J. Math. Kyoto Univ. (JMKYAZ)
45-2 (2005), 265–286

The Lax–Phillips scattering approach and singular perturbations of Schrödinger operator homogeneous with respect to scaling transformations

By

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Abstract

Spectral and scattering properties of positive self-adjoint operators that correspond to singular perturbations of the Schrödinger operator homogeneous with respect to the scaling transformations are investigated by a method based on the Lax–Phillips ideas in scattering theory.

1. Introduction

Starting from the sixties, Lax and Phillips developed a new original approach to the scattering theory, which is a convenient tool for the investigation of various scattering problems (see [22]–[24]). Later, the Lax–Phillips scattering theory was considerably developed by Adamjan [1], Arov [9], Cooper and Strauss [10], Foias [12], and Phillips [25]. In particular, a simple relationship between the analytic continuation of a Lax–Phillips scattering matrix and the characteristic function of a certain contraction operator (which, in fact, characterizes an influence of perturbation) was established. This fact enables one to apply the Foias–Sz.-Nagy theory [13] to the investigation of scattering matrices.

Nevertheless, in the contemporary literature, there are a little examples of application of the Lax–Phillips ideas (see, e.g., [11], [15], [26], [27], [29]). This is partly because that the main results of the Lax–Phillips theory [22] were obtained for a general case of an arbitrary group of unitary operators that a priori possesses outgoing and incoming subspaces and, as a result, the application of the Lax–Phillips approach to many scattering problems is impossible without a serious auxiliary work (description of incoming and outgoing subspaces, construction of spectral representations, determination of scattering matrix and so on).

In order to remove such cumbersome preparations, it is natural to develop the Lax–Phillips approach more deeply for abstract realizations of those types

2000 *Mathematics Subject Classification*(*s*). Primary 47A40; Secondary 35P25, 81U
Received May 7, 2004
Revised February 1, 2005

of evolution systems that are studied most often in the Lax–Phillips theory. Furthermore, using the specific properties of a chosen class of systems, it is necessary to establish new relationships between the nature of a perturbation and properties of the corresponding Lax–Phillips scattering matrix, which are more convenient for applications.

A lot of results in solving the problem indicated above was obtained by one of the authors (see [17]–[19] and references therein) for an abstract realization of the classical wave equation (abstract wave equation), more precisely, for evolving systems described by an operator-differential equation

$$u_{tt} = -Lu, \tag{1.1}$$

where L is a positive self-adjoint operator acting in an abstract Hilbert space $\mathfrak{H}$. The results of [17]–[19] are based on the observation that the abstract wave equation (1.1) possesses incoming and outgoing subspaces with the same properties as the classical Lax–Phillips subspaces $D_{\pm}$ for the wave equation in a space of odd dimension (see [22]) if and only if the operator L in (1.1) satisfies the following:

Condition A. *There exists a simple*[*1] *maximal symmetric operator B acting in a subspace $\mathfrak{H}_0$ of the Hilbert space $\mathfrak{H}$ such that the operator L is a positive self-adjoint extension* (*with exit in the space $\mathfrak{H}$*) *of the symmetric operator B^2, i.e.*

$$L \supset B^2 \qquad \textit{and} \qquad \mathfrak{H} \supseteq \mathfrak{H}_0.$$

We will say that an operator L is a Lax–Phillips perturbed operator if it satisfies Condition **A** for a certain choice of a simple maximal symmetric operator B.

Choosing a simple maximal symmetric operator B in various functional spaces, we get different sets of Lax–Phillips perturbed operators ([17]–[19]) that makes it possible to study various realizations of the abstract wave equation (1.1) (the partial wave equation, the wave equation in $\mathbb{R}^n$ and so on) from a unique point of view in the Lax–Phillips framework.

Furthermore, condition **A** enables one to employ methods of the extension theory of Hermitian operators for the description of Lax–Phillips perturbed operators and, as a result, to obtain new results in spectral and scattering theory of Schrödinger operators. An example of such application is considered in the present work.

The paper is organized as follows. In Section 2, assuming that L is a Lax–Phillips perturbed operator, we present the main results of [17]–[19] for the abstract wave equation (1.1) and reformulate them (due to [20], [22]) to the dual case of the abstract Schrödinger equation

$$iu_t = Lu. \tag{1.2}$$

[*1] An operator is called simple if its restriction on any nontrivial reducing subspace is not a self-adjoint operator

In Section 3, we study the formal expression

$$-\Delta + V, \qquad V : W_2^2(\mathbb{R}^3) \to W_2^{-2}(\mathbb{R}^3), \tag{1.3}$$

where a singular perturbation V of the free Schrödinger operator $-\Delta$, $(D(\Delta) = W_2^2(\mathbb{R}^3))$ is homogeneous with respect to the scaling transformations

$$(G(t)f)(x) = t^{3/2}f(tx) \quad (t > 0, \quad f(x) \in L_2(\mathbb{R}^3)). \tag{1.4}$$

Let us explain the property of homogeneity of V in more detail. To do this, we recall ([5]) that $G(t) : W_2^2(\mathbb{R}^3) \to W_2^2(\mathbb{R}^3)$ and the action of $G(t)$ on elements $\psi \in W_2^{-2}(\mathbb{R}^3)$ is defined by the relation

$$\langle G(t)\psi, u\rangle = \langle \psi, G(1/t)u\rangle \quad (\psi \in W_2^{-2}(\mathbb{R}^3),\ u(x) \in W_2^2(\mathbb{R}^3)), \tag{1.5}$$

By virtue of (1.5), $G(t)\delta(x) = t^{-3/2}\delta(x)$, where $\delta(x)$ is the delta function. Hence, if a singular perturbation V in (1.3) has the form

$$V = \alpha\langle \delta(x), \cdot\rangle \delta(x), \qquad \alpha \in \mathbb{C}, \tag{1.6}$$

then, taking into account (1.5), we get

$$G(t)Vu = t^{-3}VG(t)u \quad (\forall u \in W_2^2(\mathbb{R}^3)). \tag{1.7}$$

Thus, the singular perturbation V defined by (1.6) possesses the homogeneity property (1.7). This well-known fact is a starting point for our considerations in Section 3. Precisely, we establish that elements $\psi \in W_2^{-2}(\mathbb{R}^3)$ that, similarly to the case of delta function, satisfy the relation $G(t)\psi = t^{-3/2}\psi$ form an infinite dimensional subspace X of $W_2^{-2}(\mathbb{R}^3)$. We fix an orthonormal basis $\{\psi_j\}_1^\infty$ of X and consider an infinite-dimensional singular perturbation

$$V = \sum_{i,j=1}^{\infty} \alpha_{ij}\langle \psi_j, \cdot\rangle \psi_i, \qquad \alpha_{ij} \in \mathbb{C} \tag{1.8}$$

which, obviously, satisfies (1.7), i.e., V has the property of homogeneity with respect to the scaling transformations $G(t)$.

Reasoning in a standard way [5], we determine a symmetric operator $-\Delta_{min} = -\Delta|_{D(\Delta_{min})}$ (see (3.11)) by the formal expression (1.3), where V has the form (1.8). After that, we construct a simple maximal symmetric operator B in $L_2(\mathbb{R}^3)$ such that $-\Delta_{min} = B^2$. The latter representation plays a key role because it enables one to consider any positive self-adjoint extension of $-\Delta_{min}$ as a Lax–Phillips perturbed operator and to apply the results of Section 2 to the description of specific spectral and scattering properties of positive self-adjoint realizations of the expression (1.3), which appear due to the homogeneity of a singular perturbation V with respect to the scaling transformations.

In conclusion, we note that the theory of singular perturbations of Schrödinger operators with additional properties of homogeneity (or, more generally, symmetry, if we use unitary transformations that differ from scaling one) attracted much attention in recent years (see [5] and references therein). However, as a rule, properties of symmetry of singular perturbations have been only used for restricting the nonuniqueness of the model self-adjoint extensions and selecting some of them.

In what follows, any Hilbert space is assumed to be separable. For a Hilbert space $\mathfrak{H}$, we denote by $\|\cdot\|_{\mathfrak{H}}$ and $(\cdot,\cdot)_{\mathfrak{H}}$ its norm and scalar product, respectively. We omit the index $\mathfrak{H}$ in the case where $\mathfrak{H} = L_2(\mathbb{R}^n)$. By $L|_X$ we denote the restriction of an operator L onto a set X. The symbol $W_2^p(\mathbb{R}^n)$ ($p \in \{-2, 2\}$) denotes the usual Sobolev space, i.e. $W_2^p(\mathbb{R}^n)$ is the space of tempered distributions with a Fourier transform which is square integrable with respect to the measure with density $(1+|x|^2)^{p/2}$.

