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Lipschitz stability in the lateral Cauchy problem for elasticity system

By

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Abstract

We consider the isotropic elasticity system:

$$\rho\partial_t^2\mathbf{u} - \mu(\Delta\mathbf{u} + \nabla(\nabla^T\mathbf{u})) - \nabla(\lambda\nabla^T\mathbf{u}) - \sum_{j=1}^{3}\nabla\mu\cdot(\nabla u_j + \partial_j\mathbf{u})\mathbf{e}_j = 0 \quad \text{in} \quad \Omega\times(0,T)$$

for the displacement vector $\mathbf{u} = (u_1, u_2, u_3)$ depending on $x \in \Omega$ and $t \in (0,T)$ where Ω is a bounded domain in $\mathbb{R}^3$ with the C^2-boundary, and we assume the density $\rho \in C^2(\overline{\Omega}\times[0,T])$ and the Lamé parameters $\mu, \lambda \in C^3(\overline{\Omega}\times[0,T])$. We will give Lipschitz stability estimates for solutions $\mathbf{u}$ to the above elasticity system with the lateral boundary data

$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega\times(0,T), \qquad \partial_\nu\mathbf{u} = \mathbf{h} \quad \text{on } \Gamma\times(0,T)$$

where Γ is some part of $\partial\Omega$. Our proof is based on (1) a Carleman estimate with boundary data, (2) cut-off technique, and (3) principal diagonalization of the Lamé system.

1. Introduction and basic results

In this paper we are interested in Lipschitz stability estimates in the lateral Cauchy problem for the classical elasticity system

$$\rho\partial_t^2\mathbf{u} - \mu(\Delta\mathbf{u} + \nabla(\nabla^T\mathbf{u})) - \nabla(\lambda\nabla^T\mathbf{u}) - \sum_{j=1}^{3}\nabla\mu\cdot(\nabla u_j + \partial_j\mathbf{u})\mathbf{e}_j = 0 \quad \text{in} \quad \Omega\times(0,T) \tag{1.1}$$

for the displacement vector $\mathbf{u} = (u_1, u_2, u_3)$ depending on $x = (x_1, x_2, x_3) \in \Omega$ and $t \in (0,T)$. Here Ω is a bounded domain in $\mathbb{R}^3$ with the C^2-boundary, $Q \equiv \Omega\times(0,T)$, ν is the outward unit normal vector to $\partial\Omega$, and $\mathbf{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$,

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$\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. We will assume that the density $\rho \in C^2(\overline{Q})$ and the Lamé parameters $\mu, \lambda \in C^3(\overline{Q})$, $\mu > 0$ and $3\lambda + 2\mu > 0$, $\rho > 0$ on $\overline{Q}$. We will give stability estimates for solutions $\mathbf{u}$ to (1.1) with the lateral boundary data

$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega \times (0,T), \qquad \partial_\nu \mathbf{u} = \mathbf{h} \quad \text{on } \Gamma \times (0,T), \tag{1.2}$$

where Γ is some part of $\partial\Omega$.

We will use the following notation: $\partial_j = \partial/\partial x_j$, $\partial_t = \partial/\partial t$, $\nabla = (\partial_1, \ldots, \partial_n)$, $\nabla_{x,t} = (\partial_1, \ldots, \partial_n, \partial_t)$, $\square_a = a\partial_t^2 - \Delta$, $\Sigma = \partial\Omega \times (0,T)$. We set

$$\|u\|_2(\Omega) = \left(\int_\Omega u^2 dx \right)^{1/2} .$$

$H^s(\Omega)$, $s \in \mathbb{N}$ denotes the L^2-based Sobolev space with the norm

$$\|u\|_{(s)}^2(\Omega) = \sum_{|\alpha| \le s} \|D^\alpha u\|_2^2(\Omega) .$$

When we consider the space $H^s(\mathbb{R}^n)$, this norm can be equivalently defined by means of

$$\|u\|_{(s)}^2(\mathbb{R}^n) = (2\pi)^{-n} \int (1 + |\xi|^2)^s |\widehat{u}(\xi)|^2 d\xi,$$

where $\widehat{u}$ is the Fourier transform of u. This definition extends to $s \in \mathbb{R}$. We will drop the symbol Ω in norms and integrals when $\Omega = \mathbb{R}^n$. The norms $\|\cdot\|_2(\Gamma \times (0,T))$, $\|\cdot\|_{(1)}(\Gamma \times (0,T))$, etc. are defined similarly.

Our main results are the following theorems.

Theorem 1.1. *Assume that the functions φ, φ_1 are strongly pseudo-convex correspondingly in $\overline{Q} \setminus (x^0, T/2)$ and in $\overline{Q} \setminus (x^1, T/2)$, with respect to the differential operators $\rho\partial_t^2 - \mu\Delta$, $\rho\partial_t^2 - (2\mu + \lambda)\Delta$, x^0, x^1 are different points of Ω, $\varphi_1(x^1, T/2) < \varphi_1(x^0, T/2)$,*

$$\varphi < 0, \quad \varphi_1 < 0 \qquad \text{on} \quad \overline{\Omega} \times \{0, T\} \tag{1.3}$$

and

$$0 < \varphi, \quad 0 < \varphi_1 \qquad \text{on} \quad \overline{\Omega} \times \{T/2\}. \tag{1.4}$$

Assume that $\Gamma \subset \partial\Omega$ satisfies

$$\begin{aligned} &\{x \in \partial\Omega; (\nabla\varphi(x,t) \cdot \nu(x)) < 0, \\ &\qquad (\nabla\varphi_1(x,t) \cdot \nu(x)) < 0 \quad \text{for all } t \in [0,T]\} \supset \overline{\partial\Omega \setminus \Gamma}. \end{aligned} \tag{1.5}$$

Then there is a constant C depending only on Ω, φ, φ_1, ρ, μ, λ, Γ such that

(1.6)

$$\begin{aligned} &\|\partial_t \mathbf{u}(\cdot,t)\|_2(\Omega) + \|\mathbf{u}(\cdot,t)\|_{(1)}(\Omega) + \|\partial_t \nabla^T \mathbf{u}(\cdot,t)\|_2(\Omega) + \|\nabla^T \mathbf{u}(\cdot,t)\|_{(1)}(\Omega) \\ &\quad \le C\{\|\mathbf{h}\|_2(\Gamma \times (0,T)) + \|\mathbf{g}\|_{(1)}(\Sigma) \\ &\qquad + \|\nabla^T \mathbf{u}\|_{(1)}(\Sigma) + \|\partial_\nu \nabla^T \mathbf{u}\|_2(\Gamma \times (0,T))\} \end{aligned}$$

when $0 < t < T$.

If $x_0 \notin \overline{\Omega}$, *then we can take* $\varphi_1 \equiv \varphi$ *and in Section* 4, *we consider only such a case.*

In the case of $x_0 \in \overline{\Omega}$, a usual choice of φ (e.g., see (4.3)) does not satisfy the pseudo-convexity. Therefore we need two weight functions φ and φ_1 in Theorem 1.1. We will recall the definition of a strongly pseudo-convex function in Section 2.

We can give more explicit results in two important particular cases:

(1) $\Omega \subset B(0; R) \equiv \{x; |x| < R\}$ and $\Gamma = \partial\Omega$,

(2) $\Omega \subset \{-h < x_3 < 0,\ x_1^2 + x_2^2 < r^2\}$ and Γ is open in $\partial\Omega$.

Theorem 1.2. *Assume that the functions* $a = \rho/\mu$ *and* $a = \rho/(\lambda + 2\mu)$ *satisfy the conditions*

(1.7)
$$\theta^2 a \left(a + \frac{1}{2}\left(t - \frac{T}{2}\right)\partial_t a + a^{-1/2}\left|\left(t - \frac{T}{2}\right)\nabla a\right|\right) < a + \frac{1}{2}x \cdot \nabla a - \frac{\beta}{2}\partial_n a$$
$$\theta^2 a \le 1 \quad \text{on } \overline{Q}.$$

If in case (1) $R < \theta T/2$ *and in case* (2) Γ *contains* $\{0 \le x \cdot \nu - \beta\nu_n\} \cap \partial\Omega$, $r^2 + h^2 + 2h\beta < \theta^2 T^2/4$, *then bound* (1.6) *holds.*

We expect that the terms with $\nabla^T \mathbf{u}$ are not needed in (1.6) and can be eliminated by more sophisticated methods.

Next in place of $\partial_\nu \mathbf{u} = \mathbf{h}$ in (1.2), we take the surface traction for a stability estimate. In this paper, for brevity, we further assume that $\mathbf{g} = 0$. That is, in place of (1.2), we consider

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega \times (0,T), \qquad \sigma(\mathbf{u})\nu = \mathbf{h} \quad \text{on } \Gamma \times (0,T). \tag{1.2$'$}$$

Here the stress tensor $\sigma(\mathbf{u})$, a 3×3 matrix, is given by

$$\begin{aligned}\sigma(\mathbf{u}) = {} & \lambda(\nabla^T \mathbf{u})(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \\ & + \mu(\nabla u_1 + \partial_1 \mathbf{u},\ \nabla u_2 + \partial_2 \mathbf{u}, \nabla u_3 + \partial_3 \mathbf{u}).\end{aligned}$$

Theorem 1.3. *Assume that the functions* φ, φ_1 *are strongly pseudo-convex with respect to the differential operators* $\rho\partial_t^2 - \mu\Delta$, $\rho\partial_t^2 - (2\mu + \lambda)\Delta$ *correspondingly on* $\overline{Q} \setminus (x^0, T/2)$, *and on* $\overline{Q} \setminus (x^1, T/2)$, x^0, x^1 *are two different points of* Ω, $\varphi_1(x^1, T/2) < \varphi_1(x^0, T/2)$, *and satisfy* (1.3), (1.4), *and* (1.5). *Then there is a constant* C *depending only on* Ω, φ, φ_1, ρ, μ, λ, Γ *such that*

$$\begin{aligned}&\|\partial_t \mathbf{u}(\cdot,t)\|_2(\Omega) + \|\mathbf{u}(\cdot,t)\|_{(1)}(\Omega) + \|\partial_t \nabla^T \mathbf{u}(\cdot,t)\|_2(\Omega) \\ &\quad + \|\nabla^T \mathbf{u}(\cdot,t)\|_{(1)}(\Omega) \\ &\le C\left\{\|\mathbf{h}\|_2(\Gamma \times (0,T)) + \|\nabla \mathbf{h}\|_2(\Gamma \times (0,T))\right\}\end{aligned} \tag{1.8}$$

if $\mathbf{u} \in C^2(\overline{Q})^3$ *satisfies* (1.1) *and* (1.2)$'$, *and* $\nabla^T \mathbf{u} \in C^2(\overline{Q})^3$.

Theorem 1.4. *Assume that the functions $a = \rho/\mu$ and $a = \rho/(\lambda + 2\mu)$ satisfy conditions* (1.7)*. If in case* (1) $R < \theta T/2$ *and* $\Gamma = \partial\Omega$*, and in case* (2) Γ *contains* $\{0 \le x \cdot \nu - \beta\nu_n\} \cap \partial\Omega$, $r^2 + h^2 + 2h\beta < \theta^2 T^2/4$*, then bound* (1.8) *holds.*

Sharp uniqueness in the continuation for the static elasticity system with any first order perturbations (and smooth coefficients) was obtained by Dehman and Robbiano [6]. Using much more elementary techniques, Ang, Ikehata, Trong and Yamamoto ([2]) reduced a static system with zero order perturbations to a principally diagonal one and obtained sharp uniqueness of the continuation results under reduced smoothness assumptions. The dynamical system with constant coefficients has been considered by Bukhgeim and Kardakov [4] by using spherical means and by Alabau and Komornik [1] who transferred to this system the original approach of multipliers used by Lop Fat Ho [22] for the wave equation to obtain sharp stability estimates in energy norms (cf. Grasselli and Yamamoto [8]). The time dependent classical elasticity with variable coefficients and arbitrary first order perturbations was considered by Isakov [13] by pseudo-convexity methods. See also Isakov [14]. In the forthcoming paper (Eller, Isakov, Nakamura and Tataru [7]), sharp uniqueness results for the lateral Cauchy problem for system (1.1) and some conditional Hölder type stability estimates are proved on the basis of Tataru [25]. The results of these papers are obtained by combining known results for scalar hyperbolic equations of second order and principal diagonalization of the elasticity system and they imply approximate boundary controllability of this system. For the stabilization for the elasticity system, we can refer to Horn [11]. Yamamoto [28] applied principal diagonalization to some inverse source problems for the Maxwell system. Recently, Lebeau and Zuazua [21] obtained quite complete interior controllability results for a thermoelasticity system where Lipschitz stability is not possible. About general approaches to such estimates for scalar hyperbolic equations of second order, we refer to papers of Bardos, Lebeau and Rauch [3], Burq [5] (methods of geometrical optics) and Tataru [26] (methods of Carleman estimates).

