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The regularity of the principal symbols of systems of pseudo-differential and partial differential operators as p-evolution

To the memory of my comrade Professor Isao Higuchi

By

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Abstract

The author proposed an idea on the principal part of linear systems of pseudo-differential equations in the Cauchy problem through the (pseudo-)normal form of systems in the formal symbol class and the theory of weighted determinant in [18]. In this paper, we show the regularity of the symbols of the principal part as p-evolution. In order to show this, we consider the regularity of the p-determinant of the matrices of pseudo-differential operators.

1. Introduction

Let m be a positive integer. We consider a matrix of differential operator on D_t and pseudo-differential operators of order m on D_x in the holomorphic class:

$$P = I_{\mathtt{N}} D_t - A(t, x, D_x) \tag{1.1}$$

where, $I_{\mathtt{N}}$ is the unit matrix of order $\mathtt{N}$ and $D_t = \frac{1}{\sqrt{-1}}(\partial/\partial t)$. In the paper [18]*1, we obtained a Jordan-like pseudo-normal form of systems in the meromorphic formal symbol class (Corollary 2 in [18] = Theorem 2.2 in this

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*1 There are some mistypes in this paper. One can receive the revised version corresponding to the author.

paper):

$$(1.2)\quad \begin{aligned} &N^{-1}(t,x,\xi)\circ P(t,x,\xi;D_t)\circ N(t,x,\xi) = Q = \oplus_{1\le j\le d} Q_j\,,\\ &Q_j(t,x,\xi;D_t) = I_{n_j}D_t - \sum_{i=0}^{\infty} B_{ji}(t,x,\xi)\,, \qquad \left(\sum_{1\le j\le d} n_j = \mathtt{N}\right),\\ &B_{j0} = J(n_j)\,{\xi_1}^{m+1}\,,\\ &B_{ji} = \begin{pmatrix} O \\ *\dots* \end{pmatrix} : \textit{homogeneous of order}\; m+1-i\,, \quad (i\ge 1), \end{aligned}$$

where $J(n) = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \;:\; n\times n\;.$

Further, in [18], we defined the p-evolution by the theory of the weighted determinant of matrices of formal symbols. (See also Section 3 in this paper.) Here, p is a non-negative rational number. When, P is p-evolutional, we arrive at the following normal form (Theorem 4.1 in this paper) by a change of weight by $W = \oplus_{1\le j\le d}\mathrm{diag}\,(\xi_1^{(n_j-1)(m+1)}, \xi_1^{(n_j-2)(m+1)}, \dots, 1)$:

$$(1.3)\quad \begin{aligned} &W^{-1}\circ Q(t,x,\xi;D_t)\circ W = \tilde{Q} = \oplus_{1\le j\le d}\tilde{Q}_j\,,\\ &\tilde{Q}_j(t,x,\xi;D_t) = I_{n_j}D_t - C_j(t,x,\xi)\,, \qquad \left(\sum_{1\le j\le d} n_j = \mathtt{N}\right),\\ &C_j = \begin{pmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & 1 & \\ & & & 0 & 1 \\ c_j(1) & c_j(2) & \dots & \cdot & c_j(n_j) \end{pmatrix},\\ &\mathrm{ord}\, c_j(k) \;\textit{is an integer and at most}\; p(n_j-k+1),\\ &\textit{further, if}\; p>0,\; \textit{at lest one of}\; \mathrm{true\,ord}\, c_j(k)\; \textit{is just}\; p(n_j-k+1)\,. \end{aligned}$$

$$\begin{aligned} &\prod_{1\le j\le d}\left| I_{\mathtt{N}}\tau - \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ c^{\circ}_{j0}(1) & c^{\circ}_{j0}(2) & \dots & c^{\circ}_{j0}(n_j) \end{pmatrix}\right| \\ &= \prod_{1\le j\le d}\left\{\tau^{n_j} - \sum_{k=1}^{n_j} c^{\circ}_{j0}(k)\tau^{k-1}\right\} \end{aligned}$$

is just p-determinant of P, where $c^{\circ}_{j0}(k)$ is the highest order part of $c_j(k)$ if $\mathrm{true\,ord}\, c_j(k) = p(n_j-k+1)$ and 0 if $\mathrm{true\,ord}\, c_j(k) < p(n_j-k+1)$. Since the matrix $\tilde{Q}_j$ corresponds to the scalar operator $D_t^{n_j} - \sum_{k=1}^{n_j} c_j(k)(t, x; D_x) D_t^{k-1}$, $\tilde{Q}$ is equivalent to a direct sum of higher order scalar operators. Here, the decomposition of $\tilde{Q}$ to $\oplus_{1 \le j \le d} \tilde{Q}_j$ is not unique. We give an example which has two different decompositions in Appendix.

In this paper, by virtue of the regularity of p-det P, we shall prove that if A is a matrix of holomorphic pseudo-differential operators, $c_{j0}^{\circ}(k)(t, x, \xi)$ $(1 \le k \le n_j,\ 1 \le j \le d)$ is holomorphic in (t, x, ξ), and further, that if $A(t, x, D_x)$ is a matrix of differential operators of D_x with holomorphic coefficients on t and x in Ω, it is a polynomial of ξ and τ with holomorphic coefficients on t and x in Ω. As it is not clear whether $S_H(O)$ satisfies left or right Ore's property, the regularity of p-det P may not be obtained by the general theory.

The papers [16], [19] and this one compose a series on the normal form of systems of partial differential equations.

