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On Corestriction Principle in non abelian galois cohomology over local and global fields

By

Nguyễn Quốc Thắng*

Abstract

In this paper we prove that over local or global fields of characteristic 0, the Corestriction Principle holds for kernel and image of all maps which are connecting maps in group cohomology which extends an earlier result due to Deligne and can be considered as cohomological counterpart to a result of Lenstra and Tate.

Introduction

Let G be a linear algebraic group defined over a field k of characteristic 0. It is well-known (see e.g. [Se1]) that if G is commutative, then for any finite extension k' of k, there is the so-called corestriction map $Cores_{G,k'/k}$ (which will be denoted also by $Cores_G$ to emphasize the group G, when the fileds k', k are fixed):

$$Cores_G : \mathrm{H}^q(k', G) \to \mathrm{H}^q(k, G), \quad q \geq 0,$$

where $\mathrm{H}^q(L, H)$ denotes the Galois cohomology $\mathrm{H}^q(Gal(\bar{L}/L), H(\bar{L}))$ for a L-group H defined over a field L of characteristic 0.

However if G is not commutative, there is no such a map in general (see [RT] or Example 5) below), and, as far as we know, the most general sufficient conditions are given in [Ri1], under which such a map can be constructed. The Corestriction Theory constructed there has many applications to theory of algebras, representation theory and related questions (see also [Ri2]). In this paper we are interested in the following natural question about the corestriction map.

Assume that there is a map, which is functorial in k:

$$\alpha : \mathrm{H}^p(k, G) \to \mathrm{H}^q(k, T),$$

where T is a commutative k-group, G a non-commutative k-group, i.e., α gives rise to a morphism of functors $(k \mapsto \mathrm{H}^p(k, G)) \to (k \mapsto \mathrm{H}^q(k, T))$ (cf. also [Se2,

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Section 6.1]). By restriction, for any finite extension k'/k we have a functorial map
$$\alpha' : \mathrm{H}^p(k', G) \to \mathrm{H}^q(k', T).$$

Question. *When does* $Cores_T(\mathrm{Im}(\alpha')) \subset \mathrm{Im}(\alpha)$?

Of course, if there exists $Cores_G$ (e.g. under the conditions given in [Ri1]), which is functorial then the above question always has an affirmative answer. If the answer is affirmative for all k', we say that *the Corestriction Principle holds* for (the image of) the map α. One defines similar notion for the kernel of a map $\beta : \mathrm{H}^p(k, T) \to \mathrm{H}^q(k, G)$.

The reason that we insist on calling *Corestriction Principle* is that indeed, all the resulting "norm maps" are induced from certain corestriction maps in usual cohomology theory.

We say that the map $\alpha : \mathrm{H}^p(k, G) \to \mathrm{H}^q(k, T)$ is *connecting* if it is a connecting map obtained from the exact cohomology sequence associated with an exact sequence of k-groups involving G and T. For example, let
$$1 \to A \to B \to C \to 1,$$
be an exact sequence of k-groups, where A is considered as a normal k-subgroup of B. Then
$$\mathrm{H}^i(k, A) \to \mathrm{H}^i(k, B), \quad i = 0, 1,$$
and
$$\mathrm{H}^0(k, C) \to \mathrm{H}^1(k, A)$$
are connecting maps. In general, C is just a quotient space and may not be a group. If A is a central subgroup of G, then C is a group, and one may define a connecting map $\mathrm{H}^1(k, C) \to \mathrm{H}^2(k, A)$.

In some particular cases, the above question has an affirmative answer unconditionally and the *Norm Principle* is said to hold if it holds for $p = q = 0$ (which approves the adjective *norm*).

Example. 1) Let D be a finite dimensional central simple algebra over k, G the k-group defined by the condition $G(k) = \mathrm{GL}_n(D)$ (a k-form of the general linear group), $G' = [G, G]$ (the group defined by the condition $G'(k) = \mathrm{SL}_n(D)$). We have the following exact sequence of k-groups
$$1 \to G' \to G \xrightarrow{N} \mathbf{G}_m \to 1,$$
where N denotes the map induced from the reduced norm $\mathrm{GL}_n(D) \xrightarrow{Nrd} k^*$.

It is well-known that
$$N_{k'/k}(Nrd((D \otimes k')^*)) \subset Nrd_{D/k}(D^*),$$
which says that the Corestriction Principle holds for the image of $\alpha = N$, $p = q = 0$.

2) Let Φ be a non-degenerate J-hermitian form with values in a division k-algebra D of center k_0, which is k (resp. a separable quadratic extension of k), if the involution J of D is of the first (resp. second kind). Let $\mathrm{U}(\Phi)$ (resp. $\mathrm{GU}(\Phi)$) be the k-group defined by the unitary group (resp. by the group of similarities) of the form Φ. We have the following exact sequence of k-groups

$$1 \to \mathrm{U}(\Phi) \to \mathrm{GU}(\Phi) \stackrel{m}{\to} \mathbf{G}_m \to 1,$$

where the map m maps every similarity to its similarity factor. It is known (see [L], [Sc] for the case of quadratic forms and [T1] (and also [KMRT]) for the case of skew-hermitian forms) that the Scharlau Norm Principle holds for the group of similarity factors, so the Corestriction Principle holds for the image of $\alpha = m$ and $p = q = 0$. Notice also that since $\mathrm{SU}(\Phi)$ is the connected component of $\mathrm{U}(\Phi)$ in the Zariski topology, it follows that the Norm Principle also holds for the group of special (or proper) similarity factors.