2. Lipschitz stability for principally diagonal hyperbolic systems

In this section, we do not assume that the spatial dimension is 3. In other words, Ω is a bounded domain in an n-dimensional space.

We are considering the following system of partial differential equations

$$P(j)u_j + b_j(x,t;\nabla_{x,t}\mathbf{u},\mathbf{u}) = 0, \qquad j = 1,\dots,m \quad \text{in } Q = \Omega \times (0,T), \tag{2.1}$$

where the principal part $P(j)$ is a real second order t-hyperbolic operator with C^1-smooth coefficients, b_j are linear functions of $\nabla\mathbf{u}$ with $L^\infty(Q)$-coefficients and $\mathbf{u}$ with $L^{n+1}(Q)$-coefficient, and $\mathbf{u} = (u_1,\dots,u_m)$. We will impose the following Dirichlet lateral boundary condition

$$\mathbf{u} = \mathbf{g} \qquad \text{on} \quad \Sigma. \tag{2.2}$$

To formulate results we need a (weight) function $\varphi \in C^2(\overline{Q})$. This function is called strongly pseudo-convex on $\overline{Q}$ with respect to the operator P with the principal symbol $p(x,t;\zeta)$, $\zeta \in \mathbb{C}^{n+1}$ if $\nabla\varphi$ is not zero on $\overline{Q}$ and if for any $(x,t) \in \overline{Q}$, the equality

$$p(x,t;\zeta) = 0, \quad \zeta = \xi + i\tau\nabla\varphi, \qquad \tau \neq 0$$

or

$$p(x,t;\xi) = 0, \quad \nabla_\xi p(x,t;\xi) \cdot \nabla_{x,t}\varphi(x,t) = 0, \qquad \xi \neq 0$$

implies that

$$\sum_{j,k=1}^{n+1} \partial_j\partial_k\varphi \partial p/\partial\zeta_j \overline{\partial p/\partial\zeta_k} + \tau^{-1}\Im(\partial_k p \overline{\partial p/\partial\zeta_k}) > 0$$

(cf. Hörmander [9], [10]). Here $\overline{\alpha}$ denotes the complex conjugate and $\Im$ is the imaginary part.

Let $\Gamma_- = \{x \in \partial\Omega : (\nabla\varphi(x,t) \cdot \nu(x)) < 0 \text{ for all } t \in (0,T)\}$, $\Gamma_+ = \partial\Omega \setminus \Gamma_-$ and Γ be a neighbourhood of Γ_+ in $\partial\Omega$.

Theorem 2.1. *Assume that φ, φ_1 are strongly pseudo-convex correspondingly on $\overline{Q} \setminus (x^0, T/2)$ and on $\overline{Q} \setminus (x^1, T/2)$ with respect to $P(j)$, $j = 1, \dots, m$, and satisfy conditions* (1.3), (1.4), *and* (1.5), *x^0, x^1 are two different points of Ω, $\varphi_1(x^1, T/2) < \varphi_1(x^0, T/2)$. Let $\mathbf{u}$ satisfy* (2.1) *and* (2.2) *with* $\mathbf{g} = 0$. *Then there is a constant C depending only on Ω, φ, φ_1, $P(j)$, b_j, Γ such that*

$$\|\partial_t \mathbf{u}(\cdot,t)\|_2(\Omega) + \|\mathbf{u}(\cdot,t)\|_{(1)}(\Omega) \leq C\|\partial_\nu \mathbf{u}\|_2(\Gamma \times (0,T))$$

for all $t \in (0,T)$.

Proof. By conditions (1.3) and (1.4), there is $\varepsilon_1 > 0$ such that

$$\varphi < -\varepsilon_1, \quad \varphi_1 < -\varepsilon_1 \qquad \text{on} \quad \Omega \times ((0,\varepsilon_1) \cup (T-\varepsilon_1, T)) \tag{2.3}$$

and

$$\varepsilon_1 < \varphi, \quad \varepsilon_1 < \varphi_1 \qquad \text{on} \quad \Omega \times (T/2 - \varepsilon_1, T/2 + \varepsilon_1). \tag{2.4}$$

Henceforth we set $B(x_0, T/2; \delta) = \{(x,t) \in \mathbb{R}^{n+1}; |x-x_0|^2 + |t-T/2|^2 < \delta^2\}$. To apply Tataru's Carleman estimates ([27, Theorem 1]), we make use of cut-off functions $0 \leq \chi, \chi_1, \chi_2 \leq 1$. Let $\chi \in C^\infty(\mathbb{R})$, $\chi = 0$ on $(0, \varepsilon_1/2) \cup (T - \varepsilon_1/2, T)$, $\chi = 1$ on $(\varepsilon_1, T-\varepsilon_1)$. Let $\chi_1 \in C^\infty(\mathbb{R}^n)$, $\chi_1 = 1$ near $\partial\Omega \setminus \Gamma$ and $\chi_1 = 0$ near Γ_+. Let B_1 be the ball $B(x^0, T/2; \delta) \subset Q$ with $\overline{B}_1$ not containing the point $(x_1, T/2)$. Let $\chi_2 \in C^\infty(\mathbb{R}^{n+1})$, $\chi_2 = 1$ on $Q \setminus B_1$, $\chi_2 = 0$ on $B_0 \equiv B(x^0, T/2; \delta/2)$. In addition using the conditions of Theorem 2.1, we can choose δ so small that $\varphi_1(x^1, T/2) < \varphi_1(x,t)$ for $(x,t) \in B_1$.

We have

$$
\begin{aligned}
&\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3, \quad \text{where} \\
&\mathbf{u}_0 = (1-\chi)\chi_2\mathbf{u}, \quad \mathbf{u}_1 = \chi\chi_1\chi_2\mathbf{u}, \\
&\mathbf{u}_2 = \chi(1-\chi_1)\chi_2\mathbf{u}, \quad \mathbf{u}_3 = (1-\chi_2)\mathbf{u}.
\end{aligned}
\tag{2.5}
$$

We set $\mathbf{u}_i = (u_{i;1}, \dots, u_{i;m})$, $i = 0, 1, 2$.

Observe that $\operatorname{supp} \mathbf{u}_1 \subset K_1$ where K_1 is a compact subset of $\overline{\Omega} \times [\varepsilon_1/2, T - \varepsilon_1/2] \setminus B_0$ with $K_1 \cap \partial Q$ contained in $\Gamma_2 \times [0, T] \subset \Gamma_- \times [0, T]$ which does not intersect $\overline{\Gamma}_+$. Since $\Gamma_2 \subset \Gamma_-$ and does not intersect $\overline{\Gamma}_+$, the Dirichlet boundary operator on Γ_2 satisfies the strong Lopatinskii condition in the direction $\nabla_{x,t}\varphi$ ([27, Proposition 5.1]). Henceforth let $|\alpha| \le 1$. By Theorem 1 in [27], there exists a constant $C > 0$ such that

$$
\tau^{3-2|\alpha|}\|e^{\tau\varphi}\partial^\alpha u_{1;j}\|_2^2(Q) \le C\|e^{\tau\varphi}P(j)u_{1;j}\|_2^2(Q), \qquad 1 \le j \le m \tag{2.6}
$$

for large τ.

Similarly, $\operatorname{supp} \mathbf{u}_2 \subset K_2$ where K_2 is a compact subset of $\overline{\Omega} \times [\varepsilon_1/2, T - \varepsilon_1/2] \setminus B_0$ with $K_2 \cap \partial Q$ contained in $\Gamma \times [0, T]$. The Cauchy boundary operators always satisfy the strong Lopatinskii condition (cf. [27]) in the direction $\nabla_{x,t}\varphi$. By Theorem 1 in [27], we have

$$
\begin{aligned}
&\tau^{3-2|\alpha|}\|e^{\tau\varphi}\partial^\alpha u_{2;j}\|_2^2(Q) \\
&\quad \le C\{\|e^{\tau\varphi}P(j)u_{2;j}\|_2^2(Q) + \tau\|e^{\tau\varphi}\partial_\nu u_{2;j}\|_2^2(\Gamma \times (0,T))\}, \quad 1 \le j \le m,
\end{aligned}
\tag{2.7}
$$

for large $\tau > 0$.

Here and henceforth $C > 0$ denotes a generic constant which is independent of τ. From (2.5) and Leibniz' formula

$$
P(j)u_{1;j} = \chi\chi_1\chi_2 P(j)u_j + A_{1;j}(u_j) \equiv A_1(u; j), \qquad 1 \le j \le m,
$$

due to equations (2.1). Here $A_{1;j}$, $A(\cdot; j)$ are linear partial differential operators of first order whose coefficients of the first-order terms are in $L^\infty(Q)$ and the coefficients of the zeroth-order terms are in $L^{n+1}(Q)$ (depending on χ, χ_1 and χ_2). A similar formula holds for $u_{2;j}$, $1 \le j \le m$.

Using these formulae, adding inequalities (2.6), (2.7) and summing over $j = 1, \dots, m$ and $|\alpha| \le 1$, we arrive at

$$
\begin{aligned}
&\sum_{j=1}^m \sum_{|\alpha|\le 1} \tau^{3-2|\alpha|}\{\|e^{\tau\varphi}\partial^\alpha u_{1;j}\|_2^2(Q) + \|e^{\tau\varphi}\partial^\alpha u_{2;j}\|_2^2(Q) \\
&\qquad + \|e^{\tau\varphi}\partial^\alpha u_{0;j}\|_2^2(Q) + \|e^{\tau\varphi}\partial^\alpha u_{3;j}\|_2^2(Q)\} \\
&\quad \le C\Bigg(\sum_{j=1}^m \sum_{|\alpha|=1} \|e^{\tau\varphi}\partial^\alpha u_j\|_2^2(Q) + \sum_{j=1}^m \|q_j e^{\tau\varphi}u_j\|_2^2(Q) \\
&\qquad + \tau\|e^{\tau\varphi}\partial_\nu u_j\|_2^2(\Gamma \times (0,T)) \\
&\qquad + \tau^{3-2|\alpha|}\|e^{\tau\varphi}\partial^\alpha u_{0;j}\|_2^2(Q) + \|e^{\tau\varphi}\partial^\alpha u_{3;j}\|_2^2(Q)\Bigg).
\end{aligned}
\tag{2.8}
$$

Here $q_j u_j$ is the zeroth-order term from $b_1, \dots, b_m$. Let $n \geq 3$. Since $Q \in \mathbb{R}^{n+1}$ and $q_j \in L^{n+1}(Q)$ for $1 = 1, \dots, m$, by Hölder's inequality and the Sobolev embedding we obtain

$$\begin{aligned}\|e^{\tau\varphi} q_j u_j\|_2^2(Q) &\leq \|q_j\|^2_{L^{n+1}(Q)} \|e^{\tau\varphi} u_j\|^2_{L^{2(n+1)/(n-1)}(Q)} \leq C\|e^{\tau\varphi} u_j\|^2_{H^1(Q)} \\ &\leq C\left(\tau^2 \|e^{\tau\varphi} u_j\|_2^2(Q) + \sum_{|\alpha|\leq 1} \|e^{\tau\varphi}\partial^\alpha u_j\|_2^2(Q)\right).\end{aligned}$$

Hence by taking τ large, we can eliminate the term $\|e^{\tau\varphi} q_j u_j\|_2^2(Q)$ in the right hand of (2.8), so that equality (2.5), the triangle inequality (in the left side of (2.8)), and (2.8) yield

$$\begin{aligned}&\sum_{j=1}^m \sum_{|\alpha|\leq 1} \tau^{3-2|\alpha|} \|e^{\tau\varphi}\partial^\alpha u_j\|_2^2(Q) \\ &\leq C\left(\sum_{j=1}^m \sum_{|\alpha|\leq 1} \{\|e^{\tau\varphi}\partial^\alpha u_j\|_2^2(Q) + \tau\|e^{\tau\varphi}\partial_\nu u_j\|_2^2(\Gamma \times (0,T))\right. \\ &\qquad \left. + \tau^{3-2|\alpha|}\|e^{\tau\varphi}\partial^\alpha u_j\|_2^2(Q(1) \cup B_1)\}\right),\end{aligned} \tag{2.9}$$

where $Q(1) = \Omega \times ((0, \varepsilon_1) \cup (T - \varepsilon_1, T))$. Choosing τ large, we can eliminate the first terms $\|e^{\tau\varphi}\partial^\alpha u_j\|_2^2(Q)$ in the right side of (2.9).