The results in these papers will be applied in the forthcoming papers on *the Cauchy-Kowalevskaya theorem*. We shall show that the Cauchy problem for P is real analytic well-posed if and only if P is Kowalevskian in our sense (see Definition 4.1). The essential result was already announced in [20] and [18] with a rough sketch of the proof. In the forthcoming papers, we shall describe the condition in an invariant form using p-determinant. M. Kashiwara [13] showed the sufficiency through the cohomology theory. We shall show the convergence of the Taylor expansion of the fundamental solution. On the other hand, in order to show the necessity, assuming that P is not Kowalevskian, we shall consider the growth of the microlocal energy and lead a contradiction to an a priori estimate. Further, we shall give the necessary and sufficient condition for the C^∞-well-posedness of the first order systems with real characteristic roots of constant multiplicity (so-called *Levi condition*), assuming that all coefficients are real analytic. The proofs are similar as those on the Cauchy-Kowalevskaya theorem. In this case, for the sufficiency, it is essential that P is reduced to a direct sum of systems with only one characteristic root and each one is 0-*evolutional* (see Definition 4.1) along its bicharacteristic strip. The essential result was already announced in [17] and [18] with a rough sketch of the proof.

2. Meromorphic formal symbols and pseudo-normal form of systems

On the investigation of the differential equations in the holomorphic class, in the real analytic class or more generally in a ultradifferentiable classes, pseudo-differential operators of infinite order naturally appear. The theory of formal symbols of infinite order in the holomorphic and (or) real analytic class was started by L. Boutet de Monvel [9] and was developed by T. Aoki [4], [5], [7] and [6]. Further, it was generalized to the Gevrey classes by Y. Laurent [14]. However, in this paper, we consider only matrices of pseudo-differential operators of finite order and establish a (pseudo)-normal form through a similar transformation by a regular matrix of finite order. Therefore, through this paper, we treat only pseudo-differential operators of finite order.

We follow the results in W. Matsumoto [16], [19] and [18]. From an arbitrary asymptotic expansion of a symbol of a pseudo-differential operator in an ultradifferentiable class, a true symbol with this asymptotic expansion in the same class can be constructed and the ambiguity is of class $S^{-\infty}$ of the same class under some suitable conditions. Here, every element in $S^{-\infty}$ of this class maps ultradistributions with compact support to ultradifferentiable functions in this class. (See L. Boutet de Monvel and P. Krée [10], L. Boutet de Monvel [9] and W. Matsumoto [15]). In this paper, as we only consider the parts of finite order, we omit the details of $S^{-\infty}$. See, for example, [10] or [15] on $S^{-\infty}$ in the ultradifferentiable classes. Thus, when we consider the pseudo-differential operator whose symbol has an asymptotic expansion, the calculus of the pseudo-differential operators corresponds to that of the asymptotic expansions furnished the *operator product*.

Let $\mathbf{Z}_+$ be $\mathbf{N}\cup\{0\}$. We use the followings for α and β in ${\mathbf{Z}_+}^{1+\ell}$: $|\alpha| = \alpha_0 + \cdots + \alpha_\ell$, $\alpha! = \alpha_0!\alpha_1!\cdots\alpha_\ell!$, $\alpha+\beta = (\alpha_0+\beta_0,\dots,\alpha_\ell+\beta_\ell)$ and we denote $\beta \le \alpha$ when $\beta_k \le \alpha_k$ for $0 \le k \le \ell$. Let us set $a(t,x,\xi)^{(\beta)}_{(\alpha)} = {D_t}^{\alpha_0}{D_x}^{\alpha'}(\frac{\partial}{\partial\xi})^\beta a(t,x,\xi)$ for $\alpha \in {\mathbf{Z}_+}^{1+\ell}$ and $\beta \in {\mathbf{Z}_+}^{\ell}$ $(\alpha = (\alpha_0,\alpha'))$.

We introduce a holomorphic formal symbol and a meromorphic one. We say that a set O in $\mathbf{C}_t \times {\mathbf{C}_x}^\ell \times {\mathbf{C}_\xi}^\ell$ is conic when $(t,x,\xi) \in O$ implies $(t,x,\lambda\xi) \in O$ for arbitrary positive λ and that a subset Γ in O is conically compact in O when Γ is conic and $\Gamma \cap \{||\xi|| = 1\}$ is compact in $O \cap \{||\xi|| = 1\}$, where $||\xi|| = \sqrt{\sum_{k=1}^{\ell} |\mathrm{Re}\xi_k|^2 + |\mathrm{Im}\xi_k|^2}$. We say that Σ is a subvariety of O if it is a zero set of a holomorphic function in O.

Definition 2.1 (Meromorphic and holomorphic formal symbol)**.**
I. We say that the formal sum $a(t,x,\xi) = \sum_{i=0}^{\infty} a_i(t,x,\xi)$ is a meromorphic formal symbol (= *m.f.s.*) on O when there exist a conic subvariety Σ in O and a rational number κ such that

1) $a_i(t,x,\xi)$ is meromorphic in O, holomorphic in $O \setminus \Sigma$ and positively homogeneous of degree $\kappa - i$ on ξ , $(i \in \mathbf{Z}_+)$.

2) For an arbitrary conically compact set Γ in $O \setminus \Sigma$, there are positive constants C, R and R' and we have

$$|{a_i}^{(\beta)}_{(\alpha)}(t,x,\xi)| \le C{R'}^i R^{|\alpha|+|\beta|} i!|\alpha|!|\beta|!|\xi_1|^{\kappa-i-|\beta|} \quad on\ \Gamma\,, \qquad (i \in \mathbf{Z}_+, \alpha \in {\mathbf{Z}_+}^{1+\ell}, \beta \in {\mathbf{Z}_+}^{\ell}). \tag{2.1}$$

II. The formal sum $\sum_{i=0}^{\infty} a_i$ is called a holomorphic formal symbol (= *h.f.s.*) when it is a meromorphic formal symbol with $\Sigma = \emptyset$.

Remark 1. We use ξ_1 as a holomorphic scale of order and Σ includes $\{\xi_1 = 0\}$. Of course, ξ_1 can be replaced by another ξ_k and, in this case, Σ_k includes $\{\xi_k = 0\}$.