3) Let f be a non-degenerate quadratic form over a field k of characteristic $\neq 2$. Let $\mathrm{Spin}(f)$ (resp. $\mathrm{SO}(f)$) be the Spin (resp. special orthogonal) k-group of f. Let μ_2 be the group $\{\pm 1\}$. We have the following exact sequence

$$1 \to \mu_2 \to \mathrm{Spin}(f) \to \mathrm{SO}(f) \to 1,$$

and the exact sequence of groups deduced from this

$$\mathrm{Spin}(f)(k) \to \mathrm{SO}(f)(k) \stackrel{\delta}{\to} k^*/k^{*2}.$$

The Knebusch Norm Principle (see [L]) allows one to deduce the Corestriction Principle for the image of δ, $p = 0, q = 1$, which means that the Norm Principle holds for the spinor norms.

4) A new kind of Corestriction Principle over local and global fields has been found by P. Deligne [De, Proposition 2.4.8], which, in the case of characteristic 0 and in notations of abelian Galois cohomology ([B1], [Mi, Appendix B]), says that the Corestriction Principle for images holds for the map

$$ab^0_G : \mathrm{H}^0(k, G) \to \mathrm{H}^0_{ab}(k, G).$$

This result has been subsequently applied to various problems related with canonical models of Shimura varieties.

Also, another kind of Corestriction Principle was found by first by Gille and then by Merkurjev (see [Gi], [M1], [M2]), who establish the Norm Principle for the subgroup of all elements which are R-equivalent to the unit element in the group of rational points of connected reductive groups, and also gave some applications to rationality problem in algebraic groups.

5) Given any natural numbers $n \geq 2$, $r \geq 1$, Rosset and Tate have constructed in [RT] an example of a field E containing the group μ_n of n-th roots of 1, a finite Galois extension F of E of degree r, and an element x of $K_2(F)$, which is a symbol, such that the image of x via the trace

$$Tr_{F/E} : K_2F \to K_2E$$

is a sum of at least r symbols. From this they derive a symbol algebra of degree n over F, considered as an element of $\mathrm{H}^2(F, \mu_n)$, such that its image via the corestriction

$$Cores_{F/E} : \mathrm{H}^2(F, \mu_n) \to \mathrm{H}^2(E, \mu_n)$$

is *not* a symbol. Therefore the question above has a negative answer for the connecting map

$$\Delta : \mathrm{H}^1(E, \mathrm{PGL}_n) \to \mathrm{H}^2(E, \mu_n).$$

Despite of this, we will see that in many interesting cases, the Corestriction Principle for connecting maps hold.

The purpose of this paper is to discuss the validity of the Corestriction Principle for images and kernels of connecting maps in the case the field of definition is a local or global field of characteristic 0, and its applications. In particular, our main result of this paper (Theorem 2.5) can be considered as a generalization of the statement: *The norm of a cohomological symbol is a cohomological symbol.* In certain sense, it is a cohomological complement to the well-known result by Lenstra [Le] and Tate [Ta] that for a local or global field F, every element of $K_2(F)$ is a symbol, and it extends the result of Deligne (above) to higher dimensions.

1. Corestriction Principle in non-abelian cohomology for images

In this section we prove the validity of the Corestriction Principle for images and kernels of connecting maps for local or global base fields of characteristic 0 and consider some applications.

We assume the familiarity with the notion and results from the Borovoi-Kottwitz theory of abelian Galois cohomology of algebraic groups as presented in [B1]–[B3] (see also [Mi, Appendix B], for a survey). We recall brieflyathis notion. For a connected reductive group G defined over a field k of characteristic 0 with a maximal k-torus T, let $\tilde{G}$ be the simply connected covering of the semisimple part $G' := [G, G]$ of G with maximal torus $\tilde{T}$, which is projected into a subtorus of T via the isogeny $\tilde{G} \to G'$. One can define a complex $T^\bullet = (\tilde{T} \to T)$ of tori, where T (resp. $\tilde{T}$) is in degree 0 (resp. -1). Then

$$\mathrm{H}^i_{ab}(k, G) := \mathcal{H}^i(k, T^\bullet), \quad i \geq 0$$

where $\mathcal{H}^i$ denotes the Galois hypercohomology of the complex $T^\bullet$. Then it was shown that $\mathrm{H}^i_{ab}(k, G), i \geq 0$, satisfy usual functorial properties of a cohomology theory, and there exist functorial homomorphism and map, respectively

$$ab^0_G : \mathrm{H}^0(k, G) \to \mathrm{H}^0_{ab}(k, G), ab^1_G : \mathrm{H}^1(k, G) \to \mathrm{H}^1_{ab}(k, G).$$

This theory has its origin from a construction of Deligne [De, Sections 2.4.3 through 2.4.11] of certain Picard category.

Our first main result of this section is the following

Theorem 1.1. *Let k be a local or global field of characteristic 0, G a connected k-group, T a connected commutative k-group and $\alpha : \mathrm{H}^p(k,G) \to \mathrm{H}^q(k,T)$ a connecting map. Assume that G is a central extension of T if $p = 1, q = 2$. Then for $0 \le p \le q \le 2$ the Corestriction Principle holds for the image of α.*

Proof. We may assume that G is not commutative.

