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On phantom maps into suspension spaces

By

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Abstract

We show that there is an essential phantom map $f : K(\mathbb{Z}, n) \to \Sigma Y$ for a suitable n if $H_i(Y; \mathbb{Q}) \neq 0$ for some $i > 0$. The localized version of this problem is also considered. The ingredient of the proof is the computation of the Morava K-theories of the Eilenberg-MacLane spaces by Ravenel and Wilson.

1. Introduction

Throughout this paper we assume that a space has the homotopy type of a CW-complex with finite skeletons (or its localization) and has the base point, and that a map and a homotopy preserve the base points.

A map $f : X \to Y$ is said to be a phantom map provided that, for any finite CW-complex W and any map $j : W \to X$, the composite

$$W \xrightarrow{j} X \xrightarrow{f} Y$$

is null homotopic. By $\mathrm{Ph}(X, Y)$ we denote the subset of the pointed set $[X, Y]$ consisting of homotopy classes of phantom maps. We write $\mathrm{Ph}(-, Y) \equiv 0$ if $\mathrm{Ph}(X, Y) = *$ for any domain X, otherwise we write $\mathrm{Ph}(-, Y) \not\equiv 0$. Similarly we define $\mathrm{Ph}(X, -) \equiv 0$ and $\mathrm{Ph}(X, -) \not\equiv 0$.

In his survey paper [7] on phantom maps Roitberg asked

Question. *Is* $\mathrm{Ph}(-, \Sigma K(\mathbb{Z}, n)) \not\equiv 0$ *when* $n \geq 2$*?*

That is, he asked if the Eckmann-Hilton dual of the result that $\mathrm{Ph}(\Omega S^n, -) \equiv 0$ for $n \geq 2$ is false. In [1] we proved that $\mathrm{Ph}(\Omega X, -) \equiv 0$ for a rationally elliptic finite complex X. In this paper we will consider the dual problem.

The following theorem characterizes a space Y such that $\mathrm{Ph}(-, Y) \equiv 0$.

Theorem 1.1 (Theorem 1′ of [5]). *The following statements are equivalent.*

(i) $\mathrm{Ph}(-, Y) \equiv 0$.

(ii) $\mathrm{Ph}(K(\mathbb{Z}, n), Y) = *$ *for every* n.

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(iii) *There exists a rational homotopy equivalence from a product of* $K(\mathbb{Z}, m)$*'s to the base point component of* ΩY.

By this theorem it is not difficult to see that $\mathrm{Ph}(-, Y) \equiv 0$ if
(i) $\pi_n(Y)$ is finite for each $n > 2$, or
(ii) Y has only finitely many nonzero homotopy groups.
On the other hand, by Zabrodsky [9], $\mathrm{Ph}(-, \Omega^n Y) \not\equiv 0$ if Y is a simply connected finite complex with $\pi_i(\Omega^n Y) \otimes \mathbb{Q} \neq 0$ for some $i > 2$.

Theorem 1.2. $\mathrm{Ph}(-, \Sigma Y) \not\equiv 0$ *for a space* Y *with* $H_i(Y; \mathbb{Q}) \neq 0$ *for some* $i > 0$.

As a corollary we have the affirmative answer to the question of Roitberg.

Corollary 1.3. $\mathrm{Ph}(-, \Sigma K(\mathbb{Z}, n)) \not\equiv 0$ *for every positive integer* n.

Moreover, we have

Theorem 1.4. $\mathrm{Ph}(-, \Sigma K(\mathbb{Z}_{(p)}, n)) \not\equiv 0$ *for every prime* p *and every positive integer* n.

Needless to say, Theorem 1.4 implies Corollary 1.3 since there is a natural epimorphism

$$\mathrm{Ph}(X, \Sigma Y) \to \prod_p \mathrm{Ph}(X, \Sigma Y_{(p)})$$

for any spaces X and Y, see Section 6 of [4].

