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Fold-maps and the space of base point preserving maps of spheres

By

Yoshifumi ANDO*

Abstract

Let $f : N \to P$ be a smooth map between n-dimensional oriented manifolds which has only fold singularities. Such a map is called a fold-map. For a connected closed oriented manifold P, we shall define a fold-cobordism class of a fold-map into P of degree m under a certain cobordism equivalence. Let $\Omega_{fold,m}(P)$ denote the set of all fold-cobordism classes of fold-maps into P of degree m. Let F^m denote the space $\lim_{k\to\infty} F_k^m$, where F_k^m denotes the space of all base point preserving maps of degree m of S^{k-1}. In this paper we shall prove that there exists a surjection of $\Omega_{fold,m}(P)$ to the set of homotopy classes $[P, F^m]$, which induces many fold-cobordism invariants.

Introduction

Let N and P be smooth (C^∞) manifolds of dimension n. We shall say that a smooth map germ of (N, x) into (P, y) has a singularity of fold type at x if it is written as $(x_1, \ldots, x_{n-1}, x_n) \mapsto (x_1, \ldots, x_{n-1}, x_n^2)$ under suitable local coordinate systems on neighborhoods of $x \in N$ and $y \in P$ respectively. A smooth map $f : N \to P$ is called a *fold-map* if it has only fold singularities. In [E] Èliašberg has proved a certain "homotopy principle" (a terminology used in [G2]) for fold-maps. Let TN and $f^*(TP)$ be stably equivalent for a given map $f : N \to P$ and let an $(n-1)$-dimensional submanifold M of N be given. As an application he has given the conditions so that there is a fold-map which is homotopic to f and folds on M. For example, for any homotopy sphere of dimension n, there exists a fold-map into S^n of degree 1. These results are the motivation for the following problems. Given a connected closed oriented manifold P, consider a fold-map $f : N \to P$ of degree 1. What properties of a fold-map f represent the procedure of changing the differentiable structure of P to that of N? How is a classification of fold-maps into P together with the singularities of f related to a classification of source manifolds N? These

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problems have been studied in [An3]. This paper is its continuation and we shall study this problem in a more general situation.

Let P be a connected closed oriented smooth manifold of dimension n. For the study of this problem we shall define a fold-cobordism class of a fold-map of degree m. Namely, let $f_i : N_i \to P$ $(i = 0, 1)$ be two fold-maps of degree m, where N_i are closed oriented smooth manifolds of dimension n. We shall say that they are *fold-cobordant* when there exists a fold-map $F : (W, \partial W) \to (P \times [0,1], P \times 0 \cup P \times 1)$ of degree m such that

(i) W is oriented with $\partial W = N_0 \cup (-N_1)$ and the collar of ∂W is identified with $N_0 \times [0, \varepsilon) \cup N_1 \times (1-\varepsilon, 1]$,

(ii) $F|N_0 \times [0,\varepsilon) = f_0 \times id_{[0,\varepsilon)}$ and $F|N_1 \times (1-\varepsilon, 1] = f_1 \times id_{(1-\varepsilon,1]}$,

where ε is a sufficiently small positive number. Let $\Omega_{fold,m}(P)$ denote the set of all fold-cobordism classes of fold-maps into P of degree m.

Let F_k^m denote the space of all base point preserving maps of degree m of S^{k-1} with compact-open topology. The suspension induces the inclusion $F_k^m \to F_{k+1}^m$. Let F^m denote the space $\lim_{k\to\infty} F_k^m$. Let G_k (resp. SG_k) denote the space of all homotopy equivalences (resp. of degree 1) of S^{k-1} with compact-open topology. The suspension of a homotopy equivalence yields the inclusion $G_k \to G_{k+1}$ (resp. $SG_k \to SG_{k+1}$). We set $G = \lim_{k\to\infty} G_k$ and $SG = \lim_{k\to\infty} SG_k$ respectively. Similarly set $O = \lim_{k\to\infty} O(k)$. By considering the quotient space $G_k/O(k)$ by the action of $O(k)$ on G_k, set $G/O = \lim_{k\to\infty} G_k/O(k)$. Then we have the projection $p_{SG} : SG \to G/O$. It is well known that each F^m is weakly homotopy equivalent to SG.

The main result of this paper is the following theorem.

Theorem 1. *Let P be a connected closed oriented smooth manifold of dimension n. Then there exists a surjection $\omega_m : \Omega_{fold,m}(P) \to [P, F^m]$ for $n \geq 1$.*

Let π_n^s denote the n-th stable homotopy group of spheres, $\lim_{k\to\infty} \pi_{n+k}(S^k)$. It is known that $[S^n, F^0]$ is isomorphic to π_n^s (see, for example, [At1]). Then we have the following corollary.

Corollary 2. *There exists a surjection $\Omega_{fold,0}(S^n) \to \pi_n^s$ induced from ω_0 for $n \geq 1$.*

For example, the fold-map $S^1 \to S^1$ mapping $e^{\sqrt{-1}x} \mapsto e^{\sqrt{-1}\cos ax}$, $a \in \mathbf{Z}$, is mapped to $0 \in \pi_1^s \cong \mathbf{Z}/2\mathbf{Z}$ for odd integers a and to $1 \in \pi_1^s \cong \mathbf{Z}/2\mathbf{Z}$ for even integers $a \neq 0$ (see Proposition 5.3).

Now we recall a smooth structure on P, which refers to a homotopy equivalence $f : N \to P$ of degree 1, and the surgery obstruction in the surgery theory developed by [K-M], [Br2], [Su] and [W2]. We will say that two smooth structures on P, $f_i : N_i \to P$ $(i = 0, 1)$, are equivalent if there is a diffeomorphism $d : N_0 \to N_1$ such that f_0 is homotopic to $f_1 \circ d$. Let $\mathcal{S}(P)$ denote the set of all equivalence classes of smooth structures on P. Then there has been defined a map $\eta_n : \mathcal{S}(P) \to [P, G/O]$. Let $i_{F^1,SG} : F^1 \to SG$ be the inclusion. Then it will turn out that $(i_{F^1,SG})_* \circ \omega_1$ coincides with $\omega : \Omega_{fold,1}(P) \to [P, SG]$

defined in [An3]. As for smooth structures on P we have the following theorem (see [An3, Section 5 and Theorem 5.5]).

Theorem 3 ([An3]). *Let $n \geq 5$. Let P be a connected closed oriented smooth manifold of dimension n. If a fold-map $f : N \to P$ is a homotopy equivalence of degree 1, then we have that $(p_{SG})_* \circ \omega(f) = \eta_n(f)$.*

Furthermore, the surgery obstructions induce fold-cobordism invariants through the composition with $(p_{SG})_* \circ \omega$ ([An3, Proposition 5.1]). In particular, if P is of dimension $4k+2$ $(k \geq 1)$, then we have the Kervaire invariant $\theta_{4k+2} : [P, G/O] \to \mathbf{Z}/2\mathbf{Z}$.

Theorem 4. *Let P be a closed oriented and simply connected smooth manifold of dimension $4k+2$ $(k \geq 1)$. Then the surgery obstruction of Kervaire invariant θ_{4k+2} induces a fold-cobordism invariant $\theta_{4k+2} \circ (p_{SG})_* \circ \omega : \Omega_{fold,1}(P) \to \mathbf{Z}/2\mathbf{Z}$. In particular, if $P = S^{4k+2}$ and $k = 1,3,7$, then this invariant is not trivial.*

The latter half of Theorem 4 is a direct consequence of the results due to several authors that $\theta_{4k+2} \circ (p_{SG})_*$ for $P = S^{4k+2}$ is surjective for $k = 1, 3, 7$ (see [Br1, Corollary 1]).

Theorem 1 will make the following corollary important, in which we define other fold-cobordism invariants. As for the (co)homology groups of the space F^m, namely, SG, consult [M], [M-M] and [Tsu].

Corollary 5. *Let p be a prime number. For an element $[f]$ of $\Omega_{fold,m}(P)$, we have the homomorphism $\omega_m(f)^* : H^*(F^m; \mathbf{Z}/p\mathbf{Z}) \to H^*(P; \mathbf{Z}/p\mathbf{Z})$. Then for any element a of $H^*(F^m; \mathbf{Z}/p\mathbf{Z})$, $\omega_m(f)^*(a)$ is a fold-cobordism invariant.*

Now we shall explain the *homotopy principle* for fold-maps, which is necessary for the proof of Theorem 1. In the 2-jet space $J^2(n,n)$ we shall consider the subspace $\Omega^{10}(n,n)$ consisting of all jets of either regular germs or germs with fold singularities at the origin. In the 2-jet bundle $J^2(N,P)$ with projection $\pi^2_N : J^2(N,P) \to N$, let $\Omega^{10}(N,P)$ be its subbundle associated with $\Omega^{10}(n,n)$. A smooth map $f : N \to P$ is a fold-map if and only if the image of $j^2 f$ is contained in $\Omega^{10}(N,P)$. Let $C^\infty_\Omega(N,P)$ denote the space consisting of all fold-maps with C^∞-topology. Let $\Gamma(N,P)$ denote the space consisting of all continuous sections of the fibre bundle $\pi^2_N|\Omega^{10}(N,P) : \Omega^{10}(N,P) \to N$ with compact-open topology. Then there exists a continuous map

$$j_\Omega : C^\infty_\Omega(N,P) \to \Gamma(N,P)$$

defined by $j_\Omega(f) = j^2 f$.

We shall prove the following homotopy principle in the existence level in Section 4, where two theorems [G1, 4.1.1 Theorem] and [E, 2.2 Theorem] will play important roles. In the following theorem the manifolds N and P may not be closed or oriented.

Theorem 6. *Let $n \geq 2$. Let N and P be connected smooth manifolds of dimension n and $\partial N = \emptyset$. For any continuous section s in $\Gamma(N,P)$, there exists a fold-map $f : N \to P$ such that $j^2 f$ and s are homotopic as sections.*

In Section 1 we shall explain the well known results concerning fold singularities. In Section 2 we shall prove several results concerning Thom spaces and duality in the suspension category (see [Sp1], [Sp2] and [W1]). In Section 3 we shall review the results of [An3] and define the map ω_m. In Section 4 we shall state Propositions 4.6 and 4.7 without proofs and prove Theorem 6. In Section 5 we shall prove Theorem 1 by using Theorem 6 and give some examples. In Sections 6 and 7 we shall prove Propositions 4.6 and 4.7 respectively. In Section 8 we shall give another invariant of fold-maps, say fold-degree in $\mathbf{Z}$, which is not a fold-cobordism invariant. In odd dimensions, we shall show that many integers can be realized as fold-degrees.

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1. Preliminaries

Throughout the paper all manifolds are smooth of class C^∞. Maps are basically continuous, but may be smooth (of class C^∞) if so stated. Given a fibre bundle $\pi : E \to X$ and a subset C in X, we denote $\pi^{-1}(C)$ by $E|_C$. Let $\pi' : F \to Y$ be another fibre bundle. A map $\tilde{b} : E \to F$ is called a fibre map over a map $b : X \to Y$ if $\pi' \circ \tilde{b} = b \circ \pi$ holds. The restriction $\tilde{b}|(E|_C) : E|_C \to F$ (or $F|_{b(C)}$) is denoted by $\tilde{b}|_C$. In particular, for a point $x \in X$, $E|_x$ and $\tilde{b}|_x$ are denoted by E_x and $\tilde{b}_x : E_x \to F_{b(x)}$ respectively.

We shall review the well known results about fold singularities (see [Bo], [L1]). Let $J^k(N,P)$ denote the k-jet space of manifolds N and P. Let π_N^k and π_P^k be the projections mapping a jet to its source and target respectively. The map $\pi_N^k \times \pi_P^k : J^k(N,P) \to N \times P$ induces a structure of fibre bundle with structure group $L^k(n) \times L^k(n)$, where $L^k(n)$ denotes the group of all k-jets of local diffeomorphisms of $(\mathbf{R}^n, 0)$. The fibre $(\pi_N^k \times \pi_P^k)^{-1}(x,y)$ is denoted by $J^k_{x,y}(N,P)$.

Let $\pi_1^2 : J^2(N,P) \to J^1(N,P)$ be the canonical forgetting map. Let $\Sigma^i(N,P)$ denote the submanifold of $J^1(N,P)$ consisting of all 1-jets $z = j_x^1 f$ such that the kernel of $d_x f$ is of dimension i. We denote $(\pi_1^2)^{-1}(\Sigma^i(N,P))$ by the same symbol $\Sigma^i(N,P)$ if there is no confusion. For a 2-jet $z = j_x^2 f$ of $\Sigma^i(N,P)$, there has been defined the second intrinsic derivative $d_x^2 f : T_x N \to \mathrm{Hom}(\mathrm{Ker}(d_x f), \mathrm{Cok}(d_x f))$. Let $\Sigma^{ij}(N,P)$ denote the subbundle of $J^2(N,P)$ consisting of all jets $z = j_x^2 f$ such that $\dim(\mathrm{Ker}(d_x f)) = i$ and $\dim(\mathrm{Ker}(d_x^2 f| \mathrm{Ker}(d_x f))) = j$. In this paper we shall deal with these submanifolds only for $j \leq i \leq 1$. A jet of $\Sigma^{10}(N,P)$ will be called a fold jet. Let $\Omega^{10}(N,P)$ denote the union of $\Sigma^0(N,P)$ and $\Sigma^{10}(N,P)$ in $J^2(N,P)$. Then $\pi_N^2 \times \pi_P^2 | \Omega^{10}(N,P)$ induces a structure of an open subbundle of $\pi_N^2 \times \pi_P^2$. Let $\Omega^{10}(n,n) = \Omega^{10}(\mathbf{R}^n, \mathbf{R}^n) \cap J^2_{0,0}(\mathbf{R}^n, \mathbf{R}^n)$.

In particular, there exists a canonical diffeomorphism

$$\pi^2_{\mathbf{R}^n} \times \pi^2_{\mathbf{R}^n} \times \pi_\Omega : \Omega^{10}(\mathbf{R}^n, \mathbf{R}^n) \to \mathbf{R}^n \times \mathbf{R}^n \times \Omega^{10}(n, n).$$

Here, for a jet $z = j^2_x f \in \Omega^{10}(\mathbf{R}^n, \mathbf{R}^n)$, π_Ω is defined by $\pi_\Omega(z) = j^2_0(l(-f(x)) \circ f \circ l(x))$, where $l(a)$ denotes the parallel translation defined by $l(a)(x) = x + a$. We note that $J^2_{0,0}(\mathbf{R}^n, \mathbf{R}^n)$ is canonically identified with $\mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n) \oplus \mathrm{Hom}(S^2\mathbf{R}^n, \mathbf{R}^n)$ under the canonical basis of $\mathbf{R}^n$, where $S^2\mathbf{R}^n$ is the 2-fold symmetric product of $\mathbf{R}^n$.

Next we shall review the properties of the submanifolds $\Sigma^1(N, P)$ and $\Sigma^{10}(N, P)$ along the line of [Bo, Section 7]. Let $\mathbf{D}'$ denote the induced bundle $(\pi^2_N)^*(TN)$ over $J^2(N, P)$. Recall the homomorphism

$$\mathbf{d}^1 : \mathbf{D}' \longrightarrow (\pi^2_P)^*(TP) \qquad \text{over} \quad J^2(N, P),$$

which maps an element $\mathbf{v} = (z, \mathbf{v}') \in \mathbf{D}'_z$ with $z = j^2_x f$ to $(z, d_x f(\mathbf{v}'))$. There is a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Ker}(d_x f) & \longrightarrow & T_x N & \xrightarrow{d_x f} & f^*(TP)_x = (x, T_{f(x)}P) & \longrightarrow & \mathrm{Cok}(d_x f) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Ker}(\mathbf{d}^1_z) & \longrightarrow & \mathbf{D}'_z & \xrightarrow{\mathbf{d}^1_z} & (\pi^2_P)^*(TP)_z & \longrightarrow & \mathrm{Cok}(\mathbf{d}^1_z).
\end{array}$$

Here $\mathbf{d}^1$ is identified with a section of $\mathrm{Hom}(\mathbf{D}', (\pi^2_P)^*(TP))$ over $J^2(N, P)$. Let $\mathbf{K}$ and $\mathbf{Q}$ be the kernel bundle and the cokernel bundle of $\mathbf{d}^1$ over $\Sigma^1(N, P)$ with $\dim \mathbf{K} = \dim \mathbf{Q} = 1$ respectively. Next we have the second intrinsic derivative $\mathbf{d}^2 : \mathbf{K} \to \mathrm{Hom}(\mathbf{K}, \mathbf{Q})$ over $\Sigma^1(N, P)$, whose restriction $\mathbf{d}^2_z : \mathbf{K}_z \to \mathrm{Hom}(\mathbf{K}_z, \mathbf{Q}_z)$ with $z \in \Sigma^1(N, P)$ is nothing but the homomorphism induced from $d^2_x f : \mathrm{Ker}(d_x f) \to \mathrm{Hom}(\mathrm{Ker}(d_x f), \mathrm{Cok}(d_x f))$ by $(\pi^2_N)^*$ and $(\pi^2_P)^*$. This map is extended to the following epimorphism by [Bo, Lemma 7.4 and p. 412],

$$\mathbf{d}^2 : T(J^2(N, P))|_{\Sigma^1(N,P)} \to \mathrm{Hom}(\mathbf{K}, \mathbf{Q}) \qquad \text{over} \quad \Sigma^1(N, P),$$

where $\mathbf{D}'$ is identified with a subbundle of $T(J^2(N, P))$ corresponding to the total tangent bundle of $J^\infty(N, P)$. It has been proved in [Bo, Lemma 7.13] that there exists an exact sequence,

$$0 \longrightarrow T(\Sigma^1(N, P)) \xrightarrow{\subset} T(J^2(N, P))|_{\Sigma^1(N,P)} \xrightarrow{\mathbf{d}^2} \mathrm{Hom}(\mathbf{K}, \mathbf{Q}) \longrightarrow 0.$$

Under these notations, a 2-jet $z \in \Sigma^1(N, P)$ lies in $\Sigma^{10}(N, P)$ if and only if $\mathbf{d}^2|\mathbf{K}_z$ is an isomorphism (otherwise, z lies in $\Sigma^{11}(N, P)$). This implies that $T(\Sigma^1(N, P))_z \cap \mathbf{K}_z = \{0\}$ for any jet $z \in \Sigma^{10}(N, P)$. Hence $\mathbf{K}|_{\Sigma^{10}(N,P)}$ and $\mathrm{Hom}(\mathbf{K}, \mathbf{Q})|_{\Sigma^{10}(N,P)}$ are isomorphic to the normal bundle of $\Sigma^{10}(N, P)$ in $J^2(N, P)$.

Boardman [Bo] has first done these constructions over the infinite jet space $J^\infty(N, P)$. In particular, there has been defined the total tangent bundle $\mathbf{D}$ over $J^\infty(N, P)$, which is canonically identified with $(\pi^\infty_N)^*(TN)$. It does not seem so simple to explain how to define the extended epimorphism $\mathbf{d}^2$ and how to regard $\mathbf{K}$ as the subbundle of the tangent bundle $T(J^2(N,P))$ from the comment given in [Bo, p. 412]. The following interpretation is different from this comment. We need Riemannian metrics on N and P, which enable us to consider the exponential maps $TN \to N$ and $TP \to P$ by the Levi-Civita connections. For any points $x \in N$ and $y \in P$, we have the local coordinates $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ on convex neighborhoods of x and y associated to orthonormal basis of T_xN and T_yP respectively (see [K-N]). We shall define an embedding $\mu^2_\infty : J^2(N,P) \to J^\infty(N,P)$. Let $z \in J^2_{x,y}(N,P)$ be represented by a C^∞ map germ $f : (N,x) \to (P,y)$ such that any k-th derivative of f with $k \geq 3$ vanishes under these coordinates. Then we set $\mu^2_\infty(z) = j^\infty_x f$. It is clear that $\pi^\infty_2 \circ \mu^2_\infty = id_{J^2(N,P)}$. We can prove that $\mathbf{D}|_{\mu^2_\infty(J^2(N,P))}$ is tangent to $\mu^2_\infty(J^2(N,P))$. Indeed, for $\sigma = (\sigma_1, \ldots, \sigma_n)$ with non-negative integers σ_i, we recall the functions X_i and $Z_{j,\sigma}$ with $1 \leq i \leq n$ and $1 \leq j \leq n$ defined locally on a neighborhood of $J^\infty(N,P)$ by, for $z = j^\infty_x f$,

$$X_i(z) = x_i,$$
$$Z_{j,\sigma}(z) = \frac{\partial^{|\sigma|}(y_j \circ f)}{\partial x_1^{\sigma_1} \cdots \partial x_n^{\sigma_n}}(x),$$

which constitute the local coordinates on $J^\infty(N,P)$ as described in [Bo, Section 1]. Let Φ be a smooth function defined locally on $\mu^2_\infty(J^2(N,P))$ and let $D_i \in \mathbf{D}$ be the total tangent vector corresponding to $\partial/\partial x_i$ by the canonical identification of $\mathbf{D}$ and $(\pi^\infty_N)^*(TN)$. Then we have by [Bo, (1.8)] that

$$\begin{aligned} D_i(\Phi)(z) &= \frac{\partial(\Phi \circ j^\infty f)}{\partial x_i}(x) \\ &= \frac{\partial \Phi}{\partial X_i}(z) + \sum_{j,\sigma} \frac{\partial \Phi}{\partial Z_{j,\sigma}}(z) Z_{j,\sigma'}(z), \end{aligned}$$

where $\sigma' = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_i + 1, \sigma_{i+1}, \ldots, \sigma_n)$. If $z \in \mu^2_\infty(J^2(N,P))$, then $Z_{j,\sigma}(z)$ vanishes for $|\sigma| \geq 3$. Hence, $D_i(\Phi)$ is a smooth function defined locally on $\mu^2_\infty(J^2(N,P))$. This implies that D_i is tangent to $\mu^2_\infty(J^2(N,P))$. Since $\mathbf{D}_z$ consists of all linear combinations of $D_1, \ldots, D_n$, we have that $\mathbf{D}_z \subset T_z(\mu^2_\infty(J^2(N,P)))$.

Let $\mathbf{d}^{1,\infty} : \mathbf{D}|_{\mu^2_\infty(J^2(N,P))} \to (\pi^\infty_P)^*(TP)|_{\mu^2_\infty(J^2(N,P))}$ be the first derivative over $\mu^2_\infty(J^2(N,P))$. Let $\mathbf{K}^\infty$ and $\mathbf{Q}^\infty$ be the kernel bundle and the cokernel bundle of $\mathbf{d}^{1,\infty}$ over $\mu^2_\infty(\Sigma^1(N,P))$. Now we consider the intrinsic derivative $d(\mathbf{d}^{1,\infty}) : T(\mu^2_\infty(J^2(N,P)))|_{\mu^2_\infty(\Sigma^1(N,P))} \to \mathrm{Hom}(\mathbf{K}^\infty, \mathbf{Q}^\infty)$ of $\mathbf{d}^{1,\infty}$ (see the definition of the intrinsic derivative in [Bo, Lemma 7.4] due to I. R. Porteous). Then it induces the homomorphism $(\mu^2_\infty)^*(d(\mathbf{d}^{1,\infty})) : T(J^2(N,P))|_{\Sigma^1(N,P)} \to \mathrm{Hom}(\mathbf{K}, \mathbf{Q})$. It is clear that the restriction $(\mu^2_\infty)^*(d(\mathbf{d}^{1,\infty}))|(\mu^2_\infty)^*(\mathbf{D}) : (\mu^2_\infty)^*(\mathbf{D}) \to \mathrm{Hom}(\mathbf{K}, \mathbf{Q})$ is identified with $\mathbf{d}^2 : \mathbf{D}'|_{\Sigma^1(N,P)} \to \mathrm{Hom}(\mathbf{K}, \mathbf{Q})$, which is invariantly defined with respect to the choice of metrics on N and P, through the identification of $\mathbf{D}$ and $(\pi^\infty_N)^*(TN)$.

A smooth map $f : N \to P$ is called a fold-map when the image of $j^2 f$ is contained in $\Omega^{10}(N,P)$. Let $C^\infty_\Omega(N,P)$ and $\Gamma(N,P)$ denote the spaces defined in Introduction with the continuous map

$$j_\Omega : C^\infty_\Omega(N, P) \to \Gamma(N, P).$$

Let $\Gamma^{tr}(N, P)$ denote the subspace of $\Gamma(N, P)$ consisting of all sections s such that s is smooth on some neighborhood of $s^{-1}(\Sigma^{10}(N, P))$ and that s is transverse to $\Sigma^{10}(N, P)$. Throughout the paper $S(s)$ denotes $s^{-1}(\Sigma^{10}(N, P))$. From now on, for a point $c \in S(s)$, let $K(s)_c$ and $Q(s)_c$ denote $s^*(\mathbf{K})_c = \mathrm{Ker}(d_c f)$ and $s^*(\mathbf{Q})_c = \mathrm{Cok}(d_c f)$ respectively, where $s(c) = j^2_c f$. Let $d^1(s) : TN \to s^*(TP)$ and $d^2(s) : K(s) \to \mathrm{Hom}(K(s), Q(s))$ over $S(s)$ denote the homomorphisms induced from $\mathbf{d}^1$ and $\mathbf{d}^2$ by s respectively.

A homotopy c_λ with $\lambda \in [0, 1]$ refers to a continuous map c of $I = [0, 1]$ into a space. For example, a homotopy h_λ relative to a closed subset C of N in $\Gamma(N, P)$ refers to a continuous map $h : I \to \Gamma(N, P)$ such that $h_\lambda|C = h_0|C$ for any λ.