Remark 2. It is important that Σ is independent of i.

The number κ is called the order of the formal symbol a and denoted by "ord a". When $a_i = 0$ for $0 \le i \le i_\circ - 1$ and $a_{i_\circ} \ne 0$, $\kappa - i_\circ$ is called the *true order* of a and denoted by "true ord a". The order of 0 is posed $-\infty$. We set $S_M^\kappa(O) = \{$*the m.f.s.'s on* O *of order* $\kappa\}$, $S_H^\kappa(O) = \{$*the h.f.s.'s on* O *of order* $\kappa\}$, $S_M(O) = \cup_{\kappa\in\mathbf{R}} S_M^\kappa(O)$ and $S_H(O) = \cup_{\kappa\in\mathbf{R}} S_H^\kappa(O)$.

Corresponding to the asymptotic expansion of the symbol of the product of pseudo-differential operators, we introduce the *operator product* of formal symbols.

Definition 2.2 (Operator product). Let $a = \sum_{i=0}^\infty a_i$ and $b = \sum_{i=0}^\infty b_i$ be formal symbols. We set

$$a \circ b = \textstyle\sum_{i=0}^\infty c_i\,, \quad c_i(t,x,\xi) = \displaystyle\sum_{i_1+i_2+|\gamma|=i} \frac{1}{\gamma!} a_{i_1}^{(\gamma)}(t,x,\xi) b_{i_2(\gamma)}(t,x,\xi) \tag{2.2}$$

and call it the operator product of a and b.

By the operator product, $S_H(O)$ becomes a non-commutative ring and $S_M(O)$ does a non-commutative field. $S_H(O)$ is a subring of $S_M(O)$.

Let us consider a matrix $P = I_{\mathtt{N}} D_t - A(t,x,\xi)$, $A \in M_{\mathtt{N}}(S_M^m(O))$, $(m \in \mathbf{N})$. In [16] and [19], we obtained the following theorem.

Theorem 2.1. *We assume that $A(t,x,\xi)$ belongs to $M_{\mathtt{N}}(S_M^m(O))$ for a positive integer m and that each eigenvalue $\lambda_k(t,x,\xi)$ $(1 \le k \le d')$ of A_0 has the constant multiplicity n'_k $(\sum_{k=1}^{d'} n'_k = \mathtt{N})$. Then, there exist positive integers $\{d_k\}_{k=1}^{d'}$, positive integers $\{n_{kj}\}_{1\le j\le d_k, 1\le k\le d'}$ $(\sum_{j=1}^{d_k} n_{kj} = n'_k)$, $N'(t,x,\xi) = \sum_{i=0}^\infty N'_i(t,x,\xi)$ in $GL(\mathtt{N}; S_M(O))$, and $D_{kj}(t,x,\xi) = \sum_{i=0}^\infty D_{kji}(t,x,\xi)$ in $M_{n_{kj}}(S_M^m(O))$, such that*

$$\begin{aligned}
&N'^{-1}(t,x,\xi) \circ P(t,x,\xi;D_t) \circ N'(t,x,\xi) = \oplus_{1\le k\le d'} \oplus_{1\le j\le d_k} P_{kj}\,,\\
&P_{kj}(t,x,\xi;D_t) = I_{n_{kj}}(D_t - \lambda_k(t,x,\xi)) - \sum_{i=0}^\infty D_{kji}(t,x,\xi),\\
&D_{kj\,0} = J(n_{kj})\,\xi_1^m,\\
&D_{kj\,i} = \begin{pmatrix} \\ 0 \\ * \dots * \end{pmatrix} \quad \textit{homogeneous of degree } m-i \quad (i \ge 1),
\end{aligned} \tag{2.3}$$

where $J(n) = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$: $n \times n$.

Let us regard P as an operator of order $m+1$ on D_x. The part of order $m+1$ is the zero matrix and has the eigenvalue 0 of constant multiplicity $\mathtt{N}$. Thus, we can reduce P to the following pseudo-normal form ([18, Corollary 2]):

Theorem 2.2 (Jordan-like pseudo-normal form of systems = (1.2)). *We assume that $A(t,x,\xi)$ belongs to $M_{\mathtt{N}}(S^m_M(O))$ for a positive integer m. Then, there exist a positive integer d, positive integers $\{n_j\}_{1\le j\le d}$ ($\sum_{j=1}^d n_j = \mathtt{N}$), $N(t,x,\xi) = \sum_{i=0}^\infty N_i(t,x,\xi)$ in $GL(\mathtt{N}; S_M(O))$, and $B(t,x,\xi) = \sum_{i=0}^\infty B_{ji}(t,x,\xi)$ in $M_{n_j}(S^{m+1}_M(O))$, such that*

$$\begin{aligned}
&N^{-1}(t,x,\xi)\circ P(t,x,\xi;D_t)\circ N(t,x,\xi) = Q = \oplus_{1\le j\le d} Q_j,\\
&Q_j(t,x,\xi;D_t) = I_{n_j}D_t - \sum_{i=0}^\infty B_{ji}(t,x,\xi)\,,\\
&B_{j0} = J(n_j)\,{\xi_1}^{m+1}\,,\\
&B_{ji} = \begin{pmatrix} & & \\ & O & \\ * & \dots & * \end{pmatrix} : \textit{homogeneous of order } m+1-i\,,\ (i\ge 1)
\end{aligned} \tag{2.4}$$

3. p-determinant of matrix of pseudo-differential operators

3.1. p-determinant

On the matrix of partial differential operators, G. Hufford [12] first introduced the determinant applying the theory of J. Dieudonné [11], which is a determinant theory on a non-commutative field. M. Sato and M. Kashiwara [24][*2] generalized it to matrices of pseudo-differential operators and obtained the regularity property of the determinant. The algebraic structure of the determinant on the ring with Ore's property is well characterized by K. Adjamagbo [1] and [2]. The determinant by G. Hufford and M. Sato-M. Kashiwara is homogeneous. However, in order to consider, for example, the parabolic equations and Schrödinger type equations, we encounter inhomogeneous principal parts and need an inhomogeneous determinant.