Corollary 1.5. *For a space* Y *and a prime* p *we have* $\mathrm{Ph}(-, \Sigma Y_{(p)}) \not\equiv 0$ *if any of the following three conditions hold*:
(i) Y *is a finite complex with* $H_i(Y; \mathbb{Q}) \neq 0$ *for some* $i > 0$.
(ii) *There is an odd dimensional element* $\alpha \in \pi_{2n+1}(Y)$ *whose Hurewicz image* $\rho(\alpha) \in H_{2n+1}(Y; \mathbb{Z})$ *is of order infinite.*
(iii) *There are an even dimensional element* $\alpha \in \pi_{2n}(Y)$, $n > 0$, *and a cohomology class* $v \in H^{2n}(Y; \mathbb{Z})$ *with non-zero Kronecker product* $\langle v, \rho(\alpha) \rangle \in \mathbb{Z}$. *Moreover,* v^2 *is of order infinite.*

An example of a space which does not satisfy any of the above conditions is the homotopy fiber F of the map $u_{2n}^2 : K(\mathbb{Z}, 2n) \to K(\mathbb{Z}, 4n)$, where $u_{2n} \in H^{2n}(K(\mathbb{Z}, 2n); \mathbb{Z})$ is a generator. If we could show that $\mathrm{Ph}(-, \Sigma F_{(p)}) \not\equiv 0$, then the answer of the following question would be yes.

Question 1.6. *Is* $\mathrm{Ph}(-, \Sigma Y_{(p)}) \not\equiv 0$ *for a prime* p *and a space* Y *with* $H_i(Y; \mathbb{Q}) \neq 0$ *for some* $i > 0$?

Another problem arises from Theorem 1.2.

Question 1.7. *Let* n *be a positive integer. Is* $\mathrm{Ph}(-, \Omega^n \Sigma Y) \not\equiv 0$ *for a space* Y *with* $\pi_i(\Omega^n \Sigma Y) \otimes \mathbb{Q} \neq 0$ *for some* $i > 2$?

2. Proofs

We begin with a proof of Theorem 1.4, for which we need the localized version of Theorem 1.1.

Theorem 2.1 (Theorem 1′ of [5]). *Let Y be a nilpotent space and p be a prime. Then* $\mathrm{Ph}(-, Y_{(p)}) \equiv 0$ *if and only if there exists a rational homotopy equivalence from a product of* $K(\mathbb{Z}_{(p)}, m)$*'s to the base point component of* $\Omega Y_{(p)}$.

From now on except in the proof of Theorem 1.2 we will assume that all spaces and groups are localized at a prime p, but the notation will not be burdened with this assumption. Thus, for example, $\mathbb{Z}$ stands for $\mathbb{Z}_{(p)}$.

If $\mathrm{Ph}(-, \Sigma K(\mathbb{Z}, n)) \equiv 0$, then by the theorem above we have a rational homotopy equivalence

$$\prod_{\beta} K(\mathbb{Z}, m_\beta) \to \Omega\Sigma K(\mathbb{Z}, n).$$

On the other hand we have the homotopy equivalence

$$\Omega\Sigma K(\mathbb{Z}, n) \simeq K(\mathbb{Z}, n) \times \Omega\Sigma(K(\mathbb{Z}, n) \wedge K(\mathbb{Z}, n))$$

by Stasheff [8]. Thus we have a rational homotopy equivalence

$$\prod_{m_\beta > n} K(\mathbb{Z}, m_\beta) \to \Omega\Sigma(K(\mathbb{Z}, n) \wedge K(\mathbb{Z}, n)).$$

We will show no such a rational homotopy equivalence exists. In fact, we have

Theorem 2.2. *Let m, n and ℓ be positive integers. Then every map*

$$f : K(\mathbb{Z}, m) \to \Omega\Sigma(K(\mathbb{Z}, n) \wedge K(\mathbb{Z}, \ell))$$

induces the trivial map on rational homotopy groups.

Proof. For $n = \ell = 1$ this is well-known. According to Zabrodsky [9] f is a phantom map since $\Omega\Sigma(K(\mathbb{Z}, n) \wedge K(\mathbb{Z}, \ell)) = \Omega S^3$, and every phantom map induces the trivial map on homotopy groups.

To prove the theorem for $n > 1$ we first recall some consequences of the computation of the Morava K-theories of Eilenberg-MacLane spaces by Ravenel and Wilson [6] and the appendix of [2].

Theorem 2.3 (Corollaries 12.2 and 13.1 of [6]). *Let p be a prime and k be a positive integer, then*

$$\varinjlim_{j} K(q)_* K(\mathbb{Z}/(p^j), k) \cong K(q)_* K(\mathbb{Z}, k+1)$$

and

$$p^j_* : K(q)_* K(\mathbb{Z}, k+1) \to K(q)_* K(\mathbb{Z}, k+1)$$

is epimorphic.