Next we will eliminate the terms

$$\tau^{3-2|\alpha|}\|e^{\tau\varphi}\partial^\alpha u_j\|_2^2(\Omega \times ((0, \varepsilon_1) \cup (T - \varepsilon_1, T))).$$

Due to condition (2.4), the left side in (2.9) is greater than

$$e^{2\tau\varepsilon_1} \sum_{j=1}^m \sum_{|\alpha|\leq 1} \|\partial^\alpha u_j\|_2^2(\Omega \times (T/2 - \varepsilon_1, T/2 + \varepsilon_1)).$$

Writing the last sum as

$$\int_{T/2-\varepsilon_1}^{T/2+\varepsilon_1} E(t)dt, \qquad E(t) = \sum_{j=1}^m \sum_{|\alpha|\leq 1} \|\partial^\alpha u_j(\cdot, t)\|_2^2(\Omega)$$

and observing that by elementary properties, the integral with respect to t is not less than $2\varepsilon_1 E(\theta)$ for some $\theta \in (T/2 - \varepsilon_1, T/2 + \varepsilon_1)$, we conclude that the left side in (2.9) is not less than $2\varepsilon_1 e^{2\tau\varepsilon_1} E(\theta)$.

Since system (2.1) is t-hyperbolic, the known energy estimates (e.g., John [15]) imply that $E(t) \leq CE(\theta)$, $E(t) \leq CE(0), 0 \leq t \leq T$. Using the above bounds from (2.3) and (2.9), we conclude that

$$\begin{aligned}e^{2\tau\varepsilon_1} E(0) &\leq C(\tau\|e^{\tau\varphi}\partial_\nu \mathbf{u}\|_2^2(\Gamma \times (0,T)) + \tau^3 e^{-2\tau\varepsilon_1} E(0) \\ &\qquad + \sum_{j=1}^m \sum_{|\alpha|=1} \tau\|e^{\tau\varphi}\partial^\alpha u_j\|_2^2(B_1) + \tau^3\|e^{\tau\varphi}\mathbf{u}\|_2^2(B_1)).\end{aligned}$$

Again choosing τ large we can eliminate the term in the right side containing $E(0)$. After that we fix τ and, using that $E(t) \leq CE(0)$, we will have

$$E(t) \leq C(\|\partial_\nu \mathbf{u}\|_2^2(\Gamma \times (0,T)) + \|\mathbf{u}\|_{(1)}^2(B_1)). \tag{2.10}$$

The last step of the proof is to eliminate the norm over B_1. To do it we will use the Hölder type conditional stability estimate in the Cauchy problem from [7].

Let $Q_\varepsilon = Q \cap \{(x,t); \varphi_1(x,t) - \varphi_1(x^1, T/2) - \delta_0 > \varepsilon\}$. Due to the choice of B_1 we have $B_1 \subset Q_\varepsilon$ for some small positive ε, δ_0 and $(x^1, T/2)$ is not in $\overline{Q}_0$. We will fix such ε, δ_0. By Theorem 3.3 in [7] we have

$$\begin{aligned}\|\mathbf{u}\|_{(1)}(B_1) &\leq \|\mathbf{u}\|_{(1)}(Q_\varepsilon)\\ &\leq C\{\|\partial_\nu \mathbf{u}\|_2(\Gamma \times (0,T)) + \|\partial_\nu \mathbf{u}\|_2^\theta(\Gamma \times (0,T))\|\mathbf{u}\|_{(1)}^{1-\theta}(Q_0)\},\end{aligned}$$

where $\theta \in (0,1)$ depends on ε. Using the Hölder inequality, for any $\delta_1 > 0$, we can choose $C(\delta_1) > 0$ such that $a^\theta b^{1-\theta} \leq \delta_1 b + C(\delta_1)a$ and combining the above inequality with (2.10) without the norm over $Q(1)$ and with the obvious bound $\|\mathbf{u}\|_{(1)}^2(\Omega) \leq CE(0)$ we finally obtain

$$\|\mathbf{u}\|_{(1)}(\Omega) \leq C(\delta_1)\|\partial_\nu \mathbf{u}\|_2(\Gamma \times (0,T)) + C\delta_1\|\mathbf{u}\|_{(1)}(Q).$$

Choosing $\delta_1 < 1/C$ we eliminate the last term and complete the proof of Theorem 2.1. □

In this proof we have used general Carleman estimates of Tataru ([27]) and the device of Klibanov and Malinsky ([16]) and of Tataru ([26]) for deriving Lipschitz stability from such estimates. A new ingredient is the introduction of the cut-off function χ_1. See also Lasiecka and Triggiani [19].

Now by using the sharp bounds of Lasiecka, Lions and Triggiani ([18]) in hyperbolic problems under minimal regularity assumptions on their coefficients, we will remove the condition $\mathbf{g} = 0$ of Theorem 2.1. In the case of smooth coefficients, such estimates are due to Sakamoto ([24]).

Lemma 2.1. *Assume that equations* (2.1) *do not contain the terms with* $\partial_t \partial_k u_j$.

Then there is a constant C depending only on Ω, T, $P(j)$, b_j, $j = 1, \dots, m$ such that any solution $\mathbf{u}$ to (2.1) *and* (2.2) *with the initial conditions $\mathbf{u} = \mathbf{u}_0$ and $\partial_t \mathbf{u} = \mathbf{u}_1$ on $\Omega \times \{0\}$ satisfies the bound*

$$\begin{aligned}\|\mathbf{u}(\cdot,t)\|_{(1)}(\Omega) + \|\partial_t \mathbf{u}(\cdot,t)\|_2(\Omega) + \|\partial_\nu \mathbf{u}\|_2(\Sigma)&\\ \leq C(\|\mathbf{u}_0\|_{(1)}(\Omega) + \|\mathbf{u}_1\|_2(\Omega) + \|\mathbf{g}\|_{(1)}(\Sigma)),&\end{aligned} \tag{2.11}$$

when $0 < t < T$.

Proof. Let us consider the scalar hyperbolic problem

$$\left\{\begin{array}{l}\partial_t^2 v + Av = F \quad \text{in } Q\\ v = v_0, \quad \partial_t v = v_1 \qquad \text{on } \Omega \times \{0\}\\ v = G \quad \text{on } \Sigma, \quad \text{where } v_0 = G \text{ on } \partial\Omega \times \{0\},\end{array}\right\}, \tag{2.12}$$

where $A = -\Sigma\partial_j(a^{jk}\partial_k)$ with $a^{jk} \in C^1(\overline{Q})$ strictly uniformly positive on $\overline{Q}$. Let $T < T_0$. Let G^* be an extension of G onto $\Omega \times (0, T_0)$ with $\|G^*\|_{(1)}(\partial\Omega \times (0, T_0)) \leq C_T\|G\|_{(1)}(\Sigma)$. The existence of G^* is guaranteed by extension theorems for Sobolev spaces. Let us extend F onto $\Omega \times (T, T_0)$ as zero. By Theorem 4.1 in [18] for solutions v^* to extended problem (2.12), we have

$$\|v^*(\cdot,t)\|_{(1)}(\Omega) + \|\partial_t v^*(\cdot,t)\|_2(\Omega) + \|\partial_\nu v^*\|_2(\Sigma)$$
$$\leq C_0\left(\int_0^T \|F(\cdot,s)\|_2(\Omega)ds + \|v_0\|_{(1)}(\Omega) + \|v_1\|_2(\Omega) + \|G^*\|_{(1)}(\partial\Omega \times (0,T_0))\right)$$

where C_0 depends only on Ω, T_0, the constant of ellipticity of A and on $\|a^{l,k}\|_{(1)}(Q)$. Hence

$$\|v(\cdot,t)\|_{(1)}(\Omega) + \|\partial_t v(\cdot,t)\|_2(\Omega) + \|\partial_\nu v\|_2(\Sigma)$$
$$\leq C_0\left(\int_0^T \|F(\cdot,s)\|_2(\Omega)ds + \|v_0\|_{(1)}(\Omega) + \|v_1\|_2(\Omega) + C_T\|G\|_{(1)}(\Sigma)\right). \tag{2.13}$$

We will apply bound (2.13) to any of equations (2.1) written in form (2.12) where A is the divergent form of the principal elliptic part of $P(j)$ and F is the sum of b_j and of the remainders from the transformation of $P(j)$ into the divergent form. Observe that $\|F(\cdot,s)\|_2(\Omega) \leq C(\|\mathbf{u}(\cdot,s)\|_{(1)}(\Omega) + \|\partial_t\mathbf{u}(\cdot,s)\|_2(\Omega))$ where C depends only on the $C^1(\overline{Q})$-norms of the principal coefficients of $P(j)$ and on the $L^\infty(Q)$-norms of other coefficients. Summing these bounds over $j = 1, \ldots, m$ and applying the triangle inequality, we will have

$$\|\mathbf{u}(\cdot,t)\|_{(1)}(\Omega) + \|\partial_t\mathbf{u}(\cdot,t)\|_2(\Omega) + \|\partial_\nu\mathbf{u}\|_2(\Sigma)$$
$$\leq C_0\left(\int_0^T (\|\mathbf{u}(\cdot,s)\|_{(1)}(\Omega) + \|\partial_t\mathbf{u}(\cdot,s)\|_2(\Omega))ds\right.$$
$$\left. + \|\mathbf{u}_0\|_{(1)}(\Omega) + \|\mathbf{u}_1\|_2(\Omega) + C_T\|\mathbf{g}\|_{(1)}(\Sigma)\right)$$

for all $t \in (0,T)$. When $T < 1/C_0$ by taking supremum of the both parts of the obtained bound for $\mathbf{u}$ and using elementary properties of the integral, one eliminates the integral term. Since smallness of T needed for that is determined only by the coefficients of system (2.1) and since the energy at $t = T$ is bounded by the initial energy and the Dirichlet boundary data, we can repeat this step and in a finite number of steps to exhaust the whole initial interval $(0,T)$. Observe that T_0 is used to guarantee that the constant C_0 in (2.13) does not depend on T. The proof is complete. $\square$

This lemma permits us to eliminate the condition $\mathbf{g} = 0$ of Theorem 2.1.