We simply denote $S_M(O)$ and $S_H(O)$ by S_M and S_H, respectively. Left or right Ore's property is the necessary and sufficient condition for the existence of the quotient field. (Left Ore's property: for non-zero elements a and b, we can find non-zero c and d such that $ca = db$. Right Ore's property is also defined by the same way. See O. Ore [23] and K. Adjamagbo [1].) $S_M[D_t]$ is a non-commutative integral domain with Ore's property. On the other hand, it is not clear whether $S_H[D_t]$ has Ore's property, and then, we treat it as a subset of $S_M[D_t]$.

We fix a positive rational number p. Let us take $a(t,x,\xi,D_t) = \sum_{k=0}^m a^{\langle k\rangle}(t,x,\xi){D_t}^{m-k}$, $a^{\langle k\rangle} = \sum_{i=0}^\infty a_i^{\langle k\rangle} \in S_M$. We reset the order of $a^{\langle k\rangle}$ to its true order. Let us set

$$\begin{gathered}
p\text{-ord}\, a^{\langle k\rangle}(t,x,\xi){D_t}^{m-k} = \operatorname{ord} a^{\langle k\rangle} + p(m-k)\\
p\text{-ord}\, a = \max_{0\le k\le m} p\text{-ord}\, a^{\langle k\rangle}(t,x,\xi){D_t}^{m-k}
\end{gathered}$$

[*2] In Example in p. 19, $\beta(x)$, $\gamma(x)$ and $\delta(x)$ in *matrix* $A(x,D)$ should be replaced by $\gamma(x)$, $\delta(x)$ and $\beta(x)$, respectively. Further, δ in the first lien of $\det A(x,D)$ should be ξ.

and call them the p-order. By p-order, $S_M[D_t]$ becomes a filtered ring. We set further

$$R^{(p)}(a) = \{k \,:\, p\text{-ord}\, a^{\langle k\rangle} {D_t}^{m-k} = p\text{-ord}\, a\}$$
$$a_{p\_pr}(t,x,\xi,\tau) = \sum_{k\in R^{(p)}(a)} a_0^{\langle k\rangle}(t,x,\xi)\tau^{m-k}$$

and call the latter the p-principal symbol of a. $\cup_{p>0}\{a_0^{\langle k\rangle}(t,x,\xi)\tau^{m-k}\}_{k\in R^{(p)}(a)}$ has finite elements and brings the Newton polygon of a.

Let us take $c(t,x,\xi,\tau) = \sum_{k=0}^{m} c^{\langle k\rangle}(t,x,\xi)\tau^{m-k}$ a polynomial of τ whose coefficients are homogeneous on ξ respectively. We say that $c(t,x,\xi,\tau)$ is a p-homogeneous polynomial of τ when all $\deg c^{\langle k\rangle} + p(m-k)$ coincide each other for $0 \le k \le m$. For p-homogeneous c, we call common $\deg c^{\langle k\rangle} + p(m-k)$ the p-degree of c and denote it by p-deg$c$. Let us set

$$Y = \{p\textit{-homogeneous polynomials on } \tau\}\,.$$

$Y\setminus\{0\}$ is a commutative productive semigroup. The map σ^p from $S_M[D_t]\setminus\{0\}$ to $Y\setminus\{0\}$ defined by $\sigma^p(a) = a_{p\_pr}$ is a homomorphism of the productive semigroup. This is naturally extended to the map from $S_M[D_t]^Q\setminus\{0\}$ to $(Y\setminus\{0\})^Q$ by $\sigma^p(b^{-1}a) = a_{p\_pr}/b_{p\_pr}$ as a homomorphism of the productive group, where $S_M[D_t]^Q$ is the quotient field of $S_M[D_t]$ and $(Y\setminus\{0\})^Q$ is the quotient productive group of $Y\setminus\{0\}$. (By virtue of left Ore's property, if $b^{-1}a = b'^{-1}a'$, it holds that $a_{p\_pr}/b_{p\_pr} = a'_{p\_pr}/b'_{p\_pr}$ and the map σ^p is well defined on $S_M[D_t]^Q\setminus\{0\}$.) We set $p\text{-ord}\,(b^{-1}a) = p\text{-ord}\, a - p\text{-ord}\, b$ and $p\text{-deg}(a_{p\_pr}/b_{p\_pr}) = p\text{-deg}\, a_{p\_pr} - p\text{-deg}\, b_{p\_pr}$. We put $\sigma^p(0) = 0$, $p\text{-ord}\, 0 = -\infty$ and $p\text{-deg}\, 0 = -\infty$. Thus, we can obtain the weighted determinant theory by σ^p on $S_M[D_t]^Q$ following J. Dieudonné [11]. (See also E. Artin [8] and K. Adjamagbo [1] and [2].)

Definition 3.1 (p-determinant)**.** We call the determinant by σ^p of a matrix A with entries in $S_M[D_t]$ *p-determinant* of A and denote it by $p\text{-}\det A$.

Remark 3. 1-determinant is just Hufford and Sato-Kashiwara's determinant.

Following J. Dieudonné [11], we have obtained the elementary properties of p-determinant.

Theorem 3.1 (Elementary properties of p-determinant)**.** *We take* $A = (a^{ij})_{1\le i,j\le \mathtt{N}}$ *and* B *in* $M_{\mathtt{N}}(S_M[D_t]^Q)$.

(1) $p\text{-}\det AB = p\text{-}\det A\cdot p\text{-}\det B$.