We first begin with the case of small p, q.

a) Let $p = q = 0$. Then we may assume that the map (denoted by the same symbol) $\alpha : G \to T$, induced from $\alpha : G(k) \to T(k)$, is surjective. Then we have the following exact sequence of k-groups

$$1 \to G_1 \to G \xrightarrow{\alpha} T \to 1,$$

with $G_1 = \mathrm{Ker}(\alpha)$. It is easy to see that α is surjective on $R_u(G)(k)$, i.e., $\alpha(R_u(G)(k)) = R_u(T)(k)$, where $R_u(\cdot)$ denotes the unipotent radical of $(\cdot)$. Hence we may assume that G is reductive and T is a torus. Therefore G_1 contains $G' = [G,G]$.

Let $G = G'T'$, $F = \mathrm{Ker}(\tilde{G} \xrightarrow{\rho} G')$, where $\tilde{G}$ denotes the simply connected covering of G'. First we assume that $G_1 = G'$. By Proposition 2.4.8 of [De], there exists a corestriction map

$$Cores : G(k')/\rho(\tilde{G}(k')) \to G(k)/\rho(\tilde{G}(k)).$$

(The proof of Deligne [De] and [B1], [B3] show that in fact Deligne has proved the Corestriction Principle for ab^0 for any connected reductive group over local or global fields of characteristic 0.) We claim that this map, while restricted to the subgroup $H(k')/\rho(\tilde{G}(k'))$, where H is a connected k-subgroup of G, containing G', is the one constructed by Deligne.

Indeed, we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}^0(k',H) & \to & \mathrm{H}^0_{ab}(k',H) \\ \downarrow & & \downarrow \\ \mathrm{H}^0(k',G) & \to & \mathrm{H}^0_{ab}(k',G), \end{array}$$

where all maps are functorial (see [B1]). Then the image of $\mathrm{H}^0(k',H)$ in $\mathrm{H}^0_{ab}(k',H)$ is $H(k')/\rho(\tilde{G}(k'))$ by a result of Kottwitz [Ko] (see also [B1], [B3]). Therefore the claim follows when we project this diagram into similar diagram where k' is replaced by k and by making use of the commutativity of rectangles (or squares) in the following diagram

$$\begin{tikzcd}
 & & \mathrm{H}^0(k,H) \arrow[r] \arrow[dd] & \mathrm{H}^0(k,G) \arrow[dd] \\
\mathrm{H}^0(k',H) \arrow[r] \arrow[dd] & \mathrm{H}^0(k',G) \arrow[dd] & & \\
 & & \mathrm{H}^0_{ab}(k,H) \arrow[r] & \mathrm{H}^0_{ab}(k,G) \\
\mathrm{H}^0_{ab}(k',H) \arrow[r] \arrow[urr] & \mathrm{H}^0_{ab}(k',G) \arrow[urr] & &
\end{tikzcd}$$

In this diagram, the vertical arrows are ab^0 maps. The two skew (or south-east) arrows are the corestriction maps for abelian Galois cohomology (in fact corestriction maps for Galois hypercohomology), the existence of which follows easily from the functoriality (see, e.g., [Pe, Lemma 4.2 and its proof] for more details). (We can state in fact a more general statement, but we do not need it here.) Thus we have the following *commutative* diagram with exact rows

$$\begin{array}{ccccccc}
1 & \rightarrow & G'(k')/\rho(\tilde{G}(k')) & \rightarrow & G(k')/\rho(\tilde{G}(k')) & \rightarrow & \mathrm{H}^0(k',T) \\
 & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & G'(k)/\rho(\tilde{G}(k)) & \rightarrow & G(k)/\rho(\tilde{G}(k)) & \rightarrow & \mathrm{H}^0(k,T)
\end{array}$$

hence also the following corestriction (norm) map

$$G(k')/G'(k') \rightarrow G(k)/G'(k). \tag{1}$$

Since these two groups in (1) are respectively the images of $G(k')$ and $G(k)$ in $\mathrm{H}^0(k',T)$ and $\mathrm{H}^0(k,T)$, the assertion of the theorem follows.

Now we turn to the general case. Let us consider the following commutative diagram

$$\begin{array}{ccccccccc}
1 & \rightarrow & G' & \rightarrow & G & \rightarrow & T' & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & G_1 & \rightarrow & G & \rightarrow & T & \rightarrow & 1,
\end{array}$$

where G_1 is any k-subgroup of G containing $G' = [G, G]$. By taking the induced commutative diagram of exact cohomology sequences of these two rows and by using the fact that the Corestriction Principle holds for the image of the map $G(k) \rightarrow T'(k)$ shown above, we obtain the Corestriction Principle for the image of $G(k) \rightarrow T(k)$.

b) $p = 0, q = 1$. Let

$$1 \rightarrow T \rightarrow G_1 \rightarrow G \rightarrow 1$$

be the exact sequence of k-groups under consideration. Since $T = T_s \times T_u$, where T_s (resp. T_u) is a k-torus (resp. unipotent k-group), $\mathrm{H}^1(k, T_u) = 0$ and

$1 \to T_u \to R_u(G_1) \to R_u(G) \to 1$ is an exact sequence, we may assume that G_1, G and T are reductive.