The short exact sequence of the groups $0 \to \mathbb{Z} \xrightarrow{p^j} \mathbb{Z} \to \mathbb{Z}/(p^j) \to 0$ induces the fiber sequence

$$K(\mathbb{Z}/(p^j), k) \xrightarrow{\delta} K(\mathbb{Z}, k+1) \xrightarrow{p^j} K(\mathbb{Z}, k+1) \xrightarrow{red} K(\mathbb{Z}/(p^j), k+1).$$

Since $p^j_* : K(q)_*K(\mathbb{Z}, k{+}1) \to K(q)_*K(\mathbb{Z}, k{+}1)$ is epimorphic, $red_* : K(q)_*K(\mathbb{Z}, k+1) \to K(q)_*K(\mathbb{Z}/(p^j), k+1)$ is the trivial map, which we are now using to prove Theorem 2.2.

If $f : K(\mathbb{Z}, m) \to \Omega\Sigma(K(\mathbb{Z}, n) \wedge K(\mathbb{Z}, \ell))$ induces a non-trivial map on rational homotopy groups, then there is a map

$$g \in H^m(\Omega\Sigma(K(\mathbb{Z}, n) \wedge K(\mathbb{Z}, \ell)); \mathbb{Z}) \cong [\Omega\Sigma(K(\mathbb{Z}, n) \wedge K(\mathbb{Z}, \ell)), K(\mathbb{Z}, m)]$$

such that

$$gf = p^j \in H^m(K(\mathbb{Z}, m); \mathbb{Z}) \cong [K(\mathbb{Z}, m), K(\mathbb{Z}, m)] \cong \mathbb{Z}$$

with some non-negative integer j. Since $(gf)_* = p^j_* : K(q)_*K(\mathbb{Z}, m) \to K(q)_*K(\mathbb{Z}, m)$ is epimorphic,

$$g_* : K(q)_*(\Omega\Sigma(K(\mathbb{Z}, n) \wedge K(\mathbb{Z}, \ell))) \to K(q)_*K(\mathbb{Z}, m)$$

must be epimorphic. But as we will show below g_* is the trivial map which contradicts the fact that g_* is epimorphic since $K(q)_*K(\mathbb{Z}, m)$ is non-trivial for $q \geq m - 1$ by Theorem 12.1 of [6].

Since the Morava K-theory possesses Künneth isomorphisms, we have the following isomorphism

$$\begin{aligned} &\varinjlim K(q)_*(\Omega\Sigma(K(\mathbb{Z}/(p^j), n-1) \wedge K(\mathbb{Z}, \ell))) \\ &\qquad\qquad \cong K(q)_*(\Omega\Sigma(K(\mathbb{Z}, n) \wedge K(\mathbb{Z}, \ell))). \end{aligned}$$

Thus to prove that g_* is trivial, it is sufficient to prove that

$$h = g \circ \Omega\Sigma(\delta \wedge 1) : \Omega\Sigma(K(\mathbb{Z}/(p^j), n-1) \wedge K(\mathbb{Z}, \ell)) \to K(\mathbb{Z}, m)$$

induces the trivial map on the Morava K-theories for each positive integer j.

Proposition 2.4. *For sufficiently large* t

$$\begin{aligned} \Omega\Sigma(1 \wedge red)^* : H^m(&\Omega\Sigma(K(\mathbb{Z}/(p^j), n-1) \wedge K(\mathbb{Z}/(p^t), \ell)); \mathbb{Z}) \\ &\to H^m(\Omega\Sigma(K(\mathbb{Z}/(p^j), n-1) \wedge K(\mathbb{Z}, \ell)); \mathbb{Z}) \end{aligned}$$

is epimorphic.

Assume for the moment that this proposition is true. By this proposition there is a map

$$h' : \Omega\Sigma(K(\mathbb{Z}/(p^j), n-1) \wedge K(\mathbb{Z}/(p^t), \ell)) \to K(\mathbb{Z}, m)$$

for sufficiently large t such that $h = h' \circ \Omega\Sigma(1 \wedge red)$. Since $red : K(\mathbb{Z}, \ell) \to K(\mathbb{Z}/(p^t), \ell)$ induces the trivial map on the Morava K-theories, $\Omega\Sigma(1 \wedge red)$ induces also the trivial map on the Morava K-theories by the Künneth isomorphisms. Thus h induces the trivial map on the Morava K-theories and we complete the proof of Theorem 2.2. □