2. Thom spaces and duality in suspension category

In this section we shall give several results concerning S-dual spaces and duality maps in the suspension category. They are necessary in the arguments for defining ω_m and inducing its properties, though some of them may be known results (see [At1], [Br2], [Sp1], [Sp2] and [W1]).

In Sections 2, 3 and 5 let $k \gg n$. Let S^ℓ be the sphere with radius 1 centred at the origin in $\mathbf{R}^{\ell+1}$ and let S^ℓ be oriented so that a pair of an orthonormal basis of $T_x S^\ell$ and an inward vector at x is compatible with the canonical orientation of $\mathbf{R}^{\ell+1}$. In this section S^ℓ is identified with the wedge product $S^1 \wedge \cdots \wedge S^1$ of ℓ copies of S^1 and is oriented by coordinates $(x_1, \ldots, x_\ell)$. We denote the set of homotopy classes of maps $\alpha : A \to B$ by $[A, B]$. Let A be a space with base point. According to [Sp2], $S^\ell A$ ($S^1 A$ is written as SA for short) denotes the ℓ-th suspension $A \wedge S^\ell$. Let $S^\ell(\alpha)$ denote the ℓ-th suspension of a map α. If B is also a space with base point, then we denote the set of S-homotopy classes of S-maps by $\{A, B\}$. An element of $\{A, B\}$ represented by a map $\alpha : S^\ell A \to S^\ell B$ ($\ell \geq 0$) is written as $\{\alpha\}$. Let D^ℓ_r be the disk centred at the origin with radius r in $\mathbf{R}^\ell$ (D^ℓ_1 is often written as D^ℓ for short). For spaces A and A', let $1^\backsim : A \times A' \to A' \times A$ be the map defined by $1^\backsim(a, a') = (a', a)$.

Let A and B be connected finite polyhedrons with base points. We assume in this section that A and B are sufficiently highly connected so that we do not need to consider $S^\ell A$ and $S^\ell B$ in the following arguments. Then an *m-duality map* refers to a continuous map $v^{AB} : A \wedge B \to S^m$ such that the map $\varphi_{v^{AB}} : H_q(A; \mathbf{Z}) \to H^{m-q}(B; \mathbf{Z})$ defined by sending $z \in H_q(A; \mathbf{Z})$ to the slant product $(v^{AB})^*([S^m]^*)/z$ is an isomorphism. Let $v^{A'B'} : A' \wedge B' \to S^m$ be another m-duality map. By applying the work due to Spanier [Sp1] and [Sp2], we obtain the isomorphisms

$$(1v) \quad \mathcal{D}_m(v^{AB}, v^{A'B'}) : \{B, B'\} \to \{A', A\},$$

$$(2v) \quad \mathcal{D}(v^{AB}) : \{S^m, B\} \to \{A, S^0\},$$

$$(3v) \quad \mathcal{D}(v^{AB}) : \{S^m, B \wedge A\} \to \{A \wedge B, S^m\}.$$

We shall here recall their definitions respectively. In this paper we call isomorphisms of this type defined in [Sp2, Theorem 5.9] *duality isomorphisms*, which are often denoted simply by $\mathcal{D}$. The notation $\mathcal{D}(v^{AB})$ is different from that used in [Sp2]. The map $\mathcal{D}_m(v^{AB}, v^{A'B'})$ in $(1v)$ is defined by the isomorphism $\{B, B'\} \cong (\{B \wedge A', S^m\} \cong)\{A' \wedge B, S^m\} \cong \{A', A\}$. Namely, let $\alpha_B : B \to B'$, $\alpha_A : A' \to A$. Then the first isomorphism is defined by sending $\{\alpha_B\}$ to the element represented by the map

$$v^{A'B'} \circ (id_{A'} \wedge \alpha_B) : A' \wedge B \to A' \wedge B' \to S^m.$$

The inverse of the latter isomorphism $\{A', A\} \cong \{A' \wedge B, S^m\}$ is defined by sending $\{\alpha_A\}$ to the element represented by the map

$$v^{AB} \circ (\alpha_A \wedge id_B) : A' \wedge B \to A \wedge B \to S^m.$$

The duality isomorphisms in $(2v)$ and $(3v)$ are special cases of $(1v)$ and will be often used. As for $(2v)$, let $\{\alpha\} \in \{S^m, B\}$ be an element with $\alpha : S^m \to B$. Then $\mathcal{D}(v^{AB})(\{\alpha\})$ is defined by the element represented by the map

$$v^{AB} \circ (id_A \wedge \alpha) : A \wedge S^m \to A \wedge B \to S^m.$$

For $(3v)$, consider the map $v^{AB} \wedge (v^{AB} \circ 1^{\frown}) : A \wedge B \wedge B \wedge A \to S^m \wedge S^m = S^{2m}$. It is not difficult to see that this map is a duality map. Then, for a map $\alpha_S : S^m \to B \wedge A$, $\mathcal{D}(v^{AB})(\{\alpha_S\})$ in $(3v)$ is defined to be the element represented by the map $(v^{AB} \wedge (v^{AB} \circ 1^{\frown})) \circ (id_{A\wedge B} \wedge \alpha_S)$:

$$A \wedge B \wedge S^m \to A \wedge B \wedge B \wedge A \to A \wedge B \wedge A \wedge B \to S^m \wedge S^m = S^{2m}.$$

By the isomorphism $\mathcal{D}(v^{AB})$ in $(3v)$ we obtain a map $w^{BA} : S^m \to B \wedge A$ such that $\mathcal{D}(v^{AB})(\{w^{BA}\}) = \{v^{AB}\}$. It is not difficult to see that w^{BA} is a *duality map* in the sense of [Br2] and [W1]. In fact, the map $\varphi_{w^{BA}}$: $H^{m-q}(B; \mathbf{Z}) \to H_q(A; \mathbf{Z})$ defined by sending $z \in H^{m-q}(B; \mathbf{Z})$ to the slant product $(w^{BA})_*([S^m])\backslash z$ is an isomorphism. Similarly we obtain a duality map $w^{B'A'} : S^m \to B' \wedge A'$ such that $\mathcal{D}(v^{A'B'})(\{w^{B'A'}\}) = \{v^{A'B'}\}$. Then we define the isomorphism

$$(1w)\ \mathcal{D}(w) : \{A', A\} \to \{B, B'\}$$

as follows. The map $\mathcal{D}(w)$ in $(1w)$ is defined by $\{A', A\} \cong \{S^m, B' \wedge A\} \cong \{B, B'\}$. Namely, for a map $\alpha_A : A' \to A$, the first isomorphism is defined by sending $\{\alpha_A\}$ to the element represented by $(id_{B'} \wedge \alpha_A) \circ w^{B'A'}$. The latter isomorphism $\{B, B'\} \cong \{S^m, B' \wedge A\}$ is defined by sending $\{\alpha_B\}$ to the element represented by $(\alpha_B \wedge id_A) \circ w^{BA}$.

We prove in the following lemma that $\mathcal{D}(w) = \mathcal{D}_m(v^{AB}, v^{A'B'})^{-1}$. By this lemma we can apply the results in [Sp1] and [Sp2] to $\mathcal{D}(w)$ through $\mathcal{D}$. In particular, $\mathcal{D}(w)$ is well defined. In this paper $\mathcal{D}_m(v^{AB}, v^{AB})$ is also written as $\mathcal{D}(v^{AB})$, and we use the notation $\mathcal{D}(w^{BA})$ for $\mathcal{D}(v^{AB})^{-1}$.

Lemma 2.1. *In the cases $(1v), (1w)$ we have that $\mathcal{D}(w) = \mathcal{D}_m(v^{AB}, v^{A'B'})^{-1}$.*

Proof. For the proof, we consider the duality map $(v^{A'B'} \wedge v^{AB}) \circ (id_{A' \wedge B'} \wedge 1^{\smile}) \circ (id_{A'} \wedge 1^{\smile} \wedge id_A)$:

$$A' \wedge B \wedge B' \wedge A \to A' \wedge B' \wedge A \wedge B \to S^m \wedge S^m \cong S^{2m},$$

which is denoted by u. Furthermore, the canonical identification $S^m \wedge S^m \cong S^{2m}$ is also a duality map, which is denoted by $v^{S^{2m}}$. Then we have the duality isomorphism $\mathcal{D}_{2m}(v^{S^{2m}}, u) : \{S^m, B' \wedge A\} \to \{A' \wedge B, S^m\}$ as in (1v). We use the notation exhibited in the following diagram for the duality isomorphisms defined above to distinguish them

$$\begin{tikzcd}
\{B, B'\} \arrow[r, "\mathcal{D}^B(w)"] \arrow[d, equal] & \{S^m, B' \wedge A\} \arrow[d, "\mathcal{D}_{2m}(v^{S^{2m}}, u)"] & \{A', A\} \arrow[l, "\mathcal{D}^A(w)"'] \arrow[d, equal] \\
\{B, B'\} \arrow[r, "\mathcal{D}^B(v)"'] & \{A' \wedge B, S^m\} & \{A', A\}. \arrow[l, "\mathcal{D}^A(v)"]
\end{tikzcd}$$

We prove $\mathcal{D}_{2m}(v^{S^{2m}}, u) \circ \mathcal{D}^B(w)(\{\alpha_B\}) = \mathcal{D}^B(v)(\{\alpha_B\})$ and $\mathcal{D}_{2m}(v^{S^{2m}}, u)$ $\circ \mathcal{D}^A(w)(\{\alpha_A\}) = \mathcal{D}^A(v)(\{\alpha_A\})$. For a map $\alpha_B : B \to B'$, we have that

$$\begin{aligned}
\mathcal{D}_{2m}(v^{S^{2m}}, u) \circ \mathcal{D}^B(w)(\{\alpha_B\}) &= \mathcal{D}_{2m}(v^{S^{2m}}, u)(\{(\alpha_B \wedge id_A) \circ w^{BA}\}) \\
&= \mathcal{D}(v^{AB})(\{w^{BA}\}) \circ \mathcal{D}(\{\alpha_B \wedge id_A\}) \\
&= \{v^{AB}\} \circ \{\mathcal{D}(\{\alpha_B\}) \wedge \mathcal{D}(\{id_A\})\} \\
&= \{v^{AB}\} \circ \{\mathcal{D}^A(v)^{-1} \circ \mathcal{D}^B(v)(\{\alpha_B\}) \wedge \{id_B\}\} \\
&= \mathcal{D}^A(v) \circ \mathcal{D}^A(v)^{-1} \circ \mathcal{D}^B(v)(\{\alpha_B\}) \\
&= \mathcal{D}^B(v)(\{\alpha_B\}).
\end{aligned}$$

For a map $\alpha_A : A' \to A$ we have

$$\begin{aligned}
\mathcal{D}_{2m}(v^{S^{2m}}, u) \circ \mathcal{D}^A(w)(\{\alpha_A\}) &= \mathcal{D}_{2m}(v^{S^{2m}}, u)(\{(id_{B'} \wedge \alpha_A) \circ w^{B'A'}\}) \\
&= \mathcal{D}(v^{A'B'})(\{w^{B'A'}\}) \circ \mathcal{D}(\{id_{B'} \wedge \alpha_A\}) \\
&= \{v^{A'B'}\} \circ \{\mathcal{D}(\{id_{B'}\}) \wedge \mathcal{D}(\{\alpha_A\})\} \\
&= \{v^{A'B'}\} \circ \{\{id_{A'}\} \wedge \mathcal{D}^B(v)^{-1} \circ \mathcal{D}^A(v)(\{\alpha_A\})\} \\
&= \mathcal{D}^B(v) \circ \mathcal{D}^B(v)^{-1} \circ \mathcal{D}^A(v)(\{\alpha_A\}) \\
&= \mathcal{D}^A(v)(\{\alpha_A\}).
\end{aligned}$$

Therefore, $\mathcal{D}^A(w)$ and $\mathcal{D}^B(w)$ are isomorphisms, and hence we have

$$\begin{aligned}
\mathcal{D}(w)(\{\alpha_A\}) &= \mathcal{D}^B(w)^{-1} \circ \mathcal{D}^A(w)(\{\alpha_A\}) \\
&= \mathcal{D}^B(v)^{-1} \circ \mathcal{D}^A(v)(\{\alpha_A\}) \\
&= \mathcal{D}_m(v^{AB}, v^{A'B'})^{-1}(\{\alpha_A\}). \qquad \square
\end{aligned}$$

Let X be a connected closed oriented smooth manifold of dimension n.

Let θ^ℓ_X be the trivial bundle $X \times \mathbf{R}^\ell$. For the tangent bundle TX of X, we will denote $TX \oplus \theta^k_X$ by a symbol τ_X without specifying the number k, which is called the *stable tangent bundle* of X. Choose a smooth embedding $e : X \to \mathbf{R}^{n+k}$, and let $\nu_X(e) = T(\mathbf{R}^{n+k})|_{e(X)}/T(e(X))$ be the normal bundle of $e(X)$. The induced bundle $\nu_X = e^*(\nu_X(e))$ is also called the normal bundle of X, which has the canonical bundle map $e_{\nu_X} : \nu_X \to \nu_X(e)$. Then ν_X is a stable vector bundle, since $k \gg n$. The usual metric of $\mathbf{R}^{n+k}$ induces a splitting of the sequence

$$0 \to TX \to \theta^{n+k}_X \to \nu_X \to 0$$

by orthogonality, which yields a trivialization $t_X : \tau_X \oplus \nu_X \to \theta^{2k}_X$ with dimension of τ_X being equal to k. Let $T(\nu_X(e))$ be the Thom space. Let $\phi_X : S^{n+k} \to T(\nu_X(e))$ be the Pontrjagin-Thom construction for the embedding e of X. Then we have the homotopy class $[\alpha_X]$ of $\alpha_X = T(e^{-1}_{\nu_X}) \circ \phi_X$ in $\pi_{n+k}(T(\nu_X))$, where $[*]$ refers to the homotopy class. In this paper we also call α_X the Pontrjagin-Thom construction for the embedding e. In the following we canonically identify $T(\nu_X \oplus \theta^\ell_X)$ and $T(\nu_X \times \theta^\ell_X)$ with $T(\nu_X) \wedge S^\ell$ and $T(\nu_X) \wedge S^\ell X^0$ respectively.

It has been proved in [M-S, Lemma 2] that $T(\nu_X)$ is the S-dual space of $X^0 = X \cup *_X$, where $*_X$ is the base point. In fact, we shall construct a duality map $v_X : S^\ell X^0 \wedge T(\nu_X) \to S^{n+k+\ell}$ along the line of the arguments above by using the duality map $w_X : S^{n+k+\ell} \to T(\nu_X) \wedge S^\ell X^0$ constructed in [W1, p. 228]. Take an embedding $e : X \to \mathbf{R}^{n+k}$ with normal bundle ν_X. Consider the diagonal map $\triangle : X \to X \times X$ and the vector bundle $\nu_X \times \theta^\ell_X$ over $X \times X$. By the definition of the Whitney sum we have the bundle map $\widetilde{\triangle} : \nu_X \oplus \theta^\ell_X \to \nu_X \times \theta^\ell_X$ covering $\triangle$, which induces a map $T(\widetilde{\triangle}) : T(\nu_X \oplus \theta^\ell_X) = T(\nu_X) \wedge S^\ell \to T(\nu_X \times \theta^\ell_X) = T(\nu_X) \wedge S^\ell X^0$. Let $\widehat{e}$ be the embedding $X \to \mathbf{R}^{n+k} \times \mathbf{0} \subset \mathbf{R}^{n+k+\ell}$. Then the normal bundle of $\widehat{e}$ is identified with $\nu_X \oplus \theta^\ell_X$ and the Pontrjagin-Thom construction for the embedding $\widehat{e}$ yields the map $S^\ell(\alpha_X) : S^{n+k+\ell} \to T(\nu_X) \wedge S^\ell$. Let w_X denote the composition map

$$T(\widetilde{\triangle}) \circ S^\ell(\alpha_X) : S^{n+k+\ell} \longrightarrow T(\nu_X) \wedge S^\ell X^0.$$

It has been proved in [W1, Chapter 3] that w_X is an $(n+k+\ell)$-duality map. We shall now apply the arguments above concerning duality maps by setting $A = S^\ell X^0$, $B = T(\nu_X)$ and $w^{BA} = w_X$. Then, for $\ell \gg n$, there exists a duality map $v_X : S^\ell X^0 \wedge T(\nu_X) \to S^{n+k+\ell}$, which is defined by $\mathcal{D}(w_X)(\{v_X\}) = \{w_X\}$. This duality map induces an isomorphism

$$\mathcal{D}(v_X) : \{S^{n+k}, T(\nu_X)\} \to \{X^0, S^0\}$$

as in (2v). We should note that $\mathcal{D}(w_X)$ and $\mathcal{D}(v_X)$ are defined depending on the embedding e, although they are uniquely determined in the sense of Lemma 2.3 below.

Remark 2.2. Let e^1 be another embedding with normal bundle ν^1_X. Let α_X and α^1_X be the Pontrjagin-Thom constructions for the embeddings e

and e^1 respectively. Then there exists an isotopy of embeddings $e^\lambda : X \to \mathbf{R}^{n+k}$ with $e^0 = e$. Let ν^λ be the normal bundle of e^λ with $\nu^0 = \nu_X$ and $\nu^1 = \nu_X^1$. Let $E : I \times X \to I \times \mathbf{R}^{n+k}$ be the embedding defined by $E(\lambda, x) = (\lambda, e^\lambda(x))$. Let ν be the normal bundle of the embedding E, which yields a bundle map $B : I \times \nu_X \to \nu$ covering $id_{I\times X}$. Let $b : \nu_X \to \nu_X^1$ be the bundle map defined by $B|1 \times \nu_X : \nu_X = 1 \times \nu_X \to \nu_X^1 = \nu|_{1\times X}$ (see, for example, [An3, Proof of Lemma 4.4]). Hence, the isotopy $\widehat{e^\lambda} : X \to \mathbf{R}^{n+k} \times \mathbf{0} \subset \mathbf{R}^{n+k+\ell}$ induces homotopies $S^\ell(\alpha_X^\lambda) : S^{n+k+\ell} \to T(\nu_X^\lambda) \wedge S^\ell$ and $T(\widetilde{\Delta}^\lambda) : T(\nu_X^\lambda) \wedge S^\ell \to T(\nu_X^\lambda) \wedge S^\ell X^0$ by applying the arguments above for e^λ and ν_X^λ in place of e and ν_X.

We have the following lemma.

Lemma 2.3. *Let w_X^λ be the composition map $T(\widetilde{\Delta}^\lambda) \circ S^\ell(\alpha_X^\lambda)$ and let $\mathcal{D}(w_X^\lambda)(\{v_X^\lambda\}) = \{w_X^\lambda\}$. Then we have the following*:

(1) $w_X^1 = (T(b) \wedge id_{S^\ell X^0}) \circ w_X^0$,

(2) $\mathcal{D}_{n+\ell+k}(v_X^0, v_X^1)(\{T(b)\}) = \{id_{X^0}\}$, *where* $\mathcal{D}_{n+\ell+k}(v_X^0, v_X^1) : \{T(\nu_X), T(\nu_X^1)\} \to \{X^0, X^0\}$,

(3) $v_X^0 = v_X^1 \circ (id_{S^\ell X^0} \wedge T(b))$,

(4) *the following diagram is commutative.*

$$\begin{array}{ccc} \{S^{n+k}, T(\nu_X)\} & \xrightarrow{\mathcal{D}(v_X^0)} & \{X^0, S^0\} \\ {\scriptstyle T(b)_*}\Big\downarrow & & \Big\| \\ \{S^{n+k}, T(\nu_X^1)\} & \xrightarrow[\mathcal{D}(v_X^1)]{} & \{X^0, S^0\} \end{array}$$

Proof. By the definition of w_X^λ, we first prove (1). Indeed, we have that

$$\begin{aligned} w_X^1 &= T(\widetilde{\Delta}^1) \circ S^\ell(\alpha_X^1) \\ &= T(\widetilde{\Delta}^1) \circ S^\ell(T(b)) \circ S^\ell(\alpha_X) \\ &= (T(b) \wedge id_{S^\ell X^0}) \circ T(\widetilde{\Delta}) \circ S^\ell(\alpha_X) \\ &= (T(b) \wedge id_{S^\ell X^0}) \circ w_X^0. \end{aligned}$$

Hence, we have the commutative diagram

$$\begin{array}{ccc} S^{n+k+\ell} & \xrightarrow{w_X^0} & T(\nu_X) \wedge S^\ell X^0 \\ {\scriptstyle w_X^1}\Big\downarrow & & \Big\downarrow{\scriptstyle T(b)\wedge(id_{S^\ell X^0})} \\ T(\nu_X^1) \wedge S^\ell X^0 & \xrightarrow[id_{T(\nu_X^1)\wedge S^\ell X^0}]{} & T(\nu_X^1) \wedge S^\ell X^0. \end{array}$$

Then the assertion (2) follows from [Sp2, Theorem 5.11] or [Br2, I.4.14 Theorem] (see Lemma 2.1). Next we prove (3). By (2) we have $\mathcal{D}_{n+\ell+k}(v_X^1, v_X^0)(\{T($

$b^{-1})\}) = \{id_{X^0}\}$, where $\mathcal{D}_{n+\ell+k}(v_X^1, v_X^0) : \{T(\nu_X^1), T(\nu_X)\} \to \{X^0, X^0\}$. Since $w_X^0 = (T(b^{-1}) \wedge id_{S^\ell X^0}) \circ w_X^1$ by (1), we have

$$\begin{aligned}\{v_X^0\} &= \mathcal{D}(v_X^0)(\{w_X^0\}) \\ &= \mathcal{D}(\{w_X^1\}) \circ \mathcal{D}(\{T(b^{-1}) \wedge id_{S^\ell X^0}\}) \\ &= \mathcal{D}(\{w_X^1\}) \circ (\mathcal{D}(\{T(b^{-1})\}) \wedge \mathcal{D}(\{id_{S^\ell X^0}\})) \\ &= \{v_X^1 \circ (id_{S^\ell X^0} \wedge T(b))\},\end{aligned}$$

where we consider dualities (indicated by $\Updownarrow$) of the spaces and maps in the diagram

$$\begin{array}{ccccc} S^{n+k+\ell} & \xrightarrow{w_X^1} & T(\nu_X^1) \wedge S^\ell X^0 & \xrightarrow{T(b^{-1}) \wedge id_{S^\ell X^0}} & T(\nu_X) \wedge S^\ell X^0 \\ \Updownarrow & & \Updownarrow & & \Updownarrow \\ S^{n+k+\ell} & \xleftarrow{v_X^1} & S^\ell X^0 \wedge T(\nu_X) & \xrightarrow{id_{S^\ell X^0} \wedge T(b)} & S^\ell X^0 \wedge T(\nu_X^1). \end{array}$$

The assertion (4) follows from (3). In fact, we have

$$\begin{aligned}\mathcal{D}(v_X^1) \circ T(b)_*(\{\alpha\}) &= \{v_X^1 \circ (id_{S^\ell X^0} \wedge (T(b) \circ \alpha)\} \\ &= \{v_X^1 \circ (id_{S^\ell X^0} \wedge T(b)) \circ (id_{S^\ell X^0} \wedge \alpha)\} \\ &= \{v_X^0 \circ (id_{S^\ell X^0} \wedge \alpha)\} \\ &= \mathcal{D}(v_X^0)(\{\alpha\}).\end{aligned}$$

□

We shall say that $\{\alpha\} \in \{S^{n+k}, T(\nu_X)\}$ is of degree m if $\alpha_*([S^{n+k}]) = m([T(\nu_X)])$, where $[*]$ refers to the fundamental class of $*$. For an element $\{\beta\} \in \{X^0, S^0\}$ with $\beta : S^k X^0 \to S^k$ and a point $x \in X$, we shall define the map $\beta(x) : S^k = S^0 \wedge S^k \to S^k$ by the map $(\beta|(\{*_X, x\} \wedge S^k)) \circ (\iota_x \wedge id_{S^k})$, where $\iota_x : S^0 \to \{*_X, x\}$ is the canonical identification. Let F denote the union of all $F^m, m \in \mathbf{Z}$. Then we have the map

$$c_F : \{X^0, S^0\} \to [X, F]$$

defined by $c_F(\beta)(x) = \beta(x)$. We shall say that $\{\beta\}$ is of degree m if $c_F(\beta)(x)$ is of degree m for any $x \in X$. Let $\{S^{n+k}, T(\nu_X)\}_m$ and $\{X^0, S^0\}_m$ be the sets of all respective maps of degree m. Then c_F induces the map $c_{F^m} : \{X^0, S^0\}_m \to [X, F^m]$. Let $c_{X^0} : X^0 \to S^0$ be the base point preserving surjection mapping X to the other point. Then we have the following lemma.

Lemma 2.4. (1) $\mathcal{D}(v_X)(\{\alpha_X\}) = \{c_{X^0}\}$.
(2) *$\{\alpha\}$ is of degree m if and only if $\mathcal{D}(v_X)(\{\alpha\})$ is of degree m.*

Proof. (1) It is enough for the assertion (1) to prove that $\mathcal{D}(w_X)(\{c_{X^0}\}) = \{\alpha_X\}$. By the definition of α_X, c_{X^0} and w_X, we have the homotopy commutative diagram

$$\begin{tikzcd}
S^{n+k}\wedge S^{\ell} \arrow[r, "\alpha_X\wedge id_{S^\ell}"] \arrow[d, equal] & T(\nu_X)\wedge S^{\ell} \arrow[d, "T(\widetilde{\Delta})"] \\
S^{n+k}\wedge S^{\ell} \arrow[r, "w_X"] \arrow[d, equal] & T(\nu_X)\wedge S^{\ell}X^0 \arrow[d, "id_{T(\nu_X)}\wedge S^{\ell}(c_{X^0})"] \\
S^{n+k}\wedge S^{\ell} \arrow[r, "\alpha_X\wedge id_{S^\ell}"'] & T(\nu_X)\wedge S^{\ell}.
\end{tikzcd}$$

Since the identification $S^{n+k+\ell}=S^{n+k}\wedge S^{\ell}$ is a duality map, it follows from [Br2, I.4.14 Theorem] that $\mathcal{D}(w_X)(\{c_{X^0}\})=\{\alpha_X\}$.