Corollary 2.1. *Assume that the operators $P(j)$ do not contain terms with $\partial_t\partial_k u_j$. Let functions φ, φ_1 be strongly pseudo-convex and satisfy conditions* (1.3), (1.4), *and* (1.5) *as well as other conditions of Theorem* 2.1 *on φ, φ_1.*

Then there is a constant C depending only on Ω, T, φ, $P(j)$, b_j, Γ such that any solution to problem (2.1) *and* (2.2) *satisfies the bound*

$$\begin{aligned}(2.14)\qquad &\|\partial_t \mathbf{u}(\cdot,t)\|_2(\Omega) + \|\mathbf{u}(\cdot,t)\|_{(1)}(\Omega) + \|\partial_\nu \mathbf{u}\|_2(\Sigma)\\ &\qquad \le C(\|\mathbf{g}\|_{(1)}(\Sigma) + \|\partial_\nu \mathbf{u}\|_2(\Gamma\times(0,T))).\end{aligned}$$

Proof. To derive this result from Theorem 2.1 and Lemma 2.1, we introduce a function $\mathbf{u}_0^* \in H^1(\Omega)$ coinciding with $\mathbf{u}$ on $\partial\Omega\times\{0\}$ and such that $\|\mathbf{u}_0^*\|_{(1)}(\Omega)$ is bounded by $C\|\mathbf{g}\|_{(1)}(\Sigma)$. Solving the initial value problem for hyperbolic system (2.1) with the initial data $(\mathbf{u}^*(\cdot,0), \partial_t\mathbf{u}^*(\cdot,0)) = (\mathbf{u}_0^*, \mathbf{0})$ and the lateral Dirichlet data $\mathbf{g}$ on Σ, we obtain a function $\mathbf{u}^*$. By Lemma 2.1

$$\begin{aligned}(2.15)\qquad &\|\mathbf{u}^*(\cdot,t)\|_{(1)}(\Omega) + \|\partial_\nu \mathbf{u}^*(\cdot,t)\|_2(\Omega) + \|\partial_\nu \mathbf{u}^*\|_2(\Sigma)\\ &\qquad \le C\|\mathbf{g}\|_{(1)}(\Sigma), \qquad 0<t<T.\end{aligned}$$

Subtracting $\mathbf{u}^*$ from $\mathbf{u}$, we obtain for their difference equations (2.1) and the zero lateral Dirichlet data on Σ. Applying to this difference Theorem 2.1 and combining its bound with (2.15) by the triangle inequality, we complete the proof. □

Now we discuss cases (1) and (2) in the n-dimensional case:
(1) $\Omega \subset B(0,R)$ and $\Gamma \subset \partial\Omega$,
(2) $\Omega \subset \{-h < x_n < 0,\ x_1^2 + \cdots + x_{n-1}^2 < r^2\}$,
which are already introduced before Theorem 1.2 (the three dimensional case).

Corollary 2.2. *Assume that the coefficients $a = a(j)$, $j = 1,\ldots,m$ of the principal parts $P(j) = \Box_{a(j)}$ of system* (2.1) *satisfy conditions* (1.7).

If in case (1) *$R < \theta T/2$ and $\Gamma = \partial\Omega$ and in case* (2) *Γ contains $\{0 \le x\cdot\nu - \beta\nu_n\} \cap \partial\Omega$, $r^2 + h^2 + 2h\beta < \theta^2T^2/4$, then bound* (2.14) *holds.*

Proof. We will apply Corollary 2.1 with

$$\varphi = e^{\sigma\psi} - 1$$

and

$$\psi(x,t) = x_1^2 + \cdots + x_{n-1}^2 + (x_n-\beta)^2 - \theta^2(t-T/2)^2 - s.$$

It is known that under conditions (1.7) with respect to the principal coefficients the function $\varphi = \exp(\sigma\psi)$ for large σ is strongly pseudo-convex in $\overline{Q}\setminus\{(\beta, T/2)\}$ with respect to the wave operators $\Box_{a(j)}$ ([14, Section 3.4]).

In case (1) we let $\beta = 0$ and will make use also of the funcion $\varphi_1 = e^{\lambda\psi_1}$, $\psi_1(x,t) = \psi(x_1,\ldots,x_{n-1},x_n-\beta_0,t)$ with small β_0. Then condition (1.3) is satisfied with any $s > R^2 - \theta^2T^2/4$ and condition (1.4) is satisfied with any $s < 0$. Since $R < \theta T/2$ we can satisfy the both conditions (choosing for example $s = (1/2)(R^2 - \theta^2T^2/4)$).

In case (2) we let $s = \beta^2 - \delta$, $\delta > 0$. Then the function φ is strongly pseudo-convex on the whole $\overline{Q}$ and the function φ_1 is not needed. But to comply formally with Theorem 2.1 we can choose as x^0, x^1 any points of Ω such that $\varphi(x^1, T/2) < \varphi(x^0, T/2)$ and let $\varphi_1 = \varphi$. Condition (1.3) follows from the inequality $x_1^2 + \cdots + x_{n-1}^2 + (x_n - \beta)^2 - \theta^2 T^2/4 - \beta^2 + \delta < 0$ for all $x \in \overline{\Omega}$ which is a corollary of the inequality $r^2 + h^2 + 2h\beta < \theta^2 T^2/4$ when δ is small. Condition (1.4) is satisfied because $0 < \delta = \beta^2 - (\beta^2 - \delta) \leq \psi(x, T/2)$.

So in the both cases Corollary 2.2 follows from Corollary 2.1. □

It is not difficult to see that conditions (1.7) and $r^2 + h^2 + 2h\beta < \theta^2 T^2/4$ are compatible and mean that T is large (and θ is small, but it suffices to let $\theta = T^{-3/4}$ to satisfy the both conditions). Condition (1.7) can be satisfied when $0 < a_j + (1/2)x \cdot \nabla a_j - (\beta/2)\partial_n a_j$ on $\overline{Q}$.

3. Proofs of estimates for the elasticity system

To prove Theorems 1.1 and 1.2, we will extend system (1.1) for three unknown functions u_1, u_2, u_3 to a new one for four unknown functions by introducing $v = \nabla^T \mathbf{u}$. We refer to Eller, Isakov, Nakamura and Tataru [7], Ikehata, Nakamura and Yamamoto [12].

Lemma 3.1. *Let $v = \nabla^T \mathbf{u}$. If $\mathbf{u}$ solves* (1.1), *then*

$$\left\{ \begin{array}{l} \rho\partial_t^2 v - (\lambda + 2\mu)\Delta v + A_{1;1}(v, \mathbf{u}) = 0 \\ \rho\partial_t^2 \mathbf{u} - \mu\Delta\mathbf{u} + A_{1;2}(v, \mathbf{u}) = 0 \quad \text{in } Q, \end{array} \right\}, \tag{3.1}$$

where $A_{1;1}$ and $A_{1;2}$ are (*matrix*) *linear partial differential operators with measurable and bounded coefficients in Q.*

This lemma and the results of Section 2 imply Theorems 1.1 and 1.2 for Cauchy problem (1.1) and (1.2).

Next we have to prove Theorems 1.3 and 1.4. We recall that we take

$$\mathbf{g} = 0. \tag{3.2}$$

For simplicity we set

$$\begin{aligned} \mathcal{L}\mathbf{u} &= \mu(\Delta\mathbf{u} + \nabla(\nabla^T\mathbf{u})) - \nabla(\lambda\nabla^T\mathbf{u}) \\ &\quad - \sum_{j=1}^{3} \nabla\mu \cdot (\nabla u_j + \partial_j \mathbf{u})\mathbf{e}_j \end{aligned} \tag{3.3}$$

and $\mathbf{a}^T$ denotes the transpose of a vector $\mathbf{a}$ under consideration.

For the proof of Theorems 1.3 and 1.4, it is sufficient to prove

Lemma 3.2. *On $\partial\Omega$, we represent $\nabla\mathbf{u}$ and $\partial_\nu(\nabla^T\mathbf{u})$ by linear functions of $\sigma(\mathbf{u})\nu$ and $\nabla(\sigma(\mathbf{u})\nu)$ with bounded coefficients.*

Proof. The proof is modification of an argument in [12] where $\sigma(\mathbf{u})\nu = 0$ is assumed. Here we give the proof for completeness. □

First Step. We set

$$\nabla \mathbf{u} = \left(\frac{\partial u_i}{\partial x_j}\right)_{1\le i,j\le 3} = \begin{pmatrix} (\nabla u_1)^T \\ (\nabla u_2)^T \\ (\nabla u_3)^T \end{pmatrix}. \tag{3.4}$$

Then, under assumption (3.2), we have

$$\nabla \mathbf{u} = \{(\nabla \mathbf{u})\nu\}\nu^T \qquad \text{on} \quad \partial\Omega \times (0,T) \tag{3.5}$$

and

$$\nabla^T \mathbf{u} = (\nabla \mathbf{u})\nu \cdot \nu \qquad \text{on} \quad \partial\Omega \times (0,T). \tag{3.6}$$

In fact, setting $\mathbf{u} = (u_1, u_2, u_3)^T$ and $\nu = (\nu_1, \nu_2, \nu_3)^T$, we see that condition (3.2) implies

$$\nabla u_i = \begin{pmatrix} \frac{\partial u_i}{\partial x_1} \\ \frac{\partial u_i}{\partial x_2} \\ \frac{\partial u_i}{\partial x_3} \end{pmatrix} = (\nabla u_i \cdot \nu)\nu = \left(\frac{\partial u_i}{\partial \nu}\right)\nu, \qquad 1 \le i \le 3.$$

Therefore we have

$$\nabla \mathbf{u} = \begin{pmatrix} (\nabla u_1)^T \\ (\nabla u_2)^T \\ (\nabla u_3)^T \end{pmatrix} = \begin{pmatrix} (\nabla u_1 \cdot \nu)\nu^T \\ (\nabla u_2 \cdot \nu)\nu^T \\ (\nabla u_3 \cdot \nu)\nu^T \end{pmatrix}. \tag{3.7}$$

By definition (3.4), this means (3.5). Moreover by $\nu\nu^T = 1$, we have

$$(\nabla \mathbf{u})\nu = \begin{pmatrix} (\nabla u_1 \cdot \nu)\nu^T \\ (\nabla u_2 \cdot \nu)\nu^T \\ (\nabla u_3 \cdot \nu)\nu^T \end{pmatrix} \nu = \begin{pmatrix} (\nabla u_1 \cdot \nu) \\ (\nabla u_2 \cdot \nu) \\ (\nabla u_3 \cdot \nu) \end{pmatrix},$$

and so

$$(\nabla \mathbf{u})\nu \cdot \nu = (\nabla u_1 \cdot \nu)\nu_1 + (\nabla u_2 \cdot \nu)\nu_2 + (\nabla u_3 \cdot \nu)\nu_3 = \mathrm{Trace}\,\nabla \mathbf{u} = \nabla^T \mathbf{u}$$

by (3.7). Thus we see (3.5) and (3.6).

Second Step. In this step, we will prove that

$$\nabla \mathbf{u} = S(\sigma(\mathbf{u})\nu) \qquad \text{on} \quad \partial\Omega \times (0,T) \tag{3.8}$$

for some 3×3 matrix $S \in C(\partial\Omega \times [0,T])^{3\times 3}$.