Then by [B1], [B3] we have the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{H}^0(k, G_1) & \to & \mathrm{H}^0(k, G) & \xrightarrow{\alpha} & \mathrm{H}^1(k, T) \\
ab^0_{G_1} \downarrow & & ab^0_G \downarrow & & \downarrow \\
\mathrm{H}^0_{ab}(k, G_1) & \to & \mathrm{H}^0_{ab}(k, G) & \to & \mathrm{H}^1_{ab}(k, T),
\end{array}$$

where $\mathrm{H}^i_{ab}(\cdot, \cdot)$ denotes the i-th abelian cohomology and the vertical maps are the maps ab^0_* constructed in [B1], [B3]. Since $\mathrm{H}^1(k, T) \simeq \mathrm{H}^1_{ab}(k, T)$, it follows that if ab^0_G satisfies the Corestriction Principle (for images), then α does also. Again by Kottwitz's result ([Ko], or [B1, p. 39, Proposition 3.6]), we have

$$\mathrm{Im}(\mathrm{H}^0(k, G) \to \mathrm{H}^0_{ab}(k, G)) = G(k)/\rho(\tilde{G}(k)),$$

hence by making use of the Deligne map above the assertion is true in this case.

c) $p = q = 1$ or $p = 1, q = 2$. It is well-known that there is a canonical bijection between $\mathrm{H}^1(k, G)$ and $\mathrm{H}^1(k, L)$, where L is any Levi k-subgroup of G and that $\mathrm{H}^2(k, U) = 0$ for any commutative unipotent k-group by a theorem of Serre (see [Se1, Chapter III]). Hence we may assume that G and T are reductive. We have the following commutative diagram

$$\begin{array}{ccc}
\mathrm{H}^1(k, G) & \to & \mathrm{H}^1_{ab}(k, G) \\
\downarrow & & \downarrow \\
\mathrm{H}^q(k, T) & \to & \mathrm{H}^q_{ab}(k, T)
\end{array}$$

where $q = 1, 2$ (see [B1], [B3]). Since ab^1_G is surjective for local or global fields of charcteristic 0 ([B1], [B3]) and since $\mathrm{H}^1_{ab}(k', G) \to \mathrm{H}^1_{ab}(k, G)$ exists, the assertion of the theorem is verified. $\square$

Remark 1.2. 1) It follows from the construction of ab^2_G of [B2, p. 228], that this map satisfies the Corestriction Principle for images for any field k of characteristic 0 and any connected reductive k-group G.

2) It is desirable to modify the Borovoi-Kottwitz theory so that it can cover also the case where the characteristic of k is $p > 0$.

3) In this section we make use of abelian Galois cohomology due to its relevance to Deligne's result and its making the argument short. One can give another proof of Theorem 1.1 and also of the result of Deligne mentioned above without using abelian Galois cohomology. This follows from certain general results on Corestriction Principle for non-abelian Galois cohomology in characteristic 0 (see [T3]).

2. Corestriction Principle in non-abelian cohomology for kernels

To be complete, together with the Corestriction Principle for the *images* of connecting maps, we need also to consider the validity of this principle for *kernels* of connecting maps. Namely for a connecting map

$$\alpha : \mathrm{H}^p(k, T) \to \mathrm{H}^q(k, G),$$

where T, G are connected k-groups with T commutative, and for a finite extension k' of k with the corestriction map $Cores_T : \mathrm{H}^p(k', T) \to \mathrm{H}^p(k, T)$, we ask

Question. *When does* $Cores_T(\mathrm{Ker}(\alpha \otimes k')) \subset \mathrm{Ker}(\alpha)$*?*

By using Theorem 1.1 it is easy to see that in the case k is a global or a global field of characteristic 0, one is reduced to considering the case $p = q = 1$. We have the following affirmative result for local and global fields of characteristic 0, and it is our main result in this section.

Theorem 2.1. *Let k be a local or global field of characteristic* 0 *and T a connected commutative k-subgroup of a connected k-group G. Then the Corestriction Principle holds for the kernel of the connecting map* $\alpha : \mathrm{H}^1(k, T) \to \mathrm{H}^1(k, G)$.

Proof. As above, we may assume that T is a k-subtorus of G and G is reductive. Here the technique of abelian Galois cohomology does not seem to work and we need the following lemmas.

Lemma 2.2. *Assume that we have the following commutative diagram*

$$\begin{array}{ccccc} A' & \xrightarrow{p'} & B' & \xrightarrow{q'} & C' \\ & & \downarrow \beta & & \downarrow \gamma \\ A & \xrightarrow{p} & B & \xrightarrow{q} & C, \end{array}$$

where A, B, A', B' are groups, the left diagram is a commutative diagram of groups. Let $e' = q'(1)$, $e = q(1)$, where 1 *is the identity element of the corresponding groups. Then if* $\gamma(q'^{-1}(e')) \subset q^{-1}(e)$ *then*

$$\beta(r'^{-1}(e')) \subset r^{-1}(e),$$

with $r' = q'p'$, $r = qp$.

Proof. We have

$$\begin{aligned} r(\beta(r'^{-1}(e'))) &= q(p(\beta(r'^{-1}(e')))) \\ &= q(\gamma(p'(r'^{-1}(e')))) \\ &= q(\gamma(p'(p'^{-1}(q'^{-1}(e'))))) \\ &\subset q(\gamma(q'^{-1}(e'))) \\ &\subset q(q'^{-1}(e)) = e \end{aligned}$$

and the lemma follows. □

We recall (after Langlands) that a connected reductive k-group H is a *z-extension* of a k-group G if H is an extension of G by an *induced* k-torus Z, such that the derived subgroup (called also the *semisimple part*) $[H, H]$ of H is simply connected. For a field extension K/k and an element $x \in \mathrm{H}^1(K, G)$, a z-extension of G over k is called *x-lifting* if $x \in \mathrm{Im}\ (\mathrm{H}^1(K, H) \to \mathrm{H}^1(K, G))$.

Lemma 2.3. *Let G be a connected reductive k-group, K a finite extension of k, and x an element of $\mathrm{H}^1(K, G)$. Then there is a z-extension*

$$1 \to Z \to H \to G \to 1,$$

of G, where all groups and morphisms are defined over k, which is x-lifting.