(2) Let $\mathcal{D}(v_X)$ $(\{\alpha\})=\{\beta\}$, or $\mathcal{D}(w_X)(\{\beta\})=\{\alpha\}$. Then we have the commutative diagram

$$\begin{tikzcd}
H_\ell(S^\ell;\mathbf{Z}) \arrow[r, "\varphi_v"] & H^{n+k}(S^{n+k};\mathbf{Z}) \\
H_\ell(S^\ell X^0;\mathbf{Z}) \arrow[r, "\varphi_{v_X}"] \arrow[u, "(c_{X^0})_*"] \arrow[d, "\beta_*"'] & H^{n+k}(T(\nu_X);\mathbf{Z}) \arrow[u, "(\alpha_X)^*"'] \arrow[d, "\alpha^*"] \\
H_\ell(S^\ell;\mathbf{Z}) \arrow[r, "\varphi_v"'] & H^{n+k}(S^{n+k};\mathbf{Z}),
\end{tikzcd}$$

where v is a duality map of the identification $S^{n+k}\wedge S^{\ell}=S^{n+k+\ell}$. We note that both α_X and c_{X^0} are of degree 1. Therefore, if α is of degree m, then β must be of degree m and vice versa. $\square$

We shall recall some results about spherical fibre spaces (see [Br2], [W1] and [At2]). Let ξ be a vector bundle of dimension k with metric over a manifold X of dimension n and let $S(\xi)$ be the associated sphere bundle. A fibre map $h: S(\xi)\to S(\xi)$ covering id_X is called an *automorphism* if h is a homotopy equivalence. In this paper if ξ is oriented, then an automorphism of $S(\xi)$ is always assumed to be an orientation preserving one. Let $\mathrm{End}(\xi)$ denote the group of the homotopy classes of automorphisms of $S(\xi)$. An automorphism of $S(\xi)$ is extended to a self-fibre map of ξ by fibrewise cone construction. This self-fibre map of ξ is also called an automorphism of ξ. Let $h': S(\eta)\to S(\eta)$ be an automorphism of another vector bundle η over X. Then we can define the Whitney sum $h+h':\xi\oplus\eta\to\xi\oplus\eta$ of the fibre maps h and h' similarly as in the case of bundle maps and it yields an automorphism denoted by $h+h': S(\xi\oplus\eta)\to S(\xi\oplus\eta)$.

There is an isomorphism of $\mathrm{End}(\xi)$ to $\mathrm{End}(\xi\oplus\theta^\ell_X)$ $(\ell\geq 0)$ which maps h to $h+id_{\theta^\ell_X}$. Set $\mathcal{E}(\xi)=\lim_{\ell\to\infty}\mathrm{End}(\xi\oplus\theta^\ell_X)$. Then it follows that $\mathcal{E}(\xi)\cong\mathcal{E}(\xi\oplus\theta^\ell_X)$. Suppose that $\xi\oplus\eta$ is trivial and has its trivialization $t:\xi\oplus\eta\to\theta^{2k}_X$. Let a homomorphism $E(t):\mathrm{End}(\xi)\to\mathrm{End}(\theta^{2k}_X)$ be defined by $E(t)(h)=$

$[t \circ (h + id_\eta) \circ t^{-1}]$. Then it induces an isomorphism

$$\mathcal{E} : \mathcal{E}(\xi) \longrightarrow \mathcal{E}(\theta_X^{2k}),$$

which does not depend on the choice of a trivialization t.

Conversely, the map $\mathrm{End}(\theta_X^k) \to \mathrm{End}(\xi \oplus \theta_X^k) \cong \mathcal{E}(\xi)$ defined by mapping $h : \theta_X^k \to \theta_X^k$ to $id_\xi + h$ also induces $\mathcal{E}(\theta_X^k) \cong \mathcal{E}(\xi)$, which coincides with $\mathcal{E}^{-1}$. Therefore, an automorphism $h : S(\xi) \to S(\xi)$ has a map $\beta : X \to SG(k)$ and an automorphism $h_\beta : \theta_X^k \to \theta_X^k$ defined by $h_\beta(x, v) = (x, \beta(x)(v))$ such that $h + id_{\theta_X^k} \simeq id_\xi + h_\beta$. Furthermore, if $h : S(\xi) \to S(\xi)$ is, in particular, the associated automorphism induced from a bundle map $\xi \to \xi$ preserving the metric, then we can take β as a map $X \to SO(k)$.

If we apply this fact to the case $\xi = \nu_X$, then an automorphism $h : S(\nu_X) \to S(\nu_X)$ has a map $\beta : X \to SG(k)$ and an automorphism $h_\beta : \theta_X^k \to \theta_X^k$ such that $h + id_{\theta_X^k} \simeq id_{\nu_X} + h_\beta$.

Lemma 2.5. *Let $h\colon S(\nu_X) \to S(\nu_X)$ and $h_\beta : \theta_X^k \to \theta_X^k$ be the automorphisms given above such that $h + id_{\theta_X^k} \simeq id_{\nu_X} + h_\beta$. Consider the duality map $\mathcal{D}(v_X) : \{T(\nu_X), T(\nu_X)\} \to \{X^0, X^0\}$. Then we have $\mathcal{D}(v_X)(\{T(h)\}) = \{T(h_\beta)\}$.*

Proof. By Lemma 2.1 it is enough for the assertion to prove that $\mathcal{D}(w_X)(\{T(h_\beta)\}) = \{T(h)\}$. Since $h + id_{\theta_X^k} \simeq id_{\nu_X} + h_\beta$, we have that $T(h + id_{\theta_X^k}) \simeq T(id_{\nu_X} + h_\beta) : T(\nu_X) \wedge S^k \to T(\nu_X) \wedge S^k$. Furthermore, we have that $\widetilde{\Delta} \circ (h + id_{\theta_X^k}) \simeq (h \times id_{\theta_X^k}) \circ \widetilde{\Delta}$ and $\widetilde{\Delta} \circ (id_{\nu_X} + h_\beta) \simeq (id_{\nu_X} \times h_\beta) \circ \widetilde{\Delta}$. This implies that the following diagram is homotopy commutative, since $w_X = T(\widetilde{\Delta}) \circ (\alpha_X \wedge id_{S^k})$.

$$\begin{array}{ccc}
S^{2k} & \xrightarrow{w_X} & T(\nu_X) \wedge S^k X^0 \\
{\scriptstyle w_X}\big\downarrow & & \big\downarrow{\scriptstyle id_{T(\nu_X)} \wedge T(h_\beta)} \\
T(\nu_X) \wedge S^k X^0 & \xrightarrow[T(h) \wedge id_{S^k X^0}]{} & T(\nu_X) \wedge S^k X^0
\end{array}$$

By [Br2, I.4.14 Theorem] it follows that $\mathcal{D}(w_X)(\{T(h_\beta)\}) = \{T(h)\}$. □

The inclusion $SO \to SG$ induces a map $J : [X, SO] \to [X, SG]$. According to [Ad], its image is denoted by $J([X, SO])$. The inclusion $F^1 \to SG$ is denoted by $i_{F^1,SG}$.

Proposition 2.6. *Let $\alpha_X : S^{n+k} \to T(\nu_X)$ be the Pontrjagin-Thom construction as above and $b : \nu_X \to \nu_X$ be a bundle map over id_X. Then we have that $(i_{F^1,SG})_* \circ c_{F^1}(\mathcal{D}(v_X)(\{T(b) \circ \alpha_X\}))$ lies in $J([X, SO])$.*

Proof. Let b be a bundle map in place of h in Lemma 2.5. Then there is a bundle map b_β described above with $\beta : X \to SO(k)$. Then it follows from

Lemma 2.4 (2) that

$$\begin{aligned}(i_{F^1,SG})_* \circ c_{F^1}(\mathcal{D}(v_X)(\{T(b)\circ\alpha_X\})) \\ &= (i_{F^1,SG})_* \circ c_{F^1}(\mathcal{D}(v_X)(\{\alpha_X\})\circ\mathcal{D}(v_X)(\{T(b)\})) \\ &= (i_{F^1,SG})_* \circ c_{F^1}(\{c_{X^0}\}\circ\{T(b_\beta)\}) \\ &= J([\beta]).\end{aligned}$$

This shows the lemma. □

3. Map $\omega_m : \Omega_{fold,m}(P) \to [P, F^m]$

In this section we shall first review the results of [An2] and [An3] necessary for the definition of the map $\omega_m : \Omega_{fold,m}(P) \to [P, F^m]$ and then define the map ω_m by using the results in Section 2. We shall define the actions of $SO(n)\times SO(n)$ on $SO(n+1)$ and on $J^2(n,n)$ as follows. Let $(O', {}^tO)$ be an element of $SO(n)\times SO(n)$ and M be an element of $SO(n+1)$. Then define the actions by

$$\begin{aligned}(O', {}^tO)\cdot M &= (O'\dot{+}(1))M(O\dot{+}(1)),\\ (O', {}^tO)\cdot j_0^2 f &= j_0^2(O'\circ f\circ O),\end{aligned}$$

where O and O' are identified with the corresponding linear maps of $\mathbf{R}^n$ and $\dot{+}$ denotes the direct sum of matrices. Note that $\Omega^{10}(n,n)$ is invariant with respect to the latter action. Then we have the following theorem.

Theorem 3.1 ([An2, Theorem (ii)] and [An3, Proposition 2.4])**.** *There exists a topological embedding $i_n : SO(n+1) \to \Omega^{10}(n,n)$ such that i_n is equivariant with respect to those actions above and that the image of i_n is a deformation retract of $\Omega^{10}(n,n)$.*

Let N and P be oriented manifolds of dimension n. If we choose an orthonormal basis of $\mathbf{R}^n$, then there are canonical inclusions of $GL(n)$ into $L^2(n)$ and of $SO(n)$ into $GL(n)$. Hence, the structure group $L^2(n)\times L^2(n)$ of the fibre bundle $\Omega^{10}(N,P)$ over $N\times P$ is reduced to $SO(n)\times SO(n)$ when we provide N and P with Riemannian metrics. Let θ_N and θ_P refer to θ_N^1 and θ_P^1 respectively. Let $GL^+_{n+1}(TN\oplus\theta_N, TP\oplus\theta_P)$ and $SO_{n+1}(TN\oplus\theta_N, TP\oplus\theta_P)$ be the subbundles of $\mathrm{Hom}(TN\oplus\theta_N, TP\oplus\theta_P)$ associated with $GL^+(n+1)$ and $SO(n+1)$ respectively. Then we have the inclusion $i_{SO} : SO_{n+1}(TN\oplus\theta_N, TP \oplus\theta_P) \to GL^+_{n+1}(TN\oplus\theta_N, TP\oplus\theta_P)$, which becomes a homotopy equivalence of fibre bundles covering $id_{N\times P}$.

We define the map

$$i(N,P) : SO_{n+1}(TN\oplus\theta_N, TP\oplus\theta_P) \longrightarrow \Omega^{10}(N,P)$$

to be the map associated with i_n. Then $i(N,P)$ is a fibre homotopy equivalence. Let $(i(N,P))^{-1} : \Omega^{10}(N,P) \to SO_{n+1}(TN\oplus\theta_N, TP\oplus\theta_P)$ be the homotopy inverse of $i(N,P)$. Then we consider the fibre map

$$i_{SO}\circ(i(N,P))^{-1} : \Omega^{10}(N,P) \longrightarrow SO_{n+1}(TN\oplus\theta_N, TP\oplus\theta_P)$$

$$\longrightarrow GL^+_{n+1}(TN \oplus \theta_N, TP \oplus \theta_P)$$

giving a homotopy equivalence of fibre bundles. Then it has been shown in [An3, Proposition 3.1] that the homotopy class of the fibre map $i_{SO} \circ (i(N,P))^{-1}$ over $id_{N\times P}$ does not depend on the choice of Riemannian metrics of N and P.

The set of all continuous sections of $GL^+_{n+1}(TN \oplus \theta_N, TP \oplus \theta_P)$ over N corresponds bijectively to that of all orientation-preserving bundle maps of $TN \oplus \theta_N$ to $TP \oplus \theta_P$. Thus we have the following theorem.

Theorem 3.2 ([An3, Corollary 2]). *Given a fold-map $f : N \to P$, the section $j^2 f$ determines the homotopy class of the section $i_{SO} \circ (i(N,P))^{-1} \circ j^2 f$ of $GL^+_{n+1}(TN \oplus \theta_N, TP \oplus \theta_P)$. It induces a bundle map $\mathcal{T}(f) : TN \oplus \theta_N \to TP \oplus \theta_P$ determined up to homotopy* (*this is denoted by $\bar{f}$ in* [An3]).

Let N and P be embedded in $\mathbf{R}^{n+k}$ with the stable normal bundles ν_N and ν_P respectively. Let $\tau(f)$ denote the bundle map $\mathcal{T}(f) \oplus (f \times id_{\mathbf{R}^{k-1}})$. Then we have the following proposition.

Proposition 3.3 ([An3, Proposition 3.2]). *Let N and P be oriented manifolds of dimension n embedded in $\mathbf{R}^{n+k}$ with the trivializations $t_N : \tau_N \oplus \nu_N \to \theta^{2k}_N$ and $t_P : \tau_P \oplus \nu_P \to \theta^{2k}_P$ respectively. Then a fold-map $f : N \to P$ determines the homotopy class of a bundle map $\nu(f) : \nu_N \to \nu_P$ over f such that $t_P \circ (\tau(f) \oplus \nu(f)) \circ t_N^{-1}$ is homotopic to $f \times id_{\mathbf{R}^{2k}}$.*

Now we are ready to define the map $\omega_m : \Omega_{fold,m}(P) \to [P, F^m]$. Given a fold-map $f : N \to P$ of degree m, there is a bundle map $\tau(f) : \tau_N \to \tau_P$ and a bundle map $\nu(f) : \nu_N \to \nu_P$ determined up to homotopy by Theorem 3.2 and Proposition 3.3 respectively. Let $T(\nu(f)) : T(\nu_N) \to T(\nu_P)$ be the Thom map associated with $\nu(f)$. Then we set $\omega_m(f) = c_{F^m}(\mathcal{D}(v_P)(\{T(\nu(f)) \circ \alpha_N\}))$. Since $T(\nu(f))$ is of degree m, $\mathcal{D}(v_P)(\{T(\nu(f)) \circ \alpha_N\})$ is of degree m by Lemma 2.4 (2).

Lemma 3.4. (1) *$\omega_m(f) = c_{F^m}(\mathcal{D}(v_P)(\{T(\nu(f)) \circ \alpha_N\}))$ does not depend on the choice of embeddings of N and P into $\mathbf{R}^{n+k}$.*

(2) *$\omega_m(f)$ does not depend on the choice of a representative f of the fold-cobordism class $[f] \in \Omega_{fold,m}(P)$.*

Proof. (1) Let $e^1_N : N \to \mathbf{R}^{n+k}$ and $e^1_P : P \to \mathbf{R}^{n+k}$ be other embeddings with normal bundles ν^1_N and ν^1_P, trivializations $t^1_N : \tau_N \oplus \nu^1_N \to \theta^{2k}_N$ and $t^1_P : \tau_P \oplus \nu^1_P \to \theta^{2k}_P$ respectively and a bundle map $\nu(f)^1 : \nu^1_N \to \nu^1_P$. Then by Remark 2.2 there exist bundle maps $b_N : \nu_N \to \nu^1_N$ and $b_P : \nu_P \to \nu^1_P$ such that $b_P \circ \nu(f) \circ b_N^{-1} \simeq \nu(f)^1 : \nu^1_N \to \nu^1_P$. Then by Lemma 2.3 (4) we have that $\mathcal{D}(v^1_P) \circ T(b_P)_* = \mathcal{D}(v_P)$ and that

$$\begin{aligned}
\mathcal{D}(v^1_P)(\{T(\nu(f)^1) \circ \alpha^1_N\}) &= \mathcal{D}(v^1_P)(\{T(b_P) \circ T(\nu(f)) \circ T(b_N^{-1}) \circ T(b_N) \circ \alpha_N\}) \\
&= \mathcal{D}(v^1_P)(\{T(b_P) \circ T(\nu(f)) \circ \alpha_N\}) \\
&= \mathcal{D}(v^1_P) \circ T(b_P)_*(\{T(\nu(f)) \circ \alpha_N\}) \\
&= \mathcal{D}(v_P)(\{T(\nu(f)) \circ \alpha_N\}).
\end{aligned}$$

(2) Let $f_i : N_i \to P (i = 0, 1)$ be fold-maps of degree m, which are fold-cobordant. By the same arguments as in the proof of [An3, Lemma 4.3] we have that $\{T(\nu(f_0)) \circ \alpha_{N_0}\} = \{T(\nu(f_1)) \circ \alpha_{N_1}\}$. Hence, we have that $\omega_m(f_0) = \omega_m(f_1)$. □

In particular, if $m = 1$, then we shall see that $(i_{F^1,SG})_* \circ \omega_1$ coincides with

$$\omega : \Omega_{fold,1}(P) \longrightarrow [P, SG]$$

defined in [An3, Section 4]. Now we first review the definition of ω. Let $f : N \to P$ be a fold-map of degree 1. By Proposition 3.3, there exists a bundle map $\nu(f) : \nu_N \to \nu_P$. Then the map $T(\nu(f)) \circ \alpha_N$ gives an element of $\pi_{n+k}(T(\nu_P))$. By [Br2, I.4.19 Theorem] and [W1, Theorem 3.5], there exists an automorphism $h : S(\nu_P) \to S(\nu_P)$, which is unique up to homotopy and is extended to an automorphism $h : \nu_P \to \nu_P$ by the fibrewise cone construction satisfying the following properties. If $T(h) : T(\nu_P) \to T(\nu_P)$ is the Thom map of h, then we have that $T(\nu(f))_*([\alpha_N]) = T(h)_*([\alpha_P])$. Furthermore, there exists a map $\beta : P \to SG(k)$ and a fibre map $h_\beta : \theta^k_P \to \theta^k_P$ defined by $h_\beta(x, v) = (x, \beta(x)(v))$ such that $h + id_{\theta^k_P}$ is homotopic to $id_{\nu_P} + h_\beta$ as automorphisms. Then we have defined ω to be $\omega(f) = [\beta]$.

Lemma 3.5. *The map ω coincides with $(i_{F^1,SG})_* \circ \omega_1$.*

Proof. We shall give a sketch of a proof, since most of the arguments are similar to those found in Section 2. Since SG is weakly homotopy equivalent to F^1, we may suppose that the map β appearing in the definition of ω factors through F^1_k, namely, $\beta : P \to F^1_k \subset SG(k)$. By Lemma 2.5, we obtain that $\mathcal{D}(v_P)(\{T(h)\}) = \{T(h_\beta)\}$. Therefore, we have that

$$\begin{aligned}
(i_{F^1,SG})_* \circ \omega_1(f) &= (i_{F^1,SG})_* \circ c_{F^1}(\mathcal{D}(v_P)(\{T(h) \circ \alpha_P\})) \\
&= (i_{F^1,SG})_* \circ c_{F^1}(\mathcal{D}(v_P)(\{\alpha_P\}) \circ \mathcal{D}(v_P)(\{T(h)\})) \\
&= (i_{F^1,SG})_* \circ c_{F^1}(\{c_{P^0}\} \circ \{T(h_\beta)\}) \\
&= (i_{F^1,SG})_*([\beta]) \\
&= \omega(f).
\end{aligned}$$

□

Hence, in the rest of the paper $(i_{F^1,SG})_* \circ \omega_1$ will be written as ω.

Remark 3.6. (1) The spaces F^m and SG are weakly homotopy equivalent to the identity component of the infinite loop space $\Omega^\infty S^\infty$ (see [M-M, Corollary 3.8]). In fact, let $\hat{m} : S^1 \to S^1$ be the map defined by $x \mapsto mx$ and let $m_{(S^k)} : S^k \to S^k$ be the suspension $S^{k-1}(\hat{m})$ of degree m. Let $\vee_{S^k} : S^k \to S^k \vee S^k$ be the comultiplication and let $(\mathbf{1}, \mathbf{1}) : S^k \vee S^k \to S^k$ be the canonical map, which is the identity on each S^k. Then we have the weak homotopy equivalence $h_{F^0,F^m} : F^0 \to F^m$ (resp. $h_{F^1,F^m} : F^1 \to F^m, m \neq 0$) defined by using the homotopy equivalence $h_k : F^0_k \to F^m_k$ (resp. $\hat{h}_k : F^1_k \to F^m_k, m \neq 0$) such that $h_k(j) = (\mathbf{1}, \mathbf{1}) \circ (j \vee m_{(S^k)}) \circ \vee_{S^k}$ (resp. $\hat{h}_k(j) = j \circ m_{(S^k)}, m \neq 0$).

Since F^0, F^1 and SG are homotopy commutative H-spaces, $[P, F^0]$, $[P, F^1]$ and $[P, SG]$ have structures of an abelian group. It is well known that there is an isomorphism $[S^n, F^0] \to \pi_n^s$. In fact, we have the following (see [At1, Lemma 1.3 and (i), (ii) on p. 295]).

$$[S^n, F_k^0] \cong \pi_n(F_k^0) \cong \pi_{n+k-1}(S^{k-1}) \qquad (k > n+2)$$

(2) Many authors have contributed to the study of the very difficult structure of the algebras $H_*(SG; \mathbf{Z}/p\mathbf{Z})$ and $H^*(SG; \mathbf{Z}/p\mathbf{Z})$, where p is a prime number (consult [M-M, Chapter 6], [M, Theorem 6.1 and Conjecture 6.2] and [Tsu]).

(3) We have seen in Corollary 5 that for any element a of $H^*(F^m; \mathbf{Z}/p\mathbf{Z})$, $\omega_m(f)^*(a)$ of $H^*(P; \mathbf{Z}/p\mathbf{Z})$ is a fold-cobordism invariant. It is natural to ask how $\omega_m(f)^*(a)$ is related to the topological structure of $S(f)$ in N and $f(S(f))$ in P, where $S(f)$ is the set of fold singularities of f (see Example 8.4 (2)).

4. Homotopy principle for fold-maps

If for any section s of $\Gamma(N, P)$ there exists a fold-map $f : N \to P$ such that $j^2 f$ is homotopic to s by a homotopy in $\Gamma(N, P)$, then we shall say that the homotopy principle (a terminology used in [G2]) for fold-maps in the existence level holds. In this section we shall prove the following theorem in place of Theorem 6.

Theorem 4.1. *Let $n \geq 2$. Let N and P be connected manifolds of dimension n and $\partial N = \emptyset$. Let C be a closed subset of N. Let s be a section of $\Gamma(N, P)$ such that there exists a fold-map g defined on a neighborhood of C into P with $j^2 g|C = s|C$. Then there exists a fold-map $f : N \to P$ such that $j^2 f$ is homotopic to s relative to C by a homotopy h_λ in $\Gamma(N, P)$ with $h_0 = s$ and $h_1 = j^2 f$.*

If the closure of $N \setminus C$ has no compact connected component, then the assertion of Theorem 4.1 is a direct consequence of [G1, Theorem 4.1.1]. This theorem is a special case of [An1, Theorem 1], though the proof given there was sketchy. In particular, the proof of Proposition 4.7 below was not given. A weaker assertion where h_λ is required to be a homotopy of N into $\Omega^1(N, P)$ (not into $\Omega^{10}(N, P)$), which we can prove without Proposition 4.7, is sufficient for the proof of the main results in [An1]. Here $\Omega^1(N, P)$ denotes $\Sigma^0(N, P) \cup \Sigma^1(N, P)$ in $J^1(N, P)$. However, Theorem 4.1 above is very important for the proof of Theorem 1 in Introduction. This is the reason why a proof of Theorem 4.1 is given in detail in this paper. The following Theorem 4.2 due to Èliašberg [E] (see also [G2, 2.1.3 Theorem on p. 55]) will play an important role in the proof. We should note that Theorem 4.1 is not a generalization of Theorem 4.2.

Theorem 4.2 ([E, 2.2 Theorem])**.** *Let N and P be connected manifolds of dimension n and S be an $(n-1)$-dimensional submanifold of N. Let C be a closed subset of N such that each connected component of $N \setminus C$ has* *non-empty intersection with S. Assume that there exists an S-monomorphism $B : TN \to TP$ over a map $f_B : N \to P$, that is, a fibrewise linear map which satisfies*

(H-4.2-i) *B is of rank n outside of S and is of rank $n-1$ on S,*

(H-4.2-ii) *there exist a small tubular neighborhood $U(S)$ of S, which is identified with $S \times (-1, 1)$, and a fibre involution $i_U : U(S) \to U(S)$ such that $B \circ d(i_U)|TU(S) = B|TU(S)$ and*

(H-4.2-iii) *f_B is a fold-map on a small neighborhood of C and $df_B|_C = B|_C$.*

Then there exist a fold-map $f : N \to P$ and a homotopy of S-monomorphisms $B_\lambda : TN \to TP$ such that $B_0 = B$, $B_1 = df$ and $B_\lambda|_C = B|_C$ for any λ.