Proof of (3.8). We define a 3×3 matrix $B = B(x,t)$ by

$$B(x,t)\mathbf{a} = \lambda(x,t)(\mathbf{a} \cdot \nu(x))\nu(x) + 2\mu(x,t)\{\mathrm{Sym}\,(\mathbf{a}\nu(x)^T)\}\nu(x) \tag{3.9}$$

for $\mathbf{a} \in \mathbb{R}^3$. Here and henceforth, for square matrices $A = (a_{ij})_{1\le i,j\le 3}$ and $B = (b_{ij})_{1\le i,j\le 3}$, we set $\mathrm{Sym}\, A = (1/2)(A + A^T)$ and $A \cdot B = \sum_{i,j=1}^{3} a_{ij} b_{ij}$, $|A|^2 = \sum_{i,j=1}^{3} a_{ij}^2$. Then

$$B = B(x,t) \text{ is invertible for all } (x,t) \in \partial\Omega \times (0,T). \tag{3.10}$$

In fact, since $C\nu \cdot \mathbf{a} = C \cdot (\mathbf{a}\nu^T)$ and $(1/2)(C + C^T) \cdot C = |(1/2)(C + C^T)|^2$ for an $n \times n$ matrix C, we have

$$(\mathrm{Sym}\,(\mathbf{a}\nu^T))\nu \cdot \mathbf{a} = (\mathrm{Sym}\,(\mathbf{a}\nu^T)) \cdot (\mathbf{a}\nu^T) = |\mathrm{Sym}\,(\mathbf{a}\nu^T)|^2.$$

Hence

$$\begin{aligned}(B\mathbf{a} \cdot \mathbf{a}) &= \lambda |\mathbf{a} \cdot \nu|^2 + 2\mu |\mathrm{Sym}\,(\mathbf{a}\nu^T)|^2 \\ &= \lambda |\mathrm{Trace}\, A|^2 + 2\mu |A|^2.\end{aligned} \tag{3.11}$$

Here we set

$$A = \mathrm{Sym}\,(\mathbf{a}\nu^T)$$

and

$$C = A - \frac{\mathrm{Trace}\, A}{3} I_3, \tag{3.12}$$

where I_3 is the 3×3 identity matrix. Then $\mathrm{Trace}\, C = 0$, so that

$$C \cdot I_3 = 0 \tag{3.13}$$

by the identity $C \cdot I_3 = \mathrm{Trace}\, C$. Therefore (3.11) through (3.13) imply

$$\begin{aligned} B\mathbf{a} \cdot \mathbf{a} &= \lambda |\mathrm{Trace}\, A|^2 + 2\mu \left| \frac{\mathrm{Trace}\, A}{3} I_3 + C \right|^2 \\ &= \lambda |\mathrm{Trace}\, A|^2 + 2\mu \left(\left| \frac{\mathrm{Trace}\, A}{3} I_3 \right|^2 + |C|^2 + 2 \frac{\mathrm{Trace}\, A}{3} I_3 \cdot C \right) \\ &= \frac{3\lambda + 2\mu}{3} |\mathrm{Trace}\, A|^2 + 2\mu |C|^2 \ge \frac{\delta_0}{3} |\mathrm{Trace}\, A|^2 + \delta_0 |C|^2, \end{aligned}$$

where $\delta_0 > 0$ is a constant. By (3.12), we have $A = C + (\mathrm{Trace}\, A/3) I_3$, so that $\delta_0 |A|^2 = (\delta_0/3)|\mathrm{Trace}\, A|^2 + \delta_0 |C|^2$ by (3.13). Therefore

$$B\mathbf{a} \cdot \mathbf{a} \ge \delta_0 |\mathrm{Sym}\,(\mathbf{a}\nu^T)|^2.$$

Moreover we have

$$\begin{aligned} |\mathrm{Sym}\,(\mathbf{a}\nu^T)|^2 &= \frac{1}{4}(|\mathbf{a}\nu^T|^2 + 2(\mathbf{a}\nu^T) \cdot (\nu\mathbf{a}^T) + |\nu\mathbf{a}^T|^2) \\ &= \frac{1}{4}(|\mathbf{a}|^2 + 2|\mathbf{a} \cdot \nu|^2 + |\mathbf{a}|^2) \ge \frac{1}{2}|\mathbf{a}|^2, \end{aligned}$$

so that

$$B\mathbf{a}\cdot\mathbf{a} \ge \frac{\delta_0}{2}|\mathbf{a}|^2 \qquad \text{on} \quad \partial\Omega\times(0,T). \tag{3.14}$$

Moreover direct calculations verify that $B\mathbf{a}\cdot\mathbf{b} = B\mathbf{b}\cdot\mathbf{a}$ for every $\mathbf{a},\mathbf{b}\in\mathbb{R}^3$, which means that B is a symmetric matrix. Therefore (3.14) implies (3.10).

On the other hand, by (3.5) and (3.6) we have

$$\begin{aligned} B((\nabla\mathbf{u})\nu) &= \lambda\{(\nabla\mathbf{u})\nu\cdot\nu\}\nu + 2\mu\{\mathrm{Sym}\,(((\nabla\mathbf{u})\nu)\nu^T)\}\nu \\ &= \lambda(\nabla^T\mathbf{u})\nu + 2\mu\{\mathrm{Sym}\,(\nabla\mathbf{u})\}\nu = \sigma(\mathbf{u})\nu \qquad \text{on} \quad \partial\Omega\times(0,T). \end{aligned}$$

In view of (3.5) and (3.10), the proof of (3.8) is complete.

Third Step. Since $\mathbf{u}\in C^2(\overline{Q})^n$, we see from (3.2) and (3.8) that $\mathcal{L}\mathbf{u} = \mu(\Delta\mathbf{u}+\nabla(\nabla^T\mathbf{u})) - \nabla(\lambda\nabla^T\mathbf{u}) + F_1(S(\sigma(\mathbf{u})\nu))$ on $\partial\Omega\times(0,T)$. Here and henceforth F_i denote linear maps with continuous coefficients. Therefore we obtain

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla(\nabla^T\mathbf{u}) = F_2(\sigma(\mathbf{u})\nu) \qquad \text{on} \quad \partial\Omega\times(0,T). \tag{3.15}$$

Since $\partial\Omega$ is of C^2-class, for any $x^0 = (x_1^0, x_2^0, x_3^0)\in\partial\Omega$, we can take neighbourhoods $\mathcal{V}$ in $\mathbb{R}^3$ of x^0 and $\mathcal{U}$ in $\mathbb{R}^2$ of (x_1^0, x_2^0), a function $\chi = \chi(x_1,x_2)\in C^2(\mathcal{U})$ such that

$$(x_1,x_2,x_3)\in\mathcal{V}\cap\partial\Omega \qquad \text{if and only if} \quad x_3 = \chi(x_1,x_2). \tag{3.16}$$

We introduce a new coordinate $\eta = \eta(x) = (\eta_1,\eta_2,\eta_3)$ by

$$\eta_1 = x_1, \quad \eta_2 = x_2, \quad \eta_3 = x_3 - \chi(x_1,x_2) \tag{3.17}$$

for $(x_1,x_2)\in\mathcal{U}$. We define a set $\mathcal{W}$ of (η_1,η_2,η_3) by $\mathcal{W} = \eta(\mathcal{V}\cap\partial\Omega)\subset\{(\eta_1,\eta_2,0);\ \eta_1,\eta_2\in\mathbb{R}\}$. Henceforth we locally regard $\mathbf{u} = \mathbf{u}(x_1,x_2,x_3)$ as a function in $(\eta_1,\eta_2,\eta_3)\in\eta(\mathcal{V})$. Then boundary conditions (3.2) and (3.8) imply

$$\begin{aligned} &\mathbf{u} = 0, \quad \frac{\partial\mathbf{u}}{\partial\eta_j} = F_3(\sigma(\mathbf{u})\nu), \\ &\frac{\partial^2\mathbf{u}}{\partial\eta_i\partial\eta_j} = F_4(\sigma(\mathbf{u})\nu, \nabla(\sigma(\mathbf{u})\nu)), \quad i=1,2, \quad j=1,2,3, \qquad \text{in } \mathcal{W}. \end{aligned} \tag{3.18}$$

Then noting that $\partial_3\chi = -1$,

$$\frac{\partial\mathbf{u}}{\partial x_i} = \frac{\partial\mathbf{u}}{\partial\eta_i} - \partial_i\chi\frac{\partial\mathbf{u}}{\partial\eta_3}, \qquad i=1,2, \qquad \frac{\partial\mathbf{u}}{\partial x_3} = \frac{\partial\mathbf{u}}{\partial\eta_3},$$

we see

$$\begin{aligned} \frac{\partial^2\mathbf{u}}{\partial x_i\partial x_j} &= \frac{\partial^2\mathbf{u}}{\partial\eta_i\partial\eta_j} - \partial_j\chi\frac{\partial^2\mathbf{u}}{\partial\eta_i\partial\eta_3} - (\partial_j\partial_i\chi)\frac{\partial\mathbf{u}}{\partial\eta_3} \\ &\quad - \partial_i\chi\frac{\partial^2\mathbf{u}}{\partial\eta_j\partial\eta_3} + (\partial_i\chi)(\partial_j\chi)\frac{\partial^2\mathbf{u}}{\partial\eta_3^2}, \qquad 1\le i,\ j\le 2, \end{aligned}$$

$$\frac{\partial^2\mathbf{u}}{\partial x_3^2} = \frac{\partial^2\mathbf{u}}{\partial\eta_3^2}, \qquad \frac{\partial^2\mathbf{u}}{\partial x_i\partial x_3} = \frac{\partial^2\mathbf{u}}{\partial\eta_i\partial\eta_3} - \partial_i\chi\frac{\partial^2\mathbf{u}}{\partial\eta_3^2}, \qquad i=1,2.$$

Therefore (3.18) implies

$$\frac{\partial^2 \mathbf{u}}{\partial x_i \partial x_j} = \partial_i\chi\partial_j\chi\frac{\partial^2 \mathbf{u}}{\partial \eta_3^2} + F_5(\sigma(\mathbf{u})\nu, \nabla(\sigma(\mathbf{u})\nu)), \qquad 1 \le i,\ j \le 3 \tag{3.19}$$

in $\mathcal{W}$. We substitute (3.19) into (3.15), and we obtain

$$\begin{aligned}\mu|\nabla\chi|^2\frac{\partial^2 u_i}{\partial\eta_3^2} + (\lambda+\mu)\left(\partial_i\chi\partial_1\chi\frac{\partial^2 u_1}{\partial\eta_3^2} + \partial_i\chi\partial_2\chi\frac{\partial^2 u_2}{\partial\eta_3^2} + \partial_i\chi\partial_3\chi\frac{\partial^2 u_3}{\partial\eta_3^2}\right)&\\ = F_6(\sigma(\mathbf{u})\nu, \nabla(\sigma(\mathbf{u})\nu)), \qquad 1 \le i \le 3&\end{aligned}$$

in $\mathcal{W}$. We can rewrite the above equalities as

$$D(\eta,t)\left(\frac{\partial^2 u_1}{\partial\eta_3^2}, \frac{\partial^2 u_2}{\partial\eta_3^2}, \frac{\partial^2 u_3}{\partial\eta_3^2}\right)^T = F_7(\sigma(\mathbf{u})\nu, \nabla(\sigma(\mathbf{u})\nu)) \quad \text{in} \quad \mathcal{W}\times(0,T), \tag{3.20}$$

where we define a 3×3 matrix $D = D(\eta,t)$ by

$$D = D(\eta,t) = (\lambda+\mu)(\nabla\chi)(\nabla\chi)^T + \mu|\nabla\chi|^2 I_3.$$

For any $\mathbf{a}\in\mathbb{R}^3$, we have

$$\begin{aligned} D\mathbf{a}\cdot\mathbf{a} &= (\lambda+\mu)((\nabla\chi)(\nabla\chi)^T\mathbf{a}\cdot\mathbf{a}) + \mu|\nabla\chi|^2(\mathbf{a}\cdot\mathbf{a})\\ &= (\lambda+\mu)(\mathbf{a}\cdot\nabla\chi)^2 + \mu|\nabla\chi|^2|\mathbf{a}|^2.\end{aligned} \tag{3.21}$$

We see from $\mu > 0$ and $3\lambda + 2\mu > 0$ on $\overline{Q}$ that

$$\lambda(x,t) + 2\mu(x,t) > \delta_0, \qquad (x,t)\in\partial\Omega\times(0,T), \tag{3.22}$$

where $\delta_0 > 0$ is a constant.