Lemma 2.4. *Let $\alpha : G_1 \to G_2$ be a homomorphism of connected reductive groups, all defined over k, $x \in \mathrm{H}^1(K, G_1)$, where K is a finite extension of k. Then there exists a x-lifting z-extension $\alpha' : H_1 \to H_2$ of α, i.e., H_i is a z-extension of G_i $(i = 1, 2)$, and we have the following commutative diagram*

$$\begin{array}{ccc} H_1 & \stackrel{\alpha'}{\to} & H_2 \\ \downarrow & & \downarrow \\ G_1 & \stackrel{\alpha}{\to} & G_2, \end{array}$$

where all groups and morphisms are defined over k.

The Lemmas 2.3–2.4, in the case $K = k$, are due to Kottwitz [Ko] (see also [B1, p. 34 and p. 37]). The proofs in our case are the same: Lemma 2.4 follows from Lemma 2.3. To prove Lemma 2.3, we choose a Galois extension F/k sufficiently large so that F contains K and x is split over F (i.e. $res_{K/F}(x) = 1$, where $res_{K/F} : \mathrm{H}^1(K, G) \to \mathrm{H}^1(F, G)$), and such that there is a z-extension

$$1 \to Z \to H \to G \to 1$$

with

$$Z \simeq (\mathrm{R}_{F/k}(\mathbf{G}_m))^n$$

for some n (see [Ko], or [B1, pp. 33–34], for more details). Then one checks that the image of x in $\mathrm{H}^2(K, Z)$ is trivial. Hence $x \in \mathrm{Im}(\mathrm{H}^1(K, H) \to \mathrm{H}^1(K, G))$.

By Lemma 2.2, we may assume that T is a maximal torus of G and by Lemma 2.4, we may assume that G has simply connected semisimple part. In the case of local fields we give two arguments to prove the assertion of the theorem.

First, let $x \in \mathrm{Ker}(\alpha)$. By Lemma 2.4 there exists a x-lifting z-extension $T_1 \to G_1$ of α, all defined over k. Since T is a torus, T_1 is also a torus. It is easy to see that if the Corestriction Principle for kernels holds for any pair (T_1, G_1)

with G_1 having the simply connected semisimple part then it also holds for (T, G). So from now on we assume that $G' = [G, G]$ is simply connected.

We assume first that k is a local field. The case $k = \mathbf{R}$ is trivial, so we assume that k is a $\mathfrak{p}$-adic field. Let S be the maximal central torus of G, $G = SG'$. We have the following commutative diagram

$$\begin{array}{ccc} & & \mathrm{H}^1(k, G') \\ & & \downarrow p \\ \mathrm{H}^1(k, T) & \xrightarrow{\alpha} & \mathrm{H}^1(k, G) \\ & & \downarrow q \\ & & \mathrm{H}^1(k, G/G'). \end{array}$$

Since $\mathrm{H}^1(k, G') = 0$ by Kneser's Theorem [Kn], $\mathrm{Ker}(q) = 0$. Therefore

$$\mathrm{Ker}(\alpha) = \mathrm{Ker}(q\alpha).$$

Since G/G' is a torus, $q\alpha : T \to G/G'$ satisfies the Corestriction Principle for kernels. Hence the assertion of the theorem is verified for local fields.

Now we assume that k is either a local or a global field. By making use of a generalization of Ono's result [O] due to Sansuc (see [Sa, Lemme 1.10]), we can find a natural number m, quasi-split (induced) k-tori P, Q such that there is a central k-isogeny

$$1 \to F \to G_1 \xrightarrow{\pi} G^m \times Q \to 1,$$

where F is a finite central subgroup of a connected reductive k-group G_1, which is a direct product of P and a simply connected semisimple group G_1'. Let T_1 be the unique maximal k-torus of G_1 covering the maximal torus $T' = T^m \times Q$ of $G' = G^m \times Q$, $T_1 = \tilde{T} \times P$, $G_1 = \tilde{G}_1 \times P$, where $\tilde{T}$ is a maximal torus of the semisimple simply connected (derived) subgroup $\tilde{G}_1 = G_1'$ of G_1. It is clear that the assertion of the theorem for (T, G) is equivalent to that for (T', G'). Recall that we may assume the semisimple part of G' to be simply connected, i.e., isomorphic to $\tilde{G}_1$. Then $G' = \tilde{G}_1 P'$, where P' is the image of P. We have the following commutative diagram with exact rows

$$\begin{array}{ccccccc} \mathrm{H}^1(k, F) & \xrightarrow{\theta} & \mathrm{H}^1(k, T_1) & \xrightarrow{\beta} & \mathrm{H}^1(k, T') & \xrightarrow{\delta} & \mathrm{H}^2(k, F) \\ \downarrow = & & \downarrow \gamma & & \downarrow \alpha & & \downarrow = \\ \mathrm{H}^1(k, F) & \xrightarrow{p} & \mathrm{H}^1(k, G_1) & \xrightarrow{\pi'} & \mathrm{H}^1(k, G') & \xrightarrow{\delta} & \mathrm{H}^2(k, F). \end{array}$$

Note that if $\alpha(x) = 0$, then $0 = \delta(\alpha(x)) = \delta(x)$, hence $x \in \mathrm{Im}(\beta)$ and

$$\begin{aligned}\mathrm{Ker}(\alpha) &= \beta\ (\mathrm{Ker}(\alpha\beta)) \\ &= \beta(\{x \in \mathrm{H}^1(k, T_1) : \gamma(x) \in \mathrm{Im}(p)\}).\end{aligned}$$