We here note the following. The fibre of $\Sigma^{10}(n, n) \to \Sigma^1(n, n)$ has two connected components. Hence, if an S-monomorphism B has a fold-map f with $S(f) = S$ such that df and B are homotopic as S-monomorphisms, then the homotopy class of $j^2 f$ as a section in $\Gamma(N, P)$ is uniquely determined from B and does not depend on the choice of f.

We shall begin by proving the following proposition, which is a direct consequence of Gromov's theorem ([G1, Theorem 4.1.1]). For the fold-map g and a closed subset C in the statement of Theorem 4.1 we take a closed neighborhood $U(C)$ of C such that $Cl(\mathrm{Int}U(C)) = U(C)$ for a while, where g is defined on a neighborhood of $U(C)$. Let j_0 be the number (possibly ∞) of compact connected components of $N \backslash \mathrm{Int}(U(C))$, from each of which we choose a point q_j $(1 \le j \le j_0)$ in its interior. Using local charts of N we have embeddings $e_j : \mathbf{R}^n \to N \setminus U(C)$ with $e_j(0) = q_j$. In Sections 4, 6 and 7 we shall simply denote D^n_r by D_r.

Proposition 4.3. *Let $n \ge 1$. Let s be a section satisfying the hypothesis in Theorem 4.1. Assume that $s^{-1}(\Sigma^{10}(N, P))$ is not contained in $U(C)$. Take points $\{q_1, \dots, q_{j_0}\}$ of $N \backslash U(C)$ and embeddings e_j $(1 \le j \le j_0)$ as above. Then there exist a homotopy s_λ relative to $U(C)$ in $\Gamma(N, P)$ with $s_0 = s$ and positive numbers r_j $(1 \le j \le j_0)$ such that*

(1) *s_1 has a fold-map $f_0 : N \setminus \{q_1, \dots, q_{j_0}\} \to P$ with $j^2 f_0|(N \setminus \cup_{j=1}^{j_0} e_j(\mathrm{Int}D_{r_j})) = s_1|(N \setminus \cup_{j=1}^{j_0} e_j(\mathrm{Int}\, D_{r_j}))$,*

(2) *s_1 is transverse to $\Sigma^{10}(N, P)$ and*

(3) *$s_1^{-1}(\Sigma^{10}(N, P))$ transversely intersects $\partial e_j(D_{2r_j})$ and $\partial e_j(D_{r_j})$ for each j.*

Proof. We can take the embeddings $e_j : \mathbf{R}^n \to N \setminus U(C)$ with $e_j(0) = q_j$ so that $\pi^2_P \circ s \circ e_j(\mathbf{R}^n)$ is contained in a local chart of P. By applying [G1, Theorem 4.1.1] to the section $s|(N \setminus \{q_1, \dots, q_{j_0}\})$, we see that there exists a homotopy s'_λ relative to $U(C)$ in $\Gamma(N \setminus \{q_1, \dots, q_{j_0}\}, P)$ such that $s'_0 = s|(N \setminus \{q_1, \dots, q_{j_0}\})$ and that s'_1 has a fold-map $f_0 : N \backslash \{q_1, \dots, q_{j_0}\} \to P$ with $j^2 f_0 = s'_1$. Take a small positive number t_j for each j. By the homotopy extension property we can extend $s'_\lambda|(N \setminus \cup_{j=1}^{j_0} e_j(\mathrm{Int}D_{t_j}))$ to a homotopy s''_λ in $\Gamma(N, P)$ such that $s''_0 = s$ and $s''_\lambda|(N \setminus \cup_{j=1}^{j_0} e_j(\mathrm{Int}D_{t_j})) = s'_\lambda|(N \setminus \cup_{j=1}^{j_0} e_j(\mathrm{Int}D_{t_j}))$. Since

$j^2 f_0$ is transverse to $\Sigma^{10}(N,P)$, we can deform s''_λ to the homotopy s_λ such that

(i) $s_0 = s$,
(ii) $s_\lambda|(N \setminus \cup_{j=1}^{j_0} e_j(\mathrm{Int}D_{t_j})) = s''_\lambda|(N \setminus \cup_{j=1}^{j_0} e_j(\mathrm{Int}D_{t_j}))$ and
(iii) s_1 is transverse to $\Sigma^{10}(N,P)$.

Now recall that $S(s_1) = (s_1)^{-1}(\Sigma^{10}(N,P))$. For each j, consider the smooth map $h : S(s_1) \cap e_j(\mathbf{R}^n) \to \mathbf{R}$ defined by $h(x) = \|e_j^{-1}(x)\|$ except for the origin. The assertion (3) follows from Sard Theorem (see [H2]) for h. $\square$

Since $\mathbf{K}$ over $\Sigma^{10}(\mathbf{R}^n,P)$ is a line bundle, $S^2\mathbf{K}$ is trivial and has the canonical orientation determined by a vector $\mathbf{v} \bigcirc \mathbf{v} = (-\mathbf{v}) \bigcirc (-\mathbf{v})$, $\mathbf{v} \in \mathbf{K}$. Therefore, the intrinsic derivative $\mathbf{d}^2 : \mathbf{K} \to \mathrm{Hom}(\mathbf{K},\mathbf{Q})$ induces an orientation of $\mathbf{Q}$ over $\Sigma^{10}(\mathbf{R}^n,P)$. Throughout the paper we shall always provide $\mathbf{Q}$ with this orientation.

Let s be a section of $\Gamma^{tr}(\mathbf{R}^n,P)$. Let $\nu(s)$ denote the orthogonal normal bundle of $S(s)$ in $\mathbf{R}^n$. We set $K(s) = (s|S(s))^*\mathbf{K}$, $Q(s) = (s|S(s))^*\mathbf{Q}$ and $\theta^n(P) = (\pi_P \circ s)^*TP$. Throughout the paper we shall choose and fix a trivialization of $\theta^n(P)$ over $\mathbf{R}^n$ $(n \geq 2)$. Then we can provide $K(s)$ with the orientation induced by the exact sequence

$$0 \longrightarrow K(s) \longrightarrow T\mathbf{R}^n|_{S(s)} \xrightarrow{d^1(s)} \theta^n(P)|_{S(s)} \longrightarrow Q(s) \longrightarrow 0.$$

In fact, let $c \in S(s)$ and take an orthonormal basis $(\mathbf{m}_1,\ldots,\mathbf{m}_{n-1})$ of $K(s)_c^\perp$ in $T_c\mathbf{R}^n$ and a vector $\mathbf{v} \in Q(s)_c$ representing the orientation of $Q(s)_c$ such that $(d^1(s)(\mathbf{m}_1),\ldots,d^1(s)(\mathbf{m}_{n-1}),\mathbf{v})$ is compatible with the orientation of $\theta^n(P)_c$. Then there exists a vector $\mathbf{m}_n \in K(s)_c$ such that $(\mathbf{m}_1,\ldots,\mathbf{m}_n)$ represents the usual orientation of $\mathbf{R}^n$. We orient $K(s)_c$ by $\mathbf{m}_n$. Thus $\mathrm{Hom}(K(s),Q(s))$ is oriented and is isomorphic to the normal bundle $\nu(s)$ of $S(s)$ in $\mathbf{R}^n$ as is explained in Section 1. This induces the orientation of $\nu(s)$. On the other hand, we can provide any point x of $\mathbf{R}^n \setminus S(s)$ with sign $-$ or $+$ depending on whether the sign of the determinant of $d^1(s)_x$ is negative or positive (we note that when $n=1$, we are considering the trivialization of $\theta^1(P)$ induced from $Q(s)$ near each point c). This orientation of $\nu(s)$ coincides with the direction from the points of $\mathbf{R}^n \setminus S(s)$ with sign $-$ to those points with sign $+$. Throughout the paper we shall orient $S(s)$ so that $T(S(s)) \oplus \nu(s)$ is compatible with the usual orientation of $\mathbf{R}^n$.

Any point c of $S(s)$ has two oriented lines $\nu(s)_c$ and $K(s)_c$. Here we note the following fact concerning these orientations.

Remark 4.4. If $g : (N,x) \to (P,f(x))$ is a fold-map and x is a fold singularity, then $d_x^2 g : T_xN \to \mathrm{Hom}(K(j^2g)_x, Q(j^2g)_x)$ coincides with $d_x^2(j^2g)$ and is an epimorphism (see Section 1). Since $K(j^2g)_x \cap T_x(S(j^2g)) = \{0\}$, we may say that $K(j^2g)$ is the normal bundle of $S(j^2g)$ near x. Hence, it follows that the orientations of $\nu(j^2g)_x$ and $K(j^2g)_x$ are compatible.

For an oriented 1-dimensional subspace $L \subset \mathbf{R}^n$ we let $\mathbf{e}(L)$ denote the vector of length 1 with given orientation. Now we define the map $\mathbf{e}(s) : S(s) \to$

$S^{n-1} \times S^{n-1}$ by $\mathbf{e}(s)(c) = (\mathbf{e}(K(s)_c), \mathbf{e}(\nu(s)_c))$. Let Δ^- denote the subspace of $S^{n-1} \times S^{n-1}$ consisting of all points $(v, -v)$, $v \in S^{n-1}$. The following lemma can be proved by the standard arguments in differential topology.

Lemma 4.5. *No matter how an orientation of $\theta^n(P)$ is chosen, the subset consisting of all sections s of $\Gamma^{tr}(\mathbf{R}^n, P)$ such that $\mathbf{e}(s) : S(s) \to S^{n-1} \times S^{n-1}$ is transverse to Δ^- is open and dense.*

For the proof of Theorem 4.1 we need the following two propositions. In $\mathbf{R}^n$ let $O(p; r)$ be the open disk centered at p with radius r.

Proposition 4.6. *Let $n \geq 1$. Assume that $s \in \Gamma^{tr}(\mathbf{R}^n, P)$ satisfies the hypotheses*

(H-i) *there exists a fold-map f_0 defined on $\mathbf{R}^n \setminus \mathrm{Int}D_r$ into P such that $j^2 f_0|(\mathbf{R}^n \setminus \mathrm{Int}D_r) = s|(\mathbf{R}^n \setminus \mathrm{Int}D_r)$ and*

(H-ii) $\mathbf{e}(s)$ *is transverse to Δ^- and $\mathbf{e}(s)^{-1}(\Delta^-)$ consists of distinct points $p_1, \dots, p_m$ in $\mathrm{Int}D_r$.*

Then there exists a homotopy s_λ relative to $\mathbf{R}^n \setminus \mathrm{Int}D_{2r}$ in $\Gamma(\mathbf{R}^n, P)$ with $s_0 = s$ satisfying the following.

(1) *$s_1 \in \Gamma^{tr}(\mathbf{R}^n, P)$ and $S(s_\lambda) = S(s)$ for any λ.*

(2) *Let $\varepsilon > 0$ be any positive number such that $O(p_j; 2\varepsilon)$'s are disjoint and contained in $\mathrm{Int}D_r$. There exists a small neighborhood $U(S(s))$ of $S(s)$ such that we have a fold-map $f : ((\mathbf{R}^n \setminus \mathrm{Int}D_{2r}) \cup U(S(s))) \setminus (\cup_{j=1}^m O(p_j; \varepsilon)) \to P$ with $j^2 f = s_1$ on $((\mathbf{R}^n \setminus \mathrm{Int}D_{2r}) \cup U(S(s))) \setminus (\cup_{j=1}^m O(p_j; \varepsilon))$.*

(3) *In particular, if $\mathbf{e}(s)^{-1}(\Delta^-)$ is empty, then the fold-map f in* (2) *is defined on $(\mathbf{R}^n \setminus \mathrm{Int}D_{2r}) \cup U(S(s))$.*

Proposition 4.7. *Let $n \geq 2$. Given a section s in $\Gamma^{tr}(\mathbf{R}^n, P)$ satisfying* (H-i) *and* (H-ii) *with $m > 0$ in Proposition* 4.6, *there exists a homotopy s_λ relative to $\mathbf{R}^n \setminus \mathrm{Int}D_{2r}$ in $\Gamma(\mathbf{R}^n, P)$ with $s_0 = s$ such that $s_1 \in \Gamma^{tr}(\mathbf{R}^n, P)$, $\mathbf{e}(s_1)^{-1}(\Delta^-)$ is empty and that $S(s_1) \cap D_{2r}$ is not empty.*

The corresponding assertion for the case $n = 1$ fails (see Remark 8.5). The proofs of Propositions 4.6 and 4.7 will be given in Sections 6 and 7 respectively. Here we shall give a proof of Theorem 4.1.

Proof of Theorem 4.1. We may assume that $N \setminus C$ is not empty. From each connected compact component of $N \setminus \mathrm{Int}(U(C))$, we take a point $q_j (1 \leq j \leq j_0)$ in its interior. We first deform s by a homotopy in $\Gamma(N, P)$ so that each connected compact component of $N \setminus \mathrm{Int}(U(C))$ contains points of $S(s) \setminus C$ in its interior with q_j being excluded. Then for the section s there exists a homotopy $\overline{s_\lambda}$ with a fold-map f_0 satisfying the properties (1), (2) and (3) of Proposition 4.3. Therefore, it is enough for Theorem 4.1 to prove the special case of Theorem 4.1 where (1) $N = \mathbf{R}^n$, $C = \mathbf{R}^n \setminus \mathrm{Int}D_{2r}$ and $g = f_0$ on a neighbourhood of $\mathbf{R}^n \setminus \mathrm{Int}D_{2r}$, (2) s is transverse to $\Sigma^{10}(\mathbf{R}^n, P)$ and (3) $S(s) \cap \mathrm{Int}D_{2r}$ contains the origin. We shall prove this special case.

It follows from Lemma 4.5 and Proposition 4.7 for s that there exists a homotopy s'_λ relative to $\mathbf{R}^n \setminus \mathrm{Int}D_{2r}$ in $\Gamma(\mathbf{R}^n, P)$ with $s'_0 = s$ such that

$s_1' \in \Gamma^{tr}(\mathbf{R}^n, P)$ and $\mathbf{e}(s_1')^{-1}(\Delta^-) = \emptyset$. By applying Proposition 4.6 to the section s_1' there exists a homotopy s_λ'' relative to $\mathbf{R}^n \setminus \mathrm{Int}D_{2r}$ in $\Gamma(\mathbf{R}^n, P)$ with $s_0'' = s_1'$ such that there exists a fold-map $\hat{g} : (\mathbf{R}^n \setminus \mathrm{Int}D_{2r}) \cup U(S(s)) \to P$ with $j^2\widehat{g} = s_1''$ on $(\mathbf{R}^n \setminus \mathrm{Int}D_{2r}) \cup U(S(s))$. Therefore, we obtain a homotopy s_λ relative to $\mathbf{R}^n \setminus \mathrm{Int}D_{2r}$ in $\Gamma(N, P)$ defined by

$$s_\lambda = \begin{cases} s'_{2\lambda} & \text{for} \quad 0 \leq \lambda \leq 1/2, \\ s''_{2\lambda-1} & \text{for} \quad 1/2 \leq \lambda \leq 1. \end{cases}$$

It is clear that s_λ is well defined. We shall apply Theorem 4.2 for the section $\pi_1^2 \circ s_1$ and $\widehat{g}$. Since $J^1(N, P)$ is canonically identified with $\mathrm{Hom}(TN, TP)$, we may regard $\pi_1^2 \circ s_1$ as an $S(s_1)$-monomorphism. By Theorem 4.2 we obtain a homotopy B_λ relative to $\mathbf{R}^n \setminus \mathrm{Int}D_{2r}$ of $S(s_1)$-monomorphisms and a fold-map $f : \mathbf{R}^n \to P$ with $S(f) = S(s_1)$ such that $B_0 = \pi_1^2 \circ s_1$ and $B_1 = df$. Then this homotopy is lifted to the homotopy h_λ relative to $\mathbf{R}^n \setminus \mathrm{Int}D_{2r}$ in $\Gamma(\mathbf{R}^n, P)$ such that $h_0 = s_1$ and $h_1 = j^2 f$. Indeed, there exists a small tubular neighborhood $U(S(s_1))$ of $S(s_1)$, which is identified with $S(s_1) \times (-1, 1)$. Let $(c, t) \in S(s_1) \times (-1, 1)$. Then there exists a continuous homotopy $h_\lambda(c, t)$ in $\Gamma((\mathbf{R}^n \setminus \mathrm{Int}D_{2r}) \cup U(S(s_1)), P)$ such that

(1) $\pi_1^2 \circ h_\lambda(c, t) = (B_\lambda)_{(c,t)}$,

(2) $(d^2_{h_\lambda(c,0)}(\partial/\partial t))(\partial/\partial t) = 2\mathbf{e}(T_c(S(s_1))^\perp)$,

(3) $d^2_{h_\lambda(c,0)}$ vanishes on $T_c(S(s_1))$.

As for other second derivatives of $h_\lambda(c, t)$ we can choose them arbitrarily. We note that $S(s_1)$ is oriented in (2) and the symbol $\perp$ refers to the orthogonal complement. Since any fibre of $\pi_1^2 : \Omega^{10}(\mathbf{R}^n, P) \setminus \Sigma^1(\mathbf{R}^n, P) \to J^1(\mathbf{R}^n, P) \setminus \Sigma^1(\mathbf{R}^n, P)$ is contractible, we can extend $h_\lambda(c, t)$ to a required homotopy $h_\lambda \in \Gamma(\mathbf{R}^n, P)$. This is what we want. $\square$

Now we give an application of Theorem 4.1.

Theorem 4.8. *Let N and P be oriented manifolds of dimension n. Let $f : N \to P$ be a continuous map. Then if the tangent bundles TN and $f^*(TP)$ are stably equivalent, then there exists a fold-map homotopic to f.*

Proof. The assertion for $n = 1$ is trivial and so let $n > 1$. There exists an orientation preserving bundle map $b : TN \oplus \theta_N \to TP \oplus \theta_P$ covering f. Hence it follows that there exists a section $s \in \Gamma(N, P)$ such that $i_{SO} \circ i(N, P)^{-1} \circ s$ is homotopic to b. Then by Theorem 4.1 there exists a fold-map $g : N \to P$ such that $j^2 g$ is homotopic to s (note that $\mathcal{T}(g) \simeq b$). This is what we want. $\square$

This theorem should be compared with [E, 3.10. Theorem], from which the assertion of Theorem 4.8 follows in many cases. The converse of the theorem has been also proved in [E, 3.8 and 3.9].

5. Map ω_m is surjective

In this section we shall prove that $\omega_m : \Omega_{fold,m}(P) \to [P, F^m]$ is surjective by using Theorem 4.1.

Proof of Theorem 1. The assertion for $n = 1$ follows from Proposition 5.3 below. So let $n > 1$. Let $\beta : P \to F^m$ be a map representing an element $[\beta] \in [P, F^m]$. Take an element $\{\beta_0\} \in \{P^0; S^0\}$ such that $c_{F^m}(\{\beta_0\}) = [\beta]$. By the duality of $\mathcal{D}(v_P)$ there exists an element $\alpha_\beta \in \pi_{n+k}(T(\nu_P))$ such that $\mathcal{D}(v_P)(\{\alpha_\beta\}) = \{\beta_0\}$. Since α_β is of degree m by Lemma 2.4, we have that $U(\nu_P) \frown (\alpha_\beta)_*([S^{n+k}]) = m[P]$, where $U(\nu_P)$ refers to the Thom class of ν_P. By the Thom transversality theorem we may assume that α_β is transverse to the zero-section $P \subset T(\nu_P)$ without loss of generality. Set $N = (\alpha_\beta)^{-1}(P)$. Let $\hat{g} = \alpha_\beta | D(\nu_N)$ and $g = \alpha_\beta | N$, where $D(\nu_N)$ is the normal disk bundle to the inclusion $N \subset S^{n+k}$. Then g is of degree m. Indeed, let $[D(\nu_N)]$ be the fundamental class of $H_{n+k}(D(\nu_N), \partial D(\nu_N); \mathbf{Z})$. Let $i_N : N \to D(\nu_N)$ and $i_P : P \to D(\nu_P)$ be the inclusions to the zero sections respectively. Then we have that

$$\begin{aligned}
\hat{g}_*((i_N)_*([N])) &= \hat{g}_*(U(\nu_N) \frown [D(\nu_N)]) \\
&= \hat{g}_*(\hat{g}^*(U(\nu_P)) \frown [D(\nu_N)]) \\
&= U(\nu_P) \frown \hat{g}_*([D(\nu_N)]) \\
&= U(\nu_P) \frown (\alpha_\beta)_*([S^{n+k}]) \\
&= m((i_P)_*([P])).
\end{aligned}$$

Then we have a bundle map $b : \nu_N \to \nu_P$ over g induced from $\hat{g}$. By [An3, Proposition 3.3] there exists a bundle map $b' : \tau_N \to \tau_P$, which is uniquely determined up to homotopy so that $t_P \circ (b' \oplus b) \circ t_N^{-1}$ is homotopic to $g \times id_{\mathbf{R}^{2k}}$.

Here we choose metrics of TN and TP. Recall $SO_{n+k}(TN \oplus \theta_N^k, TP \oplus \theta_P^k)$ and $GL^+_{n+k}(TN \oplus \theta_N^k, TP \oplus \theta_P^k)$ defined in Section 3. The inclusion $GL^+_{n+1} \to GL^+_{n+k}$ induces a fibre map $i_{n+1,n+k} : GL^+_{n+1}(TN \oplus \theta_N, TP \oplus \theta_P) \to GL^+_{n+k}(TN \oplus \theta_N^k, TP \oplus \theta_P^k)$. Since $\pi_j(SO(n+k), SO(n+1)) \cong \{0\}$ for $j \leq n$ and since the canonical inclusion $SO(\ell) \to GL^+(\ell)$ is a homotopy equivalence, there exists an orientation preserving bundle map

$$b'' : TN \oplus \theta_N \to TP \oplus \theta_P \qquad \text{over} \quad g$$

such that $i_{n+1,n+k}(b'') \simeq b'$. By the fibre homotopy equivalence $i(N, P)$ we obtain the homotopy class of a section s of $\Gamma(N, P)$ such that $i_{SO} \circ i(N, P)^{-1}(s)$ is homotopic to b''. Therefore it follows from Theorem 4.1 that there exists a fold-map $f : N \to P$ of degree m such that $j^2 f$ is homotopic to s in $\Gamma(N, P)$. By the definition of $\mathcal{T}(f)$ for f, we have that $\mathcal{T}(f) \simeq b''$ and $i_{n+1,n+k}(\mathcal{T}(f)) = \tau(f)$. This implies that $\tau(f) \simeq b'$ and so $\nu(f) \simeq b$. By the definition of ω_m in Section 3 it follows that $\omega_m(f) = c_{F^m}(\mathcal{D}(v_P)(\{T(b) \circ \alpha_N\})) = c_{F^m}(\mathcal{D}(v_P)(\{\alpha_\beta\})) = [\beta]$. □

We shall prove the following proposition.

Proposition 5.1. *An element $a \in [P, SG]$ lies in $J([P, SO])$ if and only if there exists a fold-map $f : P \to P$ homotopic to id_P such that $\omega(f) = a$.*

Proof. Since $\pi_1(SO) \cong \pi_1(SG)$, the assertion for $n = 1$ follows from Proposition 5.3. Let $n > 1$. Given a fold-map $f : P \to P$ homotopic to id_P, we have a bundle map $\nu(f) : \nu_P \to \nu_P$ such that $\omega(f) = (i_{F^1,SG})_* \circ c_{F^1}(\mathcal{D}(\nu_P)(\{T(\nu(f)) \circ \alpha_P\}))$. It follows from Proposition 2.6 that $\omega(f)$ lies in $J([P, SO])$ (this has been proved in [An3, Proposition 4.5] in a slightly different way).

Next we shall prove that $a \in J([P, SO])$ has such a fold-map f with $\omega(f) = a$. The proof is parallel to that of Theorem 1.

Let $\beta : P \to SO(k)$ be a map such that $J([\beta]) = a$. The orientation preserving isomorphism $h_\beta : \theta^k_P \to \theta^k_P$ as in Lemma 2.5 has an orientation preserving isomorphism $b : \nu_P \to \nu_P$ such that $id_{\nu_P} \oplus h_\beta \simeq b \oplus id_{\theta^k_P} : \nu_P \oplus \theta^k_P \to \nu_P \oplus \theta^k_P$. By [An3, Proposition 3.3] there exists an orientation preserving isomorphism $b' : \tau_P \to \tau_P$, which is uniquely determined up to homotopy, such that $t_P \circ (b' \oplus b) \circ t_P^{-1}$ is homotopic to the identity of θ^{2k}_P. Here consider the inclusion $i_{n+1,n+k} : GL^+_{n+1}(TP \oplus \theta_P, TP \oplus \theta_P) \to GL^+_{n+k}(TP \oplus \theta^k_P, TP \oplus \theta^k_P)$, which is a homotopy equivalence. Then there exists an orientation preserving isomorphism $b'' : TP \oplus \theta_P \to TP \oplus \theta_P$ over the identity of P such that $i_{n+1,n+k}(b'') \simeq b'$. We obtain the homotopy class of a section s of $\Gamma(P, P)$ such that $i_{SO} \circ i(P, P)^{-1}(s)$ is homotopic to b'' as above. Therefore, it follows from Theorem 4.1 that there exists a fold-map $f : P \to P$ (homotopic to id_P) such that $j^2 f$ is homotopic to s in $\Gamma(P, P)$. Similarly, we obtain that $i_{n+1,n+k}(\mathcal{T}(f)) = \tau(f)$ and $\tau(f) \simeq b'$, and so $\nu(f) \simeq b$. Since

$$
\begin{aligned}
&(i_{F^1,SG})_* \circ c_{F^1}(\mathcal{D}(v_P)(\{T(b) \circ \alpha_P\})) \\
&\qquad = (i_{F^1,SG})_* \circ c_{F^1}(\mathcal{D}(v_P)(\{\alpha_P\}) \circ \{T(h_\beta)\}) \\
&\qquad = (i_{F^1,SG})_* \circ c_{F^1}(\{c_{P^0} \circ T(h_\beta)\})
\end{aligned}
$$

by Lemmas 2.4 and 2.5, we have that $\omega(f) = J([\beta]) = a$ by the definition of ω. □

We shall give some examples of fold-maps in the dimensions 1 and 2.