2. Lax–Phillips scattering theory for abstract wave equation

Let L be a positive self-adjoint operator in an abstract Hilbert space $\mathfrak{H}$. By $\mathfrak{H}_L$ we denote the completion of its domain of definition $D(L)$ with respect to the norm $\|u\|^2_{\mathfrak{H}_L} := (Lu, u)_{\mathfrak{H}}$.

The Hilbert space $H_L = \mathfrak{H}_L \oplus \mathfrak{H}$ is called the *energy space*. It is convenient to write elements of H_L as column matrices $\begin{pmatrix} u \\ v \end{pmatrix}$, where $u \in \mathfrak{H}_L$ and $v \in \mathfrak{H}$.

Put $u_t = v$ and rewrite (1.1) as

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & I \\ -L & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}.$$

The operator

$$Q = \begin{pmatrix} 0 & I \\ -L & 0 \end{pmatrix}, \qquad D(Q) = \left\{\begin{pmatrix} u \\ v \end{pmatrix} \mid \{u, v\} \subset D(L)\right\}$$

is essentially skew self-adjoint in H_L. Its closure $Q_L = \overline{Q}$ is a generator of the group of unitary (in H_L) operators $W_L(t) = e^{Q_L t}$, which determines solutions of the Cauchy problem for the abstract wave equation (1.1).

The group of operators $W_L(t)$ can be investigated in the Lax–Phillips framework if there exist orthogonal subspaces D_+ and D_- of H_L such that

(i) $$W_L(t)D_+ \subset D_+; \qquad W_L(-t)D_- \subset D_- \quad (t \geq 0);$$

(ii) $$\bigcap_{t\geq 0} W_L(t)D_+ = \bigcap_{t\geq 0} W_L(-t)D_- = \{0\}.$$

Subspaces D_+ and D_- with properties (i)–(ii) are called *outgoing* and *incoming* subspaces for the group $W_L(t)$, respectively.

Obviously, the existence of outgoing and incoming subspaces for $W_L(t)$ is due to certain properties of the operator L in (1.1). For finding such properties, we note that, in the case of wave equation

$$u_{tt}(x,t) = \Delta u(x,t) \qquad (x \in \mathbb{R}^n, \quad n \text{ odd}), \tag{2.1}$$

the corresponding outgoing and incoming subspaces $D_\pm$ (described by Lax and Phillips in [22]) satisfy the additional condition

$$JD_- = D_+, \tag{iii}$$

where J is the operator of time reversion in $H_{-\Delta}$, i.e., $J\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ -v \end{pmatrix}$. Condition (iii) can be considered as a specific property of the wave phenomena.

Theorem 2.1 ([17], [21]). *Condition* **A** *is equivalent to the existence of orthogonal outgoing and incoming subspaces* $D_\pm$ *with additional property* (iii) *for the group* $W_L(t)$ *of solutions of the Cauchy problem for the abstract wave equation* (1.1). *Furthermore, the subspaces* D_+ *and* D_- *coincide, respectively, with the closures* (*in* H_L) *of the sets*[*2]

$$\left\{\begin{pmatrix} u \\ iBu \end{pmatrix}\right\}, \qquad \left\{\begin{pmatrix} u \\ -iBu \end{pmatrix}\right\} \qquad (\forall u \in D(B^2)). \tag{2.2}$$

Thus, condition **A** enables us to study the abstract equation (1.1) in the Lax–Phillips framework by analogy with the classical wave equation (2.1).

Definition 2.1. A positive self-adjoint operator L acting in $\mathfrak{H}$ and satisfying condition **A** is called a Lax–Phillips perturbed operator.

A Lax–Phillips perturbed operator L is called 0-perturbed if the subspace $\mathfrak{H}_0$ in condition **A** coincides with $\mathfrak{H}$.

For a given operator B, we denote by Υ_B and Υ_B^0 the sets of all Lax–Phillips perturbed and 0-perturbed operators L, respectively. Obviously, these sets depend on the choice of B and $\Upsilon_B^0 \subset \Upsilon_B$. Various examples of Lax–Phillips perturbed operators can be found in [17]–[19].

Let $L \in \Upsilon_B$. We recall that the group of operators $W_L(t)$ determines a *Lax–Phillips free evolution* if its outgoing and incoming subspaces $D_\pm$ satisfy the additional condition

$$D_- \oplus D_- = H_L. \tag{iv}$$

Theorem 2.2 ([19]). *The group* $W_L(t)$ *of solutions of the Cauchy problem for the abstract wave equation* (1.1), *where* $L \in \Upsilon_B$, *determines a Lax–Phillips free evolution if and only if the operator* L *is* 0*-perturbed* (*i.e.* $L \in \Upsilon_B^0$) *and*

$$(Lu, u)_{\mathfrak{H}} = \|B^* u\|^2_{\mathfrak{H}}, \quad \forall u \in D(L). \tag{2.3}$$

Any operator $L \in \Upsilon_B$ satisfying (2.3) is called a *Lax–Phillips nonperturbed operator.* Let $\mathfrak{M}_B$ denote the set of all Lax–Phillips nonperturbed operators for a given operator B. It is clear that $\mathfrak{M}_B \subset \Upsilon_B^0 \subset \Upsilon_B$ and the set $\mathfrak{M}_B$

[*2]Without loss of generality, we assume that the operator B in condition **A** has the zero defect number in the lower half-plane.

contains at least two elements: the Friedrichs extension B^*B and the Krein - von Neumann extension BB^* of the operator B^2. A complete description of $\mathfrak{M}_B$ can be found in [19].

Let $L_0 \in \mathfrak{M}_B$. The explicit formulas (2.2) for subspaces $D_\pm$ and the well-known fact [2] that any simple maximal symmetric operator B acting in $\mathfrak{H}_0$ admits the following canonical representation:

$$B = T^{-1} i \frac{d}{ds} T, \qquad D(B) = T^{-1}\{u(s) \in W_2^1(\mathbb{R}_+, N) \mid u(0) = 0\}, \tag{2.4}$$

($\mathbb{R}_+ = (0, \infty)$), where T isometrically maps $\mathfrak{H}_0$ onto $L_2(\mathbb{R}_+, N)$ and the dimension of an auxiliary Hilbert space N is equal to the nonzero defect number of B, allow us to obtain an explicit formula for a spectral representation of the nonperturbed group $W_{L_0}(t)$ associated with $D_\pm$. Using this spectral representation and results of [9], we can obtain a simple expression for the analytic continuation of the Lax–Phillips scattering matrix for any perturbed group $W_L(t)$ $(L \in \Upsilon_B)$ (for details, see [17]–[19]).

The results pointed out above were reformulated to the case of abstract Schrödinger equation (1.2) in [20]. Now, without going into details, we recall the principal points.

Let us fix a simple maximal symmetric operator B acting in a Hilbert space $\mathfrak{H}_0$. We will say that a unitary group e^{-iL_0t} determines a free evolution if the operator L_0 belongs to $\mathfrak{M}_B$. Similarly, the group e^{-iLt} determines a perturbed evolution if $L \in \Upsilon_B$.

Proposition 2.1 ([18]). *Let $L \in \Upsilon_B$ and $L_0 \in \mathfrak{M}_B$. Then the wave operators*

$$\Omega_\pm := s - \lim_{t \to \pm\infty} e^{iLt} e^{-iL_0t}$$

exist and $\Omega_\pm : \mathfrak{H}_0 \to \mathfrak{H}$. If $L \in \Upsilon_B^0$, then the operators $\Omega_\pm$ are complete and $\Omega_\pm \mathfrak{H}_0 = \mathfrak{H}_0$.

Proposition 2.1 yields that the scattering operator $S_{(L,L_0)} = \Omega_+^* \Omega_-$ is a well-defined bounded operator in $\mathfrak{H}_0$.

In what follows, for the sake of simplicity, we assume that $L_0 = B^*B$. The general case of any $L_0 \in \mathfrak{M}_B$ has been considered in [18].