Since $\mu \in C(\overline{Q})$, by (3.22) we can choose a sufficiently small $\varepsilon > 0$ such that

$$\lambda(x,t) + (2-\varepsilon)\mu(x,t) > 0, \quad \mu(x,t) \ge \frac{\delta_0}{2}, \quad (x,t)\in\partial\Omega\times(0,T). \tag{3.23}$$

Applying Schwarz's inequality in (3.21), by (3.23), we obtain

$$\begin{aligned} D\mathbf{a}\cdot\mathbf{a} &= (\lambda+\mu)(\mathbf{a}\cdot\nabla\chi)^2 + (\mu-\mu\varepsilon)|\nabla\chi|^2|\mathbf{a}|^2 + \mu\varepsilon|\nabla\chi|^2|\mathbf{a}|^2\\ &= (\lambda+(2-\varepsilon)\mu)(\mathbf{a}\cdot\nabla\chi)^2\\ &\quad + \mu(1-\varepsilon)(|\nabla\chi|^2|\mathbf{a}|^2 - (\nabla\chi\cdot\mathbf{a})^2) + \mu\varepsilon|\nabla\chi|^2|\mathbf{a}|^2\\ &\ge \mu\varepsilon|\nabla\chi|^2|\mathbf{a}|^2 \ge \frac{\delta_0\varepsilon}{2}|\nabla\chi|^2|\mathbf{a}|^2 \ge \frac{\delta_0\varepsilon}{2}|\mathbf{a}|^2.\end{aligned} \tag{3.24}$$

At the last inequality, we use $|\nabla\chi|^2 = 1 + (\partial_1\chi)^2 + (\partial_2\chi)^2 \ge 1$. By the definition, $D(\eta,t)$ is symmetric, inequality (3.24) implies that $D = D(\eta,t)$ is invertible in $\mathcal{W}\times(0,T)$. Therefore (3.20) yields

$$\frac{\partial^2\mathbf{u}}{\partial\eta_3^2} = F_8(\sigma(\mathbf{u})\nu, \nabla(\sigma(\mathbf{u})\nu)) \qquad \text{in} \quad \mathcal{W}\times(0,T).$$

We combine (3.19) to obtain

$$\frac{\partial^2 \mathbf{u}}{\partial x_i \partial x_j}(x,t) = F_9(\sigma(\mathbf{u})\nu, \nabla(\sigma(\mathbf{u})\nu)), \qquad x \in \mathcal{V} \cap \partial\Omega,\ 0 < t < T,\ \le i,\ j \le 3.$$

Therefore, since $x^0 \in \mathcal{V}$ is an arbitrary point of $\partial\Omega$ and $\partial\Omega$ is compact,

$$\partial_\nu(\nabla^T \mathbf{u}) = F_9(\sigma(\mathbf{u})\nu, \nabla(\sigma(\mathbf{u})\nu)) \qquad \text{on} \quad \partial\Omega \times (0,T),$$

where F_9 is a linear map whose coefficients are bounded on $\partial\Omega \times (0,T)$. Thus in view of Theorem 1.1, the proof of Theorem 1.3 is complete. $\square$

4. Lipschitz stability for a principally diagonal hyperbolic system on the basis of a directly derived Carleman estimate

Theorems 1.1 through 1.4 are proved on the basis of Carleman estimates with boundary data by Tataru [27], where Carleman estimates are shown in a general setting. On the other hand, we can establish a similar Carleman estimate more directly for an operator $\Box_a \equiv a\partial_t^2 - \Delta$. This direct derivation is shown in Lavrent'ev, Romanov and Shishat·skiĭ [20] for the operator $\Box$ (i.e. $a \equiv 1$). In this section, we modify the argument in [20] for $\Box_a$. This direct way is less transparent than [27], while it admits less regular boundary $\partial\Omega$. On the other hand, the direct way gives a worse condition for T than in Sections 1 and 2, although in the case of constant a, the results in this section coincide with the results by Tataru's Carleman estimates and by the multiplier method.

In this section, we treat general spatial dimensions again except for Theorem 4.3. First for the statement of the Carleman estimate directly derived, we take $x_0 \in \mathbb{R}^n$ with $x_0 \not\in \overline{\Omega}$ and let us assume

$$a(x) > 0, \qquad 2 + \frac{(\nabla a(x) \cdot (x - x_0))}{a(x)} > 0, \qquad x \in \overline{\Omega}. \tag{4.1}$$

We take $\beta > 0$ such that

$$\left\{\begin{array}{l} \beta < \dfrac{4}{T|\nabla a(x)|}, \\ \beta < \dfrac{4a(x) + 2(\nabla a(x) \cdot (x - x_0))}{T|\nabla a(x)|(1 + a(x)) + 4a(x)^2}, \\ \beta < \dfrac{1}{a(x)} \qquad \text{on } \overline{Q}. \end{array}\right\}. \tag{4.2}$$

Remark 4.1. In the case of $a =$ constant, condition (4.2) is reduced to $\beta < (1/a)$.

Moreover we set

$$\varphi(x,t) = |x - x_0|^2 - \beta\left(t - \frac{T}{2}\right)^2 \tag{4.3}$$

and

$$Q(\delta) = \{(x,t) : x \in \Omega,\ \varphi(x,t) \geq \delta^2\} \tag{4.4}$$

with $\delta > 0$,

$$\Gamma_+ = \{x \in \partial\Omega : ((x - x_0) \cdot \nu(x)) > 0\}, \qquad \Gamma_- = \partial\Omega \setminus \Gamma_+. \tag{4.5}$$

Now we can state a Carleman estimate.

Theorem 4.1. *For $\delta > 0$, there exists a constant $C = C(\delta) > 0$ such that*

$$\begin{aligned}\int_{Q(\delta)} |\Box_a u|^2 e^{2\tau\varphi} dxdt + \int_{(\Gamma_+ \times (0,T)) \cap \partial Q(\delta)} \tau e^{2\tau\varphi} |\partial_\nu u|^2 d\sigma \\ \geq C\tau \int_{Q(\delta)} (|\nabla u|^2 + |\partial_t u|^2) e^{2\tau\varphi} dxdt + C\tau^3 \int_{Q(\delta)} u^2 e^{2\tau\varphi} dxdt\end{aligned} \tag{4.6}$$

for large $\tau > 0$ provided that

$$\left\{\begin{array}{ll} u \in H_0^1(Q(\delta)) \cap H^2(Q(\delta)), & \nabla u = 0 \quad \text{on} \quad \partial Q(\delta) \setminus (\partial\Omega \times (0,T)) \\ \partial_t u = 0 \qquad \text{on} \quad \partial Q(\delta). & \end{array}\right\}. \tag{4.7}$$

We consider

$$a_j(x)\partial_t^2 u_j - \Delta u_j + b_j(x,t; \nabla_{x,t}\mathbf{u}, \mathbf{u}) = 0, \qquad \text{in} \quad Q,\ 1 \leq j \leq m \tag{4.8}$$

and

$$\mathbf{u} = \mathbf{g} \qquad \text{on} \quad \Gamma. \tag{4.9}$$

Here $\mathbf{u} = (u_1, \ldots, u_m)$ and a_j, $1 \leq j \leq m$ satisfy (4.1), b_j are linear functions of $\nabla\mathbf{u}$ and $\mathbf{u}$ whose coefficients are in $L^\infty(Q)$ if the order of differentiation is 1 and the coefficient of the zeroth order term is in $L^{n+1}(Q)$.

On the basis of Theorem 4.1, we can prove

Theorem 4.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ whose boundary $\partial\Omega$ is of piecewise C^1. We assume*

$$T > 2 \max_{x \in \overline{\Omega}} \frac{|x - x_0|}{\sqrt{\beta}}. \tag{4.10}$$

Here $\beta > 0$ satisfies (4.2) *for all $a_1, \ldots, a_m$. Then there exists a constant $C > 0$ depending on Ω, T, x_0, a_j, b_j, $1 \leq j \leq m$, such that any solution $\mathbf{u}$ to* (4.8) *and* (4.9) *satisfies the bound*

$$\begin{aligned}&\|\mathbf{u}(\cdot,t)\|_{(1)}(\Omega) + \|\partial_t \mathbf{u}(\cdot,t)\|_2(\Omega) \\ &\quad \leq C(\|\mathbf{g}\|_{(1)}(\Sigma) + \|\partial_\nu \mathbf{u}\|_2(\Gamma_+ \times (0,T))), \qquad 0 \leq t \leq T.\end{aligned}$$

Remark 4.2. We set

$$\gamma(a)(x,t) = \min\left\{\frac{4}{T|\nabla a(x)|}, \frac{4a(x)+2(\nabla a(x)\cdot(x-x_0))}{T|\nabla a|(1+a)+4a^2}, \frac{1}{a}\right\}.$$

Then (4.2) is rewritten as

$$\beta < \min_{(x,t)\in\overline{Q}} \gamma(a)(x,t). \tag{4.2$'$}$$

In terms of $\gamma(a)$, we can rewrite (4.10) by

$$T > 2 \max_{(x,t)\in\overline{Q}, 1\le j\le m} \frac{|x-x_0|}{\sqrt{\gamma(a_j)(x,t)}}.$$

In the case where a_j, $1 \le j \le m$ are constants, then condition (4.10) for $T > 0$ follows from

$$T > 2 \max_{x\in\overline{\Omega}, 1\le j\le m} \sqrt{a_j}|x-x_0|, \tag{4.10$'$}$$

which is same as required by the multiplier method (e.g. Isakov [14], Komornik [17], Lop Fat Ho [22], Powell [23]). Therefore our result can generalize observability with constant coefficients by the multiplier method. We notice that the multiplier method can not treat b_j with terms of derivatives of the first order.

By the same argument as in Section 3, Theorem 4.2 yields

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain and let its boundary $\partial\Omega$ be of piecewise C^2. We set*

$$a_1(x) = \frac{\rho(x)}{\mu(x)}, \quad a_2(x) = \frac{\rho(x)}{2\mu(x)+\lambda(x)}, \qquad x \in \Omega. \tag{4.11}$$

In (1.1) *we assume that $\rho, \lambda, \mu \in C^3(\overline{Q})$ are independent of t, and $\mathbf{g} = 0$ in* (1.2)*. For $x_0 \notin \overline{\Omega}$, we assume that a_1 and a_2 satisfy* (4.1),

$$T > 2 \max_{x\in\overline{\Omega}} \frac{|x-x_0|}{\sqrt{\beta}}, \tag{4.12}$$

where $\beta > 0$ satisfies (4.2) *for a_1 and a_2. Then there exists a constant $C > 0$ depending on Ω, T, x_0, ρ, μ, λ, such that any solution $\mathbf{u}$ to* (1.1) *and* (1.2)$'$ *satisfies the bound*

$$\begin{aligned} &\|\mathbf{u}(\cdot,t)\|_{(1)}(\Omega) + \|\partial_t \mathbf{u}(\cdot,t)\|_2(\Omega) \\ &\quad + \|(\nabla^T \mathbf{u})(\cdot,t)\|_{(1)}(\Omega) + \|\partial_t \nabla^T \mathbf{u}(\cdot,t)\|_2(\Omega) \\ &\qquad \le C\{\|\mathbf{h}\|_2(\Gamma_+ \times (0,T)) + \|\nabla \mathbf{h}\|_2(\Gamma_+ \times (0,T))\}, \quad 0 \le t \le T. \end{aligned} \tag{4.13}$$

A similar bound holds for solutions to (1.1) *and* (1.2).

The rest part of this section is devoted to the proofs of Theorems 4.1 and 4.2.