Example 5.2. Let $f : N \to P$ be a fold-map. If $TN \oplus \theta_N$ and $TP \oplus \theta_P$ are trivial bundles with fixed trivializations, then the bundle map $\mathcal{T}(f) : TN \oplus \theta_N \to TP \oplus \theta_P$ induces a map $M(f) : N \to SO(n+1)$. Let $R(x) \in SO(2)$ be the rotation such that $R(x)\mathbf{e}_1 = {}^t(\cos x, \sin x)$. The assertions (1) and (2) below follow from [An3, Example 3.4].

(1) Let S^1 be parametrized by x of $e^{\sqrt{-1}x}$ $(0 \le x \le 2\pi)$ inducing the trivialization of TS^1. Then consider the fold-map $f^1 : S^1 \to \mathbf{R}^1$ defined by $f^1(x) = \cos 2x$. Then $M(f^1)$ is homotopic to the map $R^2 : S^1 \to SO(2)$ defined by $R^2(x) = R(2x)$.

(2) Let $S^1 \times S^1$ be parametrized by (x, y) of $(e^{\sqrt{-1}x}, e^{\sqrt{-1}y})$ $(0 \le x, y \le 2\pi)$ inducing the trivialization of $T(S^1 \times S^1)$. Consider the fold-map $f^2 : S^1 \times S^1 \to \mathbf{R}^2$ defined by $f^2(x, y) = ((3 + \cos 2y)\cos 2x, (3 + \cos 2y)\sin 2x)$. Then $M(f^2)$ is homotopic to the map $\Pi : S^1 \times S^1 \to SO(3)$ defined by $\Pi(x, y) = ((1)\dot{+}R(2y))(R(2x)\dot{+}(1))$.

By identifying $S^i \setminus \{\text{a point}\}$ with $\mathbf{R}^i$, f^i induces the fold-map into S^i of degree 0 ($i = 1, 2$). Let $\beta : S^1 \to SO(k)$ represent the generator of $\pi_1(SO(k))$. Consider the fold-map $f^{1,m} : S^1 \to S^1$ of degree m obtained by the connected sum $f^1 \sharp m_{(S^1)} : S^1 \sharp S^1 \to S^1$ for $m \neq 0$, where the two connecting points in $S^1 \sharp S^1$ should be changed from regular points of f^1 and $m_{(S^1)}$ to the fold points of $f^1 \sharp m_{(S^1)}$. It follows that $\nu(f^{1,m})$ appearing in Proposition 3.3 is homotopic to the bundle map $b^m_\beta : \theta^k_{S^1} \to \theta^k_{S^1}$ defined by $b^m_\beta(x, \mathbf{v}) = (mx, \beta(x)\mathbf{v})$ as in the case of Example 5.2 (1).

Proposition 5.3. *Let $f^i : S^i \to S^i$ and $f^{1,m} : S^1 \to S^1$ be the fold-maps given above. Then $\omega_0(f^1)$ and $\omega_0(f^2)$ are the generators of $\pi_1(F^0) \cong \mathbf{Z}/2\mathbf{Z}$ and $\pi_2(F^0) \cong \mathbf{Z}/2\mathbf{Z}$ respectively. Furthermore, $\omega_m(f^{1,m})$ is the generator of $\pi_1(F^m) \cong \mathbf{Z}/2\mathbf{Z}$ ($m \neq 0$).*

Proof. We first recall the generator of $\pi_3(S^2)$, which induces the generator of π^s_1. We identify S^3 with $\partial(D^2 \times D^2)$ and S^1 is parametrized by x as in Example 5.2 (1). If $\mu' : S^1 \times S^1 \to S^1$ is the map $\mu(x, y) = x + y$ (modulo 2π), then it induces the map $\mu : S^1 \times D^2 \cup D^2 \times S^1 \to S^2$ by the cone-wise construction, which is the generator. Note that $(\mu|S^1 \times D^2)(x, \mathbf{v}) = R(x)\mathbf{v}$.

Consider the embedding $e_{S^1\times(-1,1)} : S^1 \times (-1, 1) \to \mathbf{R}^2$ defined by $e_{S^1\times(-1,1)}(x, t) = (1-t)e^{\sqrt{-1}x}$. If we identify $T_{(x,t)}(S^1 \times (-1, 1))$ with $\mathbf{R}^2$ under the trivialization of TS^1 in Example 5.2 (1), then $d_{(x,t)}e_{S^1\times(-1,1)}$ is identified with $R(x)$. When we recall the trivialization t_{S^1} of $\tau_{S^1} \oplus \nu_{S^1}$, considered before defining duality maps in Section 2, $t_{S^1} \circ (\tau(f^1) \oplus \nu(f^1)) \circ t^{-1}_{S^1}$ must be homotopic to the identity of $\theta^{2k}_{S^1}$. Therefore, since $M(f^1)$ is homotopic to the map $x \mapsto R(2x)$, $\nu(f^1) : \nu_{S^1} \to \nu_{S^1}$ must be identified with $b^0_\beta : \theta^k_{S^1} \to \theta^k_{S^1}$.

The case $n = 1$. Consider the embedding $e : S^1 \to \mathbf{R}^{1+k}$ with normal bundle $S^1 \times D^k$. Let $b : S^1 \times D^k \to D^k$ be the bundle map defined by $b(x, \mathbf{v}) = \beta(x)\mathbf{v}$, where $\beta : S^1 \to SO(k)$ represents a generator of $\pi_1(SO)$. Then it is known from the observation above concerning the generator of π^s_1 that $\mathcal{D}(\{T(b) \circ \alpha_{S^1}\}) \in \{S^{1+k}, S^k\}$ is a generator of π^s_1. Let $\hat{b} : (1, 0) \times D^k \to S^1 \times D^k$ be the bundle map $i_{(1,0)} \times id_{D^k}$, where $(1, 0)$ is the point of S^1 and $i_{(1,0)}$ is the inclusion. Then since $\nu(f^1) \simeq \hat{b} \circ b$, we have that

$$\begin{aligned}\omega_0(f^1) &= c_{F^0}(\mathcal{D}(v_{S^1})(\{T(\hat{b} \circ b) \circ \alpha_{S^1}\})) \\ &= c_{F^0}(\mathcal{D}(v_{S^1})(\{T(b) \circ \alpha_{S^1}\}) \circ \mathcal{D}(v_{S^1})(\{T(\hat{b})\})).\end{aligned}$$

It follows from [Sp2, Theorem 6.1] that $\mathcal{D}(v_{S^1})(\{T(\hat{b})\}) \in \{(S^1)^0, S^1\}$ is represented by a base point preserving map $j_{S^1} : (S^1)^0 \to S^1$ with $j_{S^1}|S^1 = id_{S^1}$. Indeed, $(\mathcal{D}(v_{S^1})(\{T(\hat{b})\}))_* : H_1((S^1)^0) \to H_1(S^1)$ is the identity of $\mathbf{Z}$. This implies the assertion for f^1.

Next we deal with $f^{1,m}$ for $m \neq 0$. Let $m_{(S^1)^0} : (S^1)^0 \to (S^1)^0$ be the map $m_{(S^1)} \cup id_{*_{S^1}}$. Let $b^m : \theta^k_{S^1} \to \theta^k_{S^1}$ be the map defined by $b^m(x, \mathbf{v}) = (mx, \mathbf{v})$. We have that $b^m_\beta = b^m \circ b^1_\beta$. Since $\pi_1(SO) \cong \mathbf{Z}/2\mathbf{Z}$, we have that $\mathcal{D}(v_{S^1})(\{T(b^1_\beta)\}) = \{T(b^1_\beta)\}$. Since $T(b^m)$ is homotopic to $m_{(S^1)^0} \wedge id_{S^k}$, we have that $\mathcal{D}(v_{S^1})(\{T(b^m)\}) \in \{(S^1)^0, (S^1)^0\}$ is represented by a map $\Upsilon : S((S^1)^0) \to S((S^1)^0)$ by [Sp2, Theorem 6.1] such that

(1) $\Upsilon^* : H^1(S((S^1)^0); \mathbf{Z}) \to H^1(S((S^1)^0); \mathbf{Z})$ maps 1 to m,

(2) $\Upsilon^* : H^2(S((S^1)^0); \mathbf{Z}) \to H^2(S((S^1)^0); \mathbf{Z})$ maps 1 to 1.

Since $S((S^1)^0)$ is homotopy equivalent to $S^2 \vee S^1$, we may suppose that $\Upsilon|S^2 = id_{S^2}$ and that $\Upsilon|S(\{x\} \cup \{*\}) : S(\{x\} \cup \{*\}) \to S(\{x\} \cup \{*\})$ is of degree m. Thus, $\mathcal{D}(v_{S^1})(\{T(b^m)\}) \in \{(S^1)^0, (S^1)^0\}$ is represented by the map $S^{k-1}(\Upsilon)$. Hence, we have

$$\begin{aligned}\omega_m(f^{1,m}) &= c_{F^m}(\mathcal{D}(v_{S^1})(\{T(\nu(f^{1,m})) \circ \alpha_{S^1}\})) \\ &= c_{F^m}(\mathcal{D}(v_{S^1})(\{T(b^m \circ b^1_\beta) \circ \alpha_{S^1}\})) \\ &= c_{F^m}(\mathcal{D}(v_{S^1})(\{\alpha_{S^1}\}) \circ \{T(b^1_\beta)\} \circ \mathcal{D}(v_{S^1})(\{T(b^m)\})) \\ &= c_{F^m}(\{c_{(S^1)^0}\} \circ \{T(b^1_\beta) \circ S^{k-1}(\Upsilon)\}).\end{aligned}$$

Since

$$(S^{k-1}(c_{(S^1)^0}) \circ T(b^1_\beta) \circ S^{k-1}(\Upsilon))|S^k \wedge S(\{x\} \cup *_{S^1}) = T(\beta(x)) \circ m_{(S^k)},$$

we have that $\omega_0(f^{1,m}) = (h_{F^1,F^m})_*([\beta])$, where $T(\beta(x))$ is the Thom map of $\beta(x) : \mathbf{R}^k \to \mathbf{R}^k$ and β is considered as an element of $[S^1, F^1]$.

The case $n = 2$. Consider the embedding $e' : S^1 \times S^1 \to \mathbf{R}^{2+k}$ with normal bundle $S^1 \times S^1 \times D^k$. Let $B : S^1 \times S^1 \times D^k \to D^k$ be the bundle map defined by $B(x, y, \mathbf{v}) = R(x)R(y)\mathbf{v}$. Then it is known that both $\{T(B) \circ \alpha_{S^1 \times S^1}\}$ and $\mathcal{D}(v_{S^1 \times S^1})(\{T(B) \circ \alpha_{S^1 \times S^1}\})$ are the generator of π_2^s (see [To, Propositions 3.1 and 5.3]). Let $\mathbf{a} = (1, 0, 0)$, $i'_{\mathbf{a}} : \mathbf{a} \to S^2$ be the inclusion and $\hat{B} : \mathbf{a} \times D^k \to S^2 \times D^k$ be the bundle map $i'_{\mathbf{a}} \times id_{D^k}$. Then we have by Example 5.2 (2) that

$$\begin{aligned}\omega_0(f^2) &= c_{F^0}(\mathcal{D}(v_{S^1 \times S^1})(\{T(\hat{B} \circ B)\} \circ \{\alpha_{S^1 \times S^1}\})) \\ &= c_{F^0}(\mathcal{D}(v_{S^1 \times S^1})(\{T(B) \circ \alpha_{S^1 \times S^1}\}) \circ \mathcal{D}(v_{S^1 \times S^1})(\{T(\hat{B})\})).\end{aligned}$$

It follows from [Sp2, Theorem 6.1] that $\mathcal{D}(v_{S^1 \times S^1})(\{T(\hat{B})\}) \in \{(S^2)^0, S^2\}$ is represented by a base point preserving map $j_{S^2} : (S^2)^0 \to S^2$ with $j_{S^2}|S^2 = id_{S^2}$. Indeed, $\mathcal{D}(v_{S^1 \times S^1})(\{T(\hat{B})\})_* : H_2((S^2)^0) \to H_2(S^2)$ is the identity of $\mathbf{Z}$. This implies the assertion. $\square$

Remark 5.4. Let $f : N_i \to P (i = 1, 2)$ be fold-maps of degree 0. Then the disjoint union $f_1 \cup f_2 : N_1 \cup N_2 \to P$ is also a fold-map of degree 0. We define the sum $[f_1] + [f_2]$ to be $[f_1 \cup f_2]$. By this additive structure on $\Omega_{fold,0}(P)$ we can define the Grothendieck group for $\Omega_{fold,0}(P)$, which is denoted by $K(fold, 0)(P)$. Let S^n be the unit sphere in $\mathbf{R}^{n+1}$ with coordinates $(x_1, \dots, x_{n+1})$. Let $p_{S^n} : S^n \to \mathbf{R}^n$ be the projection $(x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_n)$. Let $e_{\mathbf{R}^n} : \mathbf{R}^n \to P$ be any local chart of P. Then $[e_{\mathbf{R}^n} \circ p_{S^n}]$ becomes the null element. Furthermore, the map ω_0 induces the homomorphism $K(fold, 0)(P) \to [P, F^0]$. For example, if $P = S^1$, then it is not difficult to prove that $\Omega_{fold,0}(S^1) \cong K(fold, 0)(S^1) \cong [S^1, F^0] \cong \mathbf{Z}/2\mathbf{Z}$.

Remark 5.5. For the case $P = \mathbf{R}^n$, it has been observed in [Sa, Section 5] by using [K-M] that the set of fold-cobordism classes of fold-maps into $\mathbf{R}^n$ forms a non-trivial group in many dimensions.

6. Proof of Proposition 4.6

In this section any homotopy h_λ in $\Gamma(X, P)$ refers to a homotopy h_λ relative to $X \cap (\mathbf{R}^n \setminus \mathrm{Int} D_{2r})$ in $\Gamma(X, P)$, where X is a submanifold in $\mathbf{R}^n$.

For a Riemannian manifold X without boundary, consider the exponential map $\exp_X : TX \to X$ defined by the Levi-Civita connection (see [K-N]). Let E be a subbundle of TX. Let δ be some sufficiently small positive smooth function on X. In this paper $D_\delta(E)$ always denotes the associated δ-disk bundle of E with radius δ such that $\exp_X |D_\delta(E)_x$ is an embedding for any $x \in X$.

Let L_i $(i = 1, 2)$ be two oriented lines of $\mathbf{R}^n$. If $\mathbf{e}(L_1)$ and $\mathbf{e}(L_2)$ are independent, then they uniquely determine a curve $r_\lambda(L_1, L_2)$ in $SO(n)$ defined as follows. Let θ be the angle of $\mathbf{e}(L_1)$ and $\mathbf{e}(L_2)$ less than π. Then we have the great circle of S^{n-1} through $\mathbf{e}(L_1)$ and $\mathbf{e}(L_2)$, and the rotation $r_\lambda(L_1, L_2)$ is the identity on the space orthogonal to $\mathbf{e}(L_1)$ and $\mathbf{e}(L_2)$ and rotates this great circle to the direction of $\mathbf{e}(L_1)$ to $\mathbf{e}(L_2)$ so as to carry $\mathbf{e}(L_1)$ to the point with rotated angle $\lambda\theta$, which is, in particular, equal to $\mathbf{e}(L_2)$ when $\lambda = 1$. Thus $r_1(L_1, L_2)(\mathbf{e}(L_1)) = \mathbf{e}(L_2)$. If $L_1 = L_2$ and $\mathbf{e}(L_1) = \mathbf{e}(L_2)$, then we set $r_\lambda(L_1, L_2) = E_n$ for all λ, where E_n is the unit matrix of rank n.

Lemma 6.1. *Let $s \in \Gamma^{tr}(\mathbf{R}^n, P)$ be a section satisfying* (H-i) *and* (H-ii) *of Proposition* 4.6. *For any positive number ε such that $O(p_j; 2\varepsilon)$ $(1 \le j \le m)$ are all disjoint each other, we set $S(s)_0 = S(s) \setminus (\cup_{j=1}^m O(p_j; \varepsilon))$. Then there exists a homotopy s_λ relative to $\mathbf{R}^n \setminus \mathrm{Int} D_{2r}$ in $\Gamma^{tr}(\mathbf{R}^n, P)$ with $s_0 = s$ satisfying*

(6.1.1) *$S(s_\lambda) = S(s)$ for any λ,*

(6.1.2) *for any point $c \in S(s_1)_0$ the angle of $\mathbf{e}(K(s_1)_c)$ and $\mathbf{e}(\nu(s_1)_c)$ is less than $\pi/2$,*

(6.1.3) *for any point $c \in S(s_1)_0 \cap D_r$, we have $\mathbf{e}(K(s_1)_c) = \mathbf{e}(\nu(s_1)_c)$.*

Proof. Let $\exp_{\mathbf{R}^n, x} : T_x\mathbf{R}^n \to \mathbf{R}^n$ denote the exponential map defined near $x \in \mathbf{R}^n$. Since $\nu(s)$ is a trivial bundle, its element is written as (c, t). There exists a small positive number δ such that the map

$$e : D_\delta(\nu(s))|_{S(s) \cap D_{2r}} \to \mathbf{R}^n$$

defined by $e(c, t) = \exp_{\mathbf{R}^n, c}(c, t)$ is an embedding, where $c \in S(s) \cap D_{2r}$ and $(c, t) \in D_\delta(\nu(s)_c)$ (note that $e|S(s) = id_{S(s)}$). Since for $c \notin \mathbf{e}(s)^{-1}(\Delta^-)$, we have that $\mathbf{e}(K(s)_c) \neq -\mathbf{e}(\nu(s)_c)$, we can consider the rotation $r_\lambda(\nu(s)_c, K(s)_c)$. Let $\phi : [0, \infty) \to \mathbf{R}$ be a decreasing smooth function such that $0 \le \phi(u) \le 1$, $\phi(u) = 0$ if $u \ge 3r/2$, and $\phi(u) = 1$ if $u \le r$. Let $\psi : [0, \infty) \to \mathbf{R}$ be a decreasing smooth function such that $0 \le \psi(t) \le 1$, $\psi(0) = 1$, and $\psi(t) = 0$ if $t \ge \delta$. Let ℓ_a be the parallel translation of $\mathbf{R}^n$ defined by $\ell_a(x) = x + a$.

If we represent $s(x) \in \Omega^{10}(\mathbf{R}^n, P)$ by a jet $j^2_x\sigma_x$ for a germ $\sigma_x : (\mathbf{R}^n, x) \to$

$(P, \sigma(x))$, then we define the homotopy s'_λ of $\Gamma^{tr}(\mathbf{R}^n \setminus \{p_1, \dots, p_m\}, P)$ by

$$\begin{cases} s'_\lambda(e(c,t)) = j^2_{e(c,t)}(\sigma_{e(c,t)} \circ \ell_{e(c,t)} \circ r_{\phi(\|c\|)\psi(|t|)\lambda}(\nu(s)_c, K(s)_c) \circ \ell_{-e(c,t)}) \\ \qquad\qquad\qquad \text{if } c \in S(s) \cap D_{2r} \text{ and } |t| \leq \delta, \\ s'_\lambda(x) = s(x) \quad \text{if } x \notin \mathrm{Im}(e). \end{cases}$$

If either $|t| \geq \delta$, or $\|c\| \geq 3r/2$, then we have

$$s'_\lambda(e(c,t)) = j^2_{e(c,t)}(\sigma_{e(c,t)} \circ \ell_{e(c,t)} \circ \ell_{-e(c,t)})) = j^2_{e(c,t)}(\sigma_{e(c,t)}) = s(e(c,t)).$$

Hence, s'_λ is well defined. Furthermore, we have that

(1) $\pi^2_P \circ s'_\lambda(x) = \pi^2_P \circ s(x)$,

(2) $s'_\lambda|S(s) = s|S(s)$ and $S(s'_\lambda) = S(s)$,

(3) if $c \in S(s)_0 \cap D_r$, then we have that $\mathbf{e}(K(s'_1)_c) = r_1(K(s)_c, \nu(s)_c)(\mathbf{e}(K(s)_c)) = \mathbf{e}(\nu(s)_c)$ and

(4) $s'_\lambda|\mathbf{R}^n \setminus \{p_1, \dots, p_m\}$ is transverse to $\Sigma^{10}(N, P)$.

The property (6.1.2) is satisfied for s'_1 inside of D_{2r} by the construction and outside of D_{2r} by Remark 4.4. Applying the homotopy extension property to s and $s'_\lambda|\mathbf{R}^n \setminus (\cup_{j=1}^m O(p_j;\varepsilon))$ together with the property (4), we obtain the required homotopy s_λ in $\Gamma^{tr}(\mathbf{R}^n, P)$ such that $s_0 = s$ and $s_\lambda|\mathbf{R}^n \setminus (\cup_{j=1}^m O(p_j;\varepsilon)) = s'_\lambda|\mathbf{R}^n \setminus (\cup_{j=1}^m O(p_j;\varepsilon))$. $\square$

Lemma 6.2. *Let s be a section of $\Gamma^{tr}(\mathbf{R}^n, P)$ satisfying the properties* (6.1.2) *and* (6.1.3) *for s* (*in place of s_1*) *of Lemma* 6.1. *Then there exists a homotopy s_λ relative to $\mathbf{R}^n \setminus \mathrm{Int}D_{2r}$ in $\Gamma^{tr}(\mathbf{R}^n, P)$ with $s_0 = s$ such that*

(6.2.1) *$S(s_\lambda) = S(s)$ for any λ,*

(6.2.2) *$\pi^2_P \circ s_1|S(s)_0$ is an immersion into P such that $d(\pi^2_P \circ s_1|S(s)_0) : TS(s)_0 \to TP$ is equal to $d^1(s_1)|TS(s)_0$.*

Proof. Recall $d^1(s)|TS(s)_0 : TS(s)_0 \to TP$ in Section 1. Since by the assumption (6.1.2) for s the restriction $d^1(s)|TS(s)_0$ is injective. By the Hirsch Immersion Theorem (see [H1]) we have a homotopy $b_\lambda : TS(s)_0 \to TP$ of bundle monomorphisms over $i_\lambda : S(s)_0 \to P$ relative to $S(s)_0 \setminus \mathrm{Int}D_{2r}$ such that $b_0 = d^1(s)|TS(s)_0$ and that i_1 is an immersion with $d(i_1) = b_1$.

We extend b_λ to a homotopy $m'_\lambda : T\mathbf{R}^n|_{S(s)_0} \to TP$ so that $m'_\lambda|K(s)_{S(s)_0}$ is the null-homomorphism and $m'_\lambda|TS(s)_0 = b_\lambda$. It is clear that m'_λ is of rank $n-1$. Hence, it induces a map $m'_\lambda : S(s)_0 \to \Sigma^1(\mathbf{R}^n, P)$ denoted by the same symbol m'_λ, where $\Sigma^1(\mathbf{R}^n, P)$ refers to the submanifold in $J^1(\mathbf{R}^n, P)$. By applying the covering homotopy property of the fibre bundle $\pi^2_1|\Sigma^{10}(\mathbf{R}^n, P) : \Sigma^{10}(\mathbf{R}^n, P) \to \Sigma^1(\mathbf{R}^n, P)$ to $s|S(s)_0 : S(s)_0 \to \Sigma^{10}(\mathbf{R}^n, P)$ and m'_λ, we obtain a homotopy $m_\lambda : S(s)_0 \to \Sigma^{10}(\mathbf{R}^n, P)$ such that $m_0 = s|S(s)_0$ and $\pi^2_1 \circ m_\lambda = m'_\lambda$. Since s is transverse to $\Sigma^{10}(\mathbf{R}^n, P)$, there are small tubular neighborhoods $U(S(s))$ of $S(s)$ and $U(\Sigma^{10}(\mathbf{R}^n, P))$ of $\Sigma^{10}(\mathbf{R}^n, P)$ with projections $p_S : U(S(s)) \to S(s)$ and $p_\Sigma : U(\Sigma^{10}(\mathbf{R}^n, P)) \to \Sigma^{10}(\mathbf{R}^n, P)$, which induces structures of fibre bundles with fibre $[-\delta, \delta]$ respectively so that $s|U(S(s)) : U(S(s)) \to U(\Sigma^{10}(\mathbf{R}^n, P))$ becomes a bundle map over $s|S(s)$.

By applying the covering homotopy property of the bundle map $s|p_S^{-1}(S(s)_0) : p_S^{-1}(S(s)_0) \to U(\Sigma^{10}(\mathbf{R}^n, P))$ over $s|S(s)_0$ to $s|p_S^{-1}(S(s)_0)$ and m_λ, we obtain a smooth homotopy of bundle maps $h'_\lambda : p_S^{-1}(S(s)_0) \to U(\Sigma^{10}(\mathbf{R}^n, P))$ over m_λ with $h'_0 = s|p_S^{-1}(S(s)_0)$. By the homotopy extension property applied to the bundle map $s|U(S(s))$ and the homotopy h'_λ, we can extend h'_λ to the smooth homotopy of bundle maps $h_\lambda : U(S(s)) \to U(\Sigma^{10}(\mathbf{R}^n, P))$ with $h_0 = s|U(S(s))$.