(2) $p\text{-}\det\begin{pmatrix} A\,|\,C \\ \hline O\,|\,B\end{pmatrix} = p\text{-}\det\begin{pmatrix} A\,|\,O \\ \hline D\,|\,B\end{pmatrix} = p\text{-}\det A\cdot p\text{-}\det B$

(A, B, C and D are $m\times m$, $n\times n$, $m\times n$ and $n\times m$ matrices, respectively. O is the $m\times n$ or $n\times m$ zero matrix.)

(3) *p-determinant is invariant under the similar transformation.*

(4) *If there are real numbers m_i and n_j such that p-ord $a^{ij} \leq m_i + n_j$ and the ordinary determinant $\det(\sigma^p_{m_i+n_j}(a^{ij}))_{1\leq i,j\leq \mathtt{N}}$ does not vanish, then p-det $A = \det(\sigma^p_{m_i+n_j}(a^{ij}))$, where $\sigma^p_{m_i+n_j}(a^{ij})$ is $a^{ij}_{p\text{-}pr}$ if p-ord $a^{ij} = m_i + n_j$, and is 0 if p-ord $a^{ij} < m_i + n_j$.*

(5) *We set $c_i = \max_{1\leq j\leq \mathtt{N}} p$-true ord a^{ij} and $d_j = \max_{1\leq i\leq \mathtt{N}} p$-true ord a^{ij}. Then, it holds that p-deg p-det $A \leq \min\{\sum_{1\leq i\leq \mathtt{N}} c_i, \sum_{1\leq j\leq \mathtt{N}} d_j\}$.*

Proof. The claims from (1) to (4) are well known. On (5), we prove the inequality p-deg p-det $A \leq \sum_{1\leq i\leq \mathtt{N}} c_i$. We transform A in $S_M[D_t]^Q$. If $c_1 = -\infty$, p-det $A = 0$ and then (5) holds. Let us assume that $c_1 > -\infty$. First, we make the p-order of $(1,1)$-entry greater than or equal to that of $(i,1)$-entries $(2 \leq i \leq n)$ exchanging two rows if necessary. By this transformation, p-determinant may change its sign. We eliminate $(i,1)$-entries $(2 \leq i \leq n)$ using $(1,1)$- entry. Through these transformations, $\{c_i\}_{1\leq i\leq \mathtt{N}}$ is reserved. Secondly, if $c_2 = -\infty$, p-det $A = 0$ and then (5) holds. If $c_2 > -\infty$, we make the p-order of $(2,2)$-entry greater than or equal to that of $(i,2)$-entries $(3 \leq i \leq n)$ exchanging two rows if necessary. We eliminate $(i,2)$-entries $(3 \leq i \leq n)$ using $(2,2)$-entry. Thus, finally, we obtain an upper triangular matrix A' with p-order of its (i,i)-entry less than or equal to c_i $(1 \leq i \leq \mathtt{N})$. By the relation p-det $A = \pm p$-det A' and the property (2), we see the inequality p-deg p-det $A \leq \sum_{1\leq i\leq \mathtt{N}} c_i$. We can also show the inequality p-deg p-det $A \leq \sum_{1\leq j\leq \mathtt{N}} d_j$ by the same way. □

3.2. Regularity of p-determinant

As it is not clear whether $S_H[D_t]$ satisfies left or right Ore's property, the regularity of p-det P for $A \in M_{\mathtt{N}}(S_H[D_t])$ may not be obtained by the general theory. However, using the same idea by M. Sato and M. Kashiwara [24], we can show it.

Theorem 3.2 (Regularity of p-determinant). *Let p be a positive rational number.*

(1) *For $A(t,x,\xi;D_t)$ in $M_{\mathtt{N}}(S_H(O)[D_t])$, p-det A is a p-homogeneous polynomial of τ with holomorphic coefficients on (t,x,ξ) in O.*

(2) *For a matrix of partial differential operators with holomorphic coefficients on t and x in Ω, p-det A is p-homogeneous and is a polynomial of τ and ξ with holomorphic coefficients on t and x in Ω.*

Proof. (1) We show this by a mathematical induction on the size of matrix. Let p be $\dfrac{s}{r}$ $(r,\ s \in \mathbf{N})$, where r is the common denominator of p and the orders of the formal symbols in A. (In this paper, we only consider formal symbols of rational order. See Definition 2.1.) Let $m_\circ$ be the p-degree of p-det A. We need consider the case of $m_\circ > -\infty$.

(Step 1) In case of $\mathtt{N} = 1$, the regularity is obvious.

(Step 2) We assume that p-det A is regular for $\mathtt{N} < n$ and show the regularity in case of n by a reduction to absurdity. We assume that p-det A has the singularity on $\Sigma_\circ$, which is defined by the zero set of $\alpha_\circ(t,x,\xi;\tau)$, a p-homogeneous polynomial of τ with holomorphic coefficients on (t, x, ξ). Let $\alpha(t, x, \xi; \tau)$ be one of the irreducible component of $\alpha_\circ(t, x, \xi; \tau)$ as the p-homogeneous polynomial of τ with holomorphic coefficients on (t, x, ξ). Nevertheless, we show that the definition function of the singularity of p-det A does not have $\alpha(t, x, \xi; \tau)$ as an irreducible component.