Hence it suffices to show that for the set

$$A(k) := \{x \in \mathrm{H}^1(k, T_1) : \gamma(x) \in \mathrm{Im}(p)\},$$

we have

$$Cores_{k'/k}(A(k')) \subset A(k).$$

Since P is an induced torus, we have $\mathrm{H}^1(k, P) = 0$, and

$$\mathrm{H}^1(k, T_1) = \mathrm{H}^1(k, \tilde{T}_1) \times \{0\}, \quad \mathrm{H}^1(k, G_1) = \mathrm{H}^1(k, \tilde{G}_1) \times \{0\},$$

hence $A(k)$ may be identified with the following set

$$\{x \in \mathrm{H}^1(k, \tilde{T}_1) : \pi'(\gamma(x)) = 0\},$$

where π' may be considered as the map, induced from the embedding

$$\tilde{G}_1 \hookrightarrow G' = \tilde{G}_1 P'.$$

Since $\tilde{G}_1$ is simply connected, the restriction of π on $\tilde{G}_1$ is an isomorphism. Let $F' := \tilde{G}_1 \cap P'$, $\bar{P} = P'/F'$. Then we have the following commutative diagram with exact rows

$$\begin{array}{ccccc} & & \mathrm{H}^1(k, \tilde{T}_1) & \stackrel{=}{\to} & \mathrm{H}^1(k, \tilde{T}_1) \\ & & \downarrow i & & \downarrow \\ \bar{P}(k) & \stackrel{\delta_k}{\to} & \mathrm{H}^1(k, \tilde{G}_1) & \to & \mathrm{H}^1(k, \tilde{G}_1 P'). \end{array}$$

Let $t' \in A(k')$ and (t'_s) be representative of t' and $t'_s \in \tilde{T}_1(k_s)$, respectively. Then $\pi(t'_s) = (g'p')^{-1}\ {}^s(g'p')$ for some $g' \in \tilde{G}_1(k_s)$ and $p' \in P'(k_s)$ and for all $s \in Gal(k_s/k')$. It follows that $f'_s := p'^{-1}\ {}^sp' \in F'(k_s)$ is a cocycle, representing an element f' from $\mathrm{Ker}(\mathrm{H}^1(k', F') \to \mathrm{H}^1(k', P')$ and we see that $t''_s := t'_s f'^{-1}_s = g'^{-1}\ {}^sg'$ represents an element t'' from $\mathrm{Ker}(\mathrm{H}^1(k', \tilde{T}_1) \to \mathrm{H}^1(k', \tilde{G}_1))$. Then

$$Cores_{k'/k}(t'') = Cores_{k'/k}(t') Cores_{k'/k}(f'^{-1}).$$

Since

$$Cores_{k'/k}(\mathrm{Ker}(\mathrm{H}^1(k', F') \to \mathrm{H}^1(k', P')) \subset \mathrm{Ker}(\mathrm{H}^1(k, F') \to \mathrm{H}^1(k, P')),$$

we can choose a representative $(f_r)_r$ of $Cores_{k'/k}(f')$, $r \in Gal(k_s/k)$, such that

$$f_r = p^{-1}\ {}^rp, \quad \forall r \in Gal(k_s/k).$$

Assume that

$$Cores_{k'/k}(t'') = t \in \mathrm{Ker}(\mathrm{H}^1(k, \tilde{T}_1) \to \mathrm{H}^1(k, \tilde{G}_1)),$$

$t = [(t_r)],\ t_r = g^{-1}\ {}^r g,\ g \in \tilde{G}_1(k_s),\ r \in Gal(k_s/k)$. Then

$$Cores_{k'/k}(t') = Cores_{k'/k}(t'')Cores_{k'/k}(f')$$

has a representative $(t_{1,r})_r$, where

$$t_{1,r} = g^{-1}\ {}^r g f_r = g^{-1}\ {}^r g p^{-1}\ {}^r p = (gp)^{-1}\ {}^r(gp),$$

i.e., $Cores_{k'/k}(t') \in \mathrm{Ker}(\mathrm{H}^1(k, \tilde{T}_1) \to \mathrm{H}^1(k, G'))$ as required.

Therefore we are reduced to proving the Corestriction Principle for kernels for $(\tilde{T}_1, \tilde{G}_1)$.

If k is a $\mathfrak{p}$-adic field, then the assertion now is trivial due to the fact that $\mathrm{H}^1(k, \tilde{G}_1) = 0$ by Kneser's Theorem [Kn]. If k is a number field, we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}^1(k, \tilde{T}_1) & \stackrel{\alpha_1}{\to} & \mathrm{H}^1(k, \tilde{G}_1) \\ \downarrow \lambda & & \downarrow \lambda' \\ \prod_{v\in\infty} \mathrm{H}^1(k_v, \tilde{T}_1) & \stackrel{\alpha_1'}{\to} & \prod_{v\in\infty} \mathrm{H}^1(k_v, \tilde{G}_1), \end{array}$$

where ∞ denotes the set of infinite places of k. We know by the well-known result of Kneser-Harder-Chernousov that the cohomological Hasse principle holds for H^1 of simply connected semisimple k-groups, so $\mathrm{Ker}(\lambda') = 0$, hence

$$\mathrm{Ker}(\alpha_1) = \mathrm{Ker}(\lambda'\alpha_1) = \mathrm{Ker}(\alpha_1'\lambda).$$

By Lemma 2.2 and the local field case above, the proof of Theorem 2.1 follows from the last equality. $\square$