By applying finally the homotopy extension property to s and

$$h_\lambda : (U(S(s)), \partial U(S(s))) \to (U(\Sigma^{10}(\mathbf{R}^n, P)), \partial U(\Sigma^{10}(\mathbf{R}^n, P))),$$

we obtain the extended homotopy $s_\lambda : \mathbf{R}^n \to \Omega^{10}(\mathbf{R}^n, P)$ of s. By the construction of s_λ, s_1 satisfies the required property. $\square$

Here we give two lemmas necessary for the proof of Proposition 4.6. Their proofs will be elementary and so are left to the reader.

Lemma 6.3. *Let S be a manifold of dimension $n-1$ with empty boundary. Let $f_i : S \times (-a, a) \to P$, $a > 0$ $(i = 1, 2)$ be fold-maps which fold only on $S \times 0$ such that*

(i) $f_1|S \times 0 = f_2|S \times 0$,

(ii) $d_{(c,0)} f_1 = d_{(c,0)} f_2$ *and* $d^2_{(c,0)} f_1 = d^2_{(c,0)} f_2$ *for any* $c \in S$ *and*

(iii) $K(j^2 f_i)_{(c,0)}$ *are tangent to* $c \times (-a, a)$ *and are oriented by the canonical direction of* $(-a, a)$.

Let $\eta : S \to \mathbf{R}$ be any smooth function. Then there exists a positive function $\varepsilon : S \to \mathbf{R}$ such that the map $(1-\eta)f_1 + \eta f_2$, defined by $((1-\eta)f_1 + \eta f_2)(c, t) = (1-\eta(c))f_1(c, t) + \eta(c) f_2(c, t)$ for $t \in (-\varepsilon(c), \varepsilon(c))$, is a fold-map which folds only on $S \times 0$, that $d_{(c,0)}((1-\eta)f_1 + \eta f_2) = d_{(c,0)} f_i$, and that $d^2_{(c,0)}((1-\eta)f_1 + \eta f_2) = d^2_{(c,0)} f_i$.

Lemma 6.4. *Let $E \to S$ be an oriented smooth line bundle with metric over an $(n-1)$-dimensional manifold, where S is identified with the zero-section, and let (Ω, Σ) be a pair of a smooth manifold and its submanifold of codimension 1. Let $\varepsilon : S \to \mathbf{R}$ be a positive smooth function and $D_\varepsilon(E)$ be the associated disk bundle of E with radius ε. Let $h_i : D_\varepsilon(E) \to (\Omega, \Sigma)$ $(i = 0, 1)$ be smooth maps such that $S = h_0^{-1}(\Sigma) = h_1^{-1}(\Sigma)$, $h_0|S = h_1|S$ and that h_i are transverse to Σ. Assume that for any $c \in S$, the monomorphisms $T_cE/T_cS \to T_{h_i(c)}\Omega/T_{h_i(c)}\Sigma$ induced from $d_c(h_i)$ send a unit vector to vectors with the same direction on $T_{h_i(c)}\Omega/T_{h_i(c)}\Sigma$. Then for a sufficiently small positive function $\varepsilon : S \to \mathbf{R}$, there exists a homotopy $h_\lambda : (D_\varepsilon(E), S) \to (\Omega, \Sigma)$ such that*

(1) $h_\lambda|S = h_0|S$, $h_\lambda^{-1}(\Sigma) = h_0^{-1}(\Sigma)$ *for any* λ,

(2) h_λ *is smooth and is transverse to* Σ *for any* λ.

For a vector bundle $\mathcal{F}$ over Σ and a map $\iota : S \to \Sigma$, the induced bundle map $\iota^*(\mathcal{F}) \to \mathcal{F}$ over ι is denoted by $(\iota)_\mathcal{F}$ in the proof below.

Proof of Proposition 4.6. By Lemmas 6.1 and 6.2 we may assume that s satisfies the properties (6.1.2), (6.1.3) and (6.2.2) with s_1 being replaced by s. Since s is smooth near $S(s)$ and is an embedding near $S(s)$, we can choose a Riemannian metric on $\Omega^{10}(\mathbf{R}^n, P)$ so that the induced metric by s near $S(s)$ coincides with the metric on $\mathbf{R}^n$ near $S(s)$. Take any Riemannian metric on P. Set

$\exp_\Omega = \exp_{\Omega^{10}(\mathbf{R}^n,P)}$ for simplicity. We set $E(S(s)_0) = \exp_{\mathbf{R}^n}(D_\delta(K(s)_{S(s)_0}))$, where $\delta : \Sigma^{10}(\mathbf{R}^n, P) \to \mathbf{R}$ is a sufficiently small positive function such that $\delta \circ s|S(s)_0 \cap D_{2r}$ is constant. Furthermore, if we identify $Q(s)|_{S(s)_0}$ with the orthogonal normal line bundle to the immersion $\pi_P^2 \circ s|S(s)_0 : S(s)_0 \to P$, then $\exp_P|D_\gamma(Q(s)|_{S(s)_0})$ is an immersion for some positive function γ. In the proof we represent points $E(S(s)_0)$ and $\exp_P(D_\gamma(Q(s)|_{S(s)_0}))$ as (c,t) and (c,u), where $c \in S(s)_0$, $|t| \leq \delta(s(c))$ and $|u| \leq \gamma(c)$ respectively. In the proof we say that a smooth homotopy

$$h_\lambda : (E(S(s)_0), \partial E(S(s)_0)) \to (\Omega^{10}(\mathbf{R}^n, P), \Sigma^0(\mathbf{R}^n, P))$$

has the property (C) if it satisfies that for any λ

(C-1) $h_\lambda^{-1}(\Sigma^{10}(\mathbf{R}^n, P)) = S(s)_0$ and $h_\lambda|S(s)_0 = h_0|S(s)_0$ and

(C-2) h_λ is smooth and transverse to $\Sigma^{10}(\mathbf{R}^n, P)$.

For a point $c \in S(s)_0$, the intrinsic derivative $d_c^2(s) : K(s)_c \to \mathrm{Hom}(K(s)_c, Q(s)_c)$ defines the positive function $b : S(s)_0 \to \mathbf{R}$ by the equation

$$(d_c^2(s)(\mathbf{e}(K(s)_c)))(\mathbf{e}(K(s)_c)) = 2b(c)(\mathbf{e}(Q(s)_c)).$$

If we choose δ sufficiently small compared with γ, then we can define the fold-map $g_0 : E(S(s)_0) \to P$ by

$$g_0(c,t) = (c, b(c)t^2)(= \exp_P(c, b(c)t^2)).$$

Let r_0 be a small positive real number with $r_0 < r/10$. Now we need to modify g_0 by using Lemma 6.3 so that g_0 is compatible with f_0. Let $\eta : S(s)_0 \to \mathbf{R}$ be a smooth function such that

(i) $0 \leq \eta(c) \leq 1$,

(ii) $\eta(c) = 0$ for $x \in \mathbf{R}^n \setminus \mathrm{Int}D_{2r-r_0}$,

(iii) $\eta(c) = 1$ for $x \in D_{2r-2r_0}$.

Then consider the map $G : (\mathbf{R}^n \setminus \mathrm{Int}D_{2r-r_0}) \cup E(S(s)_0) \to P$ defined by

$$\begin{cases} G(x) = f_0(x) & \text{if} \quad x \in \mathbf{R}^n \setminus \mathrm{Int}D_{2r-r_0}, \\ G(c,t) = (1-\eta(c))f_0(c,t) + \eta(c)g_0(c,t) & \text{if} \quad (c,t) \in E(S(s)_0). \end{cases}$$

It follows from Lemma 6.3 that G is a fold-map defined on a neighborhood of $(\mathbf{R}^n \setminus \mathrm{Int}D_{2r}) \cup E(S(s)_0)$, where δ is replaced by a smaller one if necessary so that $G|E(S(s)_0)$ folds only on $S(s)_0$, and that $d_c^i(G) = d_c^i(g_0)$ for any $c \in S(s)_0 \cap D_{2r}$ $(i = 1, 2)$. Furthermore, we note that if $\|c\| \geq 2r - r_0$, then $G(c,t) = f_0(c,t)$.

Next we shall construct a homotopy h_λ relative to $E(S(s)_0) \cap \mathrm{Int}(D_{2r} \setminus D_{2r-r_0})$ in $\Gamma^{tr}(E(S(s)_0) \cap \mathrm{Int}D_{2r}, P)$ satisfying the property (C) restricted to $E(S(s)_0) \cap \mathrm{Int}D_{2r}$ such that $h_0 = s$ and $h_1 = j^2G$ on $E(S(s)_0) \cap \mathrm{Int}D_{2r}$.

By applying Lemma 6.4 to the section s, we first obtain a homotopy $h'_\lambda \in \Gamma^{tr}(E(S(s))_0, P)$ with $h'_0 = s$ and $h'_1 = \exp_\Omega \circ ds \circ \exp_{\mathbf{R}^n}^{-1}$ on $E(S(s)_0)$ satisfying the properties (1) and (2) of Lemma 6.4. Since $ds|(K(s)|_{S(s)_0}) : K(s)|_{S(s)_0} \to T\Omega^{10}(\mathbf{R}^n, P)$ and $(s|S(s)_0)_{\mathbf{K}} : K(s)|_{S(s)_0} \to \mathbf{K} \subset T\Omega^{10}(\mathbf{R}^n, P)$ are homotopic by a homotopy of monomorphisms transverse to $T\Sigma^{10}(\mathbf{R}^n, P)$, we can construct a homotopy h''_λ in $\Gamma^{tr}(E(S(s)_0), P)$ satisfying the property (C) such that $h''_0 = h'_1$ and $h''_1 = \exp_\Omega \circ (s|S(s)_0)_{\mathbf{K}} \circ \exp_{\mathbf{R}^n}^{-1}$ on $E(S(s)_0)$. By pasting h'_λ and h''_λ we obtain a homotopy $h^1_\lambda \in \Gamma^{tr}(E(S(s)_0), P)$ satisfying the property (C) with $h^1_0 = s$ and $h^1_1 = \exp_\Omega \circ (s|S(s)_0)_{\mathbf{K}} \circ \exp_{\mathbf{R}^n}^{-1}$ on $E(S(s)_0)$.

Now recall the additive structure of $J^2(\mathbf{R}^n, P)$ defined by using the fixed Riemannian metric on P in [An2, Section 1]. Then we have the homotopy $j_\lambda : S(s)_0 \to J^2(\mathbf{R}^n, P)$ defined by

$$j_\lambda(c) = (1-\lambda)s(c) + \lambda j^2 G(c) \quad \text{covering} \quad i_1 : S(s)_0 \to P.$$

Since $K(s)_c = K(j^2G)_c$ and $Q(s)_c = Q(j^2G)_c$ by the construction of the immersion i_1 and the fold-map G, it follows that for any $c \in S(s)_0$ we have $K(j_\lambda)_c = K(s)_c$ and $Q(j_\lambda)_c = Q(s)_c$. Hence, we have that

$$d^i_c(j_\lambda) = (1-\lambda)d^i_c(s) + \lambda d^i_c(j^2G) = d^i_c(s) = d^i_c(j^2G).$$

This implies that j_λ is a map of $S(s)_0$ into $\Sigma^{10}(\mathbf{R}^n, P)$. Therefore, the homotopy of bundle maps $(j_\lambda)_{\mathbf{K}} : K(s)|_{S(s)_0} \to (\mathbf{K} \subset)T\Omega^{10}(\mathbf{R}^n, P)$ induces the homotopy h^2_λ satisfying the property (C) defined by

$$h^2_\lambda = \exp_\Omega \circ (j_\lambda)_{\mathbf{K}} \circ \exp_{\mathbf{R}^n}^{-1} |E(S(s)_0)$$

such that $h^2_0 = h^1_1 = \exp_\Omega \circ (s|S(s)_0)_{\mathbf{K}} \circ \exp_{\mathbf{R}^n}^{-1}$ and $h^2_1 = \exp_\Omega \circ (j^2G|S(s)_0)_{\mathbf{K}} \circ {\exp_{\mathbf{R}^n}}^{-1}$ on $E(S(s)_0)$.

By applying Lemma 6.4 to $j^2G|E(S(s)_0)$ similarly as in the case of $s|$ $E(S(s)_0)$, we have a homotopy h^3_λ satisfying the property (C) such that $h^3_0 = h^2_1 = \exp_\Omega \circ (j^2G|S(s)_0)_{\mathbf{K}} \circ \exp_{\mathbf{R}^n}^{-1}$ and $h^3_1 = j^2G$ on $E(S(s)_0)$.

Let h_λ be a homotopy in $\Gamma^{tr}(E(S(s)_0) \cap \mathrm{Int}D_{2r}, P)$ satisfying the property (C) defined by

$$h_\lambda = \begin{cases} h^1_{3\lambda}|E(S(s)_0) \cap \mathrm{Int}D_{2r} & \text{for} \quad 0 \le \lambda \le 1/3, \\ h^2_{3\lambda-1}|E(S(s)_0) \cap \mathrm{Int}D_{2r} & \text{for} \quad 1/3 \le \lambda \le 2/3, \\ h^3_{3\lambda-2}|E(S(s)_0) \cap \mathrm{Int}D_{2r} & \text{for} \quad 2/3 \le \lambda \le 1. \end{cases}$$

By modifying h_λ on $E(S(s)_0) \cap (D_{2r} \setminus \mathrm{Int}D_{2r-2r_0})$ via Lemma 6.4, we can construct a homotopy H_λ in $\Gamma^{tr}((\mathbf{R}^n \setminus \mathrm{Int}D_{2r}) \cup E(S(s)_0), P)$ satisfying the property (C) such that

(1) $H_\lambda(x) = s(x)$ for $x \in \mathbf{R}^n \setminus \mathrm{Int}D_{2r}$,
(2) $H_\lambda(c,t) = h_\lambda(c,t)$ for $(c,t) \in E(S(s)_0) \cap \mathrm{Int}D_{2r}$,
(3) $H_0(x) = s(x)$,
(4) $H_1(x) = j^2G(x)$.

By applying the homotopy extension property to s and H_λ, we obtain a homotopy

$$s_\lambda : (\mathbf{R}^n, S(s)) \to (\Omega^{10}(\mathbf{R}^n, P), \Sigma^{10}(\mathbf{R}^n, P))$$

such that

(i) $s_0 = s$,
(ii) $s_\lambda(x) = H_\lambda(x)$ for $x \in (\mathbf{R}^n \setminus \mathrm{Int}D_{2r}) \cup E(S(s)_0)$,
(iii) s_λ is transverse to $\Sigma^{10}(\mathbf{R}^n, P)$ with $s_\lambda^{-1}(\Sigma^{10}(\mathbf{R}^n, P)) = S(s)$,
(iv) if $(c,t) \in E(S(s)_0)$, then $s_1(c,t) = j^2G(c,t)$.

Hence, s_λ is a required homotopy in $\Gamma^{tr}(\mathbf{R}^n, P)$. □

7. Proof of Proposition 4.7

For a section $s \in \Gamma^{tr}(\mathbf{R}^n, P)$ given in Proposition 4.7, let $S(s) \cap D_{2r}$ be decomposed into the connected components $M_1, \dots, M_w$. In this section any one of M_j's will be often denoted by M, which may have non-empty boundary. Then by Remark 4.4 the image $\mathbf{e}(s)(\partial M)$ is contained in $S^{n-1} \times S^{n-1} \setminus \Delta^-$. Hence we can define the homomorphism

$$(\mathbf{e}(s)|M)_* : H_{n-1}(M, \partial M; \mathbf{Z}) \\ \to H_{n-1}(S^{n-1} \times S^{n-1}, S^{n-1} \times S^{n-1} \setminus \Delta^-; \mathbf{Z}) \cong \mathbf{Z}.$$

Let $[M]$ denote the fundamental class of M. The number $(\mathbf{e}(s)|M)_*([M])$ is called the degree of $\mathbf{e}(s)|M$ and denoted by $\deg(\mathbf{e}(s)|M)$. If for a point $p \in \mathbf{e}(s)^{-1}(\Delta^-)$,

$$(\mathbf{e}(s)|O(p;\varepsilon))_* : H_{n-1}(O(p;\varepsilon), \partial O(p;\varepsilon); \mathbf{Z}) \\ \to H_{n-1}(S^{n-1} \times S^{n-1}, S^{n-1} \times S^{n-1} \setminus \Delta^-; \mathbf{Z}) \cong \mathbf{Z}$$

is of degree $+1$ (resp. -1), then we shall say that the degree of $\mathbf{e}(s)$ at p is equal to $+1$ (resp. -1).

Proposition 7.1. *Let $n \geq 1$. Let s be the section of $\Gamma^{tr}(\mathbf{R}^n, P)$ given in Proposition* 4.7. *If $\deg(\mathbf{e}(s)|M_j) = 0$ $(j = 1, 2, \dots, w)$, then there exists a homotopy s_λ relative to $\mathbf{R}^n \setminus \mathrm{Int}D_r$ in $\Gamma^{tr}(\mathbf{R}^n, P)$ such that*
(1) *$S(s_\lambda)$ coincides with $S(s)$ for any λ and*
(2) *$\mathbf{e}(s_1)^{-1}(\Delta^-)$ is empty.*

Proof. We first consider the case where P is orientable and here choose the orientation of P compatible with $\theta^n(P)$, which appeared before Remark 4.4. For an element $z = j_c^2\sigma \in \Sigma^{10}(\mathbf{R}^n, P)$, let $K(z)_c$ denote the subspace $(j^2\sigma)^*(\mathbf{K}_z)$ of $T_c(\mathbf{R}^n)$, which is identified with a line of $\mathbf{R}^n$. Then we define the map $\kappa : \Sigma^{10}(\mathbf{R}^n, P) \to S^{n-1}$ by $\kappa(z) = \mathbf{e}(K(z)_c)$, which becomes a smooth fibre bundle. It is easy to see that the composition map $\kappa \circ s|S(s) : S(s) \to S^{n-1}$ satisfies $\kappa \circ s(c) = \mathbf{e}(K(s)_c)$.

Let p_1 or p_2 be the projection of $S^{n-1} \times S^{n-1}$ onto the first or second component respectively. The restriction $p_2 : S^{n-1} \times S^{n-1} \setminus \Delta^- \to S^{n-1}$ is a subbundle of p_2. Then consider the induced bundle

$$\begin{array}{ccc} (p_2 \circ \mathbf{e}(s)|M)^*(S^{n-1} \times S^{n-1} \setminus \Delta^-) & \longrightarrow & S^{n-1} \times S^{n-1} \\ \big\downarrow & & \big\downarrow p_2 \\ M & \xrightarrow{p_2 \circ \mathbf{e}(s)|M} & S^{n-1}. \end{array}$$

Here, we regard $\mathbf{e}(s)|M$ as a section of the bundle $(p_2 \circ \mathbf{e}(s)|M)^*(S^{n-1} \times S^{n-1})$. Then the unique obstruction for the section $\mathbf{e}(s)|M$ to be deformed relative to $\mathbf{R}^n \setminus \mathrm{Int}D_r$ to a section of the bundle $(p_2 \circ \mathbf{e}(s)|M)^*(S^{n-1} \times S^{n-1} \setminus \Delta^-)$ is equal to $\deg(\mathbf{e}(s)|M)$. Since $\deg(\mathbf{e}(s)|M) = 0$, there is a homotopy $\mathbf{e}_\lambda : M \to S^{n-1} \times S^{n-1}$ relative to $\mathbf{R}^n \setminus \mathrm{Int}D_r$ with $\mathbf{e}_0 = \mathbf{e}(s)|M$ such that $p_2 \circ \mathbf{e}_\lambda|M = p_2 \circ \mathbf{e}(s)|M$ for any λ and $(\mathbf{e}_1)^{-1}(\Delta^-) = \emptyset$. Then $p_1 \circ \mathbf{e}_1(c)$ is not equal to $-\mathbf{e}(\nu(s)_c)$ for any $c \in M$.

By the covering homotopy property of the fibre bundle $\kappa : \Sigma^{10}(\mathbf{R}^n, P) \to S^{n-1}$ applied to $s|S(s)$ and $p_1 \circ \mathbf{e}_\lambda$, we obtain a smooth homotopy $k_\lambda : S(s) \to \Sigma^{10}(\mathbf{R}^n, P)$ relative to $S(s) \setminus \mathrm{Int}D_r$ such that $k_0 = s|S(s)$ and $\kappa \circ k_\lambda = p_1 \circ \mathbf{e}_\lambda$.

Next consider the case where P is non-orientable and connected. In this case we need the double covering $\Upsilon_P : \widetilde{P} \to P$ associated to the first Stiefel-Whitney class $W_1(P)$. If we choose an orientation of $\widetilde{P}$, then we have the map $\widetilde{\kappa} : \Sigma^{10}(\mathbf{R}^n, \widetilde{P}) \to S^{n-1}$ defined similarly as κ. Recall that we have fixed the orientation of $\theta^n(P) = (\pi_P^2 \circ s)^*(TP)$ in Section 4, which induces a lift $\widetilde{s|S(s)} : S(s) \to \Sigma^{10}(\mathbf{R}^n, \widetilde{P})$ of $s|S(s)$. Indeed, a jet $j_c^2\sigma$ defines the jet $j_c^2\,\widetilde{\sigma}$ with map germ $\tilde{\sigma} : (\mathbf{R}^n, c) \to (\widetilde{P}, \widetilde{\sigma}(c))$ such that the orientation of $\theta^n(P)$ is compatible with that of $(\widetilde{P}, \widetilde{\sigma}(c))$. Hence, we have the following commutative diagram, where $\widetilde{\Upsilon_P}$ is induced from Υ_P.

$$\begin{array}{ccc} \Sigma^{10}(\mathbf{R}^n, \widetilde{P}) & \xrightarrow{\pi_{\widetilde{P}}} & \widetilde{P} \\ {\scriptstyle \widetilde{\Upsilon_P}}\big\downarrow & & \big\downarrow{\scriptstyle \Upsilon_P} \\ \Sigma^{10}(\mathbf{R}^n, P) & \xrightarrow[\pi_P]{} & P \end{array}$$

Therefore, by an analogous argument as above, we have a smooth homotopy $\widetilde{k}'_\lambda : S(s) \to \Sigma^{10}(\mathbf{R}^n, \widetilde{P})$ relative to $S(s) \setminus \mathrm{Int}D_r$ covering $p_1 \circ \mathbf{e}_\lambda : S(s) \to S^{n-1}$ such that $\widetilde{k}'_0 = \widetilde{s|S(s)}$ and $\widetilde{\kappa} \circ \widetilde{k}'_\lambda = p_1 \circ \mathbf{e}_\lambda$. Thus we obtain a smooth homotopy $k_\lambda : S(s) \to \Sigma^{10}(\mathbf{R}^n, P)$ defined by $k_\lambda = \widetilde{\Upsilon_P} \circ \widetilde{k}'_\lambda$ such that $k_0 = s|S(s)$, that $\kappa \circ k_\lambda = p_1 \circ \mathbf{e}_\lambda$, and that $p_1 \circ\ \mathbf{e}_1(c)$ is not equal to $-\mathbf{e}(\nu(s)_c)$ for any $c \in S(s)$.

Since s is transverse to $\Sigma^{10}(\mathbf{R}^n, P)$, there exists a bundle map $s|U(S(s)) : U(S(s)) \to U(\Sigma^{10}(\mathbf{R}^n, P))$ introduced in the proof of Lemma 6.2. By applying the homotopy extension property of this bundle map to $s|U(S(s))$ and k_λ, we have a smooth homotopy of bundle maps

$$s'_\lambda : U(S(s)) \to U(\Sigma^{10}(\mathbf{R}^n, P))$$

relative to $U(S(s)) \setminus \mathrm{Int}D_r$ covering k_λ with $s'_0 = s|U(S(s))$. By the homotopy extension property, we extend s'_λ to a homotopy s_λ relative to $\mathbf{R}^n \setminus \mathrm{Int}D_r$ in $\Gamma^{tr}(\mathbf{R}^n, P)$ by considering $s|(\mathbf{R}^n \setminus \mathrm{Int}U(S(s)))$ and $s'_\lambda|\partial U(S(s))$ into $\Omega^{10}(\mathbf{R}^n, P) \setminus \mathrm{Int}U(\Sigma^{10}(\mathbf{R}^n, P))$ such that $s_\lambda(\mathbf{R}^n \setminus \mathrm{Int}U(S(s)))$ is contained in $\Omega^{10}(\mathbf{R}^n, P) \setminus \mathrm{Int}U(\Sigma^{10}(\mathbf{R}^n, P))$. By the construction, it follows that s_1 is the required section. $\square$

By Proposition 7.1 it is enough for Proposition 4.7 to show that the given

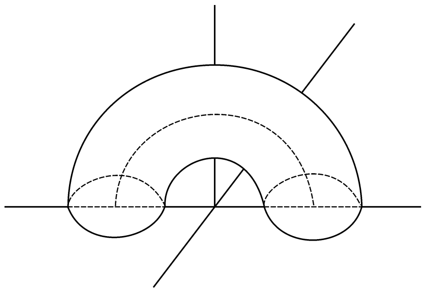

Fig. 1

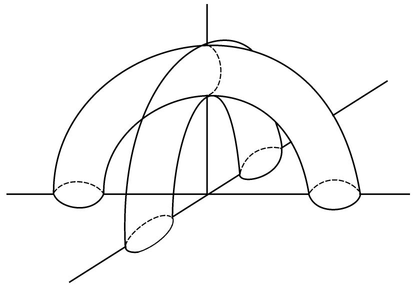

Fig.2

section s is homotopic relative to $\mathbf{R}^n \setminus \mathrm{Int}D_{2r}$ to a section s_1 in $\Gamma^{tr}(\mathbf{R}^n, P)$ such that $\deg(\mathbf{e}(s_1)|M_j)$ is equal to 0 for each j.