It is easy to verify that the operator

$$(\mathcal{F}\gamma)(\delta) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin\delta s (T\gamma)(s) ds \qquad (\forall \gamma \in \mathfrak{H}_0, \quad \delta > 0),$$

where the operator $T : \mathfrak{H}_0 \to L_2(\mathbb{R}_+, N)$ is taken from the canonical realization (2.4) of B, isometrically maps $\mathfrak{H}_0$ onto $L_2(\mathbb{R}_+, N)$ and

$$(\mathcal{F}B^*B\gamma)(\delta) = \delta^2 (\mathcal{F}\gamma)(\delta), \qquad \forall \gamma \in D(B^*B).$$

Thus, the mapping $\mathcal{F}$ determines a spectral representation of B^*B in which the action of B^*B corresponds to the multiplication by the modified spectral parameter δ^2 in $L_2(\mathbb{R}_+, N)$.

The image of the scattering operator in the spectral representation $L_2(\mathbb{R}_+, N)$ has the form $\mathbb{S}_{(L,B^*B)} = \mathcal{F}S_{(L,B^*B)}\mathcal{F}^{-1}$ and it can be realized as the multiplication by an operator-valued function $\mathbb{S}_{(L,B^*B)}(\delta)$, the values of which are bounded operators in N for almost all $\delta \in \mathbb{R}_+$. We extend $\mathbb{S}_{(L,B^*B)}(\delta)$ by the formula

$$\mathbb{S}_{(L,B^*B)}(-\delta) := \mathbb{S}^*_{(L,B^*B)}(\delta) \qquad \delta > 0$$

onto the whole real axis. The obtained operator-valued function $\mathbb{S}_{(L,B^*B)}(\delta)$ is called the *S-matrix* of the abstract Schrödinger equation (1.2).

The next theorem follows from Theorems 4.1, 4.2 in [19] and relation (12) in [20].

Theorem 2.3. *If $L \in \Upsilon_B$, then the S-matrix $\mathbb{S}_{(L,B^*B)}(\delta)$ is the boundary value in the sense of strong convergence of a contraction-valued function $\mathbb{S}(k)$ analytic in the upper half-plane $\operatorname{Im} k > 0$ and such that $\mathbb{S}_{(L,B^*B)}(-\overline{k}) = \mathbb{S}^*_{(L,B^*B)}(k)$.*

Conversely, for a given operator B, an operator-valued function $\mathbb{S}(\delta)$, the values of which are bounded operators in a Hilbert space N for almost all $\delta \in \mathbb{R}$, is the S-matrix $\mathbb{S}_{(L,B^*B)}(\delta)$ for some choice of $L \in \Upsilon_B$ if and only if the following conditions are satisfied:

(a) *the dimension of N is equal to the nonzero defect number of the operator B*;

(b) *the function $\mathbb{S}(\delta)$ is the boundary value in the sense of strong convergence of a contraction-valued function $\mathbb{S}(k)$ analytic in the upper half-plane*;

(c) *the identity $\mathbb{S}(-\overline{k}) = \mathbb{S}^*(k)$ is true for all k with $\operatorname{Im} k > 0$.*

The case where $\mathbb{S}(\delta)$ is the S-matrix for some choice of 0-perturbed operator L ($\in \Upsilon_B^0$) is specified by the following additional condition:

(d) *the identity*

$$(\operatorname{Re} k)[I - \mathbb{S}^*(k)\mathbb{S}(k)] = i(\operatorname{Im} k)[\mathbb{S}^*(k) - \mathbb{S}(k)]$$

is true for all k with $\operatorname{Im} k > 0$.

Now we recall (due to [18]) a simple operator method for finding the analytic continuation $\mathbb{S}_{(L,B^*B)}(k)$ of the S-matrix $\mathbb{S}_{(L,B^*B)}(\delta)$.

Let $L \in \Upsilon_B$. It is easy to verify that

$$PD(L) \subset D(B^{2^*}), \qquad PLf = B^{2^*}Pf \qquad (\forall f \in D(L)), \tag{2.5}$$

where P is the orthogonal projection onto $\mathfrak{H}_0$ in $\mathfrak{H}$. Since B is maximal symmetric, we get $B^{2^*} = B^{*2}$. For any k ($\operatorname{Im} k > 0$), we put

$$L_k = B^{*2}|_{D(L_k)}, \qquad D(L_k) = P(L - k^2I)^{-1}\mathfrak{H}_0. \tag{2.6}$$

By (2.5), the operators L_k are well defined in $\mathfrak{H}_0$. The set $\{L_k \mid \operatorname{Im} k > 0\}$ is called the *image set* of the Lax–Phillips perturbed operator L.

Since the negative semiaxis belongs to the resolvent set of an arbitrary operator L_k from the image set (see [18, Proposition 3.2]), the operators

$$C_k = [(L_k + I)^{-1} - (B^*B + I)^{-1}]_{\mathcal{H}} \tag{2.7}$$

are well defined bounded operators on the whole space $\mathcal{H} = \ker(B^{*2} + I)$.

Note that the dimensionalities of $\mathcal{H}$ and the auxiliary space N in the spectral representation of B^*B defined above coincide. Hence, the spaces $\mathcal{H}$ and N are unitarily equivalent and can be identified. Thus, without loss of generality, we can assume that $N = \mathcal{H}$.

Theorem 2.4 ([18]). *Let $L \in \Upsilon_B$. Then the analytic continuation into the upper half-plane of the S-matrix $\mathbb{S}_{(L,B^*B)}(\delta)$ of the abstract Schrödinger equation* (1.2) *has the form $\mathbb{S}_{(L,B^*B)}(k) = I + 4ikC_k(I - 2(1+ik)C_k)^{-1}$, where C_k are determined by the image set of L in accordance with* (2.7).

Note that general properties of C_k established in [18] imply that the operator $(I - 2(1+ik)C_k)^{-1}$ is bounded and it is defined on $\mathcal{H}$ for all k ($\operatorname{Im} k > 0$).

3. Singular perturbations of Schrödinger operator homogeneous with respect to scaling transformations

3.1. Description of elements of $\mathbf{W_2^{-2}}(\mathbb{R}^3)$ with homogeneity property

To describe all elements $\psi \in W_2^{-2}(\mathbb{R}^3)$ that possess the homogeneity property, i.e., satisfy the relation

$$G(t)\psi = t^{-3/2}\psi \qquad (\forall t > 0), \tag{3.1}$$

where the action of scaling transformations $G(t)$ on $W_2^{-2}(\mathbb{R}^3)$ is defined by (1.5), we start from some well-known facts and auxiliary notation.

Since the Sobolev space $W_2^{-2}(\mathbb{R}^3)$ can be defined as the completion of $L_2(\mathbb{R}^3)$ with respect to the norm

$$\|f\|_{W_2^{-2}(\mathbb{R}^3)} := \|(-\Delta + I)^{-1} f\|, \qquad \forall f \in L_2(\mathbb{R}^3),$$

the operator $(-\Delta+I)^{-1}$ can be continuously extended to an isometric mapping of $W_2^{-2}(\mathbb{R}^3)$ onto $L_2(\mathbb{R}^3)$. For the extended operator, we preserve the same notation: $(-\Delta + I)^{-1}$. Thus, for any $\psi \in W_2^{-2}(\mathbb{R}^3)$, the function $g(x) = (-\Delta + I)^{-1}\psi$ belongs to $L_2(\mathbb{R}^3)$ and, hence, the relation

$$\langle \psi, u \rangle := ((-\Delta + I)u(x), g(x)) = \int_{\mathbb{R}^3} (-\Delta + I)u(x)\overline{g(x)}dx \tag{3.2}$$

enables one to consider an element $\psi \in W_2^{-2}(\mathbb{R}^3)$ as a linear continuous functional on $W_2^2(\mathbb{R}^3)$.