Proof of Proposition 2.4. We first recall the mod p cohomology (resp. homology) Bockstein spectral sequence $\{E_r(X)\}$ (resp. $\{E^r(X)\}$) of a space X. $\{E_r(X)\}$ (resp. $\{E^r(X)\}$) is a spectral sequence of differential algebras (resp. coalgebras) such that $E_1(X) = H^*(X; \mathbb{Z}/(p))$ (resp. $E^1(X) = H_*(X; \mathbb{Z}/(p))$) and $E_{r+1}(X)$ (resp. $E^{r+1}(X)$) is the homology of $E_r(X)$ (resp. $E^r(X)$) with respect to the Bockstein operation β_r for $r \geq 1$. The mod p cohomology Bockstein spectral sequence $\{E_r(X)\}$ and the mod p homology Bockstein spectral sequence $\{E^r(X)\}$ are dual each other. $H^*(X; \mathbb{Z})$ (resp. $H_*(X; \mathbb{Z})$) is a direct sum of cyclic groups with one generator of order p^r for each basis element of $\mathrm{Im}(\beta_r) \subset E_r(X)$ (resp. $\mathrm{Im}(\beta_r) \subset E^r(X)$) and one generator of infinite order for each basis element of $E_\infty(X)$ (resp. $E^\infty(X)$).

Lemma 2.5. *Let $f : X \to Y$ be a map and n a positive integer. Consider the following four conditions.*

I_n: *$f_* : H_*(X; \mathbb{Z}) \to H_*(Y; \mathbb{Z})$ has a left inverse for $* \leq n$.*

II_n: *$f_* : H_*(X; \mathbb{Z}/(p)) \to H_*(Y; \mathbb{Z}/(p))$ induces monomorphisms of the Bockstein spectral sequences $f_* : E^r(X) \to E^r(Y)$ up to degree n for all r.*

III_n: *$f^* : H^*(Y; \mathbb{Z}) \to H^*(X; \mathbb{Z})$ has a right inverse for $* \leq n$.*

IV_n: *$f^* : H^*(Y; \mathbb{Z}/(p)) \to H^*(X; \mathbb{Z}/(p))$ induces epimorphisms of the Bockstein spectral sequences $f^* : E_r(Y) \to E_r(X)$ up to degree n for all r.*

Then the conditions I_n, II_n *and* IV_n *are equivalent, and* I_n *implies* III_n *and* III_{n+1} *implies* I_n.

Proof. It is easy to see that I_n implies II_n. The converse is also proved easily as follows. Let $k \leq n$ and consider the following commutative diagram.

$$\begin{array}{ccc} E^r_{k+1}(X) & \xrightarrow{f_*} & E^r_{k+1}(Y) \\ \downarrow{\scriptstyle \beta_r} & & \downarrow{\scriptstyle \beta_r} \\ E^r_k(X) & \xrightarrow{f_*} & E^r_k(Y) \end{array}$$

Let $\beta_r(x_1), \ldots, \beta_r(x_s)$ be a basis of $\mathrm{I}m(\beta_r) \subset E^r_k(X)$, then

$$f_*(\beta_r(x_1)) = \beta_r(f_*(x_1)), \ldots, f_*(\beta_r(x_s)) = \beta_r(f_*(x_s))$$

are linearly independent in $\mathrm{I}m(\beta_r) \subset E^r_k(Y)$ since $f_* : E^r_k(X) \to E^r_k(Y)$ is monomorphic by the assumption. Thus f_* maps a direct summand of all cyclic groups of order p^r in $H_*(X; \mathbb{Z})$ monomorphically into $H_*(Y; \mathbb{Z})$ as a direct summand. This is also true for a direct summand of cyclic groups of infinite order since $E_k^r(X) \cong E_k^\infty(X)$ and $E_k^r(Y) \cong E_k^\infty(Y)$ for sufficiently large r. These two facts imply I_n.

By duality II_n is equivalent to IV_n.