From Theorems 1.1. and 2.1 we derive the following

Theorem 2.5 (Corestriction Principle). *Let G, T be connected linear algebraic groups, where T is commutative, all defined over local or global field k of characteristic 0. Assume that $\alpha_k : \mathrm{H}^p(k, G) \to \mathrm{H}^q(k, T)$ (resp. $\beta_k : \mathrm{H}^q(k, T) \to \mathrm{H}^p(k, G)$) is a connecting map. Then for any finite extension k'/k we have*

$$Cores_{k'/k}(\mathrm{Im}(\alpha_{k'})) \subset \mathrm{Im}(\alpha_k),$$
$$(resp.\ Cores_{k'/k}(\mathrm{Ker}(\beta_{k'})) \subset \mathrm{Ker}(\beta_k)).$$

Remark 2.6. 1) From the proofs of Theorem 2.1 and results of [T3] one can prove the following.

The Corestriction Principle for kernels of the connecting maps $\mathrm{H}^1(k, T) \to \mathrm{H}^1(k, G)$, *where T and G are connected groups over a field k of characteristic* 0,

T is commutative, holds if and only if the same holds for all pairs (T, G) with T a maximal torus of a simply connected almost simple k-group G, all defined over k.

2) The Corestriction Principle for kernels suggests the study of the kernels of maps $\mathrm{H}^1(k, T) \to \mathrm{H}^1(k, G)$, which are little known except for the case of local or global fields. It worth noticing that the study of such kernels plays an important role in the proof of the Hasse principle for H^1 of simply connected semsimple groups, done by Harder ([Ha]). (See also further comments done by Tits [Ti]). Moreover, the proof above shows that if $\mathrm{H}^1(k, \tilde{G}) = 0$, where $\tilde{G}$ is the semisimple simply connected covering of G' (e.g., according to Bruhat-Tits, when k is a local field with residue field of cohomological dimension ≤ 1), the Corestriction Principle for kernels for $\mathrm{H}^1(k, T) \to \mathrm{H}^1(k, G)$ holds.

3) It is easy to show that for connected reductive groups G over number fields k there are norm maps $\mathrm{A}(k', G) \to \mathrm{A}(k, G)$ and $\mathrm{III}(k', G) \to \mathrm{III}(k, G)$ for all finite extensions $k \subset k'$, where $\mathrm{A}(K, G)$ denotes the (defect of weak approximation) quotient group $\prod_v G(K_v)/Cl(G(K))$, where Cl denotes the closure in the product topology of $G(K_v)$, and $\mathrm{III}(K, G)$ denotes the Tate-Shafarevich group of G. The first follows from a result of Sansuc [Sa], Theorem 3.3, or [T2], Theorem 3.9, and the second follows from a result of Borovoi [B1, Theorem 5.13] (= [B3, Theorem 5.12]).

3. Some applications and related questions

Let G be a connected reductive group over a field k. As in the case of semisimple groups, one defines the *Whitehead group of G* over k, $W(k, G) := G(k)/G(k)^+$, where $G(k)^+$ denotes the subgroup of $G(k)$ generated by k-rational points of unipotent radicals of parabolic k-subgroups of G. Note that $G(k)^+$ is a normal subgroup of $G(k)$. It is known that over any local field (resp. global field) k, the Kneser-Tits conjecture holds for all isotropic simply connected almost simple groups H (resp. except possibly for some groups of type ${}^2\mathrm{E}_6$) over k, i.e., $H(k) = H(k)^+$. Thus the Deligne's norm map gives rise to the norm map for the Whitehead groups of connected reductive groups with isotropic almost simple factors (containing no almost simple factors of type ${}^2\mathrm{E}_6$ if k is a number field). In particular the following natural question arises:

Question. *Let k be an infinite field and G be a connected reductive k-group. Is there any "norm relation" between $W(k', G)$ and $W(k, G)$ for all finite extension $k \subset k'$?*

For the case of a local or global field we will give an answer to this question in a relative form, namely modulo the image of the Whitehead group of a connected reductive k-group G_0 with semisimple part isogeneous to that of G (see the corollary below).

In the case $p = q = 0$ we have seen that there are corestriction (norm) maps for the following quotient groups of $G(k)$: $G(k)/G'(k)$ and $G(k)/\rho(\tilde{G}(k))$. It is natural to ask if there is similar map for other "intermediate" quotient groups, namely for $G(k)/\pi(G_0(k))$, where G_0 is a connected reductive k-group with a k-homomorphism $\pi : G_0 \to G$, which restricted to $G'_0 := [G_0, G_0]$ is an isogeny onto the semisimple part $G' = [G, G]$ of G. The answer is affirmative and we have the following result, which is a slight generalization of a result of Deligne [De, Proposition 2.4.8].

Theorem 3.1. *With the above notation, assume that G is a connected reductive k-group. For any finite extension k' of a local or global field k of characteristic* 0 *there is a canonical norm map*

$$G(k')/\pi(G_0(k')) \to G(k)/\pi(G_0(k)).$$

By taking the resctricted product of all such maps in the local case, as in [De], 2.4.9, we deduce from Theorem 3.1 the following

Corollary 3.2. *With above notation, we have a norm map*

$$N_{k'/k} : G(\mathbf{A}')/\pi(G_0(\mathbf{A}')) \to G(\mathbf{A})/\pi(G_0(\mathbf{A})),$$

where $\mathbf{A}'$, $\mathbf{A}$ denotes the adele ring of k', k, respectively.

Proof of Theorem 3.1. First we need the following.

