J. Math. Kyoto Univ. (JMKYAZ)
43-1 (2003), 139–163

A note on moduli of vector bundles on rational surfaces

By

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1. Introduction

Let (X, H) be a pair of a smooth rational surface X and an ample divisor H on X. Assume that $(K_X, H) < 0$. Let $\overline{M}_H(r, c_1, \chi)$ be the moduli space of semi-stable sheaves E such that $\operatorname{rk}(E) = r$, $c_1(E) = c_1$ and $\chi(E) = \chi$. The relationship between moduli spaces of different invariants is an interesting subject to be studied. If $(c_1, H) = 0$ and $\chi \leq 0$, then Maruyama [Ma2], [Ma3] studied such relations and constructed a contraction map $\phi : \overline{M}_H(r, c_1, \chi) \to \overline{M}_H(r-\chi, c_1, 0)$. Moreover he showed that the image is the Uhlenbeck compactification of the moduli space of μ-stable vector bundles. In particular, he gave an algebraic structure on the Uhlenbeck compactification which was topologically constructed before. After Maruyama's result, Li [Li] constructed the birational contraction for general cases, by using a canonical determinant line bundle, and gave an algebraic structure on the Uhlenbeck compactification. Although Maruyama's method works only for special cases, his construction is interesting of its own. Let us briefly recall his construction. Let E be a semi-stable sheaf such that $\operatorname{rk}(E) = r$, $c_1(E) = c_1$ and $\chi(E) = \chi$. Then $H^i(X, E) = 0$ for $i = 0, 2$. We consider the universal extension

$$0 \to E \to F \to H^1(X, E) \otimes \mathcal{O}_X \to 0. \tag{1.1}$$

Maruyama showed that F is a semi-stable sheaf such that $\operatorname{rk}(F) = r - \chi$, $c_1(F) = c_1$ and $\chi(F) = 0$. Then we have a map

$$\phi : \overline{M}_H(r, c_1, \chi) \to \overline{M}_H(r - \chi, c_1, 0). \tag{1.2}$$

He showed that ϕ is an immersion on the open subscheme consisting of μ-stable vector bundles and the image of ϕ is the Uhlenbeck compactification. For the proof, the rigidity of $\mathcal{O}_X$ is essential. In this note, we replace $\mathcal{O}_X$ by other rigid and stable vector bundles E_0 and show that similar results hold, if the E_0-twisted degree $\deg_{E_0}(E) := (c_1(E_0^{\vee} \otimes E), H) = 0$. If H is a general polarization, then we also show that $\operatorname{im} \phi$ is normal (Theorem 4.5).

1991 *Mathematics Subject Classification(s).* 14D20
Received June 11, 2001
Revised September 6, 2002

We are also motivated by our study of sheaves on $K3$ surfaces. For $K3$ and abelian surfaces, an integral functor called the Fourier-Mukai functor gives an equivalence of the derived categories of coherent sheaves, and under suitable conditions, we get a birational correspondence of moduli spaces (cf. [Y3], [Y5], [Y6]). For rational surfaces, we can rarely expect such an equivalence (cf. [Br]). For example, an analogue of Mukai's reflection [Mu1] (which is given by (1.1)) may lose some information. Indeed we get our contraction map $\phi : \overline{M}_H(r, c_1, \chi) \to \overline{M}_H(r-\chi, c_1, 0)$.

In Section 5, we also consider the relation between different moduli spaces in the case where $\deg_{E_0} E = 1$. Then we find some relations on (the virtual) Hodge numbers (or Betti numbers) of moduli spaces. If $X = \mathbb{P}^2$, by using known results on Hodge numbers ([E-S], [Y1]), we calculate Hodge numbers of some low dimensional moduli spaces. We also determine the boundary of the ample cones in some cases.

2. Preliminaries

2.1. Twisted stability

Let X be a smooth projective surface defined over an algebraically closed field $\mathfrak{k}$. For a point $P \in X$, $\mathfrak{k}_P$ denotes the skyscraper sheaf on X defined by the structure sheaf of P. Let $K(X)$ be the Grothendieck group of X. For $x \in K(X)$, we set

$$\gamma(x) := (\operatorname{rk} x, c_1(x), \chi(x)) \in \mathbb{Z} \oplus \operatorname{NS}(X) \oplus \mathbb{Z}. \tag{2.1}$$

Then $\gamma : K(X) \to \mathbb{Z} \oplus \operatorname{NS}(X) \oplus \mathbb{Z}$ is a surjective homomorphism and $\ker\gamma$ is generated by $\mathcal{O}_X(D) - \mathcal{O}_X$ and $\mathfrak{k}_P - \mathfrak{k}_Q$, where $D \in \operatorname{Pic}^0(X)$ and $P, Q \in X$. For $\gamma = (r, c_1, \chi) \in \mathbb{Z} \oplus \operatorname{NS}(X) \oplus \mathbb{Z}$, we set $\operatorname{rk}\gamma = r$, $c_1(\gamma) = c_1$ and $\chi(\gamma) = \chi$. For coherent sheaves E, F on X, we set

$$\chi(E, F) := \sum_{i=0}^{2} (-1)^i \dim \operatorname{Ext}^i(E, F). \tag{2.2}$$

It induces a bilinear form on $K(X)$:

$$\begin{array}{ccc} K(X) \times K(X) & \to & \mathbb{Z} \\ (x, y) & \mapsto & \chi(x, y). \end{array} \tag{2.3}$$

Lemma 2.1. *For $x, y \in K(X)$, we have*

$$\begin{aligned} \chi(x, y) = &-\operatorname{rk}(x)\operatorname{rk}(y)\chi(\mathcal{O}_X) - (c_1(x), c_1(y)) \\ &+ \operatorname{rk}(y)(K_X, c_1(x)) + \operatorname{rk}(x)\chi(y) + \operatorname{rk}(y)\chi(y). \end{aligned} \tag{2.4}$$

In particular, $\chi(x, y) = \chi(y, x) + (K_X, c_1(y^\vee \otimes x))$, where $y^\vee$ is the dual of y in $K(X)$ (that is, $y^\vee := \sum_{i=0}^{2} (-1)^i \mathcal{E}xt^i_{\mathcal{O}_X}(y, \mathcal{O}_X)$).

Proof. Let $\mathrm{ch}_2(x) \in A^2(X) \otimes \mathbb{Q}$ ([F]) be the second Chern character of x. By the Riemann-Roch theorem, we get

$$\chi(x) = \mathrm{rk}(x)\chi(\mathcal{O}_X) - (K_X, c_1(x))/2 + \int_X \mathrm{ch}_2(x). \tag{2.5}$$

Hence $\int_X \mathrm{ch}_2(x) = \chi(x) - \mathrm{rk}(x)\chi(\mathcal{O}_X) + (K_X, c_1(x))/2$. Applying the Riemann-Roch theorem to $\chi(x, y)$, we get (2.4). □

By (2.4), $\chi(\ ,\ )$ also induces a bilinear form on $\mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}$. We also denote it by $\chi(\ ,\ )$: $\chi(\gamma(x), \gamma(y)) = \chi(x, y)$.

Definition 2.1. Let $\mathcal{M}_H(\gamma)^{\mu\text{-}ss}$ (resp. $\mathcal{M}_H(\gamma)^{\mu\text{-}s}$) be the moduli stack of μ-semi-stable sheaves (resp. μ-stable sheaves) E such that $\gamma(E) = \gamma \in \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}$.

Let $Q(\gamma) := \mathrm{Quot}^{\gamma}_{\mathcal{O}_X(-l)^{\oplus N}/X/\mathfrak{k}}$ be the quot-scheme parametrizing all quotients $\mathcal{O}_X(-l)^{\oplus N} \to E$ with $\gamma(E) = \gamma$. Assume that $N = \chi(E(l))$. Let $Q_H(\gamma)^{\mu\text{-}ss}$ be the open subscheme of $Q(\gamma)$ whose points consist of quotients $\mathcal{O}_X(-l)^{\oplus N} \to E$ such that

(1) E is a μ-semi-stable sheaf with respect to H,

(2) $H^0(X, \mathcal{O}_X^{\oplus N}) \to H^0(X, E(l))$ is an isomorphism and $H^i(X, E(l)) = 0$ for $i > 0$.

The general linear group $GL(N)$ acts naturally on $Q_H(\gamma)^{\mu\text{-}ss}$. For a sufficiently large l, $\mathcal{M}_H(\gamma)^{\mu\text{-}ss}$ is described as a quotient stack:

$$\mathcal{M}_H(\gamma)^{\mu\text{-}ss} = [Q_H(\gamma)^{\mu\text{-}ss}/GL(N)]. \tag{2.6}$$

For $G \in K(X) \otimes \mathbb{Q}$ with $\mathrm{rk}\, G > 0$, we define the G-twisted rank, degree, and Euler characteristic of $x \in K(X) \otimes \mathbb{Q}$ by

$$\begin{aligned} \mathrm{rk}_G(x) &:= \mathrm{rk}(G^\vee \otimes x), \\ \deg_G(x) &:= (c_1(G^\vee \otimes x), H), \\ \chi_G(x) &:= \chi(G^\vee \otimes x). \end{aligned} \tag{2.7}$$

For $t \in \mathbb{Q}_{>0}$, we get

$$\frac{\deg_G(x)}{\mathrm{rk}_G(x)} = \frac{\deg_{tG}(x)}{\mathrm{rk}_{tG}(x)}, \qquad \frac{\chi_G(x)}{\mathrm{rk}_G(x)} = \frac{\chi_{tG}(x)}{\mathrm{rk}_{tG}(x)}. \tag{2.8}$$

We shall define the G-twisted stability as follows.

Definition 2.2 ([Y6])**.** Let E be a torsion free sheaf on X. E is G-twisted semi-stable (resp. stable) with respect to H, if

$$\frac{\chi_G(F(nH))}{\mathrm{rk}_G(F)} \le \frac{\chi_G(E(nH))}{\mathrm{rk}_G(E)}, \qquad n \gg 0 \tag{2.9}$$

for $0 \subsetneq F \subsetneq E$ (resp. the inequality is strict).

By the Riemann-Roch theorem, we see that

(2.10)
$$\frac{\chi_G(E(nH))}{\operatorname{rk}_G(E)} - \frac{\chi_G(F(nH))}{\operatorname{rk}_G(F)} = n\left(\frac{\deg(E)}{\operatorname{rk}(E)} - \frac{\deg(F)}{\operatorname{rk}(F)}\right) + \left(\frac{\chi(E)}{\operatorname{rk}(E)} - \frac{\chi(F)}{\operatorname{rk}(F)}\right) + \left(\frac{c_1(E)}{\operatorname{rk}(E)} - \frac{c_1(F)}{\operatorname{rk}(F)}, \frac{c_1(G)}{\operatorname{rk} G}\right).$$

Hence the twisted stability depends only on $\alpha := c_1(G)/\operatorname{rk} G \in \operatorname{NS}(X) \otimes \mathbb{Q}$ and it is nothing but the twisted stability due to Matsuki-Wentworth [M-W]. By (2.10), the following relations hold:
(2.11)
μ-stable $\Rightarrow$ G-twisted stable $\Rightarrow$ G-twisted semi-stable $\Rightarrow$ μ-semi-stable.

As the usual stability, we have the notion of the Harder-Narasimhan filtration and the Jordan-Hölder filtration. For a G-twisted semi-stable sheaf E, let

$$\mathbf{F} : 0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E \tag{2.12}$$

be the Jordan-Hölder filtration of E with respect to the G-twisted stability. We define the Jordan-Hölder grading by

$$\operatorname{gr}(E) := \bigoplus_{i=1}^{s} F_i/F_{i-1}. \tag{2.13}$$

As the usual stability, $\operatorname{gr}(E)$ does not depend on the choice of $\mathbf{F}$. The S-equivalence $\sim$ is the equivalence relation such that $E \sim E'$ if and only if $\operatorname{gr}(E) \cong \operatorname{gr}(E')$.