Let A in $M_n(S_H[D_t])$ be $(a_{ij}(t, x, \xi; D_t))$. We denote the true p-order of a_{ij} by d_{ij}, $\max_{1 \le i, j \le n} d_{ij}$ by M and the number of $\alpha(t, x, \xi; \tau)$ in the decomposition to the irreducible components of the true p-principal part of a_{ij} by g_{ij}. When $a_{ij} = 0$, we set $d_{ij} = -\infty$ and $g_{ij} = \infty$. Further, we set $g = \min_{1 \le i \le n} g_{i1}$. When $g_{11} > g$, we exchange the i-th row with $g_{i1} = g$ for the first row. Thus, we can assume that $g_{11} = g$. (This transformation makes a change of the sign of p-determinant but keeps its singularity. In this proof, we ignore the sign of the p-determinant.) We set $d_{11} = d_\circ$. We take $c(t, x, \xi; \tau) = a_{11p\_pr}(t, x, \xi; \tau)/\{\alpha(t, x, \xi; \tau)^g\}$, which is a p-homogeneous polynomial of τ with holomorphic coefficients on (t, x, ξ) and does not have $\alpha(t, x, \xi; \tau)$ as an irreducible component. We product $c(t, x, \xi; D_t)$ from left to i-th row as an operator and eliminate the true p-principal part of $(i, 1)$-entry using the true p-principal part of $(1, 1)$-entry for $2 \le i \le n$. Thus, we obtain a new matrix $A' = (a'_{ij})$ in $M_n(S_H[D_t])$, where

(3.1) $\quad p\text{-}\det A = c(t, x, \xi; \tau)^{-(n-1)}\, p\text{-}\det A'$,

(3.2) $\quad \text{true } p\text{-ord}\, c = d, \qquad p\text{-}\deg\, p\text{-}\det A' = m_\circ + (n-1)d,$

(3.3) $\quad \text{true } p\text{-ord}\, a'_{11} = d_\circ, \qquad p\text{-ord}\, a'_{1j} \le M\ , \qquad (2 \le j \le n),$

(3.4) $\quad p\text{-ord}\, a'_{ij} \le M + \max\{d_{i1},\, d\}\ , \qquad (2 \le i \le n,\, 2 \le j \le n),$

(3.5) $\quad p\text{-ord}\, a'_{i1} \le d_{i1} + d - \dfrac{1}{r}\ , \qquad (2 \le i \le n).$

If the minimum of new g'_{i1} of A' for $2 \le i \le n$ becomes smaller than g, we exchange the i-th row with the new minimum g' for the first row and we repeat the above procedure. Repeating this procedure, we can finally fix the first row. (For example, if we arrive at $g = 0$, we need not exchange the rows. As $m_\circ > -\infty$, the case $g = -\infty$ is impossible.)

We consider the situation after the fist row is fixed. When $F = \max_{2 \le i \le n} d_{i1}$ is greater than d we take f as $F = d + \dfrac{f}{r}$, and when F is smaller than or equal to d we set $f = 0$. Repeating the above procedure, we can obtain $A'' = (a''_{ij})$ in $M_n(S_H[D_t])$, for which the following relations hold by (3.4) and (3.5):

(3.6) $\quad p\text{-}\det A = c'(t, x, \xi; \tau)^{-(n-1)}\, p\text{-}\det A''$,

(3.7) $\quad d' = \text{true } p\text{-}\deg c', \qquad p\text{-}\deg\, p\text{-}\det A'' = m_\circ + (n-1)d',$

(3.8) $\quad \text{true } p\text{-ord}\, a''_{11} = d_\circ, \qquad p\text{-ord}\, a''_{1j} \le M\ , \qquad (2 \le j \le n),$

(3.9) $\quad p\text{-ord}\, a''_{ij} \le M + \dfrac{f(f+1)}{2r} + d'\ , \qquad (2 \le i \le n,\, 2 \le j \le n),$

(3.10) $\quad p\text{-ord}\, a''_{i1} < m_\circ - (n-1)(M' + |d'|)\ , \qquad (2 \le i \le n),$

$$M' = |M| + \frac{f(f+1)}{2r}\,,$$

where $c'(t,x,\xi;\tau)$ is a p-homogeneous polynomial of τ with holomorphic coefficients on (t,x,ξ) and does not have $\alpha(t,x,\xi;\tau)$ as an irreducible component.

Let B be the $(n-1)\times(n-1)$ matrix excluding the first row and the first column from A''. We transform B in $S_M[D_t]^Q$. First, we make the p-order of $(1,1)$-entry greater than or equal to the others exchanging two rows and (or) two columns if necessary. We eliminate $(i,1)$-entries and $(1,j)$-entries $(2\le i\le n$, $2\le j\le n)$ using $(1,1)$-entry. Secondly, we make the p-order of $(2,2)$-entry greater than or equal to the others except that of $(1,1)$-entry exchanging two rows and (or) two columns if necessary. We eliminate $(i,2)$-entries and $(2,j)$-entries $(3\le i\le n,\ 3\le j\le n)$ using $(2,2)$-entry. Thus, finally, we obtain a diagonal matrix $\tilde B = \mathrm{diag}\,(\tilde b_2,\dots,\tilde b_n)$, whose entries has p-order at most $M'+d'$. We remark that p-$\det B = p$-$\det \tilde B = \prod_{i=2}^n \tilde b_{i0}$ except the sign, where b_{i0} is the true p-principal symbol of b_i $(2\le i\le n)$. We transform A'' to $\tilde A$ by the above transformations enlarging the size to $n\times n$ in order to make the part of B in A'' diagonal. After these transformations, the followings hold

$$p\text{-}\det A = c'(t,x,\xi;\tau)^{-(n-1)}\, p\text{-}\det \tilde A\,, \tag{3.11}$$

$$p\text{-}\deg\, p\text{-}\det \tilde A = m_\circ + (n-1)d'\,, \tag{3.12}$$

$$\text{true } p\text{-ord}\,\tilde a_{11} = d_\circ, \qquad p\text{-ord}\,\tilde a_{1j} \le M, \qquad (2\le j\le n), \tag{3.13}$$

$$p\text{-ord}\,\tilde a_{ii} \le M'+d',\ \ (2\le i\le n), \qquad a_{ij}=0\,, \qquad (2\le i\ne j\le n), \tag{3.14}$$

$$p\text{-ord}\,\tilde a_{i1} < m_\circ - (n-1)(M'+|d'|)\,, \qquad (2\le i\le n). \tag{3.15}$$