Let $n(w)$ belong to the Hilbert space $L_2(S^2)$ of functions square-integrable on the unit sphere S^2 in $\mathbb{R}^3$. Then the function

$$f(y) = \frac{n(w)}{|y|^2 + 1} \qquad \left(w = \frac{y}{|y|}\right)$$

belongs to $L_2(\mathbb{R}^3)$. Hence, its Fourier transformation $\hat{f} = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ix\cdot y} f(y)dy$ belongs to $L_2(\mathbb{R}^3)$ also. Substituting $\hat{f}$ into (3.2) instead of $g(x)$, we determine a certain element of $W_2^{-2}(\mathbb{R}^3)$, which we denote by the symbol $\hat{n}(w)$. Thus,

$$(3.3)\qquad \hat{n}(w)\in W_2^{-2}(\mathbb{R}^3), \qquad \left(\frac{n(w)}{|y|^2+1}\right)^{\wedge} = (-\Delta+I)^{-1}\hat{n}(w)$$

and the action of the functional $\hat{n}(w)$ on $W_2^2(\mathbb{R}^3)$ is defined by the relation:

$$(3.4)\qquad \langle \hat{n}(w), u(x)\rangle = ((-\Delta+I)u(x), \hat{f}(x)) = \int_{\mathbb{R}^3} (|y|^2+1)\hat{u}(y)\frac{\overline{n(w)}}{|y|^2+1}dy.$$

Proposition 3.1. *An element $\psi \in W_2^{-2}(\mathbb{R}^3)$ possesses the homogeneity property* (3.1) *if and only if $\psi = \hat{n}(w)$, where $n(w) \in L_2(S^2)$.*

Proof. Let $\psi \in W_2^{-2}(\mathbb{R}^3)$. It follows from relations (1.5), (3.2), and the well-known equality

$$(3.5)\qquad G(t)\Delta u(x) = t^{-2}\Delta G(t)u(x) \qquad (\forall u \in W_2^2(\mathbb{R}^3))$$

that $\langle G(t)\psi, u\rangle = ((-\Delta+I)u(x), t^{-2}(-\Delta+t^2 I)(-\Delta+I)^{-1}G(t)g(x))$, where $g(x) = (-\Delta+I)^{-1}\psi$.

If ψ satisfies (3.1), then the latter relation and (3.2) yield that

$$(3.6)\qquad (-\Delta+t^2 I)(-\Delta+I)^{-1}G(t)g(x) = t^{1/2}g(x).$$

Let the symbol $^{\vee}$ denote the inverse Fourier transform. Then

$$(3.7)\qquad (G(t)g)^{\vee}(y) = \left(\frac{t}{2\pi}\right)^{3/2}\int_{\mathbb{R}^3} e^{-iy\cdot x}g(tx)dx = G\left(\frac{1}{t}\right)\overset{\vee}{g}(y).$$

Using (3.7), we can write (3.6) in the form

$$\frac{|y|^2+t^2}{|y|^2+1}\overset{\vee}{g}\left(\frac{y}{t}\right) = t^2\overset{\vee}{g}(y), \quad \forall t > 0.$$

This relation holds for almost all $y \in \mathbb{R}^3$. In particular, if we set $t = |y|$, then $y = tw$ $(w = y/|y|)$ and, hence,

$$\overset{\vee}{g}(y) = \frac{n(w)}{|y|^2+1} \qquad (n(w) := 2\overset{\vee}{g}(w)).$$

Furthermore, $n(w)\in L_2(S^2)$ (because $\overset{\vee}{g}(y) \in L_2(\mathbb{R}^3)$). Comparing the latter equality and (3.3), we get that $\psi = \hat{n}(w)$, where the functional $\hat{n}(w)$ is defined by (3.4).

Conversely, if $\psi = \hat{n}(w)$, where $n(w)\in L_2(S^2)$, then equalities (1.5) and (3.4) imply that

$$\langle G(t)\hat{n}(w), u\rangle = \int_{\mathbb{R}^3}(|y|^2+1)[G(1/t)u]^{\wedge}(y)\frac{\overline{n(w)}}{|y|^2+1}dy.$$

By analogy with (3.7), it is easy to see that $[G\,(1/t)\,u]^{\wedge}\,(y) = G(t)\hat{u}(y) = t^{3/2}\hat{u}(ty)$. Taking this relation into account and making the change of variables $ty = s$ in the latter integral, we get $\langle G(t)\hat{n}(w), u\rangle = t^{-3/2}\langle\hat{n}(w), u\rangle$. Thus, $\hat{n}(w)$ satisfies relation (3.1). Proposition 3.1 is proved. □

Denote by X the set of all $\psi \in W_2^{-2}(\mathbb{R}^3)$ that satisfy (3.1). By virtue of Proposition 3.1,

$$X = \{\ \hat{n}(w) \mid \forall\ n(w) \in L_2(S^2)\}. \tag{3.8}$$

Relations (3.3) imply that

$$\|\hat{n}(w)\|^2_{W_2^{-2}(\mathbb{R}^3)} = \left\|\left(\frac{n(w)}{|y|^2+1}\right)^{\wedge}\right\|^2 = \left\|\frac{n(w)}{|y|^2+1}\right\|^2 = \frac{\pi}{4}\|n(w)\|^2_{L_2(S^2)}. \tag{3.9}$$

Therefore, X is an infinite-dimensional subspace of the Hilbert space $W_2^{-2}(\mathbb{R}^3)$.

3.2. The symmetric operator $\boldsymbol{\Delta_{\min}}$ and its properties

Let us choose an orthonormal basis $\{\psi_j\}_1^\infty$ of X and consider the formal expression

$$-\Delta + \sum_{i,j=1}^{\infty}\alpha_{ij}\langle\psi_j,\cdot\rangle\psi_i, \qquad \alpha_{ij} \in \mathbb{C}. \tag{3.10}$$

Since any ψ_j satisfies (3.1), repeating the arguments presented in Introduction, we establish that the singular perturbation $V = \sum_{i,j=1}^{\infty}\alpha_{ij}\langle\psi_j,\cdot\rangle\psi_i$ satisfies (1.7), i.e., V possesses the property of homogeneity with respect to the scaling transformations $G(t)$.

Since $X \cap L_2(\mathbb{R}^3) = \{0\}$ and $\dim X = \infty$, expression (3.10) determines not a self-adjoint operator in $L_2(\mathbb{R}^3)$, but a closed densely defined symmetric operator $-\Delta_{min} = -\Delta|_{D(\Delta_{min})}$,

$$D(\Delta_{min}) = \{u(x) \in W_2^2(\mathbb{R}^3) | \langle\hat{n}(w), u(x)\rangle = 0,\ \forall\ \hat{n}(w) \in X\} \tag{3.11}$$

with infinite defect numbers.

Note that the delta function $\delta(x)$ belongs to X (it suffices to choose $n(w) \equiv 1$). Thus, expression (3.10) is an "infinite-dimensional" generalization of the classical one-point interaction $-\Delta + \alpha\langle\delta,\cdot\rangle\delta$ and the operator $-\Delta_{min}$ is a restriction of the well-known symmetric operator

$$-\Delta_0 = -\Delta|_{D(\Delta_0)}, \qquad D(\Delta_0) = \{\ u(x) \in W_2^2(\mathbb{R}^3) \mid u(0) = 0\ \}.$$

The following statement plays a key role and enables one to apply results of the Lax–Phillips theory (Section 2) to the investigation of positive self-adjoint extensions of $-\Delta_{min}$.

Theorem 3.1. *There is a simple maximal symmetric operator B in $L_2(\mathbb{R}^3)$ such that $-\Delta_{min} = B^2$ and the operator $-\Delta$ coincides with the Friedrichs extension of B^2, i.e., $-\Delta = B^*B$.*

Proof. Let $\{Y_j(w)\}_1^\infty$ be an arbitrary orthonormal basis of real spherical harmonics of the space $L_2(S^2)$. Then any function $u(x) \in L_2(\mathbb{R}^3)$ can be written in the form

$$u(x) = \frac{1}{s}\sum_{j=1}^{\infty} u_j(s)Y_j(w), \qquad w = \frac{x}{|x|}, \qquad s = |x|. \tag{3.12}$$

Here, functions $u_j(s)$ belong to $L_2(\mathbb{R}_+)$ and

$$\|u(x)\|^2 = \sum_{j=1}^{\infty} \|u_j(s)\|^2_{L_2(\mathbb{R}_+)}, \qquad s = |x|. \tag{3.13}$$

Consider the mapping

$$(Tu)(s,w) = \sum_{j=1}^{\infty} i^{d_j}[F_{sin}\Gamma_{d_j+\frac{1}{2}}u_j](s)Y_j(w), \qquad s > 0, \tag{3.14}$$

where d_j is the order of the spherical harmonic $Y_j(w)$, $u_j(s)$ are the elements of the partial decomposition (3.12), and

$$(\Gamma_\nu f)(\delta) = \int_0^\infty \sqrt{s\delta}J_\nu(s\delta)f(s)ds, \ (F_{sin}f)(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sin s\delta f(\delta)d\delta,$$

are the Hankel and sin-Fourier transformations, respectively.