Definition 2.3. For $\gamma \in \mathbb{Z} \oplus \operatorname{NS}(X) \oplus \mathbb{Z}$, let $\mathcal{M}_H^G(\gamma)^{ss}$ be the moduli stack of G-twisted semi-stable sheaves E with $\gamma(E) = \gamma$ and $\mathcal{M}_H^G(\gamma)^s$ the open substack consisting of G-twisted stable sheaves. For the usual stability, i.e., $G = \mathcal{O}_X$, we denote $\mathcal{M}_H^{\mathcal{O}_X}(\gamma)^{ss}$ by $\mathcal{M}_H(\gamma)^{ss}$.

Let $Q_H^G(\gamma)^{ss}$ be the open subscheme of $Q_H(\gamma)^{\mu\text{-}ss}$ in (2.6) such that the quotient sheaf E is G-twisted semi-stable. Then

$$\mathcal{M}_H^G(\gamma)^{ss} = [Q_H^G(\gamma)^{ss}/GL(N)]. \tag{2.14}$$

Theorem 2.2 ([M-W])**.**
(1) *There is a coarse moduli scheme* $\overline{M}_H^G(\gamma)$ *of* S*-equivalence classes of* G*-twisted semi-stable sheaves* E *with* $\gamma(E) = \gamma$*.*
(2) $\overline{M}_H^G(\gamma)$ *is a projective scheme over* $\mathfrak{k}$*.*

For a μ-semi-stable sheaf E, let

$$\mathbf{F} : 0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E \tag{2.15}$$

be the Jordan-Hölder filtration of E with respect to the μ-stability. We set $E_i := F_i/F_{i-1}$. We define a Jordan-Hölder grading with respect to the μ-stability by:

$$\operatorname{gr}^{\mu}_{\mathbf{F}}(E) := \bigoplus_{i=1}^{s} E_i. \tag{2.16}$$

Unfortunately $\operatorname{gr}^{\mu}_{\mathbf{F}}(E)$ depends on the choice of the filtration $\mathbf{F}$. In order to get an invariant of E itself, we set

$$\sigma_{\mathbf{F}}(E) := \bigoplus_{i=1}^{s} E_i^{\vee\vee} \oplus \bigoplus_{i=1}^{s} \operatorname{gr}(E_i^{\vee\vee}/E_i), \tag{2.17}$$

where $E_i^{\vee\vee} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(E_i, \mathcal{O}_X), \mathcal{O}_X)$ is the double dual of E_i and $\operatorname{gr}(E_i^{\vee\vee}/E_i)$ is the Jordan-Hölder grading of the semi-stable sheaf $E_i^{\vee\vee}/E_i$.

Remark 1. Every 0-dimensional coherent sheaf E is semi-stable in the sense of Simpson [S] and every 0-dimensional stable sheaf is the skyscraper sheaf $\mathfrak{k}_x$, $x \in X$.

Lemma 2.3. *$\sigma_{\mathbf{F}}(E)$ does not depend on the choice of $\mathbf{F}$. Hence we may denote $\sigma_{\mathbf{F}}(E)$ by $\sigma(E)$.*

Proof. We shall prove our claim by induction on $\operatorname{rk} E$. We may assume that E is properly μ-semi-stable. It is easy to see that

$$\sigma_{\mathbf{F}}(E) = \sigma_{\widetilde{\mathbf{F}}}(E^{\vee\vee}) \oplus \operatorname{gr}(E^{\vee\vee}/E), \tag{2.18}$$

where $\widetilde{\mathbf{F}}$ is the Jordan-Hölder filtration of $E^{\vee\vee}$ induced by the filtration $\mathbf{F}$. Hence we may assume that E is locally free. Let

$$\mathbf{F}^i : 0 \subset F_1^i \subset F_2^i \subset \cdots \subset F_{s_i}^i = E, \qquad i = 1, 2 \tag{2.19}$$

be two Jordan-Hölder filtrations of E. By the induction hypothesis, we may assume that $F_1^1 \neq F_1^2$. Since F_1^1 and F_1^2 are μ-stable, we see that $F_1^1 + F_1^2 = F_1^1 \oplus F_1^2$. We take the Jordan-Hölder filtrations of E

$$\mathbf{F}^i : 0 \subset F_1^i \subset F_2^i \subset \cdots \subset F_t^i = E, \qquad i = 3, 4 \tag{2.20}$$

such that

$$\begin{aligned} &F_1^3 = F_1^1,\ F_1^4 = F_1^2,\\ &F_2^3 = F_2^4 = F_1^1 + F_1^2,\\ &F_j^3 = F_j^4,\ j \geq 3. \end{aligned} \tag{2.21}$$

Obviously $\sigma_{\mathbf{F}^3}(E) = \sigma_{\mathbf{F}^4}(E)$. Let $\overline{\mathbf{F}}^i$, $1 \leq i \leq 4$ be the induced filtration of $\mathbf{F}^i$ on E/F_1^i. By the induction hypothesis, we get

$$\begin{aligned} \sigma_{\overline{\mathbf{F}}^1}(E/F_1^1) &= \sigma_{\overline{\mathbf{F}}^3}(E/F_1^1),\\ \sigma_{\overline{\mathbf{F}}^2}(E/F_1^2) &= \sigma_{\overline{\mathbf{F}}^4}(E/F_1^2). \end{aligned} \tag{2.22}$$

Hence we see that

$$\begin{aligned}\sigma_{\mathbf{F}^1}(E) &= F_1^1 \oplus \sigma_{\overline{\mathbf{F}}^1}(E/F_1^1)\\ &= F_1^1 \oplus \sigma_{\overline{\mathbf{F}}^3}(E/F_1^1)\\ &= \sigma_{\mathbf{F}^3}(E)\\ &= \sigma_{\mathbf{F}^4}(E) = \sigma_{\mathbf{F}^2}(E).\end{aligned} \tag{2.23}$$

□

3. Construction of the contraction map

From now on, we assume that (X, H) is a pair of a rational surface X defined over $\mathfrak{k}$ and an ample divisor H on X. Then $\gamma : K(X) \to \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}$ is an isomorphism. Throughout this note, we assume that

$$(K_X, H) < 0. \tag{3.1}$$

By this assumption and the Serre duality, we get the following lemma.

Lemma 3.1. *Let E and F be torsion free sheaves such that $\deg E/\operatorname{rk} E = \deg F/\operatorname{rk} F$. Assume that E and F are μ-semi-stable with respect to H. Then $\mathrm{Ext}^2(E, F) = 0$.*

Definition 3.1. A coherent sheaf E on a rational surface X is exceptional, if

$$\begin{cases}\mathrm{Hom}(E, E) = \mathfrak{k},\\ \mathrm{Ext}^1(E, E) = 0,\\ \mathrm{Ext}^2(E, E) = 0.\end{cases} \tag{3.2}$$

Example 3.1. $\mathcal{O}_X$ is an exceptional sheaf. Let E be a stable torsion free sheaf with respect to H. If E is rigid, that is, there is no infinitesimal deformation, then by Lemma 3.1, we see that E is an exceptional vector bundle. For more details on exceptional vector bundles, see [D1], [D-L].

Let E_0 be an exceptional vector bundle which is stable with respect to H. Let $e_0 \in K(X)$ be the class of E_0 in $K(X)$. We set $\gamma_0 := \gamma(E_0)$ and $\omega := \gamma(\mathfrak{k}_P)$, $P \in X$. We define homomorphisms $L_{e_0}, R_{e_0} : K(X) \to K(X)$ by

$$\begin{aligned} L_{e_0}(x) &:= x - \chi(x, e_0)e_0, \qquad x \in K(X),\\ R_{e_0}(x) &:= x - \chi(e_0, x)e_0, \qquad x \in K(X).\end{aligned} \tag{3.3}$$

It is easy to see the following equality.

Lemma 3.2. *$\chi(x, R_{e_0}(y)) = \chi(L_{e_0}(x), y)$ for all $x, y \in K(X)$.*

3.1. Existence of a μ-stable vector bundle

In this subsection, we shall give a sufficient condition for $\mathcal{M}_H(r\gamma_0 - a\omega)^{\mu\text{-}s}$ to be non-empty.

Lemma 3.3. *Assume that* $\mathcal{M}_H(r\gamma_0 - a\omega)^{\mu\text{-}ss} \neq \emptyset$. *Then* $\mathcal{M}_H(r\gamma_0 - a\omega)^{\mu\text{-}ss}$ *is smooth and* $\dim \mathcal{M}_H(r\gamma_0 - a\omega)^{\mu\text{-}ss} = 2ra \operatorname{rk} E_0 - r^2$.

Proof. For $E \in \mathcal{M}_H(r\gamma_0 - a\omega)^{\mu\text{-}ss}$, Lemma 3.1 implies that

$$\operatorname{Ext}^2(E, E) = 0. \tag{3.4}$$

Hence $\mathcal{M}_H(r\gamma_0 - a\omega)^{\mu\text{-}ss}$ is smooth and

$$\begin{aligned} \dim \mathcal{M}_H(r\gamma_0 - a\omega)^{\mu\text{-}ss} &= \dim \operatorname{Ext}^1(E, E) - \dim \operatorname{Hom}(E, E) \\ &= -\chi(E, E) = 2ra \operatorname{rk} E_0 - r^2. \end{aligned} \tag{3.5}$$

□

Lemma 3.4. *If* $\mathcal{M}_H^{E_0}(r\gamma_0 - a\omega)^s \neq \emptyset$, *then* $r = 1$ *and* $a = 0$, *or* $a \operatorname{rk} E_0 - r \geq 0$.

Proof. Let E be an element of $\mathcal{M}_H^{E_0}(r\gamma_0 - a\omega)^s$. Since E is simple and $\operatorname{Ext}^2(E, E) = 0$, we get

$$\begin{aligned} 1 &\geq \dim \operatorname{Hom}(E, E) - \dim \operatorname{Ext}^1(E, E) \\ &= \chi(E, E) = r^2 - 2ra \operatorname{rk} E_0. \end{aligned} \tag{3.6}$$

Hence $a \geq (1/(2 \operatorname{rk} E_0))(r - 1/r) \geq 0$. Assume that $\chi(E_0, E) = r - a \operatorname{rk} E_0 > 0$. Then there is a non-zero homomorphism $E_0 \to E$. Since $c_1(E)/\operatorname{rk} E = c_1(E_0)/\operatorname{rk} E_0$, the E_0-twisted stability of E implies that

$$\frac{1}{\operatorname{rk} E_0} = \frac{\chi(E_0, E_0)}{\operatorname{rk} E_0} \leq \frac{\chi(E_0, E)}{r \operatorname{rk} E_0} = \frac{r - a \operatorname{rk} E_0}{r \operatorname{rk} E_0} \tag{3.7}$$

and the inequality is strict, unless $r = 1$. Therefore $a = 0$ and $r = 1$. □

Lemma 3.5. *Let* E *be a* μ*-semi-stable sheaf of* $\deg_{E_0}(E) = 0$. *Then the canonical evaluation homomorphism* $ev : \operatorname{Hom}(E_0, E) \otimes_{\mathfrak{k}} E_0 \to E$ *is injective and* $\operatorname{coker}(ev)$ *is* μ*-semi-stable.*

