)$ a non-zero element of $(\mathcal{P}_a^+)_{2i+1} \cap \operatorname{Ker} w_{-1}$ of dimension one. Denote by $\mathcal{P}_a^+[2i+1]$ the $\mathfrak{g}$-module generated by v_{2i+1}^a. Then,*

$$\dim \left(\mathcal{P}_a^+[2i+1]\right)_l = \left[\frac{l-1}{2} - i\right]. \tag{4.12}$$

Now the $\mathfrak{g}$-module structure of $\mathcal{P} = \mathcal{P}_{(2)}$ is completely clear. $\mathcal{P}$ is decomposed into two invariant indecomposable subspaces as $\mathcal{P} = \mathcal{P}_s \oplus \mathcal{P}_a$. They contains submodules $\mathcal{P}_s'$ and $\mathcal{P}_a'$ respectively:

$$\mathcal{P}_s' \cong V(0) \subset \mathcal{P}_s, \qquad \mathcal{P}_a{}' \cong V(1) \subset \mathcal{P}_a.$$

Here, $\mathcal{P}_s'$ has a one-dimensional submodule $\mathbb{C}$, and ${\mathcal{P}_a}'$ is irreducible. For $i \in \mathbb{N}$, $(\mathcal{P}_s^+)_{2i} = (\mathcal{P}_s/\mathcal{P}_s')_{2i}$ has an element v_{2i}^s killed by w_{-1}.

Finally, we have the following Jordan-Hölder series of submodules.

Theorem 4.5. *Let $\mathfrak{g} = W_1$ and $\mathcal{P} = \mathcal{P}_{(2)}$.*
(1) *The top level $\mathcal{P}_s^+ = \mathcal{P}_s/V(0)$ is an indecomposable $\mathfrak{g}$-module with*

$$\mathcal{P}_s^+ = \mathcal{P}_s^+[2] \supset \mathcal{P}_s^+[4] \supset \mathcal{P}_s^+[6] \supset \cdots \supset \mathcal{P}_s^+[2k] \supset \mathcal{P}_s^+[2k+2] \supset \cdots, \tag{4.13}$$

where, $\mathcal{P}_s^+[2i]$ is a $\mathbb{Z}$-graded $\mathfrak{g}$-module generated by an element $v_{2i}^s \in (\mathcal{P}_s^+)_{2i} \cap \operatorname{Ker} w_{-1}$. The composition factors are

$$\mathcal{P}_s^+[2i]/\mathcal{P}_s^+[2i+2] \cong V(2i) \qquad (i \geq 1). \tag{4.14}$$

(2) *The top level $\mathcal{P}_a^+ = \mathcal{P}_a/V(1)$ is an indecomposable $\mathfrak{g}$-module with*

$$\mathcal{P}_a^+ = \mathcal{P}_a^+[3] \supset \mathcal{P}_a^+[5] \supset \mathcal{P}_a^+[7] \supset \cdots \supset \mathcal{P}_a^+[2i+1] \supset \mathcal{P}_a^+[2i+3] \supset \cdots, \tag{4.15}$$

where $\mathcal{P}_a^+[2i+1]$ is a $\mathbb{Z}$-graded $\mathfrak{g}$-module generated by $v_{2i+1}^a \in (\mathcal{P}_a^+)_{2i+1} \cap \operatorname{Ker} w_{-1}$. The composition factors are

$$\mathcal{P}_a^+[2i+1]/\mathcal{P}_a^+[2i+3] \cong V(2i+1) \qquad (i \geq 1). \tag{4.16}$$

5. The commutant algebra $\mathfrak{C}_2$

The commutant algebra $\mathfrak{C}_2$ of the W_1-module $\mathcal{P}_{(2)}$ was studied in [2].
Let $D \in \operatorname{End}[\mathbb{C}[x_1, x_2]]$ be an operator on $\mathcal{P}_{(2)}$ defined as

$$D({x_1}^k {x_2}^l) = \frac{l{x_1}^{k+l} + k{x_2}^{k+l}}{k+l} \quad (k, l \geq 0, k+l > 0), \qquad D(1) = 1.$$

Then, the commutant algebra $\mathfrak{C}_2$ is generated by a semigroup ring $\mathbb{C}[\mathfrak{M}_2]$ and an operator D (cf. [2]). Here, $\mathfrak{M}_2$ consists of four elements, $1, \tau_1, \tau_2$ and σ, and 1 acts on $\mathcal{P}_{(2)}$ as the identity operator and

$$\tau_i(x_j) = x_i \ (i, j = 1, 2); \quad \sigma(x_1) = x_2,\ \sigma(x_2) = x_1.$$

Put $\tau_s = (\tau_1 + \tau_2)/2$, $\tau_a = (\tau_1 - \tau_2)/2$, then,

$$\tau_s({x_1}^k {x_2}^l) = \frac{1}{2}\left(\tau_1({x_1}^k {x_2}^l) + \tau_2({x_1}^k {x_2}^l)\right) = \frac{1}{2}({x_1}^{k+l} + {x_2}^{k+l}),$$
$$\tau_a({x_1}^k {x_2}^l) = \frac{1}{2}\left(\tau_1({x_1}^k {x_2}^l) - \tau_2({x_1}^k {x_2}^l)\right) = \frac{1}{2}({x_1}^{k+l} - {x_2}^{k+l}).$$

We have $\mathbb{C}[\mathfrak{M}_2] = \mathbb{C}[\mathfrak{M}_2']$ with $\mathfrak{M}_2' = \{1, \tau_s, \tau_a, \sigma\}$. All the elements in $\mathcal{P}_s$ and $\mathcal{P}_a$ are linear combinations of

$$x_s(k, l) := {x_1}^k {x_2}^l + {x_1}^l {x_2}^k \quad (k, l \geq 0),$$
$$x_a(k, l) := {x_1}^k {x_2}^l - {x_1}^l {x_2}^k \quad (k, l \geq 0, k+l > 0),$$

respectively. The upper part of the following table shows the action of $\mathfrak{M}_2'$ on these elements in $\mathcal{P}_s$ and $\mathcal{P}_a$.

The lower part of the table shows the action of elements of the commutant algebra $\mathfrak{C}_2$ on each representation space. Elements 1 and σ act as $\pm$ (identity). Elements τ_s and τ_a kill the whole spaces $\mathcal{P}_a$ and $\mathcal{P}^+$. Elements τ_s and D map $\mathcal{P}_s$ into its submodule $\mathcal{P}_s'$, and act as identities on $\mathcal{P}_s'$. Element τ_a maps $\mathcal{P}_s$ into $\mathcal{P}_a'$.

	1	σ	τ_s	τ_a	D
$x_s(k,l)$	$x_s(k,l)$	$x_s(k,l)$	$x_s(k+l,0)$	$x_a(k+l,0)$	$x_s(k+l,0)$
$x_a(k,l)$	$x_a(k,l)$	$-x_a(k,l)$	0	0	$\frac{l-k}{l+k}\cdot x_a(k+l,0)$
$\mathcal{P}_s$	1	1	$\mathcal{P}_s'$	$\mathcal{P}_a'$	$\mathcal{P}_s'$
$\mathcal{P}_s'$	1	1	1	$\mathcal{P}_a'$	1
$\mathcal{P}_s^+$	1	-1	0	0	0
$\mathcal{P}_a$	1	-1	0	0	$\mathcal{P}_a'$
$\mathcal{P}_a'$	1	-1	0	0	-1
$\mathcal{P}_a^+$	1	-1	0	0	0

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