Since the transformations F_{sin} and Γ_ν are unitary operators in $L_2(\mathbb{R}_+)$, relations (3.13) and (3.14) imply that T maps isometrically $L_2(\mathbb{R}^3)$ onto the space $L_2(\mathbb{R}_+, L_2(S^2))$. But then, using (2.4), we can define a simple maximal symmetric operator

$$B = T^{-1}i\frac{d}{ds}T, \quad D(B) = T^{-1}\{\mathfrak{u}(s) \in W_2^1(\mathbb{R}_+, L_2(S^2)) \mid \mathfrak{u}(0) = 0\} \tag{3.15}$$

and its adjoint

$$B^* = T^{-1}i\frac{d}{ds}T, \quad D(B^*) = T^{-1}W_2^1(\mathbb{R}_+, L_2(S^2)) \tag{3.16}$$

acting in the space $L_2(\mathbb{R}^3)$.

It is well known that, for all $u(x) \in W_2^2(\mathbb{R}^3)$,

$$-\Delta u(x) = \frac{1}{s}\sum_{j=1}^{\infty} L_j u_j(s)Y_j(w) \qquad \left(s = |x|, \ w = \frac{x}{|x|}\right),$$

where

$$L_j f = -\frac{d^2}{ds^2}f(s) + \frac{d_j(d_j+1)}{s^2}f(s), \quad D(L_j) = \{f \in L_2(\mathbb{R}_+) \mid L_j f \in L_2(\mathbb{R}_+)\}$$

(if $d_j > 0$) and $D(L_j) = \{f(s) \in W_2^2(\mathbb{R}_+) \mid f(0) = 0\}$ (if $d_j = 0$) are self-adjoint operators in $L_2(\mathbb{R}_+)$.

Applying (3.14) and taking into account that $\Gamma_{d_j+\frac{1}{2}} L_j f = \delta^2 \Gamma_{d_j+\frac{1}{2}} f$ for all $f \in D(L_j)$, we obtain

$$-(T\Delta u)(s,w) = \sum_{j=1}^{\infty} i^{d_j}[F_{sin}\delta^2\Gamma_{d_j+\frac{1}{2}} u_j](s) Y_j(w) = \mathfrak{L}(Tu)(s,w), \tag{3.17}$$

where the operator

$$\mathfrak{L} = -\frac{d^2}{ds^2}, \qquad D(\mathfrak{L}) = \{\mathfrak{u}(s) \in W_2^2(\mathbb{R}_+, L_2(S^2)) \mid \mathfrak{u}(0) = 0\} \tag{3.18}$$

is self-adjoint in $L_2(\mathbb{R}_+, L_2(S^2))$.

Relations (3.15), (3.16) and (3.17), (3.18) imply that $-\Delta = B^*B$.

To prove the equality $-\Delta_{min} = B^2$, we note that the domain of definition $D(\Delta_{min})$ is not changed if instead of all functions $n(w) \in L_2(S^2)$ in (3.11) we consider only elements of the basis $\{Y_j(w)\}_{j=1}^{\infty}$.

In accordance with (3.2) (for $\psi = \hat{Y}_j(w)$) and (3.17), the following equality is true for all $u \in W_2^2(\mathbb{R}^3)$:

$$\langle \hat{Y}_j(w), u(x)\rangle = ((-\Delta+I)u(x), g_j(x)) = ((\mathfrak{L}+I)Tu, Tg_j)_{L_2(\mathbb{R}_+, L_2(S^2))}, \tag{3.19}$$

where $g_j(x) = (-\Delta+I)^{-1}\hat{Y}_j(w)$. Furthermore, it follows from (3.3) (for $n(w) = Y_j(w)$) and the well-known properties of the Fourier transform (see [28]) that

$$g_j(x) = \left(\frac{Y_j(w)}{|y|^2+1}\right)^{\wedge} = \frac{i^{d_j}}{|x|}\left[\Gamma_{d_j+\frac{1}{2}} \frac{|y|}{|y|^2+1}\right](|x|) Y_j(w). \tag{3.20}$$

By virtue of the evident relation

$$F_{sin} \frac{|y|}{|y|^2+1} = \sqrt{\frac{\pi}{2}} e^{-s}$$

and the fact that $\Gamma_{d_j+1/2}$ is a unitary and self-adjoint operator in $L_2(\mathbb{R}_+)$, equalities (3.14) and (3.20) imply that

$$(Tg_j)(s,w) = (-1)^{d_j}\sqrt{\frac{\pi}{2}} e^{-s} Y_j(w). \tag{3.21}$$

Substituting (3.21) into (3.19), we get

$$\langle \hat{Y}_j(w), u(x)\rangle = (-1)^{d_j}\sqrt{\frac{\pi}{2}}\int_0^{\infty}(-\mathfrak{u}_j''(s) + \mathfrak{u}_j(s))e^{-s} ds = (-1)^{d_j}\sqrt{\frac{\pi}{2}}\mathfrak{u}_j'(0), \tag{3.22}$$

where $\mathfrak{u}_j(s) = i^{d_j}[F_{sin}\Gamma_{d_j+1/2} u_j](s)$ is an element of the partial decomposition (3.14) of the function $\mathfrak{u} = (Tu)(s,w) \in D(\mathfrak{L})$.

By virtue of (3.11) and (3.22), a function $u(x)$ belongs to $D(-\Delta_{min})$ if and only if all elements $\mathfrak{u}_j(s)$ of decomposition (3.14) satisfy the condition $\mathfrak{u}_j'(0) = 0$. Thus, taking into account (3.17) and (3.18), we get

$$D(-\Delta_{min}) = T^{-1}\{\mathfrak{u}(s) \in D(\mathfrak{L}) \mid \mathfrak{u}'(0) = 0\}.$$

But then, in view of (3.15), we obtain $-\Delta_{min} = B^2$. Theorem 3.1 is proved. □

3.3. Description of Lax–Phillips 0-perturbed operators corresponding to the formal expression (3.10)

By Theorem 3.1 and Definition 2.1, the set of positive self-adjoint extensions of $-\Delta_{min}$ in $L_2(\mathbb{R}^3)$ coincides with the set Υ_B^0 of Lax-Phillips 0-perturbed operators, where B is defined by (3.15). Thus, any element of Υ_B^0 can be regarded as a self-adjoint operator realization of (3.10) in the space $L_2(\mathbb{R}^3)$.

Let us describe elements of Υ_B^0. It follows from the general statements of the Birman-Krein-Vishik extension theory ([7], [14]) that the domain of definition $D(L)$ of any self-adjoint extension L of B^2 such that $-1 \in \rho(L)$ coincides with the set

$$D(L) = \{\, f(x) = u(x) + CP_{\mathcal{H}}(-\Delta + I)u(x) \mid \forall u(x) \in W_2^2(\mathbb{R}^3)\}, \tag{3.23}$$

where C is a bounded self-adjoint operator in $\mathcal{H} = \ker(B^{*2} + I)$ and $P_{\mathcal{H}}$ is the orthogonal projection onto $\mathcal{H}$ in $L_2(\mathbb{R}^3)$. Furthermore,

$$Lf(x) = -\Delta u(x) - CP_{\mathcal{H}}(-\Delta + I)u(x), \qquad \forall f \in D(L). \tag{3.24}$$

Conversely, for any self-adjoint bounded operator C, formulas (3.23) and (3.24) determine a self-adjoint operator L in $L_2(\mathbb{R}^3)$ such that $L \supset B^2$ and $-1 \in \rho(L)$.

In what follows, to underline the connection between L and C, we will use the notation L_C for an operator L defined by (3.23) and (3.24).

Lemma 3.1. *An operator L_C belongs to Υ_B^0 if and only if the corresponding self-adjoint operator parameter C in* (3.23), (3.24) *satisfies the condition*

$$0 \le C \le \frac{1}{2}I. \tag{3.25}$$

Proof. Using the well-known result [16] on extremal properties of the Friedrichs B^*B and Krein–von Neumann BB^* extensions of B^2, we arrive at the conclusion that a self-adjoint extension L_C of B^2 is nonnegative if and only if

$$(B^*B + I)^{-1} \le (L_C + I)^{-1} \le (BB^* + I)^{-1}.$$

By Theorem 3.1, $-\Delta = B^*B$. Furthermore, it is easy to see that the operator C in (3.23), (3.24) can be written in the form

$$C = (L_C + I)^{-1} - (-\Delta + I)^{-1}. \tag{3.26}$$

Therefore, L_C is nonnegative if and only if

$$0 \leq C \leq C_N = (BB^* + I)^{-1} - (B^*B + I)^{-1}.$$

It follows from [17, Lemma 3.5 in Chapter 4] that[*3] $C_N = 1/2I$. Thus, condition (3.25) describes the set of all self-adjoint nonnegative extensions L_C of B^2.