By the universal coefficient theorem it is easy to see that I_n implies III_n and III_{n+1} implies I_n. □

By Lemma 2.5 to prove Proposition 2.4 it is sufficient to prove that if $t > j$, then

$$\begin{aligned}\Omega\Sigma(1 \wedge red)_* : H_*(\Omega\Sigma(K(\mathbb{Z}/(p^j), n-1) \wedge K(\mathbb{Z}, \ell)); \mathbb{Z}/(p)) \\ \to H_*(\Omega\Sigma(K(\mathbb{Z}/(p^j), n-1) \wedge K(\mathbb{Z}/(p^t), \ell)); \mathbb{Z}/(p))\end{aligned}$$

induces monomorphisms of the Bockstein spectral sequences

$$\begin{aligned}\Omega\Sigma(1 \wedge red)_* : E^r(\Omega\Sigma(K(\mathbb{Z}/(p^j), n-1) \wedge K(\mathbb{Z}, \ell))) \\ \to E^r(\Omega\Sigma(K(\mathbb{Z}/(p^j), n-1) \wedge K(\mathbb{Z}/(p^t), \ell)))\end{aligned}$$

up to degree $2p^{t+1-j}$ for all r. For any space X we have

$$E^r(\Omega\Sigma X) \cong T(\tilde{E}^r(X)),$$

where $T(A)$ denotes the tensor algebra generated by a module A and $\{\tilde{E}^r(X)\}$ denotes the Bockstein spectral sequence associated with $\tilde{H}_*(X; \mathbb{Z})$. Thus it is sufficient to prove that if $t > j$, then

$$\begin{aligned}(1 \wedge red)_* : H_*(K(\mathbb{Z}/(p^j), n-1) \wedge K(\mathbb{Z}, \ell); \mathbb{Z}/(p)) \\ \to H_*(K(\mathbb{Z}/(p^j), n-1) \wedge K(\mathbb{Z}/(p^t), \ell); \mathbb{Z}/(p))\end{aligned}$$

induces monomorphisms of the Bockstein spectral sequences

$$\begin{aligned}(1 \wedge red)_* : E^r(K(\mathbb{Z}/(p^j), n-1) \wedge K(\mathbb{Z}, \ell)) \\ \to E^r(K(\mathbb{Z}/(p^j), n-1) \wedge K(\mathbb{Z}/(p^t), \ell))\end{aligned}$$

up to degree $2p^{t+1-j}$ for all r. By duality we show that if $t > j$, then

$$\begin{aligned}(2.6) \quad (1 \wedge red)^* : E_r(K(\mathbb{Z}/(p^j), n-1) \wedge K(\mathbb{Z}/(p^t), \ell)) \\ \to E_r(K(\mathbb{Z}/(p^j), n-1) \wedge K(\mathbb{Z}, \ell))\end{aligned}$$

are epimorphic up to degree $2p^{t+1-j}$ for all r. Since (2.6) is epimorphic for $r \le t$ by Theorem 10.4 of [3], it is sufficient to show that

$$\begin{aligned}\tilde{E}_{t+1}(K(\mathbb{Z}/(p^j), n-1) \wedge K(\mathbb{Z}, \ell)) \\ \cong \tilde{E}_{t+1}(K(\mathbb{Z}/(p^j), n-1)) \otimes \tilde{E}_{t+1}(K(\mathbb{Z}, \ell)) = 0\end{aligned}$$

up to degree $2p^{t+1-j}$. For $n = 2$ clearly $\tilde{E}_{t+1}(K(\mathbb{Z}/(p^j), 1)) = 0$ since $t > j$. For $n > 3$ $\tilde{E}_{t+1}(K(\mathbb{Z}/(p^j), n-1)) = 0$ up to degree $2p^{t+1-j}$ by Theorem 10.4

of [3]. Thus we complete the proof of Proposition 2.4 and, therefore, the proof of Theorem 1.4. □

Proof of Corollary 1.5. If the condition (i) holds, then $\mathrm{Ph}(-, \Sigma Y) \not\equiv 0$ by a localized version of the theorem of Zabrodsky [9].

We suppose that the condition (ii) holds, then there is a map $g : Y \to K(\mathbb{Z}, 2n+1)$ such that the composite

$$S^{2n+1} \xrightarrow{\alpha} Y \xrightarrow{g} K(\mathbb{Z}, 2n+1)$$

is a rational equivalence. Therefore the composite

$$\Omega S^{2n+2} \to \Omega\Sigma Y \to \Omega\Sigma K(\mathbb{Z}, 2n+1)$$

is also a rational equivalence. If $\mathrm{Ph}(-, \Sigma Y) \equiv 0$, then there is a map

$$K(\mathbb{Z}, 4n+2) \to \Omega\Sigma Y \to \Omega\Sigma K(\mathbb{Z}, 2n+1)$$

which induces an essential map on rational homotopy groups by Theorem 2.1. But this contradicts Theorem 2.2 since $\Omega\Sigma K(\mathbb{Z}, 2n+1) \simeq K(\mathbb{Z}, 2n+1) \times \Omega\Sigma(K(\mathbb{Z}, 2n+1) \wedge K(\mathbb{Z}, 2n+1))$, and so the proof follows.