Proof. We set $G := \ker(ev)$. Then G is locally free and $\deg_{E_0}(G) = 0$. Assume that $G \neq 0$. Let G_0 be a μ-stable locally free subsheaf of G such that $\deg_{E_0} G_0 = 0$. Then we have a non-zero homomorphism $\phi : G_0 \to E_0$. Since G_0 is locally free and $\deg_{E_0}(G_0) = 0$ means $\deg(G_0)/\operatorname{rk} G_0 = \deg(E_0)/\operatorname{rk} E_0$, ϕ must be an isomorphism. Hence $\operatorname{Hom}(E_0, G_0) \neq 0$. On the other hand, ev induces an isomorphism

$$\operatorname{Hom}(E_0, \operatorname{Hom}(E_0, E) \otimes_{\mathfrak{k}} E_0) \to \operatorname{Hom}(E_0, E). \tag{3.8}$$

Hence $\operatorname{Hom}(E_0, G) = 0$, which is a contradiction. Therefore $G = 0$. We next show that $I := \operatorname{coker}(ev)$ is μ-semi-stable. Assume that I has a torsion submodule T. Then $J := \ker(E \to I/T)$ is a submodule of E containing $\operatorname{im}(ev)$. By the μ-semi-stability of E, $0 \geq \deg_{E_0}(J) = \deg_{E_0}(T)$. Hence T is of dimension 0. Since $\operatorname{im}(ev)$ is locally free, $J = \operatorname{im}(ev)$. Thus I is torsion free. Then it is easy to see that $\operatorname{coker}(ev)$ is μ-semi-stable. □

Corollary 3.6. *If $\mathcal{M}_H(r\gamma_0 - a\omega)^{\mu\text{-}ss} \neq \emptyset$, then $a \geq 0$.*

Proof. If $a < 0$, then $\dim \operatorname{Hom}(E_0, E) > r$ for all $E \in \mathcal{M}_H(r\gamma_0 - a\omega)^{\mu\text{-}ss}$. By Lemma 3.5, we get a contradiction. □

Proposition 3.7. *$\mathcal{M}_H(r\gamma_0 - a\omega)^{\mu\text{-}s} \neq \emptyset$, if $r - a \operatorname{rk} E_0 \leq 0$. Moreover, there is a μ-stable locally free sheaf E with $\gamma(E) = r\gamma_0 - a\omega$.*

Proof. Let W be a closed substack of $\mathcal{M}_H(r\gamma_0 - a\omega)^{\mu\text{-}ss}$ such that E belongs to W if and only if there is a quotient $E \to G$ such that $(c_1(G)/\operatorname{rk} G, H) = (c_1(E_0)/\operatorname{rk} E_0, H)$ but $c_1(G)/\operatorname{rk} G \neq c_1(E_0)/\operatorname{rk} E_0$. Let $f : E_0^{\oplus r} \to \bigoplus_{i=1}^{a} \mathfrak{k}_{x_i}$, $x_i \in X$ be a surjective homomorphism. Then $E := \ker f$ is μ-semi-stable and does not belong to W. Hence $\mathcal{M}_H(r\gamma_0 - a\omega)^{\mu\text{-}ss} \setminus W$ is a non-empty open substack of $\mathcal{M}_H(r\gamma_0 - a\omega)^{\mu\text{-}ss}$. For pairs of integers (r_1, a_1) and (r_2, a_2) such that $r_1, r_2 > 0$, $a_1, a_2 \geq 0$ and $(r_1 + r_2, a_1 + a_2) = (r, a)$, let $N(r_1, a_1; r_2, a_2)$ be the substack of $\mathcal{M}_H(r\gamma_0 - a\omega)^{\mu\text{-}ss}$ consisting of E which fits in an exact sequence:

$$0 \to E_1 \to E \to E_2 \to 0, \tag{3.9}$$

where E_1 is a μ-stable sheaf with $\gamma(E_1) = r_1\gamma_0 - a_1\omega$ and E_2 is a μ-semi-stable sheaf with $\gamma(E_2) = r_2\gamma_0 - a_2\omega$. By Lemma 3.1, we get $\operatorname{Ext}^2(E_2, E_1) = 0$. By [D-L, Section 1] or [Y4, Lemma 5.2], we see that

$$\begin{aligned} \operatorname{codim}_{\mathcal{M}_H(r\gamma_0 - a\omega)^{\mu\text{-}ss}} N(r_1, a_1; r_2, a_2) &\geq -\chi(E_1, E_2) \\ &= (a_1 r_2 + a_2 r_1) \operatorname{rk} E_0 - r_1 r_2. \end{aligned} \tag{3.10}$$

By Lemma 3.4, $(a_1 + a_2) \operatorname{rk} E_0 - (r_1 + r_2) \geq 0$. Hence if $a_1 = 0$ or $a_2 = 0$, then we get $(a_1 r_2 + a_2 r_1) \operatorname{rk} E_0 - r_1 r_2 \geq 0$. If $a_1, a_2 > 0$, then by using Lemma 3.4 again, we see that $(a_1 r_2 + a_2 r_1) \operatorname{rk} E_0 - r_1 r_2 \geq a_2 r_1 \operatorname{rk} E_0 > 0$. Therefore $N(r_1, a_1; r_2, a_2)$ is a proper substack of $\mathcal{M}_H(r\gamma_0 - a\omega)^{\mu\text{-}ss} \setminus W$, which implies that $\mathcal{M}_H(r\gamma_0 - a\omega)^{\mu\text{-}s} \neq \emptyset$. By [Y1, Theorem 0.4], the locus of non-locally free sheaves is of codimension $r \operatorname{rk} E_0 - 1 > 0$ (use (4.7) in Section 4.1). Hence $\mathcal{M}_H(r\gamma_0 - a\omega)^{\mu\text{-}s}$ contains a locally free sheaf. □

3.2. Universal extension and the contraction map

We define a coherent sheaf $\mathcal{E}$ on $X \times X$ by the following exact sequence

$$0 \to \mathcal{E} \to p_1^*(E_0^\vee) \otimes p_2^*(E_0) \xrightarrow{ev} \mathcal{O}_\Delta \to 0. \tag{3.11}$$

Then $\mathcal{E}$ is p_2-flat and $\mathcal{E}_x := \mathcal{E}_{|\{x\} \times X}$ is an E_0-twisted stable sheaf with $\gamma(\mathcal{E}_x) = \operatorname{rk}(E_0)\gamma(E_0) - \omega$. In particular $\chi(E_0, \mathcal{E}_x) = 0$. Let E be a coherent sheaf on X. By (3.11), we have an exact sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & p_{2*}(\mathcal{E} \otimes p_1^*(E)) & \longrightarrow & \operatorname{Hom}(E_0, E) \otimes_{\mathfrak{k}} E_0 & \xrightarrow{ev} & E \\ & \longrightarrow & R^1 p_{2*}(\mathcal{E} \otimes p_1^*(E)) & \longrightarrow & \operatorname{Ext}^1(E_0, E) \otimes_{\mathfrak{k}} E_0 & \longrightarrow & 0 \\ & \longrightarrow & R^2 p_{2*}(\mathcal{E} \otimes p_1^*(E)) & \longrightarrow & \operatorname{Ext}^2(E_0, E) \otimes_{\mathfrak{k}} E_0 & \longrightarrow & 0. \end{array} \tag{3.12}$$

Lemma 3.8. *For a μ-semi-stable sheaf E of $\deg_{E_0}(E) = 0$, we have*

$$p_{2*}(\mathcal{E} \otimes p_1^*(E)) = R^2 p_{2*}(\mathcal{E} \otimes p_1^*(E)) = 0. \tag{3.13}$$

Proof. For $E \in \mathcal{M}_H(\gamma)^{\mu\text{-}ss}$, Lemma 3.5 implies that $ev : \mathrm{Hom}(E_0, E) \otimes_{\mathfrak{k}} E_0 \to E$ is injective. Hence $p_{2*}(\mathcal{E} \otimes p_1^*(E)) = 0$. By Lemma 3.1, we get $\mathrm{Ext}^2(E_0, E) = 0$. Therefore $R^2 p_{2*}(\mathcal{E} \otimes p_1^*(E)) \cong \mathrm{Ext}^2(E_0, E) \otimes_{\mathfrak{k}} E_0 = 0$. □

The following is our main theorem of this section.

Theorem 3.9. *Let $e \in K(X)$ be a class such that $\mathrm{rk}\, e > 0$ and $\deg_{E_0}(e) = 0$. Then,*

(1) *we have a morphism $\phi_{\gamma(e)} : \overline{M}_H(\gamma(e)) \to \overline{M}_H^{E_0}(\gamma(\hat{e}))$ sending E to the S-equivalence class of $R^1 p_{2*}(\mathcal{E} \otimes p_1^*(E))$, where $\hat{e} := R_{e_0}(e)$.*

(2) *The restriction of $\phi_{\gamma(e)}$ to $M_H(\gamma(e))^{\mu\text{-}s,loc}$ is an immersion, where $M_H(\gamma(e))^{\mu\text{-}s,loc}$ is the open subscheme consisting of μ-stable vector bundles.*

(3) *$\phi_{\gamma(e)}(E) = \phi_{\gamma(e)}(E')$ if and only if $\sigma(E) = \sigma(E')$.*

In order to prove this theorem, we prepare some lemmas.

Lemma 3.10.

$$\mathbf{R} p_{2*}(\mathcal{E} \otimes p_1^*(E_0)) = 0. \tag{3.14}$$

Proof. We note that $\deg_{E_0}(E_0) = 0$. Since ev is isomorphic and

$$\mathrm{Ext}^1(E_0, E_0) = 0, \tag{3.15}$$

by using (3.12), we get that $R^1 p_{2*}(\mathcal{E} \otimes p_1^*(E_0)) = 0$. This and Lemma 3.8 imply our claim. □

Lemma 3.11. *For a μ-semi-stable sheaf E of $\deg_{E_0}(E) = 0$, we have*

$$\mathrm{Hom}(E_0, R^1 p_{2*}(\mathcal{E} \otimes p_1^*(E))) = 0. \tag{3.16}$$

Proof. By the Leray spectral sequence and the projection formula, we get

$$\mathrm{Hom}(E_0, R^1 p_{2*}(\mathcal{E} \otimes p_1^*(E))) = H^1(X \times X, \mathcal{E} \otimes p_1^*(E) \otimes p_2^*(E_0^{\vee})). \tag{3.17}$$

Since $\mathbf{R} p_{1*}(\mathcal{E} \otimes p_2^*(E_0^{\vee})) = 0$,

$$\mathbf{R} p_{1*}(\mathcal{E} \otimes p_1^*(E) \otimes p_2^*(E_0^{\vee})) = \mathbf{R} p_{1*}(\mathcal{E} \otimes p_2^*(E_0^{\vee})) \overset{\mathbf{L}}{\otimes} E = 0, \tag{3.18}$$

which implies our claim. □

For simplicity, we set $\widehat{E} := R^1 p_{2*}(\mathcal{E} \otimes p_1^*(E))$.

Proposition 3.12. *For a μ-semi-stable sheaf E of $\deg_{E_0}(E) = 0$, $\widehat{E}$ is an E_0-twisted semi-stable sheaf with $\chi(E_0, \widehat{E}) = 0$.*

Proof. By (3.12) and Lemma 3.5, $\widehat{E}$ fits in an exact sequence

$$0 \to \operatorname{Hom}(E_0, E) \otimes_{\mathfrak{k}} E_0 \xrightarrow{ev} E \to \widehat{E} \to \operatorname{Ext}^1(E_0, E) \otimes_{\mathfrak{k}} E_0 \to 0. \tag{3.19}$$

By using Lemma 3.5 again, we see that $\widehat{E}$ is μ-semi-stable. It is easy to see that $\chi(E_0, \widehat{E}) = 0$. Assume that $\widehat{E}$ is not semi-stable and let G be a destabilizing subsheaf. Then $\deg_{E_0}(G) = 0$ and $\chi(E_0, G)/\operatorname{rk} G > 0$. By Lemma 3.1, we get $\operatorname{Ext}^2(E_0, G) = 0$. Hence $\operatorname{Hom}(E_0, G) \neq 0$, which contradicts Lemma 3.11. □

Remark 2. If E is an E_0-twisted semi-stable sheaf such that $\chi(E_0, E) \leq 0$, then $\widehat{E}$ fits in an exact sequence

$$0 \to E \to \widehat{E} \to \operatorname{Ext}^1(E_0, E) \otimes_{\mathfrak{k}} E_0 \to 0. \tag{3.20}$$

By Lemma 3.11, (3.20) is the universal extension.