Since B is a simple maximal symmetric operator, $\ker B^* = \{0\}$ (see [17]) and, hence, any nonnegative extension L_C of B^2 is positive. Lemma 3.1 is proved. $\square$

The definition of L_C by (3.23) and (3.24) requires an additional efforts for finding $CP_{\mathcal{H}}(-\Delta + I)u(x)$. To avoid cumbersome calculations we use the following procedure:

Let $\{Y_j(w)\}_1^\infty$ be an arbitrary orthonormal basis of real spherical harmonics of $L_2(S^2)$. Then, by virtue of (3.8) and (3.9), the elements[*4]

$$\psi_j = (-1)^{d_j}\frac{2}{\sqrt{\pi}}\hat{Y}_j(w), \qquad 1 \leq j \leq \infty \tag{3.27}$$

form an orthonormal basis of X. But then, relations (3.2) and (3.11) imply that the vectors $\gamma_j(x) = (-\Delta + I)^{-1}\psi_j$ form an orthonormal basis $\{\gamma_j(x)\}_1^\infty$ of $\mathcal{H} = \ker(\Delta^*_{min} + I) = \ker(B^{*2} + I)$. Furthermore, it follows from (3.20) and (3.21) that

$$\gamma_j(x) = (-1)^{d_j}\frac{2}{\sqrt{\pi}}g_j(x) = \sqrt{2}T^{-1}[e^{-s}Y_j(w)]. \tag{3.28}$$

Using (3.2), (3.19) and (3.28), we get

$$\langle\psi_j, u\rangle = ((-\Delta + I)u, \gamma_j) = (-1)^{d_j}\frac{2}{\sqrt{\pi}}\langle\hat{Y}_j(w), u(x)\rangle \tag{3.29}$$

for any $u(x) \in W_2^2(\mathbb{R}^3)$. Hence, $P_{\mathcal{H}}(-\Delta + I)u(x) = \sum_{j=1}^\infty \langle\psi_j, u\rangle\gamma_j(x)$ and

$$CP_{\mathcal{H}}(-\Delta + I)u(x) = \sum_{i,j=1}^\infty c_{ij}\langle\psi_j, u\rangle\gamma_i(x),$$

where $\langle\psi_j, u\rangle$ are defined by (3.29) and the coefficients c_{ij} are determined by the expansion

$$C\gamma_j(x) = \sum_{i=1}^\infty c_{ij}\gamma_i(x), \qquad 1 \leq j \leq \infty.$$

(The infinite-dimensional matrix $(c_{ij})_{i,j=1}^\infty$ is called *the matrix decomposition* of the operator C with respect to the basis $\{\gamma_j(x)\}_1^\infty$.)

Substituting the obtained expression of $CP_{\mathcal{H}}(-\Delta + I)u(x)$ into (3.23) and (3.24) and using Lemma 3.1, we obtain the following description of Υ_B^0:

[*3] It is easy to verify this fact directly, using formulas (3.15) and (3.16)

[*4] The factors $(-1)^{d_j}$, where d_j is the order of the spherical harmonic $Y_j(w)$, are not essential and they are used to simplify the formulas.

Theorem 3.2. *An operator L_C belongs to Υ_B^0, where B is defined by* (3.15) *if and only if*

$$D(L_C) = \left\{ f(x) = u(x) + \sum_{i,j=1}^{\infty} c_{ij}\langle\psi_j, u\rangle\gamma_i(x) \;\middle|\; \forall u(x) \in W_2^2(\mathbb{R}^3) \right\}$$

and

$$L_C f(x) = -\Delta u(x) - \sum_{i,j=1}^{\infty} c_{ij}\langle\psi_j, u\rangle\gamma_i(x), \qquad \forall f(x) \in D(L_C),$$

where $(c_{ij})_{i,j=1}^{\infty}$ is the matrix decomposition of C with respect to the basis $\{\gamma_j(x)\}_1^{\infty}$ and the self-adjoint operator C satisfies (3.25).

In conclusion of Subsection 3.3, we outline some remarks concerning relationship between parameters c_{ij} in Theorem 3.2 and coefficients α_{ij} of the singular perturbation $V = \sum_{i,j=1}^{\infty}\alpha_{ij}\langle\psi_j,\cdot\rangle\psi_i$ in (3.10), where, without loss of generality, we suppose that the singular elements ψ_j are defined by (3.27). Furthermore, we assume that the coefficient matrix $(\alpha_{ij})_{i,j=1}^{\infty}$ is the matrix decomposition (with respect to the basis $\{\gamma_j(x)\}_1^{\infty}$) of a self-adjoint operator A bounded in $\mathcal{H}$.

To define an operator realization of (3.10) in $L_2(\mathbb{R}^3)$ we use an approach suggested initially by Albeverio and Kurasov [3], [4] for the case of finite rank singular perturbations and its generalization to the infinite dimensional case [8]. The main idea consists in the construction of some regularization

$$-\Delta_V := -\Delta^c + \sum_{i,j=1}^{\infty}\alpha_{ij}\langle\psi_j^{ext},\cdot\rangle\psi_i \tag{3.30}$$

of $-\Delta + V$, where $-\Delta^c : L_2(\mathbb{R}^3) \to W_2^{-2}(\mathbb{R}^3)$ is the continuation of $-\Delta$ onto $L_2(\mathbb{R}^3)$ and ψ_j^{ext} are the extensions of functionals ψ_j onto $D(-\Delta_{min}^*)$, which can be defined in different ways. The operator $-\Delta_V$ maps $D(-\Delta_{min}^*)$ to $W_2^{-2}(\mathbb{R}^3)$ and the corresponding operator realization L_A of (3.10) has the form

$$L_A = -\Delta_V|_{D(L_A)}, \qquad D(L_A) = \{f \in D(-\Delta_{min}^*) \mid -\Delta_V f \in L_2(\mathbb{R}^3)\}. \tag{3.31}$$

The principal point here is the construction of the extended functionals ψ_j^{ext}. These functionals are uniquely determined by a self-adjoint bounded operator R acting in $\mathcal{H}$ (or, that is equivalent, by the matrix decomposition $(r_{ij})_{i,j=1}^{\infty}$ of R with respect to the basis $\{\gamma_j(x)\}_1^{\infty}$, where, by definition, $r_{ij} = \langle\psi_i^{ext},\gamma_j\rangle$). In applications, the most appropriate choice of R has to be determined by imposing additional requirements related to physical nature of the singular perturbation V.

Denote

$$\Gamma_0 f = P_1 f, \qquad \Gamma_1 f = P_{\mathcal{H}}(-\Delta + I)(I - P_1)f, \qquad f \in D(-\Delta_{min}^*), \tag{3.32}$$

where P_1 is the projector onto $\mathcal{H}$ with respect to the decomposition $D(-\Delta^*_{min}) = W_2^2(\mathbb{R}^3)\dot{+}\mathcal{H}$ and $P_{\mathcal{H}}$ is the orthoprojector onto $\mathcal{H}$ in $L_2(\mathbb{R}^3)$.

Using (3.32), it is easy to verify that an operator $L_C \in \Upsilon^0_B$ defined by (3.23) and (3.24) coincides with restriction of $-\Delta^*_{min}$ onto

$$D(L_C) = \{f \in D(-\Delta^*_{min}) \mid C\Gamma_1 f = \Gamma_0 f\}. \tag{3.33}$$

On the other hand, repeating the reasoning of [3], [6], we can establish that the operator realization L_A of $-\Delta + V$ (see (3.31)) coincides with restriction of $-\Delta^*_{min}$ onto

$$D(L_A) = \{f \in D(-\Delta^*_{min}) \mid A(\Gamma_1 + R\Gamma_0)f = -\Gamma_0 f\}. \tag{3.34}$$

Let us assume that $\|R\| < 2$. In this case, by virtue of Lemma 3.1, we can state that $(I + RC)^{-1}$ is a bounded operator defined on $\mathcal{H}$ for any choice of $L_C \in \Upsilon^0_B$. Using this fact and comparing equalities (3.33) and (3.34), we arrive at the conclusion that an element $L_C \in \Upsilon^0_B$ can be regarded as an operator realization L_A of (3.10), where $A = -C(I + RC)^{-1}$. Going over to the matrix decompositions with respect to $\{\gamma_j(x)\}_1^\infty$ in the latter equality, we get the relationship between parameters c_{ij} in Theorem 3.2 and coefficients α_{ij} in the formal expression (3.10).

3.4. Spectral and scattering properties

It is natural to expect that self-adjoint realizations $L_C \in \Upsilon^0_B$ of the formal expression (3.10) possess specific spectral and scattering properties, which appear due to the homogeneity of a singular perturbation with respect to the scaling transformations.