We assume that the condition (iii) holds. Let $g : Y \to K(\mathbb{Z}, 2n)$ represent the class v. Then it is easy to see that the map

$$\Omega\Sigma Y \xrightarrow{\Omega\Sigma g} \Omega\Sigma K(\mathbb{Z}, 2n)$$

induces a non-trivial map on $\pi_{4n}(-) \otimes \mathbb{Q}$. Now we complete the proof by the similar argument as above. □

Proof of Theorem 1.2. In this proof spaces and groups are *not* localized at any prime. By Corollary 1.5 it suffices to prove the theorem for a space Y such that there are an even dimensional element $\alpha \in \pi_{2n}(Y)$, $n > 0$, and a cohomology class $v \in H^{2n}(Y; \mathbb{Z})$ with non-zero Kronecker product $\langle v, \rho(\alpha) \rangle \in \mathbb{Z}$ and $v^2 = 0$.

Let

$$F_{2n} \xrightarrow{i} K(\mathbb{Z}, 2n) \xrightarrow{u_{2n}^2} K(\mathbb{Z}, 4n)$$

be the fibration, where $u_{2n} \in H^{2n}(K(\mathbb{Z}, 2n); \mathbb{Z})$ is a generator. Then there is a map $f : Y \to F_{2n}$ such that $\Sigma f : \Sigma Y \to \Sigma F_{2n}$ induces an epimorphism on the rational homotopy groups. By Theorem 2 of [5] Σf induces an epimorphism

$$\mathrm{Ph}(X, \Sigma Y) \to \mathrm{Ph}(X, \Sigma F_{2n})$$

for any space X. Therefore it suffices to prove the theorem for the space F_{2n}. $i^*(u_{2n})$ is a generator of $H^{2n}(F_{2n}; \mathbb{Z}) \cong \mathbb{Z}$ and $i^*(u_{2n})^2 = 0$ by the definition of F_{2n}. Let $v_{2n} \in H^{2n}(\Omega\Sigma F_{2n}; \mathbb{Z}) \cong \mathbb{Z}$ be a generator. Then it is easy to see that $v_{2n}^p = 0$ in $H^{2pn}(\Omega\Sigma F_{2n}; \mathbb{Z}/(p))$ for any prime p. If $\mathrm{Ph}(-, \Sigma F_{2n}) \equiv 0$, then by Theorem 1 there is a rational equivalence

$$K(\mathbb{Z}, 2n) \xrightarrow{h} \Omega\Sigma F_{2n}.$$

Let $g : \Omega\Sigma F_{2n} \to K(\mathbb{Z}, 2n)$ represent the cohomology class v_{2n}. Then $hg = a \neq 0$ and, therefore, there is a prime p which is coprime to a. We have

$$a^p u_{2n}^p = (au_{2n})^p = ((gh)^*(u_{2n}))^p = h^*(v_{2n})^p = h^*(v_{2n}^p) = 0$$

in $H^*(K(\mathbb{Z}, 2n); \mathbb{Z}/(p))$. Since a^p is a unit in $\mathbb{Z}/(p)$, this contradicts the fact that $u_{2n}^p \neq 0$. We complete the proof of Theorem 1.2. □

Remark. For $F = F_2$ we have the Atiyah-Hirzebruch-Serre spectral sequence

$$E^2 \cong H_*(K(\mathbb{Z}, 2); K(q)_*K(\mathbb{Z}, 3)) \Longrightarrow K(q)_*F$$

associated with the fibration

$$K(\mathbb{Z}, 3) \to F \xrightarrow{i} K(\mathbb{Z}, 2).$$

Since $K(q)_*K(\mathbb{Z}, 3)$ is concentrated in even dimensions by Theorem 12.1 of [6], the above spectral sequence has no nontrivial differentials for dimensional reasons and hence collapses. This implies that $i_* : K(q)_*F \to K(q)_*K(\mathbb{Z}, 2)$ is epimorphic. It is easy to show that for any nontrivial map $g : \Omega\Sigma F \to K(\mathbb{Z}, 2)$

$$g_* : K(q)_*(\Omega\Sigma F) \to K(q)_*K(\mathbb{Z}, 2)$$

is also epimorphic. Thus the argument in the proof of Theorem 1.4 does *not* apply to this case.

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