Lemma 3.13. *Let E be a μ-stable vector bundle of $\deg_{E_0}(E) = 0$. Then $\widehat{E}$ is an E_0-twisted stable vector bundle.*

Proof. We may assume that $E \neq E_0$. Then $\widehat{E}$ fits in the universal extension

$$0 \to E \to \widehat{E} \to E_0^{\oplus h} \to 0, \tag{3.21}$$

where $h = \dim \operatorname{Ext}^1(E_0, E)$. Hence $\widehat{E}$ is locally free. Assume that $\widehat{E}$ is not E_0-twisted stable. By Proposition 3.12, there is an E_0-twisted stable subsheaf G_1 of $\widehat{E}$ such that $\deg_{E_0}(G_1) = \chi(E_0, G_1) = 0$ and $G_2 := \widehat{E}/G_1$ is an E_0-twisted semi-stable sheaf with $\deg_{E_0}(G_2) = \chi(E_0, G_2) = 0$. If E is contained in G_1, then we get a homomorphism $E_0^{\oplus h} \to G_2$. Since $\chi(E_0, G_2)/\operatorname{rk} G_2 = 0 < \chi(E_0, E_0^{\oplus h})/h \operatorname{rk} E_0$, we get a contradiction. Hence E is not contained in G_1. Since E is μ-stable, we get $E \cap G_1 = 0$. Hence $G_1 \to E_0^{\oplus h}$ is injective. Let G' be a μ-stable locally free subsheaf of G_1. Then we see that $G' \cong E_0$, which implies that G_1 is not E_0-twisted stable. Therefore $\widehat{E}$ is E_0-twisted stable. □

Proof of Theorem 3.9. Let $\{\mathcal{F}_s\}_{s \in S}$ be a flat family of μ-semi-stable sheaves of $\deg_{E_0}(\mathcal{F}_s) = 0$. Then Lemma 3.8 and Proposition 3.12 imply that $\{\widehat{\mathcal{F}_s}\}_{s \in S}$ is also a flat family of E_0-twisted semi-stable sheaves (cf. [Mu2, Theorem 1.6]). Hence we get a morphism $\phi_{\gamma(e)} : \overline{M}_H(\gamma(e)) \to \overline{M}_H(\gamma(\hat{e}))$. Let E be a μ-stable vector bundle of $\deg_{E_0}(E) = 0$ and $\varphi : E \to T$ a quotient such that T is of dimension 0. Then for $F := \ker \varphi$, we get an exact sequence

$$0 \to p_{2*}(\mathcal{E} \otimes p_1^*(T)) \to \widehat{F} \to \widehat{E} \to 0. \tag{3.22}$$

Let

$$0 \subset T_1 \subset T_2 \subset \cdots \subset T_n = T \tag{3.23}$$

be a filtration such that $T_i/T_{i-1} \cong \mathfrak{k}_{x_i}$, $x_i \in X$ (i.e, the Jordan-Hölder filtration with respect to Simpson's stability). Then $G := p_{2*}(\mathcal{E} \otimes p_1^*(T))$ has a filtration

$$0 \subset G_1 \subset G_2 \subset \cdots \subset G_n = G \tag{3.24}$$

such that $G_i/G_{i-1} \cong \mathcal{E}_{x_i}$. Since $\widehat{E}$ is an E_0-twisted stable sheaf with

$$\deg_{E_0}(\widehat{E}) = \chi(E_0, \widehat{E}_0) = 0, \tag{3.25}$$

$\widehat{F}$ is S-equivalent to $\widehat{E} \oplus \bigoplus_{i=1}^n \mathcal{E}_{x_i}$.

For a μ-semi-stable sheaf E of $\deg_{E_0}(E) = 0$, let $\operatorname{gr}_{\mathbf{F}}^{\mu}(E) = \bigoplus_{i=1}^n E_i$ be the Jordan-Hölder grading of E with respect to the μ-stability. We set $\operatorname{gr}(E_i^{\vee\vee}/E_i) = \bigoplus_j \mathfrak{k}_{x_{i,j}}$. Then $\sigma(E) = \bigoplus_{i=1}^n (E_i^{\vee\vee} \oplus \oplus_j \mathfrak{k}_{x_{i,j}})$ and $\operatorname{gr}(\widehat{E}) = \bigoplus_{i=1}^n (\widehat{E_i^{\vee\vee}} \oplus \oplus_j \mathcal{E}_{x_{i,j}})$. Since $\widehat{E_i^{\vee\vee}}$ are locally free, the set of pinch points of $\operatorname{gr}(\widehat{E})$ is $\{x_{i,j}\}_{i,j}$. By Proposition 3.14 and Remark 3 below, $E_i^{\vee\vee}$ is uniquely determined by $\widehat{E_i^{\vee\vee}}$. Hence $\sigma(E)$ is determined by $\operatorname{gr}(\widehat{E})$. Hence the claim (3) holds. The second claim follows from Remark 3 (the proof is left to the reader). □

Proposition 3.14. *Let F be an E_0-twisted stable sheaf such that*

$$\deg_{E_0}(F) = \chi(E_0, F) = 0. \tag{3.26}$$

Then

(1) *$F = \mathcal{E}_x$, $x \in X$, or*

(2) *F fits in an exact sequence*

$$0 \to E \to F \to E_0^{\oplus n} \to 0, \tag{3.27}$$

where E is a μ-stable locally free sheaf.

Proof. Assume that F is a μ-stable non-locally free sheaf. Since

$$\chi(E_0, F^{\vee\vee}) = \chi(E_0, F^{\vee\vee}/F) > 0 \tag{3.28}$$

and $\operatorname{Ext}^2(E_0, F^{\vee\vee}) = 0$, we see that $F^{\vee\vee} \cong E_0$. Since $\chi(E_0, F) = 0$, we see that $\operatorname{rk} E_0 = 1$ and $F \cong \mathcal{E}_x$, $x \in X$. If F is a μ-stable locally free sheaf, then F satisfies (2) with $n = 0$. Assume that F is not μ-stable and there is an exact sequence

$$0 \to G_1 \to F \to G_2 \to 0, \tag{3.29}$$

where G_1 is a μ-stable sheaf of $\deg_{E_0}(G_1) = 0$ and G_2 is a μ-semi-stable sheaf of $\deg_{E_0}(G_2) = 0$. Then we get an exact sequence

$$0 \to \widehat{G_1} \to \widehat{F} \to \widehat{G_2} \to 0. \tag{3.30}$$

Since F is E_0-twisted stable and $\chi(E_0, F) = 0$, we get $\operatorname{Hom}(E_0, F) = 0$, and hence we also get $\operatorname{Ext}^1(E_0, F) = 0$. By using (3.12) and Lemma 3.8, we see that $\widehat{F} = F$. In particular $\widehat{F}$ is E_0-twisted stable. By the stability of F, we get $\chi(E_0, G_1) < 0$. In particular we get $\operatorname{Ext}^1(E_0, G_1) \neq 0$. Combining this with (3.12), we get $\widehat{G_1} \neq 0$. Therefore $\widehat{G_1} \cong \widehat{F}$ and $\widehat{G_2} = 0$. By using (3.12) and Lemma 3.8 again, we see that $\operatorname{Hom}(E_0, G_2) \otimes_{\mathfrak{k}} E_0 \to G_2$ is an isomorphism. We note that $\widehat{G_1}$ fits in an exact sequence

$$0 \to p_{2*}(\mathcal{E} \otimes p_1^*(G_1^{\vee\vee}/G_1)) \to \widehat{G_1} \to \widehat{G_1^{\vee\vee}} \to 0. \tag{3.31}$$

By the stability of F, (i) $G_1^{\vee\vee}/G_1 = 0$, or (ii) $G_1^{\vee\vee}/G_1 = \mathfrak{k}_x$, $x \in X$ and $\widehat{G_1^{\vee\vee}} = 0$. Therefore G_1 is locally free, or $F = \mathcal{E}_x$. $\square$

Remark 3. If F fits in the exact sequence (3.27), then

$$E = \ker(F \to \operatorname{Hom}(F, E_0)^{\vee} \otimes_{\mathfrak{k}} E_0). \tag{3.32}$$

Thus E is uniquely determined by F.

Example 3.2. Assume that $(X, H) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ and $E_0 = \Omega_X(1)$. Then we have a contraction

$$\overline{M}_H(2, -H, -n) \to \coprod_{0 \le k \le n} M_H(2, -H, -k)^{\mu\text{-}s,loc} \times S^{n-k}X \tag{3.33}$$

sending E to $\sigma(E) = (E^{\vee\vee}, \operatorname{gr}(E^{\vee\vee}/E))$.

Remark 4. For a μ-semi-stable sheaf E of $\deg_{E_0}(E) = 0$, $\mathcal{H}(E) := \operatorname{Ext}^1_{p_1}(p_2^*(E), \mathcal{E})$ is an $E_0^{\vee}$-twisted semi-stable sheaf such that $\deg_{E_0^{\vee}}(\mathcal{H}(E)) = 0$ and $\chi(E_0^{\vee}, \mathcal{H}(E)) = 0$. Indeed, it is easy to see that $\mathcal{H}(E)$ is a μ-semi-stable sheaf such that $\deg_{E_0^{\vee}} \mathcal{H}(E) = 0$ and $\chi(E_0^{\vee}, \mathcal{H}(E)) = 0$. Since

$$\operatorname{Hom}(E_0^{\vee}, \mathcal{H}(E)) = \operatorname{Ext}^1(p_2^*(E), \mathcal{E} \otimes p_1^*(E_0)) = 0, \tag{3.34}$$

$\mathcal{H}(E)$ is $E_0^{\vee}$-twisted semi-stable. Hence we have a morphism

$$\psi_{\gamma(e)} : \overline{M}_H^{E_0}(\gamma(e)) \to \overline{M}_H^{E_0^{\vee}}(\gamma(\hat{e}^{\vee})). \tag{3.35}$$

It is easy to see that $\psi_{\gamma(\hat{e})}$ is an isomorphism and we get a commutative diagram.

$$\begin{array}{ccccc}
 & \overline{M}_H^{E_0}(\gamma(e)) & & & \overline{M}_H^{E_0^{\vee}}(\gamma(e^{\vee})) \\
 \swarrow^{\phi_{\gamma(e)}} & & \searrow^{\psi_{\gamma(e)}} & \swarrow^{\phi_{\gamma(e^{\vee})}} & \\
\overline{M}_H^{E_0}(\gamma(\hat{e})) & \xrightarrow{\psi_{\gamma(\hat{e})}} & \overline{M}_H^{E_0^{\vee}}(\gamma(\hat{e}^{\vee})) & &
\end{array}$$