Theorem 3.3. *For any $L_C \in \Upsilon^0_B$, the following statements are valid*:

(i) *the spectrum of L_C coincides with* $[0,\infty)$ *and it is purely absolutely continuous with the same multiplicity at each point of* $[0,\infty)$;

(ii) *the wave operators $\Omega_\pm = \lim_{t\to\pm\infty} e^{iL_Ct}e^{i\Delta t}$ exist and are unitary operators in $L_2(\mathbb{R}^3)$*;

(iii) *the S-matrix $\mathbb{S}_{(L_C,-\Delta)}(\delta)$ of the Schrödinger equation $iu_t = L_C u$ admits the following analytic contraction-valued continuation* (*in the sense of strong convergence*) *in the upper half-plane*:

$$\mathbb{S}_{(L_C,-\Delta)}(k) = I + 4ikC(I - 2(1 + ik)C)^{-1} \qquad (\mathrm{Im}\ k > 0). \tag{3.35}$$

Proof. Statement (ii) follows from Proposition 2.1. Statement (i) is a direct consequence of (ii). To prove statement (iii), we note that the relation $L_C \in \Upsilon^0_B$ imply that operators L_k from the image set of L_C (see (2.6)) coincide with L_C. Taking (2.7) (for $B^*B = -\Delta$) and (3.26) into account, we obtain $C_k = C$ for any k $(\mathrm{Im}\, k > 0)$. Thus, statement (iii) is a particular case of Theorem 2.4. Theorem 3.3 is proved. □

3.5. Inverse problem

Theorems 2.3 and 3.3 enable one to get a simple algorithm for recovering an operator $L_C \in \Upsilon_B^0$ in terms of the analytic continuation of an S-matrix (see for details [18, Corollaries 3.1, 3.2]). For the convenience of readers we recall the principal points.

Let an operator-valued function $\mathbb{S}(\delta)$, the values of which are bounded operators in a Hilbert space N be given. First, we should use Theorem 2.3 in order to check whether the function $\mathbb{S}(\delta)$ is an S-matrix of Eq. (1.2) under a certain choice of L_C.

If conditions $(a)-(d)$ of Theorem 2.3 are satisfied, then $\mathbb{S}(\delta)$ is the boundary value in the sense of strong convergence of an operator-valued function $\mathbb{S}(k)$ analytic in the upper half-plane, the values of which are contractions in N. Furthermore, in view of condition (a) and relations (3.15), (3.16), $\dim N = \dim \mathcal{H} = \infty$. Thus, we can identify the spaces N and $\mathcal{H}$. Note that such identification can be realized by different ways and its choice has to be determined by additional specific requirements of the problem (we illustrate this fact below).

By Theorem 3.3, the analytic continuation $\mathbb{S}(k)$ of an S-matrix, admits the representation (3.35). Using (3.35), we determine the bounded (in $\mathcal{H}$) operator

$$C = \frac{1}{2}(I - \mathbb{S}(k))[I - \mathbb{S}(k) - ik(I + \mathbb{S}(k))]^{-1}. \tag{3.36}$$

Note that $0 \in \rho(I - \mathbb{S}(k) - ik(I + \mathbb{S}(k)))$, because

$$I - \mathbb{S}(k) - ik(I + \mathbb{S}(k)) = -(ik+1)(\mathbb{S}(k) - \theta I), \quad \theta = \frac{1-ik}{1+ik},$$

where $\|\mathbb{S}(k)\| \le 1$ and $|\theta| > 1$. Thus, the definition of C by (3.36) is well posed. Substituting C in (3.23) and (3.24), we determine a self-adjoint operator $L_C \in \Upsilon_B^0$ such that the function $\mathbb{S}(\delta)$ coincides with the S-matrix $\mathbb{S}_{(L_C,-\Delta)}(\delta)$.

Example 3.1. One-dimensional interaction. The function

$$\mathbb{S}(\delta) = \frac{a+i\delta}{a-i\delta} \qquad (\delta \in \mathbb{R}, \quad a > 0)$$

can be considered as an operator-valued function with values in the Hilbert space $N = \mathbb{C}$. Obviously, under the choice of B in the form (3.15), the function $\mathbb{S}(\delta)$ satisfies only conditions (b)–(d) of Theorem 2.3. To satisfy condition (a), we identify N with one-dimensional subspace $\langle \gamma_k(x) \rangle$ of $\mathcal{H}$ generated by an element $\gamma_k(x)$ of the basis $\{\gamma_j(x)\}_1^\infty$ of $\mathcal{H}$ (see (3.28)) and consider the following modified variant of $\mathbb{S}(\delta)$:

$$\mathbb{S}^{\mathsf{m}}(\delta)\gamma_k := \frac{a+i\delta}{a-i\delta}\gamma_k, \quad \mathbb{S}^{\mathsf{m}}(\delta)\gamma := \gamma, \quad \forall \gamma \in \mathcal{H}' = \mathcal{H} \ominus \langle \gamma_k \rangle.$$

For $\mathbb{S}^{\mathsf{m}}(\delta)$ all conditions of Theorem 2.3 are true and, hence, $\mathbb{S}^{\mathsf{m}}(\delta)$ coincides with the S-matrix $\mathbb{S}_{(L_C,-\Delta)}(\delta)$ under some choice of $L_C \in \Upsilon_B^0$.

In accordance with (3.36),

$$C\gamma_k = \frac{1}{2(a+1)}\gamma_k \qquad \text{and} \qquad C\gamma = 0 \qquad \forall \gamma \in \mathcal{H}'.$$

The entries of matrix decomposition $(c_{ij})_{i,j=1}^{\infty}$ of C with respect to $\{\gamma_j(x)\}_1^{\infty}$ have only one non-zero element $c_{kk} = 1/2(a+1)$. Hence, by virtue of Theorem 3.2 and relation (3.29), the required operator L_C is defined by the relations

$$L_C f(x) = -\Delta u(x) - \frac{(-1)^{d_k}}{\sqrt{\pi}} \frac{\langle \hat{Y}_k(w), u(x)\rangle}{(a+1)}\gamma_k(x),$$

$$f(x) = u(x) + \frac{(-1)^{d_k}}{\sqrt{\pi}} \frac{\langle \hat{Y}_k(w), u(x)\rangle}{(a+1)}\gamma_k(x) \qquad (\forall u(x) \in W_2^2(\mathbb{R}^3)), \tag{3.37}$$

where, using (3.20) and (3.28), we can clarify the form of $\gamma_k(x)$:

$$\gamma_k(x) = \frac{2}{i^{d_k}\sqrt{\pi}} \frac{1}{|x|} \left[\Gamma_{d_k+\frac{1}{2}} \frac{|y|}{|y|^2+1}\right](|x|)Y_k(w). \tag{3.38}$$

Choosing different $Y_k(w)$ in (3.38) (or, that is equivalent, identifying N to different one-dimensional subspaces of $\mathcal{H}$), we obtain solutions L_C of the inverse problem describing interactions in different partial waves. In order to obtain the classical δ-interaction in s-wave, we have to choose $Y_k(w)$ in the form of normalized spherical harmonic of zero order $Y_k(w) = 1/2\sqrt{\pi}$. Then $\hat{Y}_k = \pi\sqrt{2}\delta(x)$, $\gamma_k(x) = \frac{1}{\sqrt{2\pi}}\frac{e^{-|x|}}{|x|}$ and formula (3.37) reduces to the well-known relation

$$L_C f(x) = -\Delta u(x) - \frac{u(0)}{a+1}\frac{e^{-|x|}}{|x|} \qquad (\forall u(x) \in W_2^2(\mathbb{R}^3)).$$

3.6. Preservation of the initial homogeneity

By virtue of (1.7) and (3.5), the nonperturbed operator $-\Delta$ and its singular perturbation V in (3.10) are homogeneous with respect to scaling transformations $G(t)$ with *different coefficients* of homogeneity: t^{-2} and t^{-3}, respectively. Thus, any operator $L_C \in \Upsilon_B^0$ that satisfies the equality

$$G(t)L_C f(x) = t^{-2} L_C G(t) f(x) \qquad (t > 0, \quad \forall f(x) \in D(L_C)) \tag{3.39}$$

and, hence, preserves the initial homogeneity of $-\Delta$, can be interpreted as a self-adjoint operator realization of (3.10) that is "transparent" with respect to singular perturbations considered in (3.10). It is natural to assume that such operators are close to $-\Delta$ in a certain sense. Indeed, as follows from Theorem 3.4 presented below, any "transparent" operator L_C does not differ from $-\Delta$ from the point of view of scattering theory (i.e., the corresponding S-matrix $\mathbb{S}_{(L_C,-\Delta)}(\delta)$ has no singularities).