We set $\tilde d_i = p$-true ord $\tilde b_i$. If one of $\{\tilde d_i\}$ might be less than $m_\circ - d - (n-2)(M'+|d'|)$, p-deg p-det $\tilde A$ became less than $m_\circ + (n-1)d'$ by (5) in Theorem 3.1. Thus, we see that $d_\circ + \sum_{2\le i\le n}\tilde d_i = m_\circ + (n-1)d'$ and p-$\det\tilde A = a''_{11p\_pr}\cdot\prod_{i=2}^n \tilde b_{i0} (= a''_{11p\_pr}\cdot p$-$\det B)$ except the sign by (4) in Theorem 3.1 taking $n_1 = d$ and $n_j = M$ $(2\le j\le n)$, $m_1 = 0$ and $m_i = \tilde d_i - M$ $(2\le i\le n)$. By the assumption of the induction, p-det B is regular and p-det A has the singular factor $c'(t,x,\xi;\tau)$, which does not include $\alpha(t,x,\xi;\tau)$ as an irreducible component. This is a contradiction to our assumption.

Thus, also in case of n, p-det A is a polynomial of τ with holomorphic coefficients on (t,x,ξ).

(2) In case of $\ell=1$, we consider A as a matrix of differential operators on $\Omega\times\mathbf{C}\subset\mathbf{C}^2$. Taking the both of scales of order ξ_1 and ξ_2, we see that the singular set of p-det P has the codimension at least 2. This implies p-det P is holomorphic in $\Omega\times\mathbf{C}^{1+\ell}$. Thus, p-det P is a polynomial of τ with holomorphic coefficients in $\Omega\times\mathbf{C}^\ell_\xi$ of polynomial order in ξ. This implies that p-det P is a polynomial of ξ and τ with holomorphic coefficients in Ω. $\square$

Remark 4. In spite of the simplicity of the proof, the regularity of p-determinant is delicate in the concrete calculation. (See [18, Example 5].)

4. Normal form of system and the regularity of the p-principal part

4.1. p-evolution

On the matrix of the form $P = I_{\mathrm{N}} D_t - A$, $A \in M_{\mathrm{N}}(S^m_M)$, we give the representation of p-determinant using the element of the normal form in Theorem 2.2.

In Theorem 2.2, let us set

$$\sum_{i=1}^{\infty} B_{ji}(t,x,\xi) = \begin{pmatrix} & O & \\ b_j(1) & \dots & b(n_j) \end{pmatrix}.$$

We set

$$\begin{aligned} &\operatorname{true\,ord} b_j(h) = r^j_h, \\ (4.1)\qquad &M^p_j = \max_{1\le h\le n_j} \{r^j_h + (m+1)(n_j - h) + p(h-1)\}, \\ &R^p_j = \{h : r^j_h + (m+1)(n_j - h) + p(h-1) = M^p_j\}. \end{aligned}$$

Applying the property (4) in Theorem 3.1, we have the following.

Proposition 4.1 (Relation between normal form and p-determinant)**.**

(4.2)

$$p\text{-}\det P = \prod_{j=1}^{d} p\text{-}\det Q_j\,,$$

$$\begin{aligned} p\text{-}\det Q_j &= \begin{cases} \tau^{n_j} & (pn_j > M_j)\,, \\ \tau^{n_j} - \displaystyle\sum_{h\in R^p_j} b_j(h)_0(t,x,\xi)\,\xi_1^{(m+1)(n_j-h)}\,\tau^{h-1}\,, & (pn_j = M_j)\,, \\ -\displaystyle\sum_{h\in R^p_j} b_j(h)_0(t,x,\xi)\,\xi_1^{(m+1)(n_j-h)}\,\tau^{h-1}\,, & (pn_j < M_j)\,, \end{cases} \\ &= \textit{the highest } p\textit{-degree part of the ordinary determinant of } Q_j \end{aligned}$$

Definition 4.1 ($p_\circ$-evolution and Kowalevskian)**.** By Proposition 4.1, we have only two cases; 1) there is a unique positive rational $p_\circ$ for which $p_\circ$-$\det P$ has the term τ^{N} and other terms, 2) p-$\det P$'s are always τ^{N} for all $p > 0$.

(1) In the case 1), we say that P is $p_\circ$-*evolutional* and in the case 2), we do that P is 0-*evolutional*.

(2) If P is $p_\circ$-evolutional for $p_\circ \le 1$, we say that P is *Kowalevskian*.

0-evolutional operator is essentially an ordinary differential operator on D_t. Our definition of "Kowalevskian system" is different from that in S. Mizohata [22] and in M. Miyake [21]. For p-evolutional P with $p > 1$, if every root of p-$\det P = 0$ has the positive imaginary part, we say that P is p-*parabolic* and if every root is real, we do that P is of p-*Schrödinger type*.

4.2. Regularity of the p-principal part

When the system is p-evolutional, from the pseudo-normal form in Theorem 2.2, we can obtain the final Jordan-like normal form:

Theorem 4.1 (Jordan-like normal form of systems = (1.3))**.** *We assume that $A(t,x,\xi)$ belongs to $M_{\mathtt{N}}(S^m_M(O))$ for a positive integer m and P is p-evolutional for $p \geq 0$. Then, there exist a positive integer d, positive integers $\{n_j\}_{1\leq j\leq d}$ ($\sum_{j=1}^d n_j = \mathtt{N}$), $\tilde{N}(t,x,\xi) = \sum_{i=0}^\infty \tilde{N}_i(t,x,\xi)$ in $GL(\mathtt{N}; S_M(O))$, and $C(t,x,\xi)$ in $M_{n_j}(S_M(O))$, such that*

$$N^{-1} \circ Q(t,x,D_t,\xi) \circ N = \tilde{Q} = \oplus_{1\leq j\leq d} \tilde{Q}_j \,,$$

$$\tilde{Q}_j(t,x,\xi;D_t) = I_{n_j} D_t - C_j(t,x,\xi) \,, \qquad \left(\sum_{1\leq j\leq d} n_j = \mathtt{N} \right),$$

$$C_j = \begin{pmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & 1 & \\ & & & 0 & 1 \\ c_j(1) & c_j(2) & \dots & \cdot & c_j(n_j) \end{pmatrix}, \tag{4.3}$$