4. The image of the contraction

4.1. Brill-Noether locus

We set $\widehat{\gamma} := m\gamma_0 - c\omega$, where $m = c \operatorname{rk} E_0$. Assume that H is general with respect to $\widehat{\gamma}$, that is, for every μ-semi-stable sheaf F with $\gamma(F) = \widehat{\gamma}$ and a subsheaf F' of F,

$$\frac{\deg(F')}{\operatorname{rk} F'} = \frac{\deg(F)}{\operatorname{rk} F} \quad \text{if and only if} \quad \frac{c_1(F')}{\operatorname{rk}(F')} = \frac{c_1(F)}{\operatorname{rk} F} \tag{4.1}$$

(cf. [M-W], [Y2], [Y4]). Hence we get $\mathcal{M}_H^{E_0}(\widehat{\gamma})^{ss} = \mathcal{M}_H(\widehat{\gamma})^{ss}$ (cf. (2.10)). We define the Brill-Noether locus by

$$\mathcal{M}_H(\widehat{\gamma}, n) := \{F \in \mathcal{M}_H(\widehat{\gamma})^{\mu\text{-}ss} | \dim \operatorname{Hom}(F, E_0) \geq n\} \tag{4.2}$$

and the open substack $\mathcal{M}_H(\widehat{\gamma}, n)_0 := \mathcal{M}_H(\widehat{\gamma}, n) \setminus \mathcal{M}_H(\widehat{\gamma}, n+1)$. By using a determinantal ideal, we see that $\mathcal{M}_H(\widehat{\gamma}, n)$ has a substack structure. Indeed, let $Q_H(\widehat{\gamma})^{\mu\text{-}ss}$ be the standard open covering of $\mathcal{M}_H(\widehat{\gamma})^{\mu\text{-}ss}$ in (2.6). We may assume that

$$H^i(X, E_0(l)) = 0, \qquad i > 0. \tag{4.3}$$

Let $\mathcal{O}_{Q_H(\widehat{\gamma})^{\mu\text{-}ss} \times X}(-l)^{\oplus N} \to \mathcal{Q}$ be the universal quotient and $\mathcal{K}$ the universal subsheaf. We set

$$\begin{aligned} V &:= \operatorname{Hom}_{p_{Q_H(\widehat{\gamma})^{\mu\text{-}ss}}}(\mathcal{O}_{Q_H(\widehat{\gamma})^{\mu\text{-}ss} \times X}(-l)^{\oplus N}, \mathcal{O}_{Q_H(\widehat{\gamma})^{\mu\text{-}ss}} \otimes_{\mathfrak{k}} E_0), \\ W &:= \operatorname{Hom}_{p_{Q_H(\widehat{\gamma})^{\mu\text{-}ss}}}(\mathcal{K}, \mathcal{O}_{Q_H(\widehat{\gamma})^{\mu\text{-}ss}} \otimes_{\mathfrak{k}} E_0). \end{aligned} \tag{4.4}$$

Since $\operatorname{Ext}^2(\mathcal{Q}_q, E_0) = \operatorname{Hom}(E_0, \mathcal{Q}_q(K_X))^\vee = 0$, $q \in Q_H(\widehat{\gamma})^{\mu\text{-}ss}$, (4.3) implies that $\operatorname{Ext}^i(\mathcal{K}_q, E_0) = 0$ for all $i > 0$ and $q \in Q_H(\widehat{\gamma})^{\mu\text{-}ss}$. Hence V and W are locally free sheaves on $Q_H(\widehat{\gamma})^{\mu\text{-}ss}$ and we have an exact sequence

$$0 \to \operatorname{Hom}(\mathcal{Q}_q, E_0) \to V_q \to W_q \to \operatorname{Ext}^1(\mathcal{Q}_q, E_0) \to 0, \quad q \in Q_H(\widehat{\gamma})^{\mu\text{-}ss}. \tag{4.5}$$

Therefore we shall define the stack structure on $\mathcal{M}_H(\widehat{\gamma}, n)$ as the zero locus of $\wedge^s V \to \wedge^s W$, where $s = \dim V - n + 1 = \dim W - n + 1$.

Let $\mathcal{M}_H(\widehat{\gamma}, n\gamma_0)$ be the moduli stack of isomorphism classes of $F \to E_0^{\oplus n}$ such that $F \in \mathcal{M}_H(\widehat{\gamma})^{\mu\text{-}ss}$ and $\operatorname{Hom}(E_0^{\oplus n}, E_0) \to \operatorname{Hom}(F, E_0)$ is injective. We have a natural projection $\mathcal{M}_H(\widehat{\gamma}, n\gamma_0) \to \mathcal{M}_H(\widehat{\gamma}, n)$. Let $\mathcal{M}_H(\widehat{\gamma}, n\gamma_0)_0$ be the open substack of $\mathcal{M}_H(\widehat{\gamma}, n\gamma_0)$ consisting of F with an isomorphism $\operatorname{Hom}(E_0^{\oplus n}, E_0) \to \operatorname{Hom}(F, E_0)$. By [ACGH, Chapter II Section 2, 3], $\mathcal{M}_H(\widehat{\gamma}, n\gamma_0)_0$ is isomorphic to $\mathcal{M}_H(\widehat{\gamma}, n)_0$.

We shall show that $\mathcal{M}_H(\widehat{\gamma}, n)$ is Cohen-Macaulay and normal. By [F, Theorem 14.3] (or [ACGH, Chapter II Proposition (4.1)]), if $\mathcal{M}_H(\widehat{\gamma}, n)$ has an expected codimension, that is, $\operatorname{codim}_{\mathcal{M}_H(\widehat{\gamma})^{\mu\text{-}ss}} \mathcal{M}_H(\widehat{\gamma}, n) = n^2$, then $\mathcal{M}_H(\widehat{\gamma}, n)$ is Cohen-Macaulay. We shall estimate the dimension of the substack $\mathcal{M}_H(\widehat{\gamma}; n, p, a)$ of $\mathcal{M}_H(\widehat{\gamma})^{\mu\text{-}ss}$ consisting of $F \in \mathcal{M}_H(\widehat{\gamma})^{\mu\text{-}ss}$ such that $\dim F^{\vee\vee}/F = p$ and $F^{\vee\vee}$ fits in an exact sequence

$$0 \to E \to F^{\vee\vee} \to G \to 0, \tag{4.6}$$

where E is a μ-semi-stable sheaf with $\gamma(E) = r\gamma_0 - b\omega$, $G^{\vee\vee} \cong E_0^{\oplus n}$ and $\gamma(G) = n\gamma_0 - a\omega$.

Lemma 4.1. $\operatorname{codim}_{\mathcal{M}_H(\widehat{\gamma})^{\mu\text{-}ss}} \mathcal{M}_H(\widehat{\gamma}; n, p, a) \geq n^2 + (r \operatorname{rk} E_0 - 1)(a+p)$.

Proof. For a locally free sheaf L, let $\operatorname{Quot}^{a\omega}_{L/X/\mathfrak{k}}$ be the quot-scheme parametrizing all quotients $L \to A$ with $\gamma(A) = a\omega$. Then [Y1, Theorem 0.4] implies that

$$\dim \operatorname{Quot}^{a\omega}_{L/X/\mathfrak{k}} = (\operatorname{rk} L + 1)a. \tag{4.7}$$

Let N be the substack of $\mathcal{M}_H((r+n)\gamma_0 - (a+b)\omega)^{\mu\text{-}ss}$ consisting of F which fits in an exact sequence

$$0 \to E \to L \to G \to 0 \tag{4.8}$$

where E is a μ-semi-stable sheaf with $\gamma(E) = r\gamma_0 - b\omega$, $G^{\vee\vee} \cong E_0^{\oplus n}$ and $\gamma(G) = n\gamma_0 - a\omega$. By [Y4, Lemma 5.2], we see that

(4.9)
$$\begin{aligned}
\dim N &\leq \dim \mathcal{M}_H(r\gamma_0 - b\omega)^{\mu\text{-}ss} + \dim([\operatorname{Quot}^{a\omega}_{E_0^{\oplus n}/X/\mathfrak{k}} / \operatorname{Aut}(E_0^{\oplus n})]) - \chi(G, E)\\
&= (2rb \operatorname{rk} E_0 - r^2) + ((n \operatorname{rk} E_0 + 1)a - n^2) + ((ra + nb) \operatorname{rk} E_0 - rn))\\
&= (r+n)((a+b)\operatorname{rk} E_0 - (r+n)) + n(r+n) + a + br \operatorname{rk} E_0 - n^2.
\end{aligned}$$

Hence by using (4.7) and the assumption $(a+b+p)\operatorname{rk} E_0 = r+n$, we see that

$$\begin{aligned}
\dim \mathcal{M}_H(\widehat{\gamma}; n, p, a) &= \dim N + ((r+n)\operatorname{rk} E_0 + 1)p\\
&\leq n(r+n) + a + p + br \operatorname{rk} E_0 - n^2.
\end{aligned} \tag{4.10}$$

Therefore we get

(4.11)
$$\begin{aligned}
\operatorname{codim}_{\mathcal{M}_H(\widehat{\gamma})^{\mu\text{-}ss}} \mathcal{M}_H(\widehat{\gamma}; n, p, a) &\geq (r+n)(2(a+b+p)\operatorname{rk} E_0 - (r+n))\\
&\quad - (n(r+n) + a + p + br \operatorname{rk} E_0 - n^2)\\
&= (r+n)^2 - n(r+n) - (a + p + br \operatorname{rk} E_0 - n^2)\\
&= n^2 + (r \operatorname{rk} E_0 - 1)(a+p).
\end{aligned}$$

□

Corollary 4.2. *If $r := m - n \geq 1$, then $\mathcal{M}_H(\widehat{\gamma}; n)$ is Cohen-Macaulay.*

Assume that $r \operatorname{rk} E_0 \geq 2$. Then $\operatorname{codim}_{\mathcal{M}_H(\widehat{\gamma};n)} \mathcal{M}_H(\widehat{\gamma}; n+1) \geq 2n+1$. Thus, by Serre's criterion, it is enough for the normality of $\mathcal{M}_H(\widehat{\gamma}; n)$ to show that $\mathcal{M}_H(\widehat{\gamma}; n)_0 \cong \mathcal{M}_H(\widehat{\gamma}, n\gamma_0)_0$ is regular in codimension 1. For an element $F \to E_0^{\oplus n}$ of $\mathcal{M}_H(\widehat{\gamma}, n\gamma_0)_0$, the obstruction for smoothness belongs to $\operatorname{Ext}^2(F, F \to E_0^{\oplus n})$.

Lemma 4.3. *If $F \to E_0^{\oplus n}$ is surjective or F is locally free, then*

$$\operatorname{Ext}^2(F, F \to E_0^{\oplus n}) = 0. \tag{4.12}$$

Proof. We have an exact sequence

$$\operatorname{Ext}^2(F, E) \to \operatorname{Ext}^2(F, F \to E_0^{\oplus n}) \to \operatorname{Ext}^2(F, G \to E_0^{\oplus n}), \tag{4.13}$$

where $E := \ker(F \to E_0^{\oplus n})$ and $G := \operatorname{im}(F \to E_0^{\oplus n})$. By Lemma 3.1, we get $\operatorname{Ext}^2(F, E) = 0$. Since $\operatorname{Ext}^2(F, G \to E_0^{\oplus n}) = \operatorname{Ext}^1(F, E_0^{\oplus n}/G)$, we get our claim. □

If $a + p \geq 2$, then $\operatorname{codim}_{\mathcal{M}_H(\widehat{\gamma})^{\mu\text{-}ss}} \mathcal{M}_H(\widehat{\gamma}; n, p, a) \geq 2$. If $a + p \leq 1$, then $a = 0$ or $p = 0$. Assume that $F \in \mathcal{M}_H(\widehat{\gamma}, n)_0$ belongs to $\mathcal{M}_H(\widehat{\gamma}; n, p, a)$. If $p = 1$ and $a = 0$, then $F^{\vee\vee}$ fits in an exact sequence (4.6). For a general quotient map $f : F^{\vee\vee} \to \mathfrak{k}_x$, $x \in X$, $\ker f \cap E \neq E$. This means that $F \to E_0^{\oplus n}$ is surjective. If $p = 0$, then F is locally free. Therefore by using Lemma 4.3, we see that $\mathcal{M}_H(\widehat{\gamma}, n)$ is regular in codimension 1.