Derivation of Theorem 4.2 from Theorem 4.1. The deduction is similar to the proof of Theorem 2.1. By $x_0 \not\in \overline{\Omega}$, we note that we can take $\varphi_1 = \varphi$. In terms of Lemma 2.1, we may assume that $\mathbf{g} = 0$. We set $t_0 = T/2$ in (4.3), that is,

$$\varphi(x,t) = |x - x_0|^2 - \beta\left(t - \frac{T}{2}\right)^2, \tag{4.14}$$

where $\beta > 0$ is chosen later. For the proof, it is sufficient to verify that φ defined by (4.14) satisfies (1.3) and (1.4) with $\beta > 0$ satisfying (4.2). In fact, condition (1.4) is straightforward from $x_0 \not\in \overline{\Omega}$ and (1.3) follows directly from (4.10). $\square$

Proof of Theorem 4.1. We will extend the proof in [20] (pp. 123–128) where $a(x) \equiv 1$ is considered. For convenience, we denote:

$$\begin{aligned} &x = (x_1, \ldots, x_n), \quad x_{n+1} = t, \quad A_1 = \cdots = A_n = 1, \quad A_{n+1} = -a, \\ &w(x,t) = e^{\tau\varphi(x,t)} \end{aligned} \tag{4.15}$$

and will make use of the function

$$\psi(x,t) = 4 - 2\beta|\nabla a(x)|\left|t - \frac{T}{2}\right| - 2n - 2\beta a(x) - \varepsilon. \tag{4.16}$$

Letting

$$v = wu, \tag{4.17}$$

we obtain

$$\begin{aligned} w^2(\Box_a u)^2 &= \left\{\sum_{j=1}^{n+1} A_j\{\partial_j^2 v - 2\tau\partial_j\varphi\partial_j v + (\tau^2(\partial_j\varphi)^2 - \tau\partial_j^2\varphi)v\}\right\}^2 \\ &\ge -2(\Box_a v)\tau\psi v - 4\left(\sum_{j=1}^{n+1} A_j\partial_j^2 v\right)\left(\sum_{j=1}^{n+1} A_j\tau\partial_j\varphi\partial_j v\right) \\ &\quad + \left(2\tau\psi\sum_{j=1}^{n+1} A_j(\tau^2(\partial_j\varphi)^2 - \tau\partial_j^2\varphi)v^2 - \tau^2\psi^2 v^2\right) \\ &\quad - 4\left(\sum_{j=1}^{n+1} A_j\tau\partial_j\varphi\partial_j v\right)\left(\sum_{k=1}^{n+1} A_k(\tau^2(\partial_k\varphi)^2 - \tau\partial_k^2\varphi)v\right) \\ &\equiv S_1 + S_2 + S_3 + S_4, \end{aligned} \tag{4.18}$$

where we have used an elementary inequality:

$$\begin{aligned}(B-C+D)^2 &= (B+D-\tau\psi v)^2 + C^2 + 2B\tau\psi v \\ &\quad - 2BC + 2\tau\psi v D - \tau^2\psi^2 v^2 - 2CD \\ &\geq 2B\tau\psi v - 2BC + 2\tau\psi v D - \tau^2\psi^2 v^2 - 2CD\end{aligned}$$

with the obvious choice of B, C, D and S_1, S_2, S_3, S_4.

Henceforth the sums are over $j, k = 1, \ldots, n+1$, unless specified.

Multiplying the terms in S_2, we yield

$$\begin{aligned}S_2 &= -4\tau \sum A_j A_k \partial_k \varphi \partial_j^2 v \partial_k v \\ &= -4\tau \sum \partial_j (A_j A_k \partial_k \varphi \partial_j v \partial_k v) + 2\tau \sum \partial_k (A_j A_k \partial_k \varphi (\partial_j v)^2) \\ &\quad - 2\tau \sum \partial_k (A_j A_k \partial_k \varphi)(\partial_j v)^2 + 4\tau \sum \partial_j (A_j A_k \partial_k \varphi) \partial_j v \partial_k v,\end{aligned}$$

where we have used $2\partial_j v(\partial_k \partial_j v) = \partial_k((\partial_j v)^2)$.

Substituting $v = wu$, observing that $\partial_j v = w(\partial_j u + \tau(\partial_j \varphi)u)$ and generically denoting by R_0 the terms containing the factor u, and by R, R_1, R_2, R_3, $\ldots$. the terms bounded by $Cw^2(|\nabla u|^2 + \tau^2 u^2)$ (where a constant $C > 0$ is independent of τ), we have

$$\begin{aligned}S_2 &= 2\tau \sum \partial_j (w^2(A_j A_k \partial_j \varphi (\partial_k u)^2 - 2A_j A_k \partial_k \varphi \partial_j u \partial_k u + R_0)) \\ &\quad - 2\tau w^2 \sum \partial_k (A_j A_k \partial_k \varphi)(\partial_j u)^2 - 2\tau^2 w^2 \sum \partial_k (A_j A_k \partial_k \varphi) \partial_j \varphi \partial_j (u^2) \\ &\quad - 2\tau^3 w^2 \sum \partial_k (A_j A_k \partial_k \varphi)(\partial_j \varphi)^2 u^2 \\ &\quad + 4\tau w^2 \sum \partial_j (A_j A_k \partial_k \varphi) \partial_j u \partial_k u \\ &\quad + 2\tau^2 w^2 \sum \partial_j (A_j A_k \partial_k \varphi)(\partial_j \varphi \partial_k (u^2) + \partial_k \varphi \partial_j (u^2)) \\ &\quad + 4\tau^3 w^2 \sum \partial_j (A_j A_k \partial_k \varphi) \partial_j \varphi (\partial_k \varphi) u^2.\end{aligned}$$

We note

$$\begin{aligned}&\tau^2 w^2 \sum \partial_k (A_j A_k \partial_k \varphi) \partial_j \varphi \partial_j (u^2) \\ &= \sum \partial_j (\tau^2 w^2 \partial_k (A_j A_k \partial_k \varphi)(\partial_j \varphi) u^2) - \tau^2 \partial_j (w^2) \partial_k (A_j A_k \partial_k \varphi)(\partial_j \varphi) u^2 \\ &\quad - \tau^2 w^2 \partial_j \partial_k (A_j A_k \partial_k \varphi)(\partial_j^2 \varphi) u^2 \\ &= -2\tau^3 w^2 (\partial_j \varphi)^2 \partial_k (A_j A_k \partial_k \varphi) u^2 + \sum \partial_k R_1 + R_2 \\ &= \sum \partial_k R_0 - 2\tau^3 w^2 \sum (\partial_j \varphi)^2 \partial_k (A_j A_k \partial_k \varphi) u^2 + R.\end{aligned}$$

Moreover using similar relations for the terms with $\partial_k(u^2)$, $\partial_j(u^2)$ and collecting the terms with the factors τw^2 and $\tau^3 w^2$, we have

$$
\begin{aligned}
S_2 &= 2\tau \sum \partial_j (w^2(A_j A_k \partial_j\varphi(\partial_k u)^2 - 2A_jA_k\partial_k\varphi\partial_j u\partial_k u) + R_0) \\
&\quad + 2\tau w^2 \sum \{2\partial_j(A_jA_k\partial_k\varphi)\partial_j u\partial_k u - \partial_k(A_jA_k\partial_k\varphi)(\partial_j u)^2\} \\
&\quad + 2\tau^3 w^2 \sum \{\partial_k(A_jA_k\partial_k\varphi)(\partial_j\varphi)^2 \\
&\quad - 2\partial_j(A_jA_k\partial_k\varphi)\partial_j\varphi\partial_k\varphi\}u^2 + R + \sum \partial_j R_3.
\end{aligned} \tag{4.19}
$$

Similarly

$$
\begin{aligned}
S_3 &= 2\tau\psi w^2 \sum A_j(\tau^2(\partial_j\varphi)^2 - \tau\partial_j^2\varphi)u^2 - \tau^2 w^2 \varphi^2 u^2 \\
&= 2\tau^3 w^2 \psi \sum A_j(\partial_j\varphi)^2 u^2 + R_4
\end{aligned} \tag{4.20}
$$

and

$$
\begin{aligned}
S_4 &= -2\tau \sum A_jA_k\partial_j\varphi(\tau^2(\partial_k\varphi)^2 - \tau\partial_k^2\varphi)\partial_j(v^2) \\
&= 2\tau^3 w^2 \sum \partial_j(A_jA_k\partial_j\varphi(\partial_k\varphi)^2)u^2 + \sum \partial_j R_5 + R_6.
\end{aligned} \tag{4.21}
$$

Noting $\partial_j v = (\partial_j u + \tau(\partial_j\varphi)u)w$, we see

$$
-2\tau \sum A_j\psi(\partial_j v)^2 = -2\tau w^2 \sum A_j\psi(\partial_j u + \tau(\partial_j\varphi)u)^2
$$

and

$$
\begin{aligned}
&- 4\tau^2 w^2 A_j\psi(\partial_j u)(\partial_j\varphi)u = -2\tau^2 w^2 (A_j\psi)(\partial_j(u^2))(\partial_j\varphi) \\
&\quad = -2\tau^2\partial_j(w^2(A_j\psi\partial_j\varphi)u^2) + 2\tau^2\partial_j(w^2)A_j\psi(\partial_j\varphi)u^2 + 2\tau^2 w^2\partial_j(A_j\psi\partial_j\varphi)u^2,
\end{aligned}
$$

we have

$$
\begin{aligned}
S_1 &= -2(\Box_a v)\tau\psi v \\
&= 2\tau \sum \partial_j(A_j\psi(\partial_j v)v) - \tau\sum\partial_j(A_j\psi)\partial_j(v^2) - 2\tau\sum A_j\psi(\partial_j v)^2 \\
&= \sum \partial_j R_0 + R - 2\tau w^2 \sum A_j\psi(\partial_j u)^2 + 2\tau^3 w^2 \sum A_j\psi(\partial_j\varphi)^2 u^2.
\end{aligned} \tag{4.22}
$$

By (4.19) through (4.22), we have

$$
\begin{aligned}
w^2(\Box_a u)^2 \ge\ & 2\tau\sum \partial_j(w^2(A_jA_k(\partial_j\varphi(\partial_k u)^2 - 2\partial_k\varphi\partial_j u\partial_k u)) + R_0) \\
& + 2\tau w^2 \sum \{2\partial_j(A_jA_k\partial_k\varphi)\partial_j u\partial_k u \\
& - \partial_k(A_jA_k\partial_k\varphi)(\partial_j u)^2 - A_j\psi(\partial_j u)^2\} \\
& + 2\tau^3 w^2 \sum \{2A_j\psi(\partial_j\varphi)^2 + \partial_k(A_jA_k\partial_k\varphi)(\partial_j\varphi)^2 \\
& - 2\partial_j(A_jA_k\partial_k\varphi)\partial_j\varphi\partial_k\varphi + \partial_j(A_jA_k\partial_j\varphi(\partial_k\varphi)^2)\}u^2 \\
& + R + \sum \partial_j R_0.
\end{aligned} \tag{4.23}
$$

Now we consider the integral of $w^2(\Box_a u)^2$ over $Q(\delta)$. First we estimate the integral of the divergence terms in the right side of (4.23). Since R_0 contains u as factor, the divergence theorem yields

$$\int_{Q(\delta)} \partial_j R_0 dxdt = 0$$

by $u_{|\partial Q(\delta)} = 0$. Moreover, since $u = 0$ on $\partial Q(\delta)$, as the surface integrals, we have only

$$2\tau \int_{\partial Q(\delta)\cap(\Gamma\times(0,T))} \sum w^2 A_j A_k \{\partial_j\varphi(\partial_k u)^2 - 2\partial_k\varphi\partial_j u\partial_k u\}\nu_j d\sigma.$$

When $k = n+1$, the integral is zero because $\partial_{n+1}u = 0$ on $\Gamma\times(0,T)$ and when $j = n+1$, it is zero because $\nu_{n+1} = 0$ there. Due to the condition $u = 0$ on $\Gamma\times(0,T)$, we have

$$\partial_j u = \nu_j \partial_\nu u.$$

Hence the integral over $\partial Q(\delta)\cap(\Gamma\times(0,T))$ is

$$\begin{aligned} 2\tau \int_{\partial Q(\delta)\cap(\Gamma\times(0,T))} & w^2 \sum((\partial_j\varphi)\nu_k^2 - 2(\partial_k\varphi)\nu_j\nu_k)(\partial_\nu u)^2\nu_j d\sigma \\ &= -2\tau \int_{\partial Q(\delta)\cap(\Gamma\times(0,T))} w^2 \sum^n (\partial_j\varphi)\nu_j(\partial_\nu u)^2 d\sigma, \end{aligned}$$

because $\sum^n \nu_j^2 = 1$. According to our definition of φ, we have

$$\sum^n \partial_j\varphi\cdot\nu_j = \sum^n 2(x-x_0)_j\cdot\nu_j \le 0$$

on Γ_-. Summing up, we obtain

$$\begin{aligned} &[\text{the surface integrals from the right side of (4.23)}] \\ &\qquad\qquad \ge -2\tau C\int_{\partial Q(\delta)\cap(\Gamma_+\times(0,T))} w^2(\partial_\nu u)^2 d\sigma \end{aligned}$$

with some constant $C > 0$. Henceforth, generically by C, we denote a positive constant independent of τ.