$\operatorname{ord} c_j(k)$ *is an integer and at most* $p(n_j - k + 1)$*,*

further, if $p > 0$, at lest one of $\operatorname{true\,ord} c_j(k)$ *is just* $p(n_j - k + 1)$*.*

We set $c^\circ_{j0}(k) = c_{j0}(k)$ if $\operatorname{true\,ord} c_j(k) = p(n_j - k + 1)$ *and* 0 *if* $\operatorname{true\,ord} c_j(k) < p(n_j - k + 1)$*. $c^\circ_{j0}(k)$ is holomorphic in O if A belongs to $M_{\mathtt{N}}(S_H(O))$, and further, is a polynomial of ξ with holomorphic coefficients on t and x in Ω if A is a matrix of differential operators with holomorphic coefficients on t and x in Ω.*

Proof. Through the similar transformation by $W = \oplus_{1\leq j\leq d} \operatorname{diag}(\xi_1^{(n_j-1)(m+1)}, \xi_1^{(n_j-2)(m+1)}, \dots, 1)$ on Q in Theorem 2.2, we obtain Theorem 4.1 except the regularity.

As p-det P is holomorphic (Theorem 3.2) for A in $M_{\mathtt{N}}(S_H(O))$ and p-det P $= \prod_{1\leq j\leq d} \left\{\tau^{n_j} - \sum_{k=1}^{n_j} c^\circ_{j0}(k)\, \tau^{k-1}\right\}$, $c^\circ_{j0}(k)$ may have algebraic singularities but it can have only poles. This means that $c^\circ_{j0}(k)(t,x,\xi)$ is holomorphic. Further, if every entry in A is a differential polynomial with holomorphic coefficients in Ω in $\mathbf{C}_t \times \mathbf{C}^\ell_x$, it is a polynomial of ξ by the same reason of the proof of (2) in Theorem 3.2. □

It is natural to call

$$\oplus_{1\leq j\leq d} \begin{pmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & 1 & \\ & & & 0 & 1 \\ c^\circ_{j0}(1) & c^\circ_{j0}(2) & \dots & c^\circ_{j0}(n_j-1) & c^\circ_{j0}(n_j) \end{pmatrix} \tag{4.4}$$

the p-principal part of $P(t,x,D_x,D_t)$. The regularity of the symbol of A reflects it. (The decomposition to a direct sum is not unique. See Appendix. On the other hand, p-determinant is invariant.)

A higher order scalar equation with a scaler unknown function

$$(D_t)^{\mathtt{N}}v-\sum_{k=1}^{\mathtt{N}}a(k)(t,x,D_x)(D_t)^{k-1}v=f(t,x)$$

is reduced to a first order system on D_t taking $u={}^t(v,\,D_tv,\dots,D_t^{\mathtt{N}-1}v)$ and $F={}^t(0,\dots,0,f)$:

$$D_tu-\begin{pmatrix}0&1&&&\\&0&\ddots&&\\&&\ddots&1&\\&&&0&1\\a(1)&a(2)&\dots&\cdot&a(\mathtt{N})\end{pmatrix}u=F(t,x)\,,$$

where this system is p-evolutional for $p=\max_{1\le k\le \mathtt{N}}\{\mathrm{true\,ord}\,a(k)/(\mathtt{N}-k+1)\}$ if this p is positive. Therefore, Theorem 4.1 say that a system is reduced to a direct sum of p-evolutional scalar equations outside of a subvariety of O modulo $S^{-\infty}$.

5. Appendix

In Jordan's theory , the distribution of the sizes of Jordan blocks is invariant. However, in our Jordan-like theory, the sizes of blocks in the (pseudo-) normal form is not invariant.

Example 5.1. Let a be a non-zero constant.

$$P=I_3\,D_t-\left(\begin{array}{cc|c}0&1&\\&0&\\\hline&&0\end{array}\right)\xi_1-\left(\begin{array}{cc|c}a&0&\\\hline&&0\end{array}\right)\tag{5.1}$$

has one block of size 2 and one block of size 1. Let us take

$$\begin{aligned}N&=\begin{pmatrix}0&0&(1/a)\xi_1\\0&1&0\\-a\xi_1^{-1}&0&1\end{pmatrix}\\&=\begin{pmatrix}1&&\\&1&\\&&\xi_1^{-1}\end{pmatrix}\begin{pmatrix}0&0&(1/a)\\0&1&0\\-a&0&1\end{pmatrix}\begin{pmatrix}1&&\\&1&\\&&\xi_1\end{pmatrix}.\end{aligned}$$

Its inverse is

$$N^{-1} = \begin{pmatrix} 1 & 0 & -(1/a)\xi_1 \\ 0 & 1 & 0 \\ a\xi_1^{-1} & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & \\ & 1 & \\ & & \xi_1^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & -(1/a) \\ 0 & 1 & 0 \\ a & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & \xi_1 \end{pmatrix}.$$

The similar transformation by N brings

$$N^{-1} \circ P \circ N = I_3 D_t - \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix} \xi_1 - \begin{pmatrix} & & \\ & & \\ 0 & a & 0 \end{pmatrix}, \tag{5.2}$$

which has only one block of size 3. The above transformation is composed by two changes of weight and one homogeneous similar transformation.

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