Lemma 3.3. *There is a canonical norm map*

$$G'(k')/\pi(G'_0(k')) \to G'(k)/\pi(G'_0(k)).$$

To see this, we consider the following commutative diagram

$$\begin{array}{ccccc} \tilde{G}(k') & \xrightarrow{p} & G'(k') & \xrightarrow{\delta'} & \mathrm{H}^1(k', F) \\ \downarrow & & \downarrow & & \downarrow \gamma \\ G'_0(k') & \xrightarrow{\pi} & G'(k') & \xrightarrow{\delta} & \mathrm{H}^1(k', B), \end{array}$$

where $F = \mathrm{Ker}(\tilde{G} \to G)$ and $B = \mathrm{Ker}(\tilde{G} \to G'_0)$. (Recall that $\tilde{G}$ is the simply connected covering for both G'_0 and G'.) Now $G'(k')/\pi(G'_0(k'))$ is the image of $G'(k')$ in $\mathrm{H}^1(k', B)$ which is equal to $\gamma(\delta'(G'(k'))$. Since δ' and γ satisfy the Corestriction Principle for images (see Theorem 1.1) the same holds for δ.

Now we come to the proof of the theorem. First we prove the theorem when $G_0 = G'_0$, i.e., the central torus part of G_0 is trivial. We have the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & G'(k')/\rho(\tilde{G}(k')) & \to & G(k')/\rho(\tilde{G}(k')) & \to & G(k')/G'(k') & \to & 1 \\ & & \downarrow \alpha & & \downarrow \beta & & =\downarrow & & \\ 1 & \to & G'(k')/\pi(G'_0(k')) & \to & G(k')/\pi(G'_0(k')) & \to & G(k')/G'(k') & \to & 1. \end{array}$$

We denote by α', β', γ the corresponding canonical corestriction maps for $G'(k')/\rho(\tilde{G}(k'))$, $G(k')/\rho(\tilde{G}(k'))$ and $G(k')/G'(k')$, which exist by what we have proved above. Consider the following maps:

$$\alpha'' : G'(k)/\rho(\tilde{G}(k)) \to G'(k)/\pi(G_0(k)),$$
$$\beta'' : G(k)/\rho(\tilde{G}(k)) \to G(k)/\pi(G_0(k)).$$

Let $b \in G(k')/\pi(G_0(k'))$, $b_1 \in G(k')/\rho(\tilde{G}(k'))$ such that $\beta(b_1) = b$. It is natural to define the image of b in $G(k)/\pi(G_0(k))$ by $\beta''(\beta'(b_1))$. If $\beta(b_2) = b$, then $b_1 = b_2 a$, $a \in \mathrm{Ker}(\alpha) = \mathrm{Ker}(\beta)$. Hence

$$\begin{aligned}\beta''(\beta'(b_1)) &= \beta''(\beta'(b_2 a))\\ &= \beta''(\beta'(b_2)\beta'(a)).\end{aligned}$$

Since $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\beta)$, one sees that $\beta'(\mathrm{Ker}(\beta)) = \alpha'(\mathrm{Ker}(\alpha)) \subset \mathrm{Ker}(\alpha'') = \mathrm{Ker}(\beta'')$. Thus

$$\beta''(\beta'(b_1)) = \beta''(\beta'(b_2))$$

as required.

In the general case, let $G_0 = G'_0 S$, where S is a central connected (torus) part of G_0. We have the following "conjectural" commutative diagram

$$\begin{array}{ccccccccc} 1 \to & \pi(G'_0(k'))/\pi(G'_0(k')) & \to & G(k')/\pi(G'_0(k')) & \to & G(k')/\pi(G_0(k')) & \to 1 \\ & \downarrow (?)\eta & & \downarrow \mu & & (?)\zeta \downarrow & \\ 1 \to & \pi(G_0(k))/\pi(G'_0(k)) & \to & G(k)/\pi(G'_0(k)) & \to & G(k)/\pi(G_0(k)) & \to 1, \end{array}$$

where (?) means a map to be proved existing. It is clear that ζ will exist if we can prove that η exists. Thus we are reduced to proving the existence of the following conjectural commutative diagram

$$\begin{array}{ccccccccc} 1 \to & F(k')G'_0(k')/G'_0(k') & \to & G_0(k')/G'_0(k')) & \to & \pi(G_0(k'))/\pi(G'_0(k')) & \to 1 \\ & \downarrow (?)\theta & & \downarrow \epsilon & & (?)\kappa \downarrow & \\ 1 \to & F(k)G'_0(k)/G'_0(k) & \to & G_0(k)/G'_0(k) & \to & \pi(G_0(k))/\pi(G'_0(k)) & \to 1, \end{array}$$

thus also to the existence of θ, since the existence of ϵ is known due to the proof of case a) of Theorem 1.1 (see (1)). Since for any extension $k \subset K$ we have $F(K)G'_0(K)/G'_0(K) = F(K)/F_0(K)$ where $F_0 := F \cap G'_0$ is a finite central k-subgroup of G'_0, θ is nothing else than the norm map induced from that of F and F_0. □

From this theorem we deduce immediately the following

Corollary 3.4. *Let the notation be as above. Then π induces a canonical norm homomorphism*

$$W(k', G)/\pi_*(W(k', G_0)) \to W(k, G)/\pi_*(W(k, G_0)),$$

where π_* *denotes the homomorphism* $W(\cdot, G_0) \to W(\cdot, G)$ *induced from* π.

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INSTITUTE OF MATHEMATICS
P.O. BOX 631
BO HO, HANOI, VIETNAM

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