We break the factor of $2\tau w^2$ in (4.23) into the sums over $j,k = 1,\dots,n$; $j = 1,\dots,n, k = n+1$; $k = 1,\dots,n, j = n+1$; $j = k = n+1$ to conclude that this factor is

$$
\begin{aligned}
&\sum^{n}\{2\partial_j^2\varphi(\partial_j u)^2-\partial_k^2\varphi(\partial_j u)^2-\psi(\partial_j u)^2\}\\
&\quad+\sum^{n}\{2(\partial_j(-a)\partial_{n+1}\varphi)\partial_j u\partial_{n+1}u-\partial_{n+1}((-a)\partial_{n+1}\varphi)(\partial_j u)^2\}\\
&\quad+\sum^{n}\{2\partial_{n+1}(-a\partial_k\varphi)\partial_{n+1}u\partial_k u+\partial_k(a\partial_k\varphi)(\partial_{n+1}u)^2\}+a\psi(\partial_{n+1}u)^2\\
&\quad+\partial_{n+1}(a^2\partial_{n+1}\varphi)(\partial_{n+1}u)^2\\
&\quad=(4-2n-\psi)|\nabla_x u|^2+\sum^{n}\left\{4\beta\left(t-\frac{T}{2}\right)\partial_j a\partial_j u\partial_t u-2\beta a(\partial_j u)^2\right\}\\
&\quad+\sum 2(\partial_j a(x-x_0)_j+na)(\partial_t u)^2+a\psi(\partial_t u)^2-2\beta a^2(\partial_t u^2)
\end{aligned}
$$

where we have used definition (4.14) of φ, the equality $\partial_{n+1}=\partial_t$ and time independence of a. By using the Schwarz inequality for the scalar product $\nabla_x a\cdot\nabla_x u$ and then the inequality $2|\nabla_x u||\partial_t u|\le|\nabla_x u|^2+|\partial_t u|^2$, we conclude that the last sum is not less than

$$
\begin{aligned}
&\left\{4-2n-\psi-2\beta|\nabla_x a|\left|t-\frac{T}{2}\right|-2\beta a\right\}|\nabla_x u|^2\\
&\quad+\left\{-2\beta\left|t-\frac{T}{2}\right||\nabla_x a|+2(\nabla_x a\cdot(x-x_0)+na)+a\psi-2\beta a^2\right\}(\partial_t u)^2\\
&\ge c|\nabla_x u|^2\\
&\quad+\left\{-2\beta|\nabla a|\left|t-\frac{T}{2}\right|(1+a)+2\nabla a\cdot(x-x_0)+(4-\varepsilon)a-4\beta a^2\right\}(\partial_t u)^2
\end{aligned}
$$

due to choice (4.16) of ψ. Summing up, we can claim that for some small ε, in view of (4.1), the factor of τw^2 in (4.23) is not less than

$$c|\nabla_x u|^2+c|\partial_t u|^2,$$

with some constant $c>0$, provided that

$$\beta<\frac{2(\nabla a\cdot(x-x_0))+4a}{T|\nabla a|(1+a)+4a^2}\qquad\text{on}\quad\overline{Q}.$$

Similarly the coefficient of $\tau^3 w^2$ in (4.23) is

$$
\begin{aligned}
&[\text{the coefficient of } 2\tau^3 w^2 u^2] \\
&\quad = \sum \{2A_j\psi(\partial_j\varphi)^2 + \partial_k(A_kA_j\partial_k\varphi)(\partial_j\varphi)^2 \\
&\qquad - 2\partial_j(A_jA_k\partial_k\varphi)\partial_j\varphi\partial_k\varphi + \partial_j(A_jA_k\partial_j\varphi(\partial_k\varphi)^2)\} \\
&\quad = 8\left\{(\psi + 2n + 2a\beta)|x-x_0|^2 - (a\psi\beta^2 + 2na\beta^2 + 2a^2\beta^3)\left|t-\frac{T}{2}\right|^2\right\} \\
&\quad = 8\left\{\left(4-\varepsilon-2\beta|\nabla a|\left|t-\frac{T}{2}\right|\right)|x-x_0|^2\right. \\
&\qquad \left. - \left(\beta^2(4-\varepsilon)a - 2a\beta^3|\nabla a|\left|t-\frac{T}{2}\right|\right)\left|t-\frac{T}{2}\right|^2\right\} \\
&\quad > 8\left\{2\beta\left|t-\frac{T}{2}\right|^2\left(a\beta^2|\nabla a|\left|t-\frac{T}{2}\right|\right.\right. \\
&\qquad \left. - \left(|\nabla a|\left|t-\frac{T}{2}\right| + a\left(2-\frac{\varepsilon}{2}\right)\right)\beta + \left(2-\frac{\varepsilon}{2}\right)\right) \\
&\qquad \left. + \left(4-\varepsilon-2\beta|\nabla a|\left|t-\frac{T}{2}\right|\right)\delta^2\right\}.
\end{aligned}
$$

Here we have used $\varphi(t,x) > \delta^2$ in $Q(\delta)$. The first and the third conditions in (4.2) imply that

$$
a\beta^2|\nabla a|\left|t-\frac{T}{2}\right| - \left(|\nabla a|\left|t-\frac{T}{2}\right| + 2a\right)\beta + 2 > 0 \qquad \text{on } \overline{Q}.
$$

Then for sufficiently small $\varepsilon > 0$, we obtain

$$
\begin{aligned}
a\beta^2|\nabla a|\left|t-\frac{T}{2}\right| - \left(|\nabla a|\left|t-\frac{T}{2}\right| + a\left(2-\frac{\varepsilon}{2}\right)\right)\beta& \\
+\left(2-\frac{\varepsilon}{2}\right) \geq 0 \qquad \text{on } \overline{Q}.&
\end{aligned}
$$

Therefore, under the first condition in (4.2),

$$
\begin{aligned}
[\text{the coefficient of } 2\tau^3 w^2 u^2] &\geq 8\left(4-\varepsilon-2\beta|\nabla a|\left|t-\frac{T}{2}\right|\right)\delta^2 \\
&\geq 8(4-\varepsilon-T\beta|\nabla a|)\delta^2.
\end{aligned}
$$

Hence we choose a sufficiently small $\varepsilon > 0$, so that we can complete the proof of Theorem 4.1. □

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References

[1] F. Alabau and V. Komornik, *Observabilité, contrôlabilité et stabilisation frontière du système d'élasticité linéare*, C. R. Acad. Sci. Paris Sér. I. Math. **324** (1997), 519–524.

[2] D. D. Ang, M. Ikehata, D. D. Trong and M. Yamamoto, *Unique continuation for a stationary isotropic Lamé system with variable coefficients*, Comm. Partial Differential Equations **23** (1998), 371–385.

[3] C. Bardos, G. Lebeau and J. Rauch, *Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary*, SIAM J. Control Optim. **30** (1992), 1024–1065.

[4] A. Bukhgeim and V. Kardakov, *Solution of the inverse problem for the equation of elastic waves by the method of spherical means*, Siberian Math. J. **19** (1978), 528–535.

[5] N. Burq, *Contrôlabilité exacte des ondes dans des ouverts peu réguliers*, Asymptot. Anal. **14** (1997), 157–191.

[6] B. Dehman and L. Robbiano, *La propriété du prolongement unique pour un système elliptique. le système de Lamé*, J. Math. Pures Appl. **72** (1993), 475–492.

[7] M. Eller, V. Isakov, G. Nakamura and D. Tataru, *Uniqueness and stability in the Cauchy problem for Maxwell and elasticity systems*, Stud. Math. Appl. **31**, Elsevier, Amsterdam, 2002, pp. 329–349.

[8] M. Grasselli and M. Yamamoto, *Identifying a spatial body force in linear elastodynamics via traction measurements*, SIAM J. Control Optim. **36** (1998), 1190–1206.

[9] L. Hörmander, *Linear Partial Differential Operators*, Springer-Verlag, Berlin, 1963.

[10] ______, *The Analysis of Linear Partial Differential Operators I-IV*, Springer-Verlag, Berlin, 1983.

[11] M. A. Horn, *Implication of sharp trace regularity results on boundary stabilization of the system of linear elasticity*, J. Math. Anal. Appl. **223** (1998), 126–150.

[12] M. Ikehata, G. Nakamura and M. Yamamoto, *Uniqueness in inverse problems for the isotropic Lamé system*, J. Math. Sci. Univ. Tokyo **5** (1998), 627–692.

[13] V. Isakov, *A nonhyperbolic Cauchy problem for $\Box_b\Box_c$ and its applications to elasticity theory*, Comm. Pure Appl. Math. **39** (1986), 747–767.

[14] ______, *Inverse Problems for Partial Differential Equations*, Springer-Verlag, Berlin, 1998.

[15] F. John, *Partial Differential Equations*, Springer-Verlag, Berlin, 1986.

[16] M. V. Klibanov and J. Malinsky, *Newton-Kantorowich method for three-dimensional potential inverse scattering problem and stability of the hyperbolic Cauchy problem with time-dependent data*, Inverse problems **7** (1991), 577–596.

[17] V. Komornik, *Exact Controllability and Stabilization, the Multiplier Method*, Masson, Paris, 1994.

[18] I. Lasiecka, J.-L. Lions and R. Triggiani, *Non-homogeneous boundary value problems for second order hyperbolic operators*, J. Math. Pures Appl. **65** (1986), 149–192.

[19] I. Lasiecka and R. Triggiani, *Carleman estimates and exact boundary controllability for a system of coupled, nonconservative second-order hyperbolic equations*, In: *Partial Differential Equation Methods in Control and Shape Analysis*, Lecture Notes in Pure Appl. Math. **188**, Marcel Dekker, Inc., New York, 1997, pp. 215–243.

[20] M. M. Lavrent'ev, V. G. Romanov and S. P. Shishat·skiĭ, *Ill-posed Problems of Mathematical Physics and Analysis*, English translation of the original Russian version in 1980, American Mathematical Society, Providence, Rhode Island, 1986.

[21] G. Lebeau and E. Zuazua, *Null-controllability of a system of linear thermoelasticity*, Arch. Rational Mech. Anal. **141** (1998), 297–329.

[22] Lop Fat Ho, *Observabilité frontière de l'équation des ondes*, C. R. Acad. Sci. Paris Sér. I. Math. **302** (1986), 443–446.

[23] J. Powell, *An unconditional estimate for solutions of a wave equation*, J. Math. Anal. Appl. **179** (1993), 179–186.

[24] R. Sakamoto, *Mixed problems for hyperbolic equations, I. Energy inequalities*, J. Math. Kyoto Univ. **10** (1970), 349–373.

[25] D. Tataru, *Unique continuation for solutions to PDE's*; *between Hörmander's theorem and Holmgren's theorem*, Comm. Partial Differential Equations **20** (1995), 855–884.

[26] ———, *Boundary controllability for conservative PDEs*, Appl. Math. Optim. **31** (1995), 257–295.

[27] ———, *Carleman estimates and unique continuation for solutions to boundary value problems*, J. Math. Pures Appl. **75** (1996), 367–408.

[28] M. Yamamoto, *On an inverse problem of determining source terms in Maxwell's equations with a single measurement*, In: *Inverse Problems, Tomography, and Image Processing*, Plenum Press, New York, 1998, pp. 241–256.