We begin by defining several spaces in $\mathbf{R}^n$. Let $\mathcal{S}_2^{i-1}$ denote the $(i-1)$-sphere of radius 2 in $\mathbf{R}^i \times \mathbf{0}_{n-i}$, which consists of all points $a = (a_1, \dots, a_i, 0, \dots, 0)$ with $\|a\| = 2$. Let $\mathcal{D}_2^i$ denote the upper hemi-sphere of $\mathbf{R}^i \times \mathbf{0}_{n-i-1} \times \mathbf{R}$, which consists of all points $a = (a_1, \dots, a_i, 0, \dots, 0, a_n)$ with $\|a\| = 2$ and $a_n \geq 0$. Let $U(\mathcal{S}_2^{i-1})$ denote the tubular neighborhood of $\mathcal{S}_2^{i-1}$ in $\mathbf{R}^{n-1} \times 0$, which consists of all points $(x_1, \dots, x_{n-1}, 0)$ such that $x_j = (1 + t/2)a_j$ $(1 \leq j \leq i)$ with $a \in \mathcal{S}_2^{i-1}$ and $\|(x_{i+1}, \dots, x_{n-1}, t)\| \leq 1$. Let $H(\mathcal{D}_2^i)$ denote the i-handle, which consists of all points $(x_1, \dots, x_n)$ such that $x_j = (1 + t/2)a_j$ $(1 \leq j \leq i$ or $j = n)$ with $a \in \mathcal{D}_2^i$, $x_n \geq 0$ and $\|(x_{i+1}, \dots, x_{n-1}, t)\| \leq 1$.

For the cases where $n \geq 3$ and $1 \leq i < n-1$, we consider the union $\mathbf{R}^{n-1} \times 0 \cup \partial H(\mathcal{D}_2^i) \setminus \mathrm{Int}U(\mathcal{S}_2^{i-1})$. Let H^i denote the submanifold of codimension 1 in $\mathbf{R}^n$ obtained from this union by rounding the corners by a slight deformation. We should note that H^i is connected (see Fig. 1).

For the case $n = 3$ and $i = 2$, let $\mathcal{D}_2^{1\prime}$ denote the upper hemi-sphere of $0 \times \mathbf{R}^2$, which consists of all points $b = (0, b_2, b_3)$ with $\|b\| = 2$ and $b_3 \geq 0$. Let $\mathcal{S}_2^{0\prime}$ denote the boundary of $\mathcal{D}_2^{1\prime}$. Let $U(\mathcal{S}_2^{0\prime})$ denote the tubular neighborhood of $\mathcal{S}_2^{0\prime}$ in $\mathbf{R}^2 \times 0$, which consists of all points $(x_1, x_2, 0)$ with $x_1^2 + (x_2 - 2)^2 \leq 1$ or $x_1^2 + (x_2 + 2)^2 \leq 1$. Let $H(\mathcal{D}_2^{1\prime})$ denote the 1-handle, which consists of all points (x_1, x_2, x_3) such that $x_j = (1 + t/2)b_j$ $(j = 2, 3)$ with $b \in \mathcal{D}_2^{1\prime}$, $x_3 \geq 0$ and $x_1^2 + t^2 \leq 1$. Then consider the union $\mathbf{R}^2 \times 0 \cup \partial(H(\mathcal{D}_2^1) \cup H(\mathcal{D}_2^{1\prime})) \setminus \mathrm{Int}(U(\mathcal{S}_2^0) \cup U(\mathcal{S}_2^{0\prime}))$. Let H' denote the submanifold of $\mathbf{R}^3$ obtained from this union by rounding the corners by a slight modification. We should note that H' is connected (see Fig. 2).

We shall explain an outline of the proof of Proposition 4.7 for $n \geq 3$ and $1 \leq i < n-1$. We start with the fold-map $\sigma : \mathbf{R}^n \to \mathbf{R}^n$ defined by $\sigma(x_1, \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1}, x_n^2)$. Then $S(\sigma)$ coincides with $\mathbf{R}^{n-1} \times 0$, which we orient by $(x_1, \dots, x_{n-1})$. The usual surgery of $\mathbf{R}^{n-1} \times 0$ by the embedded sphere $\mathcal{S}_2^{i-1}$ and the handle $H(\mathcal{D}_2^i)$ induces a new connected and oriented manifold H^i, that is, $\mathbf{R}^{n-1} \times 0 \cup \partial H(\mathcal{D}_2^i) \setminus \mathrm{Int}U(\mathcal{S}_2^{i-1})$ with rounded corners. This procedure of the surgery is realized by a homotopy σ_λ in $\Gamma(\mathbf{R}^n, \mathbf{R}^n)$ with

$\sigma_0 = j^2\sigma$ such that

(1) $S(\sigma_0) = \mathbf{R}^{n-1}$ and $S(\sigma_1) = \mathsf{H}^i$,

(2) $\mathbf{e}(\sigma_1)^{-1}(\triangle^-)$ consists of a single point $(0, \dots, 0, 1)$,

(3) $\deg(\mathbf{e}(\sigma_1)|S(\sigma_1)) = (-1)^i$.

Next for the given section s in Proposition 4.7 we take disjoint embeddings $e_\ell : (\mathbf{R}^n, \mathbf{R}^{n-1} \times 0) \to (\mathbf{R}^n \setminus (\cup_{j=1}^m O(p_j; \varepsilon)), S(s))$ such that $\pi_P \circ s \circ e_\ell(\mathbf{R}^n)$ is contained in a local chart of P $(1 \le \ell \le |\deg(\mathbf{e}(s)|M)|)$. Then we can deform s on each $e_\ell(\mathbf{R}^n)$ by using σ_λ so that the degrees become 0. The proof of the case $n = 3$ and $i = 2$ is similar, though the case $n = 2$ is very exceptional.

Let $\mu : \mathbf{R}^n \to \mathbf{R}$ be a smooth map such that 0 is a regular value and $\mu(x) = 2x_n$ outside of D^n_4. We can orient $\mu^{-1}(0)$ by using $\mathrm{grad}\mu$. Then we can consider the map $\mathbf{e}(\mu) : (\mu^{-1}(0), \mu^{-1}(0) \setminus D^n_4) \to (S^{n-1}, \mathbf{e}_n)$ defined by

$$\mathbf{e}(\mu)(c) = (\mathrm{grad}\mu)(c)/\|(\mathrm{grad}\mu)(c)\|, c \in \mu^{-1}(0).$$

We define the degree of $\mathbf{e}(\mu)$ by $\mathbf{e}(\mu)_*([\mu^{-1}(0)]) = \deg\mathbf{e}(\mu)[S^{n-1}]$, where $[\mu^{-1}(0)]$ is the fundamental class of $H^{n-1}(\mu^{-1}(0), \mu^{-1}(0) \setminus D^n_4; \mathbf{Z})$.

Lemma 7.2. *Let $n \ge 3$. For $i = 1, \dots, n-1$, there exist functions $\mu^i_\lambda : \mathbf{R}^n \to \mathbf{R}$, $\lambda \in \mathbf{R}$, which are smooth with respect to the variables $x_1, \dots, x_n$ and λ such that*

(1) *$\mu^i_\lambda(x) = 2x_n$ if $\lambda \le -1/2$ or $\|(x_1, \dots, x_n)\| \ge 4$,*

(2) *$\mu^i_\lambda(x) = \mu^i_1(x)$ if $\lambda \ge 1/2$,*

(3) *if $|\lambda| \ge 1/2$, then 0 is a regular value of μ^i_λ,*

(4) *if $n \ge 3$ and $1 \le i < n-1$ (resp. $n = 3$ and $i = 2$), then the oriented manifold $(\mu^i_1)^{-1}(0)$ coincides with the connected and oriented manifold H^i (resp. H') and*

(5) *μ^i_1 has a unique point $(0, \dots, 0, 1)$ such that $\mathbf{e}(\mu^i_1)(0, \dots, 0, 1) = -\mathbf{e}_n$ and the degree of $\mathbf{e}(\mu^i_1)$ is equal to $(-1)^i$ (resp. 1).*

Proof. In $\mathbf{R}^{n+1}$ with coordinates $(x_1, \dots, x_n, \lambda)$, consider the subspace $\mathcal{H}$, which is the union

$$\begin{gathered}\mathbf{R}^{n-1} \times 0 \times (-\infty, 0] \cup H(\mathcal{D}^i_2) \times 0 \\ \cup \{\mathbf{R}^{n-1} \times 0 \cup \partial H(\mathcal{D}^i_2) \setminus \mathrm{Int}U(\mathcal{S}^{i-1}_2)\} \times [0, \infty).\end{gathered}$$

We shall round the corner of $\mathcal{H}$ by a slight modification, which is denoted by the same letter $\mathcal{H}$, so that $\mathcal{H} \cap (\mathbf{R}^n \times \lambda) = \mathsf{H}^i \times \lambda$, for $\lambda \ge 1/2$. Let $\nu_{\mathcal{H}}$ denote the orthogonal normal bundle of $\mathcal{H}$. Then $\mathcal{H}$ has the Riemannian metric and $\nu_{\mathcal{H}}$ has the metric, which are induced from the metric on $\mathbf{R}^{n+1}$. Then we have the embedding $\exp_{\mathbf{R}^{n+1}} | D_\varepsilon(\nu_{\mathcal{H}}) : D_\varepsilon(\nu_{\mathcal{H}}) \to \mathbf{R}^{n+1}$ for a small positive number ε, which preserves the metrics. Since $\nu_{\mathcal{H}}$ is trivial, we can choose a trivialization $t(\nu_{\mathcal{H}}) : \nu_{\mathcal{H}} \to \mathcal{H} \times \mathbf{R}$ preserving the metrics of the vector bundles. Let $p_2 : \mathcal{H} \times \mathbf{R} \to \mathbf{R}$ be the projection onto the second component. Then we set

$$\mu' = 2p_2 \circ t(\nu_{\mathcal{H}}) \circ \exp^{-1}_{\mathbf{R}^{n+1}} | \exp_{\mathbf{R}^{n+1}}(D_\varepsilon(\nu_{\mathcal{H}})).$$

This map satisfies that $\mu'(x_1,\dots,x_n,\lambda)=2x_n$ if $\lambda<-1/2$ or $\|(x_1,\dots,x_n)\|\geq 4$, and $|x_n|<\varepsilon$. Furthermore, if $\lambda>1/2$, then we have $D_\varepsilon(\nu_{\mathcal{H}}|_{\mathsf{H}^i\times\lambda}) = D_\varepsilon(\nu_{\mathsf{H}^i})\times\lambda$ and $\mathcal{H}\cap(\mathbf{R}^n\times\lambda)=\mathsf{H}^i\times\lambda$. Hence, $\mu'|\exp_{\mathbf{R}^{n+1}}(D_\varepsilon(\nu_{\mathsf{H}^i}\times\lambda))$ is regular on $\mathsf{H}^i\times\lambda$ with regular value 0 for $\lambda>1/2$.

Now we can extend μ' to the map $\mu:\mathbf{R}^{n+1}\to\mathbf{R}$ so that $\mu(x_1,\dots,x_n,\lambda)=2x_n$ for any $\lambda<-1/2$ or $\|(x_1,\dots,x_n)\|\geq 4$ and that $\mu^{-1}(0)=(\mu')^{-1}(0)$. Set $\mu_\lambda(x)=\mu(x,\lambda)$. Then μ_λ is the required map. The assertions (1) to (4) have already been proved. Since $x_j=(1+t/2)a_j$ for $1\leq j\leq i$ and $j=n$, the length of the vector

$$(x_1,\dots,x_n)-(a_1,\dots a_i,0,\dots,0,a_n)$$
$$=(ta_1/2,\dots,ta_i/2,x_{i+1},\dots,x_{n-1},ta_n/2)$$

is equal to $\sqrt{x_{i+1}^2+\cdots+x_{n-1}^2+t^2}$. Hence, $\mu_1(x_1,\dots,x_n)$ is equal to $2(\sqrt{x_{i+1}^2+\cdots+x_{n-1}^2+t^2}-1)$ on a neighborhood of H^i with $x_n>0$ except for the rounded corners. Furthermore, we have $t=\sqrt{x_1^2+\cdots+x_i^2+x_n^2}-2$. Hence,
$\partial\mu_1(x_1,\dots,x_n)/\partial x_j$ is equal to

$$\begin{cases} \dfrac{2t}{\sqrt{x_{i+1}^2+\cdots+x_{n-1}^2+t^2}}\cdot\dfrac{x_j}{\sqrt{x_1^2+\cdots+x_i^2+x_n^2}} & \text{for} \quad 1\leq j\leq i \text{ or } j=n,\\ \dfrac{2x_j}{\sqrt{x_{i+1}^2+\cdots+x_{n-1}^2+t^2}}, & \text{for} \quad i+1\leq j\leq n-1. \end{cases}$$

If the gradient vector of μ_1 on a point $(x_1,\dots,x_n)\in\mu_1^{-1}(0)$ is equal to $(0,\dots,0,-1)$ up to length, then we have that $(x_1,\dots,x_n)=(0,\dots,0,1)$. We should note here that $(-x_1,x_2,\dots,x_{n-1})$ can be oriented local coordinates of both spaces $\mu^{-1}(0)$ and S^{n-1} near the point $(0,\dots,0,1)$, since the normal vectors at the point $(0,\dots,0,1)$ are directed to $-\mathbf{e}_n$.

Therefore, we calculate the gradient vector of μ_1 on those points of $t=-1$ and obtain that the degree of $\mathbf{e}(\mu_1)$ is equal to $(-1)^i$. This proves the assertion except for the case $n=3$ and $i=2$.

If $n=3$ and $i=2$, then H^2 is not connected. This is the reason why we need to consider H' defined before. Here we define the subspace $\mathcal{H}'$ of $\mathbf{R}^4$ to be the union

$$\mathbf{R}^2\times 0\times(-\infty,0]\cup(H(\mathcal{D}_2^1)\cup H(\mathcal{D}_2'))\times 0$$
$$\cup\{\mathbf{R}^2\times 0\cup\partial(H(\mathcal{D}_2^1)\cup H(\mathcal{D}_2'))\setminus\mathrm{Int}(U(\mathcal{S}_0)\cup U(\mathcal{S}_0'))\}\times[0,\infty).$$

We can round the corner of $\mathcal{H}'$ by a slight modification to be a smooth submanifold, which is denoted by the same symbol, so that $\mathcal{H}'\cap\mathbf{R}^3\times\lambda=\mathsf{H}'\times\lambda$ for $\lambda\geq 1/2$. The rest of the proof in this case is quite analogous to the proof given above. Therefore it is left to the reader. □

Proposition 7.3. *Let $n\geq 3$. Consider the fold-map $\sigma:\mathbf{R}^n\to\mathbf{R}^n$ defined by $\sigma(x_1,\dots,x_{n-1},x_n)=(x_1,\dots,x_{n-1},x_n^2)$. Then there exists a homotopy σ_λ relative to $\mathbf{R}^n\setminus\ \mathrm{Int}D_{4r}$ in $\Gamma(\mathbf{R}^n,\mathbf{R}^n)$ such that*

(1) $\sigma_0 = j^2\sigma$,

(2) σ_1 *is a smooth section transverse to* $\Sigma^{10}(\mathbf{R}^n, \mathbf{R}^n)$ *and* $S(\sigma_1)$ *is connected,*

(3) $\mathbf{e}(\sigma_1)^{-1}(\Delta^-)$ *consists of a single point such that* $\deg(\mathbf{e}(\sigma_1)|S(\sigma_1))$ *is equal to any one of* 1 *and* -1.

Proof. Recall the identifications

$$\pi^2_{\mathbf{R}^n} \times \pi^2_{\mathbf{R}^n} \times \pi_\Omega : \Omega^{10}(\mathbf{R}^n, \mathbf{R}^n) \to \mathbf{R}^n \times \mathbf{R}^n \times \Omega^{10}(n, n),$$

$$J^2(n, n) \cong \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n) \oplus \mathrm{Hom}(S^2\mathbf{R}^n, \mathbf{R}^n)$$

in Section 1. Then

$$j^2_x\sigma = (x, \sigma(x), E_{n-1} \dot{+} (2x_n), (\overbrace{\mathbf{0}, \dots, \mathbf{0}}^{n-1}, \Delta(0, \dots, 0, 2)),$$

where $\mathbf{0}$ denotes the zero $n \times n$-matrix and $\Delta(0, \dots, 0, 2)$ denotes the diagonal $n \times n$-matrix with diagonal components $(0, \dots, 0, 2)$. Let $\mu^i_\lambda(x)$ be the function considered in Lemma 7.2. Then we define the required homotopy σ_λ with $\sigma_0 = j^2\sigma$ by

$$\sigma_\lambda(x) = (x, \sigma(x), E_{n-1} \dot{+} (\mu^i_\lambda(x)), (\mathbf{0}, \dots, \mathbf{0}, \Delta(0, \dots, 0, 2)).$$

It is clear that $S(\sigma_1) = \mu_1^{-1}(0)$. On any point $c \in S(\sigma_1)$, the 2-jet $\pi_\Omega \circ \sigma_1(c)$ is represented by the germ $\sigma : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$. Hence, $\mathbf{Q}_{\sigma_1(c)}$ and $\mathbf{K}_{\sigma_1(c)}$ are generated and oriented by $\mathbf{e}_n$. Therefore, $\mathrm{Hom}(\mathbf{K}_{\sigma_1(c)}, \mathbf{Q}_{\sigma_1(c)}) \cong \mathbf{R}$ and by the definition of the intrinsic derivative we have that $d_c(\mu^i_1)$ is identified with $\mathbf{d}^2_{\sigma_1(c)} \circ d_c\sigma_1 : T_c\mathbf{R}^n \to \mathrm{Hom}(\mathbf{K}_{\sigma_1(c)}, \mathbf{Q}_{\sigma_1(c)}) \cong \mathbf{R}$. This shows that $\mathbf{e}(\sigma_1)^{-1}(\Delta^-) = \mathbf{e}(\mu^i_1)^{-1}(-\mathbf{e}_n)$, which consists of a single point $(0, \dots, 0, 1)$ by Lemma 7.2 (5). Furthermore, we have that the degrees of $\mathbf{e}(\sigma_1)$ and $\mathbf{e}(\mu^i_1)$ are equal to $(-1)^i$. This proves the proposition. $\square$

Proof of the case $n \geq 3$ *of Proposition* 4.7. We give a proof for the case $n \geq 3$. Let M be any one of M_j's. For the given section s, we take distinct points $c_\ell \in M$ and disjoint embeddings $e_\ell : \mathbf{R}^n \to \mathbf{R}^n \setminus (\cup_{j=1}^m O(p_j; \varepsilon))$ with $e_\ell(0) = c_\ell$ such that $\pi_P \circ s \circ e_\ell(\mathbf{R}^n)$ is contained in a local chart of P, which can be identified with $\mathbf{R}^n$ in the proof ($1 \leq \ell \leq |\deg(\mathbf{e}(s)|M)|$). By Proposition 4.6 (2) we may suppose that $s \circ e_\ell$ coincides with $j^2\sigma$, where $\sigma(x_1, \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1}, x_n^2)$. For each $e_\ell(\mathbf{R}^n)$, we can construct the homotopy $\sigma(e_\ell)_\lambda \in \Gamma(e_\ell(\mathbf{R}^n), P)$ defined by $\sigma(e_\ell)_\lambda(x) = \sigma_\lambda(e_\ell^{-1}(x))$. By Proposition 7.3 we can take σ_λ so that

$$\deg(\mathbf{e}(\sigma(e_\ell)_1)) = -\frac{\deg(\mathbf{e}(s)|M)}{|\deg(\mathbf{e}(s)|M)|}.$$

By using $\sigma(e_\ell)_\lambda$ for each M_j, we have a homotopy s'_λ in $\Gamma(\mathbf{R}^n, P)$ defined by $s'_\lambda|e_\ell(\mathbf{R}^n) = \sigma(e_\ell)_\lambda$ on each $e_\ell(\mathbf{R}^n)$ and $s'_\lambda|(\mathbf{R}^n \setminus \cup_{\ell=1}^{|\deg(\mathbf{e}(s)|M)|} e_\ell(\mathbf{R}^n)) =$

$s|(\mathbf{R}^n \setminus \cup_{\ell=1}^{|\deg(\mathbf{e}(s)|M)|} e_\ell(\mathbf{R}^n))$ outside of all $e_\ell(\mathbf{R}^n)$ for all M_j's. Then it is easy to see from the additive property of the degree that the degree of $\mathbf{e}(s_1')$ on each connected component M_j is equal to 0. By Proposition 7.1 we obtain the required homotopy s_λ. □

Next we shall prove the case $n = 2$ of Proposition 4.7. This case is very exceptional and the arguments above for $n \geq 3$ are not available. We need to use the properties of the embedding $i_2 : SO(3) \to \Omega^{10}(2,2)$ in Theorem 3.1 described in Remark 7.4 and Proposition 7.7 below.

Remark 7.4. We interpret the following properties concerning the embedding $i_2 : SO(3) \to \Omega^{10}(2,2)$. Let $\Sigma^0_+(2,2)$ and $\Sigma^0_-(2,2)$ be the subsets of $\Sigma^0(2,2)$ consisting of all regular jets preserving and reversing the orientation respectively. According to [An2], there exists a deformation retraction $R_\lambda : \Omega^{10}(2,2) \to \Omega^{10}(2,2)$ such that $R_0 = id_{\Omega^{10}(2,2)}$, the image of R_1 coincides with the image of i_2 and that R_λ preserves $\Sigma^0_+(2,2)$, $\Sigma^0_-(2,2)$ and $\Sigma^{10}(2,2)$.

Let $\pi : SO(3) \to SO(3)/SO(2) \times (1) \cong S^2$ be the fibre bundle defined by mapping $M \mapsto M\mathbf{e}_3$. Let D_+, D_- and $S^1 \times 0$ be the subsets consisting of all points ${}^t(x_1, x_2, x_3) \in S^2$ with $x_3 \geq 0$, $x_3 \leq 0$ and $x_3 = 0$ respectively. Let $q : \Sigma^{10}(2,2) \to S^1 \times 0$ be defined by $q(j_0^2 f) = \mathbf{e}(\mathrm{Im}(d_0 f)^\perp)$. Then the embedding i_2 has the properties ([An2, Proposition 3.4 and Section 4]):

(i) $i_2^{-1}(\Sigma^0_+(2,2)) = \pi^{-1}(\mathrm{Int}D_+)$, $i_2^{-1}(\Sigma^0_-(2,2)) = \pi^{-1}(\mathrm{Int}D_-)$ and $i_2^{-1}(\Sigma^{10}(2,2)) = \pi^{-1}(S^1 \times 0)$,

(ii) i_2 is smooth near $\pi^{-1}(S^1 \times 0)$ and is transverse to $\Sigma^{10}(2,2)$,

(iii) we have that $q \circ i_2 = \pi$ on $\pi^{-1}(S^1 \times 0)$ and,

(iv) there exists a trivialization $t : \pi^{-1}(S^1 \times 0) \to S^1 \times SO(2)$ such that if $t(M) = (x, U)$ and $i_2(M) = j_0^2 f$, then we have that ${}^tU\mathbf{e}(\mathrm{Im}(d_0 f)^\perp) = \mathbf{e}(\mathrm{Ker}(d_0 f))$ and $x = \mathbf{e}(\mathrm{Im}(d_0 f)^\perp)$.

We should note that $\pi^{-1}(D_-)$ and $\pi^{-1}(D_+)$ are pasted by the transformation $T : \pi^{-1}(\partial D_-) \to \pi^{-1}(\partial D_+)$ defined by $T((\cos\theta, \sin\theta), U) = ((\cos\theta, \sin\theta), R(-2\theta)U)$ by [Ste, 23.4 Theorem and 27.2 Theorem], where

$$R(-2\theta) = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}.$$

In the following we use the maps $GL^\pm(2) \to S^1$ sending $U \mapsto U\mathbf{e}_2/\|U\mathbf{e}_2\|$ in dealing with degrees.