Theorem 3.4. *Let $L_C \in \Upsilon_B^0$, where B is defined by* (3.15). *Then the following statements are equivalent*:

(i) *the operator L_C satisfies equality* (3.39);

(ii) L_C *belongs to the set* $\mathfrak{M}_B$ *of Lax–Phillips nonperturbed operators*;

(iii) *the S-matrix $\mathbb{S}_{(L_C,-\Delta)}(\delta)$ of the Schrödinger equation $iu_t = L_C u$ coincides with a unitary operator constant U (i.e., $\mathbb{S}_{(L_C,-\Delta)}(\delta) = U$ for all $\delta \in \mathbb{R}$, where U is a unitary operator in $\mathcal{H}$).*

Proof. By virtue of (3.15), $B^2 = T^{-1}\mathfrak{L}_0 T$, where

$$\mathfrak{L}_0 = -\frac{d^2}{ds^2}, \qquad D(\mathfrak{L}_0) = \{\mathfrak{u}(s) \in W_2^2(\mathbb{R}_+, L_2(S^2)) \mid \mathfrak{u}(0) = \mathfrak{u}'(0) = 0\}$$

is a symmetric operator in $L_2(\mathbb{R}_+, L_2(S^2))$. Hence, relations

$$\mathfrak{L}_{\mathfrak{C}} = TL_C T^{-1} \quad \text{and} \quad \mathfrak{C} = TCT^{-1} \tag{3.40}$$

establish a one-to-one correspondence between self-adjoint extensions of B^2 and $\mathfrak{L}_0$, which act in the spaces $L_2(\mathbb{R}^3)$ and $L_2(\mathbb{R}_+, L_2(S^2))$, respectively.

By virtue of definitions (3.14) and (1.4) of T and $G(t)$, respectively, it is easy to verify that the operators $\mathfrak{G}(t) := TG(t)T^{-1}$ are defined by the formula

$$\mathfrak{G}(t)\mathfrak{f}(s) = t^{1/2}\mathfrak{f}(ts), \qquad t > 0, \quad \forall \mathfrak{f}(s) \in L_2(\mathbb{R}_+, L_2(S^2)) \tag{3.41}$$

and, hence, these operators can be considered as scaling transformations in $L_2(\mathbb{R}_+, L_2(S^2))$. Furthermore, in view of (3.40), equality (3.39) is equivalent to the relation

$$\mathfrak{G}(t)\mathfrak{L}_{\mathfrak{C}}\mathfrak{f}(s) = t^{-2}\mathfrak{L}_{\mathfrak{C}}\mathfrak{G}(t)\mathfrak{f}(s), \qquad t > 0, \quad \forall \mathfrak{f}(s) \in D(\mathfrak{L}_{\mathfrak{C}}). \tag{3.42}$$

Thus, we reduce the problem of determination of $L_C \in \Upsilon_B^0$ in (3.39) to the similar problem for positive self-adjoint extensions $\mathfrak{L}_{\mathfrak{C}}$ of $\mathfrak{L}_0$ in (3.42).

Taking (3.41) into account, it is easy to verify that the operator

$$\mathfrak{L}_0^* = -\frac{d^2}{ds^2}, \qquad D(\mathfrak{L}_0^*) = W_2^2(\mathbb{R}_+, L_2(S^2))$$

satisfies the relation $\mathfrak{G}(t)\mathfrak{L}_0^* = t^{-2}\mathfrak{L}_0^*\mathfrak{G}(t)$. Since any $\mathfrak{L}_{\mathfrak{C}}$ is the restriction of $\mathfrak{L}_0^*$ to the domain of definition $D(\mathfrak{L}_{\mathfrak{C}})$, we arrive at the conclusion that $\mathfrak{L}_{\mathfrak{C}}$ satisfies (3.42) if and only if

$$\mathfrak{G}(t) : D(\mathfrak{L}_{\mathfrak{C}}) \to D(\mathfrak{L}_{\mathfrak{C}}), \qquad t > 0. \tag{3.43}$$

It follows from (3.17), (3.23), (3.24), and (3.40) that

$$D(\mathfrak{L}_{\mathfrak{C}}) = \{\mathfrak{f}(s) = \mathfrak{u}(s) + \mathfrak{C}P_{T\mathcal{H}}(\mathfrak{L} + I)\mathfrak{u}(s) \mid \forall \mathfrak{u}(s) \in D(\mathfrak{L})\}, \tag{3.44}$$

where $\mathfrak{L}$ is defined by (3.18), $P_{T\mathcal{H}}$ is the orthogonal projection onto the subspace $T\mathcal{H}$ of $L_2(\mathbb{R}_+, L_2(S^2))$.

By virtue of (3.28),

$$T\mathcal{H} = T\ker(B^{*2} + I) = \{e^{-s}n(w) \mid s \geq 0, \ \forall n(w) \in L_2(S^2)\}.$$

This simple presentation of $T\mathcal{H}$ enables one to calculate directly the expression $P_{T\mathcal{H}}(\mathfrak{L}+I)\mathfrak{u}(s)$ in (3.44) and, as a result, we get[*5]

$$D(\mathfrak{L}_{\mathfrak{C}}) = \{\mathfrak{f}(s) = \mathfrak{u}(s) + 2e^{-s}[\mathfrak{C}\mathfrak{u}'(0)] \mid \forall \mathfrak{u}(s) \in D(\mathfrak{L})\}. \tag{3.45}$$

By (3.41) and (3.45), the action of $\mathfrak{G}(t)$ on any $\mathfrak{f}(s)\in D(\mathfrak{L}_{\mathfrak{C}})$ has the form

$$\mathfrak{G}(t)\mathfrak{f}(s) = \mathfrak{w}(s) + 2t^{1/2}e^{-s}[\mathfrak{C}\mathfrak{u}'(0)], \tag{3.46}$$

where $\mathfrak{w}(s) = t^{1/2}\mathfrak{u}(ts) + 2t^{1/2}e^{-ts}[\mathfrak{C}\mathfrak{u}'(0)] - 2t^{1/2}e^{-s}[\mathfrak{C}\mathfrak{u}'(0)]$ belongs to $D(\mathfrak{L})$.

On the other hand, (3.45) and (3.46) imply that the inclusion $\mathfrak{G}(t)\mathfrak{f}(s) \in D(\mathfrak{L}_{\mathfrak{C}})$ is equivalent to the following representation of $\mathfrak{G}(t)\mathfrak{f}(s)$:

$$\mathfrak{G}(t)\mathfrak{f}(s) = \mathfrak{w}(s) + 2e^{-s}[\mathfrak{C}\mathfrak{w}'(0)]. \tag{3.47}$$

Since $\mathfrak{w}'(0) = t^{3/2}\mathfrak{u}'(0) - 2t^{3/2}[\mathfrak{C}\mathfrak{u}'(0)] + 2t^{1/2}[\mathfrak{C}\mathfrak{u}'(0)]$ we get, equating (3.46) and (3.47), that the representation (3.47) is possible if and only if the operator $\mathfrak{C}$ satisfies the condition

$$t(\mathfrak{C} - 2\mathfrak{C}^2) = (\mathfrak{C} - 2\mathfrak{C}^2), \qquad \forall t > 0.$$

Thus, $\mathfrak{L}_{\mathfrak{C}}$ satisfies (3.42) if and only if $\mathfrak{C}(I - 2\mathfrak{C}) = 0$. Using (3.40) and taking Lemma 3.1 into account, we arrive at the dual statement on the equivalence of the equality $C(I - 2C) = 0$ and statement (i) of Theorem 3.4.

Since C is self-adjoint in $\mathcal{H}$, the latter equality is equivalent to the decomposition $\mathcal{H} = \ker C \oplus \ker(C - 1/2I)$, which, by Lemma 3.5 in [17, Chapter 4], is equivalent to statement (ii). The equivalence of (ii) and (iii) follows from [18]. Theorem 3.4 is proved. □

Acknowledgements. The authors thank L. Nizhnik for valuable discussions and the referee for remarks which led to some improvements in the paper. The first author (S.K.) expresses their gratitude to CRDF, Grant No. UM1-2567-OD-03 and DFFD of Ukraine, project No. 01.07/027 for the support.

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[*5] Without loss of generality, we assume that the operator $\mathfrak{C}$ acts in the space $L_2(S^2)$.

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