Proposition 4.4. *Assume that $r \operatorname{rk} E_0 \geq 2$. Then $\mathcal{M}_H(\widehat{\gamma}; n)$, $n := m - r$ is normal and a general member F fits in an exact sequence*

$$0 \to E \to F \to E_0^{\oplus n} \to 0, \tag{4.14}$$

where $E \in \mathcal{M}_H(\widehat{\gamma} - n\gamma_0)^{\mu\text{-}s,loc}$ and $\operatorname{Hom}(E, E_0) = 0$.

The following is a partial answer to [Ma3, Question 6.5].

Theorem 4.5. *Assume that $r \operatorname{rk} E_0 \geq 2$. For $n := m - r$, we set*

$$\overline{M}_H(\widehat{\gamma}; n) := \{F \in \overline{M}_H(\widehat{\gamma}) \mid \dim \operatorname{Hom}(F, E_0) \geq n\}. \tag{4.15}$$

Then $\overline{M}_H(\widehat{\gamma}; n)$ is normal, $\overline{M}_H(\widehat{\gamma}; n) = \phi_\gamma(\overline{M}_H(\gamma))$ and we have an identification

$$\overline{M}_H(\widehat{\gamma}; n) = \coprod_{r_i, a_i, n_i, l} \prod_i S^{n_i} M_H(r_i\gamma_0 - a_i\omega)^{\mu\text{-}s,loc} \times S^l X, \tag{4.16}$$

where r_i, a_i, n_i, l satisfy that $a_i \operatorname{rk} E_0 \geq r_i$, $(r_i, a_i) \neq (r_j, a_j)$ for $i \neq j$, $l + \sum_i n_i a_i = a$ and $\sum_i n_i r_i \leq r = m - n$. Therefore $\phi_\gamma(\overline{M}_H(\gamma))$ is normal.

Proof. By Proposition 4.4, we get that $\overline{M}_H(\widehat{\gamma}; n)$ is normal. Moreover $\phi_\gamma(M_H(\gamma)^{\mu\text{-}s,loc})$ is a dense subset of $\overline{M}_H(\widehat{\gamma}; n)$. Hence $\overline{M}_H(\widehat{\gamma}; n) = \phi_\gamma(\overline{M}_H(\gamma))$. Let F be a poly-stable sheaf with $\gamma(F) = \widehat{\gamma}$, i.e., F is a direct sum of E_0-twisted stable sheaves. By Proposition 3.14, there are μ-stable locally free sheaves E_i, $1 \leq i \leq k$ with $\gamma(E_i) = r_i\gamma_0 - a_i\omega$ and points $x_j \in X$, $1 \leq j \leq l$ such that $F = \bigoplus_{i=1}^k \widehat{E_i} \oplus \bigoplus_{j=1}^l \mathcal{E}_{x_j}$. Since $\dim \operatorname{Hom}(\widehat{E_i}, E_0) = a_i \operatorname{rk} E_0 - r_i$ and $\dim \operatorname{Hom}(\mathcal{E}_{x_j}, E_0) = \operatorname{rk} E_0$, we see that

$$\begin{aligned} \dim \operatorname{Hom}(F, E_0) &= \sum_i (a_i \operatorname{rk} E_0 - r_i) + l \operatorname{rk} E_0 \\ &= a \operatorname{rk} E_0 - \sum_i r_i = m - \sum_i r_i. \end{aligned} \tag{4.17}$$

Hence F belongs to $\overline{M}_H(\widehat{\gamma}; n)$ if and only if $\sum_i r_i \le m - n = r$. Then the last claim follows from this. □

5. The case where $\deg_{E_0}(E) = 1$

5.1. Twisted coherent systems and correspondences

In this section, we shall treat the case where the E_0-twisted degree is 1, where E_0 is the exceptional bundle in Section 3. This case was highly motivated by Ellingsrud and Strømme's paper [E-S]. In this section, we assume that

$$(\operatorname{rk} E_0)(-K_X, H) > 1. \tag{5.1}$$

Let e be a class in $K(X)$ such that $\operatorname{rk} e > 0$ and $\deg_{e_0}(e) = 1$. We set $\gamma := \gamma(e)$ and $\gamma_0 := \gamma(e_0) = \gamma(E_0)$. Then every μ-stable sheaf E with $\gamma(E) = \gamma$ is μ-stable. Thus the G-twisted stability does not depend on the the choice of G.

Lemma 5.1. *Assume that there is a stable sheaf E with $\gamma(E) = \gamma$. Then $-\chi(\gamma, \gamma_0) \ge 0$.*

Proof. For a stable sheaf E with $\gamma(E) = \gamma$, $\operatorname{Hom}(E, E_0) = 0$. Since $\deg_{E_0}(E(K_X)) = \deg_{E_0}(E) + \operatorname{rk} E \operatorname{rk} E_0 (K_X, H) < 0$, we get $\operatorname{Ext}^2(E, E_0) = \operatorname{Hom}(E_0, E(K_X))^\vee = 0$. Hence $-\chi(E, E_0) \ge 0$. □

Proposition 5.2. *$M_H(\gamma)$ is projective and there is a universal family on $M_H(\gamma) \times X$.*

Proof. Since $\deg_{e_0}(e) = \operatorname{rk} e_0 (c_1(e), H) - \operatorname{rk} e (c_1(e_0), H) = 1$, $\operatorname{rk} e$ and $(c_1(e), H)$ are relatively prime. Hence by [Ma1], there is a universal family. □

In order to construct a correspondence, we consider E_0-twisted coherent systems. Let $\operatorname{Syst}(E_0^{\oplus n}, \gamma)$ be the moduli space of E_0-twisted coherent systems:

$$\operatorname{Syst}(E_0^{\oplus n}, \gamma) := \{(E, V) \mid E \in M_H(\gamma), V \subset \operatorname{Hom}(E_0, E), \dim V = n\}. \tag{5.2}$$

$\operatorname{Syst}(E_0^{\oplus n}, \gamma)$ is a projective scheme over $M_H(\gamma)$ (cf. [Le]).

We set

$$M_H(\gamma)_i := \{E \in M_H(\gamma) \mid \dim \operatorname{Hom}(E_0, E) = i\}. \tag{5.3}$$

If $i \ge n$, then $\operatorname{Syst}(E_0^{\oplus n}, \gamma) \times_{M_H(\gamma)} M_H(\gamma)_i \to M_H(\gamma)_i$ is a $Gr(i, n)$-bundle, where $Gr(i, n)$ is the Grassmann variety of n-dimensional subspaces of an i-dimensional vector space.

Lemma 5.3 ([Y3, Lemma 2.1]). *For $E \in M_H(\gamma)$ and $V \subset \operatorname{Hom}(E_0, E)$, the following (1) or (2) occurs:*

(1) *$ev : V \otimes_{\mathfrak{k}} E_0 \to E$ is injective and* $\operatorname{coker}(ev)$ *is stable.*

(2) *$ev : V \otimes_{\mathfrak{k}} E_0 \to E$ is surjective in codimension 1 and* $\ker(ev)$ *is stable.*

Lemma 5.4. *Keep notation as above. If $ev : V \otimes_{\mathfrak{k}} E_0 \to E$ is surjective in codimension 1, then*

(1) $D(E) := \mathcal{E}xt^1_{\mathcal{O}_X}(V \otimes_{\mathfrak{k}} E_0 \to E, \mathcal{O}_X)$ *is a stable sheaf of* $\deg_{E_0^\vee} D(E) = 1$.

(2) $\operatorname{Ext}^1(E_0, E) = 0$.

In particular $\chi(\gamma_0, \gamma) \geq n$.

Proof. We have an exact sequence

$$\begin{aligned}\mathcal{E}xt^1_{\mathcal{O}_X}(\operatorname{im}(ev) \to E, \mathcal{O}_X) &\to \mathcal{E}xt^1_{\mathcal{O}_X}(V \otimes_{\mathfrak{k}} E_0 \to E, \mathcal{O}_X)\\ &\to \mathcal{H}om_{\mathcal{O}_X}(\ker(ev), \mathcal{O}_X) \to \mathcal{E}xt^2_{\mathcal{O}_X}(\operatorname{im}(ev) \to E, \mathcal{O}_X).\end{aligned} \tag{5.4}$$

By Lemma 5.3, $\ker(ev)$ is stable and $\operatorname{coker}(ev)$ is 0-dimensional. Then

$$\mathcal{E}xt^1_{\mathcal{O}_X}(\operatorname{im}(ev) \to E, \mathcal{O}_X) \cong \mathcal{E}xt^1_{\mathcal{O}_X}(\operatorname{coker}(ev), \mathcal{O}_X) = 0 \tag{5.5}$$

and $\mathcal{E}xt^2_{\mathcal{O}_X}(\operatorname{im}(ev) \to E, \mathcal{O}_X) \cong \mathcal{E}xt^2_{\mathcal{O}_X}(\operatorname{coker}(ev), \mathcal{O}_X)$ is 0-dimensional. Hence $D(X)$ is stable.

We next show that $\operatorname{Ext}^1(E_0, E) = 0$. Since $\ker(ev)$ is stable, we get

$$\operatorname{Ext}^2(E_0, \ker(ev)) = \operatorname{Hom}(\ker(ev), E_0(K_X))^\vee = 0. \tag{5.6}$$

Combining the fact $\operatorname{Ext}^1(E_0, E_0) = 0$ with this, we see that $\operatorname{Ext}^1(E_0, \operatorname{im}(ev)) = 0$. Since $\operatorname{Ext}^1(E_0, \operatorname{coker}(ev)) = 0$, we get $\operatorname{Ext}^1(E_0, E) = 0$. □

Proposition 5.5. $\operatorname{Syst}(E_0^{\oplus n}, \gamma)$ *is smooth and*

$$\dim \operatorname{Syst}(E_0^{\oplus n}, \gamma) = \dim M_H(\gamma) - n(n - \chi(\gamma_0, \gamma)). \tag{5.7}$$

Proof. Let $(E, V) \in \operatorname{Syst}(E_0^{\oplus n}, \gamma)$ be an E_0-twisted coherent system. Since $V \subset \operatorname{Hom}(E_0, E)$, we have a homomorphism

$$\operatorname{Hom}(V \otimes_{\mathfrak{k}} E_0, V \otimes_{\mathfrak{k}} E_0) \to \operatorname{Hom}(V \otimes_{\mathfrak{k}} E_0, E) \to \operatorname{Ext}^1(V \otimes_{\mathfrak{k}} E_0 \to E, E). \tag{5.8}$$

Then the cokernel is the Zariski tangent space of $\operatorname{Syst}(E_0^{\oplus n}, \gamma)$ at (E, V) and the obstruction space for the smoothness is $\operatorname{Ext}^2(V \otimes_{\mathfrak{k}} E_0 \to E, E)$. If $\operatorname{rk}(\gamma - n\gamma_0) \geq 0$, then $\operatorname{Ext}^2(V \otimes_{\mathfrak{k}} E_0 \to E, E) \cong \operatorname{Ext}^2(\operatorname{coker}(ev), E) = 0$. If $\operatorname{rk}(\gamma - n\gamma_0) < 0$, then by using Lemma 5.4 and an exact sequence

$$\operatorname{Ext}^1(V \otimes_{\mathfrak{k}} E_0, E) \to \operatorname{Ext}^2(V \otimes_{\mathfrak{k}} E_0 \to E, E) \to \operatorname{Ext}^2(E, E), \tag{5.9}$$

we see that $\operatorname{Ext}^2(V \otimes_{\mathfrak{k}} E_0 \to E, E) = 0$. Hence $\operatorname{Syst}(E_0^{\oplus n}, \gamma)$ is smooth. Then we see that

$$\begin{aligned}\dim \operatorname{Syst}(E_0^{\oplus n}, \gamma) &= \dim \operatorname{Ext}^1(V \otimes_{\mathfrak{k}} E_0 \to E, E) - \dim PGL(V)\\ &= -\chi(E, E) + n\chi(E_0, E) - (n^2 - 1)\\ &= \dim M_H(\gamma) - n(n - \chi(\gamma_0, \gamma)).\end{aligned} \tag{5.10}$$

□

Proposition 5.6. *We set $m := -\chi(\gamma, \gamma_0)$.*

(1) If $\operatorname{rk}\gamma \geq n \operatorname{rk}\gamma_0$, then $\operatorname{Syst}(E_0^{\oplus n}, \gamma)$ is a $Gr(m+n,n)$-bundle over $M_H(\gamma - n\gamma_0)$.