Lemma 7.5. *Let D^2 be the disk centred at the origin with radius 1 in $\mathbf{R}^2$ and let $\mathbf{r} : D^2 \to D^2$ be the map defined by $\mathbf{r}(x_1, x_2) = (-x_1, x_2)$. Let $h : D^2 \to \mathbf{R}^2$ be the fold-map defined by $h(x_1, x_2) = e^{(-x_1^2 - x_2^2)}(-x_1, x_2)$. Then we have that*

(1) *h folds only on the circle $S^1_{1/\sqrt{2}}$ with radius $1/\sqrt{2}$,*

(2) *h preserves the orientation outside of $S^1_{1/\sqrt{2}}$ and reverses the orientation inside of $S^1_{1/\sqrt{2}}$ and*

(3) *if we canonically identify $T_x\mathbf{R}^2$ with $\mathbf{R}^2$, then the maps $T^\pm(dh)$:*

$\partial D^2 \to GL^{\pm}(2)$ *defined by* $T^+(dh)(x) = d_x h$ *and* $T^-(dh)(x) = d_x(h \circ \mathbf{r})$ *are of degree* -2 *and* 2 *respectively.*

Proof. We have that

$$d_{(x_1,x_2)}h = e^{(-x_1^2-x_2^2)}\begin{pmatrix} -1+2x_1^2 & 2x_1x_2 \\ -2x_1x_2 & 1-2x_2^2 \end{pmatrix},$$

whose determinant is equal to $e^{-2(x_1^2+x_2^2)}(2(x_1^2+x_2^2)-1)$. Therefore, h folds only on $S^1_{1/\sqrt{2}}$ and $T^+(dh)(\cos\theta, \sin\theta)$ is equal to the matrix $e^{-1}R(-2\theta)$. Hence, the degree of $T^+(dh)$ is equal to -2. The assertion for $T^-(dh)$ is similar. $\square$

For a positive real number A, let $C(A)$ be the subspace of $\mathbf{R}^2$ consisting of all points $y = (y_1, y_2)$ with $|y_i| \le A$ $(i = 1, 2)$. Let $J = [-A, A]$ and δ be a sufficiently small positive real number with $\delta < A/4$. Let $\iota = 1$ or -1. We need the fold-map $\sigma : C(A) \to \mathbf{R}^2$ defined by $\sigma(y_1, y_2) = (y_1, y_2^2)$. Suppose that $\omega \in \Gamma^{tr}(C(A), \mathbf{R}^2)$ satisfies the properties:

(i) $S(\omega) = J \times 0$, $(\pi_\Omega \circ \omega)^{-1}(\Sigma^0_+(2,2)) = J \times (0, A]$ and $(\pi_\Omega \circ \omega)^{-1}(\Sigma^0_-(2,2)) = J \times [-A, 0)$.

(ii) $\omega|(J \times [-2\delta, 2\delta] \setminus C(A/2)) = j^2\sigma|(J \times [-2\delta, 2\delta] \setminus C(A/2))$.

(iii) The degree of $\mathbf{e}(\omega)|S(\omega)$ is ι and $(\mathbf{e}(\omega)|S(\omega))^{-1}(\Delta^-)$ consists of a single point $(0, 0)$.

(iv) Let $p_2 : \pi^{-1}(\partial D_- \times 0) \to SO(2)$ be the projection through the trivialization t. The degree of $p_2 \circ i_2^{-1} \circ R_1 \circ \pi_\Omega \circ \omega|J \times 0 : J \times 0 \to SO(2)$ is equal to d.

By (ii), (iii), $K(\omega)_{(-A,0)}$ and $K(\omega)_{(A,0)}$ are generated and oriented by $\mathbf{e}_2$. Since the point $(0, 0)$ lies in $(\mathbf{e}(\omega)|S(\omega))^{-1}(\Delta^-)$, $\nu(\omega)_{(0,0)}$ and $K(\omega)_{(0,0)}$ are generated and oriented by $\mathbf{e}_2$ and $-\mathbf{e}_2$ respectively. We can consider the degree of $\pi_\Omega \circ \omega|J \times \{\pm\delta\} : (J \times \{\pm\delta\}, \partial J \times \{\pm\delta\}) \to (\Sigma^0_\pm(2,2), \pi_\Omega(\omega(\pm A, \pm\delta)))$ by noting $\pi_1(\Sigma^0_\pm(2,2)) \cong \pi_1(GL^{\pm}(2)) \cong \mathbf{Z}$.

Lemma 7.6. *Let* ω *be the section given above. Then the degree* $\pi_\Omega \circ \omega|J \times \{-\delta\} : J \times \{-\delta\} \to \Sigma^0_-(2,2) \simeq GL^-(2)$ *is equal to* d *and the degree of* $\pi_\Omega \circ \omega|J \times \delta : J \times \delta \to \Sigma^0_+(2,2) \simeq GL^+(2)$ *is equal to* $-d - 2\iota$.

Proof. By Remark 7.4 (iv), the degree of $(q \circ \pi_\Omega \circ \omega)|S(\omega)$ is equal to $d + \iota$. The degree of the map $S^1 \to S^1$ sending $(\cos\theta, \sin\theta)$ to $R(-2\theta)\mathbf{e}_2$ is equal to -2. By the properties of i_2 and [Ste, 23.4 Theorem] stated in Remark 7.4, it follows that $\deg(\pi_\Omega \circ \omega|J \times \delta) = d + (-2)(d + \iota) = -d - 2\iota$. $\square$

Proposition 7.7. *Let* ω^ι *be the section* ω *given above for* $d = 1 - \iota$ $(\iota = 1$ *or* $-1)$. *Then there exists a homotopy* ω^ι_λ *relative to* $C(A) \setminus C(A/2)$ *in* $\Gamma(C(A), \mathbf{R}^2)$ *such that* $\omega^\iota_0 = \omega^\iota$, $\omega^\iota_1 \in \Gamma^{tr}(C(A), \mathbf{R}^2)$ *and that* $S(\omega^\iota_1)$ *is the disjoint union of* $J \times 0$ *and a circle* L *in* $\mathrm{Int}C(A/2)$ *with* $(\mathbf{e}(\omega^\iota_1)|J \times 0)^{-1}(\Delta^-) = \emptyset$ *and* $(\mathbf{e}(\omega^\iota_1)|L)^{-1}(\Delta^-) = \emptyset$.

Proof. Let C^+ (resp. C^-) be the subspace consisting of all points (y_1, y_2) with $|y_1| \le A/2$ and $\delta \le y_2 \le 2\delta$ (resp. $-2\delta \le y_2 \le -\delta$). We first construct a map $v_1^\iota : C(A) \to \Omega^{10}(2,2)$ as in (i) through (iv) below. Since $\pi_2(\Omega^{10}(2,2)) \cong \pi_2(SO(3)) \cong \{0\}$ by Theorem 3.1, we have a homotopy $v_\lambda^\iota : C(A) \to \Omega^{10}(2,2)$ relative to $C(A) \setminus C(A/2)$ with $v_0^\iota = \pi_\Omega \circ \omega^\iota$. Then we obtain a required homotopy ω_λ^ι by $\omega_\lambda^\iota = (\pi^2_{\mathbf{R}^2} \circ \omega^\iota, \pi^2_{\mathbf{R}^2} \circ \omega^\iota, v_\lambda^\iota)$.

(i) $v_1^\iota(y_1, y_2) = \pi_\Omega \circ \omega^\iota(y_1, y_2)$ outside of $[-A/2, A/2] \times [-2\delta, 2\delta]$.

(ii) $v_1^\iota(y_1, y_2) = \pi_\Omega \circ j^2\sigma(y_1, y_2)$ for $(y_1, y_2) \in J \times [-\delta, \delta]$.

(iii) Let $\iota = 1$. Since the degrees of $\pi_\Omega \circ \omega^1 | J \times \{-\delta\}$ and $\pi_\Omega \circ j^2\sigma | J \times \{-\delta\}$ in $GL^-(2)$ are equal to 0, we can find an extension $v_1^1 | C^- : C^- \to \Sigma^0_-(2,2)$.

The degree of the map $\partial C^+ \to \Sigma^0_+(2,2)$ is equal to 2, which is the sum of $-\deg(\pi_\Omega \circ \omega^1 | (\partial C^+ \setminus [-A/2, A/2] \times \delta))(= 2)$ and $\deg(\pi_\Omega \circ j^2\sigma | [-A/2, A/2] \times \delta)$ $(= 0)$. Hence, if we identify C^+ with D^2_2, then we can paste the map $\pi_\Omega \circ \omega^1 |$ ∂C^+ and the map $\pi_\Omega \circ j^2 h \circ \mathbf{r}$ defined on D^2 in C^+ by a homotopy $D^2_2 \setminus D^2 \to \Sigma^0_+(2,2)$. The circle L becomes $S^1_{1/\sqrt{2}}$. Thus we obtain a map $v_1^1 | C^+ : C^+ \to \Omega^{10}(2,2)$. Since $d\mathbf{r}$ reverses the orientation of TD^2, we should note that $K(j^2 h \circ \mathbf{r}) = (d\mathbf{r})^{-1}(K(j^2 h))$, which is different from $\mathbf{r}^*(K(j^2 h))$. Hence, we have that $\nu(j^2 h \circ \mathbf{r}) = K(j^2 h \circ \mathbf{r})$, and so $(\mathbf{e}(\omega_1^1)|L)^{-1}(\Delta^-) = \emptyset$.

(iv) Let $\iota = -1$. Since the degrees of $\pi_\Omega \circ \omega^{-1} | J \times \delta$ and $\pi_\Omega \circ j^2\sigma | J \times \delta$ in $GL^+(2)$ are equal to 0, we can find an extension $v_1^{-1} | C^+ : C^+ \to \Sigma^0_+(2,2)$.

The degree of the map $\partial C^- \to \Sigma^0_-(2,2)$ is the sum of the degree of $\pi_\Omega \circ \omega^{-1} | (\partial C^- \setminus [-A/2, A/2] \times \{-\delta\})(= 2)$ and the degree of $\pi_\Omega \circ j^2\sigma | [-A/2, A/2] \times \{-\delta\}(= 0)$. Hence, if we identify C^- with D^2_2, then we can paste the map $\pi_\Omega \circ \omega^{-1} | \partial C^-$ and the map $\pi_\Omega \circ j^2(h \circ \mathbf{r})$ defined on D^2 in C^- by a homotopy $D^2_2 \setminus D^2 \to \Sigma^0_-(2,2)$. Thus we obtain a map $v_1^{-1} | C^- : C^- \to \Omega^{10}(2,2)$. $\square$

Proof of the case $n = 2$ of Proposition 4.7. By Remark 7.4, $\Sigma^{10}(2,2)$ is homotopy equivalent to $\pi^{-1}(S^1 \times 0) = S^1 \times SO(2)$. Let p be any one of the points p_j's. Since the normal bundle of $S(s)$ is trivial as is explained in Section 4, we can take local coordinates $y = (y_1, y_2)$ under which we consider $C(A)$ such that $y(p) = (0,0)$ and that $S(s) \cap C(A)$ is on the line $y_2 = 0$. If ε is sufficiently small in Proposition 4.6, then we may deform s so that $O(p;\varepsilon)$ is contained in $C(A/2)$ and that s coincides with $j^2\sigma$ on $J \times [-2\delta, 2\delta] \setminus O(p;\varepsilon)$. That is, $K(s)_{(-A,0)}$ and $K(s)_{(A,0)}$ are generated and oriented by $\mathbf{e}_2$. Since $(\mathbf{e}(s)|J \times 0)^{-1}(\Delta^-)$ consists of a single point $(0,0)$, $\nu(s)_{(0,0)}$ and $K(s)_{(0,0)}$ are generated and oriented by $\mathbf{e}_2$ and $-\mathbf{e}_2$ respectively. Recall the fibre bundle $\kappa : \Sigma^{10}(2,2) \to S^1$ sending $j_0^2 f$ to $\mathbf{e}(K(j_0^2 f))$ in the proof of Proposition 7.1, which is a trivial bundle by Remark 7.4 (iv). Since A is sufficiently small and J is an interval, we can deform s so that the degree of $p_2 \circ i_2^{-1} \circ R_1 \circ \pi_\Omega \circ s | J \times 0$ is equal to $1 - \iota$ without changing $\kappa \circ \pi_\Omega \circ s | J \times 0$. This implies that the degree of $\pi_\Omega \circ s | J \times \{-\delta\} : (J \times \{-\delta\}, \partial J \times \{-\delta\}) \to (\Sigma^0_-(2,2), \pi_\Omega(s(\pm A, -\delta)))$ is equal to $1 - \iota$. Now we again apply Proposition 4.6 to this deformed section s. Thus we may assume that s satisfies the assumption of Proposition 7.7. Consequently, we obtain a homotopy $s_\lambda | C(A) \in \Gamma(C(A), \mathbf{R}^2)$ such that $s_1 | C(A) \in \Gamma^{tr}(C(A), \mathbf{R}^2)$ and that $S(s_1 | C(A))$ is the union of $J \times 0$ and a circle L contained in $\mathrm{Int}C(A/2)$ and that $(\mathbf{e}(s_1)|J \times 0)^{-1}(\Delta^-)$ and $(\mathbf{e}(s_1)|L)^{-1}(\Delta^-)$ are empty.

For any point p_j, we consider the homotopy $(s_\lambda | C(A))_j \in \Gamma(C(A), \mathbf{R}^2)$, which is the homotopy $s_\lambda|C(A)$ defined above for p. Now we are ready to construct a homotopy h_λ of s. We set $h_\lambda = s$ outside of the union of all $C(A)_j$'s for any $\lambda \in [0,1]$ and $h_\lambda = s_\lambda|C(A)_j$ on any one of $C(A)_j$'s. By construction, h_λ satisfies the required properties. $\square$

8. Fold-degree and Gauss maps

Let ξ be an oriented vector bundle of dimension $n+1$ with metric over a space X and $S^n(\xi)$ be its associated n-sphere bundle over X. The fibre $S^n(\xi_x)$ over x of X is canonically identified with the space of all oriented n-subspaces of ξ_x. For an oriented n-space a of ξ_x, we shall write the corresponding point of $S^n(\xi_x)$ by $[a]$. Let N be connected, closed and oriented, and P be oriented in this section. Let $f : N \to P$ be a fold-map. We shall construct two continuous sections of $S^n(f^*(TP \oplus \theta_P))$ over N as follows. For any point x of N, the space $T_{f(x)}P$ gives a point of $S^n(T_{f(x)}P \oplus \mathbf{R})$ and so we define the first section $s_0(f)$ by

$$s_0(f)(x) = (x, [T_{f(x)}P]).$$

Next the homomorphism $\mathcal{T}(f) : TN \oplus \theta_N \to TP \oplus \theta_P$ given in Theorem 3.2 defines the second section $s_1(\mathcal{T}(f))$ by

$$s_1(\mathcal{T}(f))(x) = (x, [\mathrm{Im}(\mathcal{T}(f)|T_xN)]).$$

By applying the obstruction theory of fibre bundles for these two sections, it follows from [Ste, 37.5 Classification Theorem] that the difference cocycle $d(s_0(f), s_1(\mathcal{T}(f)))$ defines an element of $H^n(N, \pi_n(S^n)) \cong \mathbf{Z}$. We shall call this number the *fold-degree* of f, which is denoted by $\mathrm{D}^{\mathrm{fold}}(f)$.

We have another interpretation of the fold-degree in the case where P is $\mathbf{R}^n$ or S^n. In this case the associated homomorphism $\mathcal{T}(f)$ of a fold-map f determines a monomorphism $\mathcal{T}(f)|TN$ into $T(P \times \mathbf{R})$. Here if P is S^n, then $P \times \mathbf{R}$ is canonically embedded in $\mathbf{R}^{n+1}$ as the tubular neighborhood of the unit sphere. By applying the Hirsch Immersion Theorem ([H1]) to $\mathcal{T}(f)|TN$ we obtain an immersion of N into $P \times \mathbf{R}$ and its Gauss map $N \to S^n$, which is denoted by $G(f)$. If P is $\mathbf{R}^n$ (resp. S^n), then the degree of $G(f)$ is nothing but $\mathrm{D}^{\mathrm{fold}}(f)$ (resp. $\mathrm{D}^{\mathrm{fold}}(f) + \deg(f)$). In fact, if $P = S^n$, then let $c_0(f)$ be the map defined by $c_0(f)(x) = (x, [\mathbf{R}^n \times 0])$. The degree of $G(f)$ is equal to the difference cocycle $d(c_0(f), s_1(\mathcal{T}(f))) = d(c_0(f), s_0(f)) + d(s_0(f), s_1(\mathcal{T}(f)))$ and $d(c_0(f), s_0(f))$ is equal to the degree of f. It is known that if n is even, then the degree of $G(f)$ is equal to $(1/2)\chi(N)$ (see, for example, [L2, Theorem 2]).

We shall show that the fold-degree is nontrivial in odd dimensions. Let $p : SO(n+1) \to S^n$ be the map sending a rotation T of $SO(n+1)$ onto its first column vector. The following lemma is well known ([Ste, 8.6 Theorem and 23.5 Corollary]).

Lemma 8.1. *The image of* $(p_*)_n : \pi_n(SO(n+1)) \to \pi_n(S^n) = \mathbf{Z}$ *is the whole integers* $\mathbf{Z}$ *if* $n = 1, 3$ *or* 7 *and is* $2\mathbf{Z}$ *if* n *is odd other than* $1, 3$ *and* 7.

Proposition 8.2. *Let N and P be the manifolds as above of odd dimension n other than 1 and $f : N \to P$ be a fold-map. Then we have the following.*

(1) *If n is not $1, 3$ or 7, then any integer of $\mathrm{D}^{\mathrm{fold}}(f) + 2\mathbf{Z}$ can be a fold-degree of a fold-map homotopic to f.*

(2) *If n is 3 or 7, then any integer of $\mathbf{Z}$ can be a fold-degree of a fold-map homotopic to f.*

Proof. Let m be any integer (resp. even integer) for the case (2) (resp. (1)). There exists a section s of $S^n(f^*(TP \oplus \theta_P))$ such that the difference cocycle $d(s_1(\mathcal{T}(f)), s) = m$ by [Ste, 37.5]. By the assumption there is a map $m' : N \to SO(n+1)$ with degree of $p \circ m'$ being m by Lemma 8.1. We here have a bundle map $b_m : TN \oplus \theta_N \to TN \oplus \theta_N$ coming from m'. For the bundle homomorphism $\mathcal{T}(f) : TN \oplus \theta_N \to TP \oplus \theta_P$, consider the composition $\mathcal{T}(f) \circ b_m : TN \oplus \theta_N \to TP \oplus \theta_P$ such that $s_1(\mathcal{T}(f) \circ b_m)$ is homotopic to s. By Theorem 4.1 there is a fold-map g such that $\mathcal{T}(g)$ is homotopic to $\mathcal{T}(f) \circ b_m$ and that $\mathrm{D}^{\mathrm{fold}}(g) = \mathrm{D}^{\mathrm{fold}}(f) + m$ by

$$\begin{aligned}\mathrm{D}^{\mathrm{fold}}(g) &= d(s_0(g), s_1(\mathcal{T}(g))) \\ &= d(s_0(f), s_1(\mathcal{T}(f))) + d(s_1(\mathcal{T}(f)), s) \\ &= \mathrm{D}^{\mathrm{fold}}(f) + m.\end{aligned}$$ □

Corollary 8.3. *Suppose $N = P$ in addition to the hypothesis of Proposition 8.2. Consider the identity of P. Then we have the following.*

(1) *If n is not $1, 3$ or 7, then any even integer can be a fold-degree of a fold-map homotopic to the identity of P.*

(2) *If n is 3 or 7, then any integer can be a fold-degree of a fold-map homotopic to the identity of P.*

Proof. By Proposition 8.2 it is enough to prove that the fold-degree of id_P is equal to 0. This follows from the fact that $\mathcal{T}(id_P)$ is homotopic to the identity of $TP \oplus \theta_P$, which is a consequence of the property that $i_n(E_{n+1})$ is equal to $j_0^2\sigma$ with $\sigma(x_1, \ldots, x_n) = (1/n)(x_1, \ldots, x_n)$ (see [An3, Section 2]). □

Example 8.4. A 2-jet $z = j_0^2 f \in \Omega^{10}(1,1)$ is represented by the coordinates $(f'(0), f''(0)) \in \mathbf{R}^2 \setminus \{(0,0)\}$. Recall the embedding $i_1 : SO(2) \to \Omega^{10}(1,1)$, which sends $R(\theta)$ to $(\cos\theta, -\sin\theta) \in \Omega^{10}(1,1)$ by [An3, Section 2]. We here consider fold-maps f of S^1 into S^1 or $\mathbf{R}$. The map $\mathcal{T}(f)$ is identified with the following map $R \circ \theta : S^1 \to SO(2)$. First $\pi_\Omega(j_x^2 f) \in \Omega^{10}(1,1)$ has the coordinates $(f'(x), f''(x))$. Define the angle $\theta(x)$ by $(\cos\theta(x), -\sin\theta(x)) = (f'(x), f''(x))/\|(f'(x), f''(x))\|$. Then it follows from the definition of $i_1 : SO(2) \to \Omega^{10}(1,1)$ in [An2, Section 5] and [An3, Section 2] that $R \circ \theta : S^1 \to SO(2)$ is homotopic to ${i_1}^{-1} \circ R_1 \circ \pi_\Omega \circ j^2 f : S^1 \to SO(2)$ with

$$R \circ \theta(x) = \begin{pmatrix} \cos\theta(x) & -\sin\theta(x) \\ \sin\theta(x) & \cos\theta(x) \end{pmatrix}.$$

(1) If $f : S^1 \to \mathbf{R}$ is defined by $f(x) = \cos x$, then $\theta(x) = \pi/2 + x$. Hence, we have that $\mathrm{D}^{\mathrm{fold}}(f) = 1$.

(2) Let $f : S^1 \to S^1$ be a fold-map of degree 1. Let a_1 be the generator of $H^1(F^1; \mathbf{Z}/2\mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}$. We can prove that $\mathrm{D}^{\mathrm{fold}}(f)$ or $\sharp S(f)/2$ modulo 2, where $\sharp$ denotes the number of fold singularities, is equal to $\omega_1(f)^*(a_1) \in H^1(S^1; \mathbf{Z}/2\mathbf{Z})$ in Corollary 5. More generally, consider a fold-map $f : N \to P$ of degree 1. The element $\omega_1(f)^*(a_1)$ is identified with the element of $\mathrm{Hom}(H_1(P; \mathbf{Z}/2\mathbf{Z}), \mathbf{Z}/2\mathbf{Z})$. Any element $u \in H_1(P; \mathbf{Z}/2\mathbf{Z})$ has an embedding $i_u : S^1 \to P$ with $(i_u)_*([S^1]) = u$ such that i_u is transverse to $f(S(f))$ and does not intersect with the subset in $f(S(f))$ consisting of double points of $f|S(f)$. Let $S_u = i_u(S^1)$ and S_N be the manifold $f^{-1}(S_u)$, which may not be connected. Then $i_u^{-1} \circ f|S_N : S_N \to S^1$ is a fold-map of degree 1. Then we have that $\omega_1(f)^*(a_1)(u)$ is equal to $\sharp S(i_u^{-1} \circ f|S_N)/2$ modulo 2.

We shall give an outline of the proof. Recall the notations in Section 3 and the definition of ω exactly before Lemma 3.5. Let $\nu_{S_u \subset P}$ be the normal bundle of S_u in P. We identify $D(\nu_{S_u \subset P})$ with a tubular neighborhood of S_u in P. Similarly we have the normal bundle $\nu_{S_N \subset N}$ and a tubular neighborhood $D(\nu_{S_N \subset N})$ of S_N in N with natural bundle maps $\nu_{S_N \subset N} \to \nu_{S_u \subset P}$ and $D(\nu_{S_N \subset N}) \to D(\nu_{S_u \subset P})$ induced from f. We can construct the collapsing maps $\mathbf{a}_N : T(\nu_N) \to T(\nu_N|_{S_N} \oplus \nu_{S_N \subset N})$ and $\mathbf{a}_P : T(\nu_P) \to T(\nu_P|_{S_u} \oplus \nu_{S_u \subset P})$ by collapsing $T(\nu_N|_{N \setminus \mathrm{Int} D(\nu_{S_N \subset N})})$ and $T(\nu_P|_{P \setminus \mathrm{Int} D(\nu_{S_u \subset P})})$ respectively. Let $h : \nu_P \to \nu_P$ be an automorphism such that $T(h)_*([\alpha_P]) = T(\nu(f))_*([\alpha_N])$ and that $h \oplus id_{\theta_P^k} \simeq id_{\nu_P} \oplus h_\beta$. Then we have that $h|_{S_u} \oplus id_{\nu_{S_u \subset P}} \oplus id_{\theta_{S_u}^k} \simeq id_{\nu_{S_u}} \oplus h_{\beta \circ i_u}$ and that

$$
\begin{aligned}
(\mathbf{a}_P)_* \circ T(h)_*([\alpha_P]) &= T(h|_{S_u} \oplus id_{\nu_{S_u \subset P}})_* \circ (\mathbf{a}_P)_*([\alpha_P]) \\
&= T(h|_{S_u} \oplus id_{\nu_{S_u \subset P}})_*([\alpha_{S_u}]), \\
(\mathbf{a}_P)_* \circ T(\nu(f))_*([\alpha_N]) &= T(\nu(f)|_{S_N} \oplus id_{\nu_{S_N \subset N}})_* \circ (\mathbf{a}_N)_*([\alpha_N]) \\
&= T(\nu(f)|_{S_N} \oplus id_{\nu_{S_N \subset N}})_*([\alpha_{S_N}]).
\end{aligned}
$$

Since $\omega(f) = [\beta]$ by the definition of ω, we have that

$$(i_u)^* \circ \omega(f) = i_u^*([\beta]) = [\beta \circ i_u] = \omega(i_u^{-1} \circ f|S_N) \in [S^1, SG],$$

where $i_u^* : [P, SG] \to [S^1, SG]$. Furthermore, $\omega(i_u^{-1} \circ f|S_N)^*(a_1)([S^1])$ is identified with $\sharp S(i_u^{-1} \circ f|S_N)/2$ modulo 2.

Remark 8.5. Since the C^∞-equivalence classes of fold-germs in $\Omega^{10}(1,1)$ are $x \mapsto \pm x^2$, it follows that the fold-degree of f must be positive. This positiveness is essentially suggested to the author by Professor O. Saeki.

Department of Mathematics
Faculty of Science
Yamaguchi University
Yamaguchi 753-8512, Japan
e-mail: andoy@po.cc.yamaguchi-u.ac.jp

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