(2) If $\operatorname{rk}\gamma < n \operatorname{rk}\gamma_0$, then $\operatorname{Syst}(E_0^{\oplus n}, \gamma) \cong \operatorname{Syst}((E_0^\vee)^{\oplus n}, n\gamma_0^\vee - \gamma^\vee)$, where $\gamma_0^\vee = \gamma(E_0^\vee)$ and $\gamma^\vee = \gamma(e^\vee)$. In particular $\operatorname{Syst}(E_0^{\oplus n}, \gamma)$ is a $Gr(m+n,n)$-bundle over $M_H(n\gamma_0^\vee - \gamma^\vee)$.

Proof. We first assume that $\operatorname{rk}\gamma \geq n \operatorname{rk}\gamma_0$. For $(E,V) \in \operatorname{Syst}(E_0^{\oplus n}, \gamma)$, Lemma 5.3 implies that $ev : V \otimes_{\mathfrak{k}} E_0 \to E$ is injective and $\operatorname{coker}(ev)$ is stable. Thus we have a morphism $\pi_n : \operatorname{Syst}(E_0^{\oplus n}, \gamma) \to M_H(\gamma - n\gamma_0)$. Conversely for $G \in M_H(\gamma - n\gamma_0)$ and an n-dimensional subspace U of $\operatorname{Ext}^1(G, E_0)$, we have an extension

$$0 \to U^\vee \otimes_{\mathfrak{k}} E_0 \to E \to G \to 0 \tag{5.11}$$

whose extension class corresponds to the inclusion $U \hookrightarrow \operatorname{Ext}^1(G, E_0)$. Then E is stable. Since

$$\begin{aligned} \dim \operatorname{Ext}^1(G, E_0) &= -\chi(G, E_0) \\ &= -\chi(\gamma - n\gamma_0, \gamma_0) = m + n \end{aligned} \tag{5.12}$$

and there is a universal family, we see that π_n is a (Zariski locally trivial) $Gr(m+n,n)$-bundle. Therefore we get our claim.

We next treat the second case. For $(E,V) \in \operatorname{Syst}(E_0^{\oplus n}, \gamma)$, $D(E) := \mathcal{E}xt^1_{\mathcal{O}_X}(V \otimes_{\mathfrak{k}} E_0 \to E, \mathcal{O}_X)$ fits in an exact sequence

$$0 \to E^\vee \to (V \otimes_{\mathfrak{k}} E_0)^\vee \to D(E) \to \mathcal{E}xt^1_{\mathcal{O}_X}(E, \mathcal{O}_X) \to 0. \tag{5.13}$$

Hence $(V \otimes_{\mathfrak{k}} E_0)^\vee \to D(E)$ defines a point of $\operatorname{Syst}((E_0^\vee)^{\oplus n}, n\gamma_0^\vee - \gamma^\vee)$. Thus we get a morphism

$$\psi : \operatorname{Syst}(E_0^{\oplus n}, \gamma) \to \operatorname{Syst}((E_0^\vee)^{\oplus n}, n\gamma_0^\vee - \gamma^\vee). \tag{5.14}$$

Conversely for $(F,U) \in \operatorname{Syst}((E_0^\vee)^{\oplus n}, n\gamma_0^\vee - \gamma^\vee)$, we get a homomorphism

$$U^\vee \otimes_{\mathfrak{k}} E_0 \to \mathcal{E}xt^1_{\mathcal{O}_X}(U \otimes_{\mathfrak{k}} E_0^\vee \to F, \mathcal{O}_X). \tag{5.15}$$

It gives the inverse of ψ (for more details, see [K-Y, Proposition 5.128]). □

Lemma 5.7.

(1) If $\operatorname{rk}(\gamma - \chi(\gamma_0, \gamma)\gamma_0) \geq 0$, then $\overline{M_H(\gamma)_i} = \emptyset$ for $\operatorname{rk}(\gamma - i\gamma_0) < 0$.

(2) If $\operatorname{rk}(\gamma - \chi(\gamma_0, \gamma)\gamma_0) < 0$, then $M_H(\gamma)_{\chi(\gamma_0,\gamma)} = M_H(\gamma)$.

Proof. If $\dim(E_0, E) = i$ with $\operatorname{rk}(\gamma - i\gamma_0) < 0$, then Lemma 5.4 implies that $\chi(\gamma_0, \gamma) \geq i$. Hence $\operatorname{rk}(\gamma - \chi(\gamma_0, \gamma)\gamma_0) < 0$. By Lemma 5.4, $\operatorname{Ext}^1(E_0, E) = 0$ for all $E \in M_H(\gamma)$. Hence $M_H(\gamma)_{\chi(\gamma_0,\gamma)} = M_H(\gamma)$. □

By using Proposition 5.6, we get the following theorem.

Theorem 5.8. *We set* $s := -(K_X, c_1(e_0^\vee \otimes e))$ *and* $\zeta := \gamma(L_{e_0}(e)) = \gamma - \chi(\gamma,\gamma_0)\gamma_0$. *Assume that* $n := -\chi(\gamma,\gamma_0) > 0$. *Then* $M_H(\gamma) \cong \mathrm{Syst}(E_0^{\oplus n}, \zeta)$ *and we get a morphism* $\lambda_{\gamma_0,\gamma} : M_H(\gamma) \to M_H(\zeta)$ *by sending* E *to the universal extension*

$$0 \to E_0 \otimes_{\mathfrak{k}} \mathrm{Ext}^1(E, E_0)^\vee \to \lambda_{\gamma_0,\gamma}(E) \to E \to 0. \tag{5.16}$$

Hence we have a stratification

$$M_H(\gamma) = \coprod_{i \geq s} \lambda_{\gamma_0,\gamma}^{-1}(M_H(\zeta)_i) \tag{5.17}$$

such that $\lambda_{\gamma_0,\gamma}^{-1}(M_H(\zeta)_i) \to M_H(\zeta)_i$ *is a* $Gr(i,n)$*-bundle. In particular,*

$$M_H(\gamma)_0 \to M_H(\zeta)_n \tag{5.18}$$

is an isomorphism for $n \geq s$.

Corollary 5.9. *If* $0 > \chi(e_0, e) = -k \geq -s$, *then*

$$M_H(\gamma(e)) \to M_H(\gamma(L_{e_0}(e))) \tag{5.19}$$

is birationally $Gr(s,k)$*-bundle. In particular, if* $\chi(e_0,e) = -s$, *then* $M_H(\gamma(e)) \to M_H(\gamma(L_{e_0}(e)))$ *is a birational map.*

Example 5.1. Assume that $(X,H) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,n))$, $n > 0$. We set $L := \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1, n+1)$. Then $(L,H) = 1$, $s = (L, -K_X) = 2n$ and $\chi(L) = 0$. Hence $M_H(1+r, L, r) \cong Gr(2n, r)$.

5.2. Virtual Hodge polynomial

From now on, we assume that $\mathfrak{k} = \mathbb{C}$. For a variety Y over $\mathbb{C}$, the cohomology group with compact support $H_c^*(Y,\mathbb{Q})$ has a natural mixed Hodge structure. Let $e^{p,q}(Y) := \sum_k (-1)^k h^{p,q}(H_c^k(Y))$ be the virtual Hodge number and $e(Y) := \sum_{p,q} e^{p,q}(Y) x^p y^q$ the virtual Hodge polynimial of Y. The virtual Hodge polynomial satisfies the following properties (cf. [D-K]):

(1) If Y is a smooth projective variety, then $e(Y)$ is the usual Hodge polynomial of Y:

$$e(Y) = \sum_{p,q} (-1)^{p+q} h^{p,q}(Y) x^p y^q.$$

(2) For a closed subset $Z \subset Y$, $e(Y) = e(Z) + e(Y \setminus Z)$.

(3) For a Zariski locally trivial fiber space $Y \to Z$ with a fiber F, $e(Y) = e(X)e(F)$.

We set $a := -\chi(\gamma,\gamma_0)$. Assume that $\mathrm{rk}(\gamma - \chi(\gamma_0,\gamma)\gamma_0) \geq 0$. We shall consider the vitrual Hodge polynomial of $M_H(\gamma + k\gamma_0)_i$ for some k, i. We set $t := xy$. Then

$$\begin{aligned} e(M_H(\gamma + k\gamma_0)_j) &= e(Gr(a+j-k, j))e(M_H(\gamma + (k-j)\gamma_0)_0) \\ &= \frac{[a+j-k]!}{[a-k]![j]!} e(M_H(\gamma + (k-j)\gamma_0)_0), \end{aligned} \tag{5.20}$$

where

$$[n] := \frac{t^n - 1}{t - 1}, \quad [n]! := [n][n-1]\cdots[1]. \tag{5.21}$$

By summing up all $e(M_H(\gamma + k\gamma_0)_k)$, we get

$$\begin{aligned} &\sum_k [a-k]! e(M_H(\gamma + k\gamma_0)) y^k \\ &= \left(\sum_j \frac{1}{[j]!} y^j\right) \left(\sum_l [a-l]! e(M_H(\gamma + l\gamma_0)_0) y^l\right). \end{aligned} \tag{5.22}$$

Since

$$\left(\sum_j \frac{1}{[j]!} y^j\right)^{-1} = \sum_j \frac{(-1)^j t^{j(j-1)/2}}{[j]!} y^j, \tag{5.23}$$

we get that

Lemma 5.10. *If* $\mathrm{rk}(\gamma - \chi(\gamma_0, \gamma)\gamma_0) \geq 0$, *then*

$$e(M_H(\gamma + l\gamma_0)_0) = \sum_{j \geq 0} (-1)^j t^{j(j-1)/2} \frac{[a+j-l]!}{[a-l]![j]!} e(M_H(\gamma + (l-j)\gamma_0)). \tag{5.24}$$

In particular

$$\begin{aligned} &e(M_H(\gamma + k\gamma_0)_i) \\ &= \sum_{j \geq 0} (-1)^j t^{j(j-1)/2} \frac{[a-k+i+j]!}{[a-k]![i]![j]!} e(M_H(\gamma + (k-i-j)\gamma_0)). \end{aligned} \tag{5.25}$$

Since $M_H(\gamma + l\gamma_0)_0 = \emptyset$ *for* $a - s < l \leq a$, *we also get the following relations*:

$$\sum_{j \geq 0} (-1)^j t^{j(j-1)/2} \frac{[a+j-l]!}{[a-l]![j]!} e(M_H(\gamma + (l-j)\gamma_0)) = 0 \tag{5.26}$$

for $a - s < l \leq a$.

5.3. Examples on $\mathbb{P}^2$

From now on, we assume that X is $\mathbb{P}^2$. Then $s = -(K_X, \mathcal{O}_X(1)) = 3$. Hence we get the following relations:

$$\begin{aligned} &\sum_{j \geq 0} (-1)^j t^{j(j-1)/2} e(M_H(\gamma + (a-j)\gamma_0)) = 0, \\ &\sum_{j \geq 0} (-1)^j t^{j(j-1)/2} [j+1] e(M_H(\gamma + (a-1-j)\gamma_0)) = 0, \\ &\sum_{j \geq 0} (-1)^j t^{j(j-1)/2} \frac{[j+2][j+1]}{[2]!} e(M_H(\gamma + (a-2-j)\gamma_0)) = 0. \end{aligned} \tag{5.27}$$

By a simple calculation, we get

Proposition 5.11.

$$(5.28)\quad \begin{aligned} & e(M_H(\gamma+(a-2)\gamma_0)) \\ & \quad = \sum_{j\geq 0}(-1)^j t^{(j+1)j/2}\frac{[j+3][j+2]}{[2]!}e(M_H(\gamma+(a-3-j)\gamma_0)), \\ & e(M_H(\gamma+(a-1)\gamma_0)) \\ & \quad = \sum_{j\geq 0}(-1)^j t^{(j+1)j/2}[j+3][j+1]e(M_H(\gamma+(a-3-j)\gamma_0)), \\ & e(M_H(\gamma+a\gamma_0)) \\ & \quad = \sum_{j\geq 0}(-1)^j t^{(j+1)j/2}\frac{[j+2][j+1]}{[2]!}e(M_H(\gamma+(a-3-j)\gamma_0)). \end{aligned}$$

Assume that $E_0 := \mathcal{O}_X$. We set $\gamma := \gamma(\mathcal{O}_X(1))$. Then

$$(5.29)\quad \begin{aligned} M_H(\gamma - a\omega - \gamma_0) &= \{\mathcal{O}_l(1-a) \mid l \text{ is a line on } \mathbb{P}^2\} \\ &\cong \mathbb{P}^2. \end{aligned}$$

Hence $M_H(\gamma - a\omega)_1$, $a \geq 2$ is a $\mathbb{P}^a$-bundle over $\mathbb{P}^2$. By the morphism

$$(5.30)\quad M_H(\gamma - a\omega) \to M_H(\gamma - a\omega + a\gamma_0),$$

the fibers of $M_H(\gamma - a\omega)_1 \to \mathbb{P}^2$ are contracted.

Example 5.2. If $a = 2$, then $M_H(\gamma - 2\omega + 2\gamma_0) \cong M_H(\gamma^2 - \gamma_0) \cong \mathbb{P}^2$. That is, $E \in M_H(\gamma - 2\omega + 2\gamma_0)$ fits in a universal extension

$$(5.31)\quad 0 \to \mathcal{O}_X^{\oplus 3} \to E \to \mathcal{O}_l(-1) \to 0.$$

Moreover we see that $M_H(\gamma - 2\omega + i\gamma_0)$, $i = 0, 1$ are $\mathbb{P}^2$-bundle over $M_H(\gamma - 2\omega + 2\gamma_0) \cong \mathbb{P}^2$.

Example 5.3. If $a = 3$, then $M_H(\gamma - 3\omega) \to M_H(\gamma - 3\omega + 3\gamma_0)$ is the blow-up along $M_H(\gamma - 3\omega + 3\gamma_0)_4 \cong M_H(\gamma - 3\omega - \gamma_0)$. This was obtained by Drezet [D3, IV].

By [E-S] and [Y1], we know $e(M_H(r, H, \chi))$ for $r = 1, 2$. By using Proposition 5.11, we get the following:

$$(5.32)\quad \begin{aligned} e(M_H(1, H, 0)) &= 1 + 2t + 5t^2 + 6t^3 + 5t^4 + 2t^5 + t^6, \\ e(M_H(2, H, 1)) &= 1 + 2t + 6t^2 + 9t^3 + 12t^4 + 9t^5 + 6t^6 + 2t^7 + t^8, \\ e(M_H(3, H, 2)) &= 1 + 2t + 5t^2 + 8t^3 + 10t^4 + 8t^5 + 5t^6 + 2t^7 + t^8, \\ e(M_H(4, H, 3)) &= 1 + t + 3t^2 + 3t^3 + 3t^4 + t^5 + t^6. \end{aligned}$$

$$
\begin{aligned}
e(M_H(1,H,-1)) &= 1+2t+6t^2+10t^3+13t^4+10t^5+6t^6+2t^7+t^8,\\
e(M_H(2,H,0)) &= 1+2t+6t^2+13t^3+24t^4+35t^5+41t^6\\
&\quad +35t^7+24t^8+13t^9+6t^{10}+2t^{11}+t^{12},\\
e(M_H(3,H,1)) &= 1+2t+6t^2+12t^3+24t^4+38t^5+54t^6+59t^7\\
&\quad +54t^8+38t^9+24t^{10}+12t^{11}+6t^{12}+2t^{13}+t^{14},\\
e(M_H(4,H,2)) &= 1+2t+5t^2+10t^3+18t^4+28t^5+38t^6+42t^7\\
&\quad +38t^8+28t^9+18t^{10}+10t^{11}+5t^{12}+2t^{13}+t^{14},\\
e(M_H(5,H,3)) &= 1+t+3t^2+5t^3+8t^4+10t^5+12t^6\\
&\quad +10t^7+8t^8+5t^9+3t^{10}+t^{11}+t^{12}.
\end{aligned}
\tag{5.33}
$$

If $E_0 := \Omega_X(1)$, then $\deg_{E_0}(\mathcal{O}_X) = 1$. We set $\gamma = \gamma(\mathcal{O}_X)$. Then

- $M_H(\gamma - a\omega) \to M_H(\gamma - a\omega + 2a\gamma_0)$ is a closed immersion for $a \geq 2$.
- If $a = 2$, then $M_H(\gamma - 2\omega + \gamma_0) \to M_H(\gamma - 2\omega + 4\gamma_0)$ is the blow-up along $M_H(\gamma - 2\omega)$.

Here we remark that Drezet showed that $M_H(\gamma - 2\omega + 4\gamma_0) = M_H(9, -4H, -1) \cong Gr(6,2)$ (see [D1, Appendice]). Since $e(M_H(1,0,-1)) = 1 + 2t + 3t^3 + 2t^3 + t^4$ and $e(M_H(3,-H,-1)) = e(M_H(3,H,2))$, Proposition 5.11 implies that

$$
\begin{aligned}
e(M_H(3,-H,-1)) &= 1+2t+5t^2+8t^3+10t^4+8t^5+5t^6+2t^7+t^8,\\
e(M_H(5,-2H,-1)) &= 1+2t+5t^2+8t^3+13t^4+14t^5\\
&\quad +13t^6+8t^7+5t^8+2t^9+t^{10},\\
e(M_H(7,-3H,-1)) &= 1+2t+4t^2+6t^3+9t^4+10t^5\\
&\quad +9t^6+6t^7+4t^8+2t^9+t^{10},\\
e(M_H(9,-4H,-1)) &= 1+t+2t^2+2t^3+3t^4+2t^5+2t^6+t^7+t^8\\
&\quad (= e(Gr(6,2))).
\end{aligned}
\tag{5.34}
$$

5.3.1. Line bundles on $M_H(\gamma)$

Let $p_{M_H(\gamma(e))} : M_H(\gamma(e)) \times X \to M_H(\gamma(e))$ and $q : M_H(\gamma(e)) \times X \to X$ be the projections, and let $\mathcal{E}$ be a universal family on $M_H(\gamma(e)) \times X$. We define a homomorphism $\theta_e : e^\perp \to \mathrm{Pic}(M_H(\gamma(e)))$ by

$$
\theta_e(x) := \det p_{M_H(\gamma(e))!}(\mathcal{E}^\vee \otimes q^*(x)), \tag{5.35}
$$

where $e^\perp := \{x \in K(X) \mid \chi(e,x) = 0\}$ and $\mathcal{E}^\vee$ is the dual of $\mathcal{E}$ in $K(M_H(\gamma(e)) \times X)$. The following is a special case of Drezet's results.

Theorem 5.12 ([D2]). *Assume that* $\dim M_H(\gamma(e)) = 1 - \chi(e,e) > 0$. *Then* θ_e *is surjective and*

(1) θ_e *is an isomorphism, if* $\chi(e,e) < 0$,

(2) $\ker \theta_e = \mathbb{Z} e_0$, *if* $\chi(e,e_0) = 0$.

We set $\tilde{e} := L_{e_0}(e)$. By a simple calculation, we see that the following diagram is commutative:

$$
\begin{array}{ccc}
e^{\perp} & \xleftarrow{R_{e_0}} & \tilde{e}^{\perp}/e_0 \\
{\scriptstyle \theta_e}\downarrow & & \downarrow{\scriptstyle \theta_{\tilde{e}}} \\
\operatorname{Pic}(M_H(\gamma(e))) & \xleftarrow[\lambda^*_{\gamma(e_0),\gamma(e)}]{} & \operatorname{Pic}(M_H(\gamma(\tilde{e})))
\end{array}
\tag{5.36}
$$

We set $\alpha_e := -(\operatorname{rk} e)\mathcal{O}_H + \chi(e, \mathcal{O}_H)\mathbb{C}_P$. Then it gives a map to the Uhlenbeck compactification [Li]. $\beta_e := R_{e_0}(\alpha_{\tilde{e}})$ gives the map $\lambda_{\gamma(e_0),\gamma(e)} : M_H(\gamma(e)) \to M_H(\gamma(\tilde{e}))$.

• If $E_0 = \mathcal{O}_X$, $\operatorname{rk} e > 0$ and $\chi(e, e_0) < 0$, then the nef. cone of $M_H(\gamma(e))$ is generated by α_e and β_e.

This is a generalization of [S].

For $\gamma := (3, H, 5 - a)$, we set $\gamma_0 := (1, 0, 1)$, $\gamma_1 := \gamma(\Omega_X(1)) = (2, -H, 0)$, $\delta := \gamma + a\gamma_0$ and $\eta := \gamma^{\vee} + (2a - 3)\gamma_1$. $N_H(\gamma)$ denotes the Uhlenbeck compactification of $M_H(\gamma)^{\mu\text{-}s,loc}$. Then we get the following diagram:

$$
\begin{array}{ccccccc}
 & M_H(\gamma) & \longleftarrow & \cdots\cdots & \longrightarrow & M_H(\gamma^{\vee}) & \\
{\scriptstyle \lambda_{\gamma_0,\gamma}}\swarrow & & \searrow & & \swarrow & & \searrow{\scriptstyle \lambda_{\gamma_1,\gamma^{\vee}}} \\
M_H(\delta) & & & N_H(\gamma) & & & M_H(\eta)
\end{array}
$$

$M_H(\gamma^{\vee})$ contains $\mathbb{P}^{2a-3}$-bundle over $M_H(1, 0, 2 - a)$ and $\lambda_{\gamma_0,\gamma^{\vee}}$ contracts the fibers. $\lambda_{\gamma_0,\gamma|M_H(\gamma)_i}$ is a $Gr(a - 2 + i, a - 2)$-bundle over $M_H(\delta)_{a-2+i} \cong M_H(\gamma - i\gamma_0)_0$. Then it is easy to see that $M_H(3, H, 5 - a) \not\cong M_H(3, -H, 2 - a)$.

Acknowledgements. A starting point of this note is [E-S]. I would like to thank M. Maruyama for giving me a preprint version of [E-S].

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