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Bour's theorem in Minkowski 3-space

By

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Abstract

In this study, we show that a generalized helicoid with null axis is isometric to a rotation surface with null axis so that helices on the helicoid correspond to parallel circles on the rotation surface in three dimensional Minkowski space. Moreover, we obtained that these surfaces are minimal. An addition, if these surfaces have the same Gauss map, we can determine them.

1. Introduction

In classical surface geometry in Euclidean space, it is well known that the right helicoid (resp. catenoid) is the only ruled (resp. rotation) surface which is minimal. Moreover, a pair of these two surfaces has interesting properties. That is, they are both members of a one-parameter family of isometric minimal surfaces and have the same Gauss map. This pair is a typical example for minimal surfaces. On the other hand, the pair of the right helicoid and the catenoid has following generalization.

Bour's Theorem. *A generalized helicoid is isometric to a rotation surface so that helices on the helicoid correspond to parallel circles on the rotation surface* [1], [6].

In this generalization, original properties that they are minimal and preserve the Gauss map are not generally kept.

In [4], T. Ikawa showed that a generalized helicoid and a rotation surface have isometric relation by Bour's theorem in Euclidean 3-space. He determined pairs of surfaces with an additional conditional that they have the same Gauss map on Bour's theorem. About helicoidal surfaces in Euclidean 3-space, M. P. do Carmo and M. Dajczer [2] proved that, by using a result of E. Bour [1], there exists a two-parameter family of helicoidal surfaces isometric to a given helicoidal surface. By making use of this parametrization, they found a representation formula for helicoidal surfaces with constant mean curvature. Furthermore they proved that the associated family of Delaunay surfaces is made up by helicoidal surfaces of constant mean curvature.

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J. Hano and K. Nomizu [3] classified the spacelike rotation surfaces in $\mathbb{R}^3_1$ that have constant mean curvature and they proved that the profile curve of a rotation surface with nonzero constant mean curvature in $\mathbb{R}^3_1$ can be described as the locus of the focus when a quadratic curve is rolled along the axis of rotation. In addition, N. Sasahara [7] studied spacelike helicoidal surfaces with constant mean curvature in Minkowski 3-space.

On the other hand, Ikawa classified the spacelike and timelike surfaces as (axis, profile curve)-type in [5]. He proved an isometric relation between a spacelike (timelike) generalized helicoid and a spacelike (timelike) rotation surface of spacelike (timelike) axis on Bour's theorem. In [5], Ikawa gave Bour's theorem on surfaces with lightlike axis which is spanned by (0, 1, 1) vector in Minkowski 3-space.

In this study, we give Bour's theorem on surfaces with lightlike axes in Minkowski 3-space. Then, we give Bour's theorem in Minkowski 3-space and determine pairs of surfaces under an additional condition that the pair has zero mean curvature (*minimal* or a spacelike surface with vanishing mean curvature is called a *maximal surface*) and the same Gauss map.

Let $\mathbb{R}^3_1$ be a 3-dimensional Minkowski space with natural Lorentzian metric $\langle , \rangle = dx^2 + dy^2 - dz^2$. A vector w in $\mathbb{R}^3_1$ is called spacelike (resp. timelike) if $\langle w, w \rangle > 0$ or $w = 0$ (resp. $\langle w, w \rangle < 0$). If $w \neq 0$ satisfies $\langle w, w \rangle = 0$, then w is called lightlike. A surface in Minkowski 3-space $\mathbb{R}^3_1$ is called a *spacelike* (resp. *timelike*, *degenere* (lightlike)) if the induced metric on the surface is a positive definite Riemannian (resp. Lorentzian, degenere) metric.

Now we define a non degenerate rotation surface and generalized helicoid in $\mathbb{R}^3_1$. For an open interval $I \subset \mathbb{R}$, let $\gamma : I \longrightarrow \Pi$ be a curve in a plane Π in $\mathbb{R}^3_1$, and let ℓ be a straight line in Π which does not intersect the curve γ. A *rotation surface* in $\mathbb{R}^3_1$ is defined as a non degenerate surface rotating a curve γ around a line ℓ (these are called the *profile curve* and the *axis*, respectively). Suppose that when a profile curve γ rotates around the axis ℓ, it simultaneously displaces parallel to ℓ so that the speed of displacement is proportional to the speed of rotation. Then the resulting surface is called the *generalized helicoid* with axis ℓ and *pitch* a.

We classified a surface by types of axis and profile curve, and write as (axis's type, profile curve's type)-type; for example, (L, S)-type mean that the surface has a lightlike axis and a spacelike profile curve.

2. Rotation and helicoidal surfaces with lightlike axis

In this section, we will obtain some rotation and helicoidal surfaces with lightlike axis. In the rest of this paper we shall identify a vector (a, b, c) with its transpose $(a, b, c)^{\ t}$.

If the axis l is lightlike in Minkowski 3-space $\mathbb{R}^3_1$, then we may suppose that l is the line spanned by the vector $(0, 1, 1)$ (resp. $(1, 0, 1)$, $(0, 1, -1)$ and $(1, 0, -1)$). The semi$-$orthogonal matrices given as follows are the subgroup of the Lorentzian group that fixes the above vectors as invariant

$$A_1 = \begin{pmatrix} 1 & -v & v \\ v & 1-\frac{v^2}{2} & \frac{v^2}{2} \\ v & -\frac{v^2}{2} & 1+\frac{v^2}{2} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1-\frac{v^2}{2} & v & \frac{v^2}{2} \\ -v & 1 & v \\ -\frac{v^2}{2} & v & 1+\frac{v^2}{2} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1 & -v & -v \\ v & 1-\frac{v^2}{2} & -\frac{v^2}{2} \\ -v & \frac{v^2}{2} & 1+\frac{v^2}{2} \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1-\frac{v^2}{2} & v & -\frac{v^2}{2} \\ -v & 1 & -v \\ \frac{v^2}{2} & -v & 1+\frac{v^2}{2} \end{pmatrix}$$

where, $\epsilon = diag(1,1,-1)$, $A_i^t \epsilon A_i = \epsilon$ and $\det A_i = +1$ $(i = 1,2,3,4)$ for $v \in \mathbb{R}$.

Suppose that the axis of rotating is lightlike line, or equvalently the line of the plane x_2x_3 spanned by the vector $(0,1,1)$. Since the surface is non degenerate, we may assume that the profile curve γ_1 lies in the x_2x_3-plane without loss of generality and its parametrization is given by $\gamma_1(u) = (0, \varphi(u)+u, \varphi(u)-u)$, where $\varphi(u)+u$ and $\varphi(u)-u$ are functions on I such that $\varphi(u)+u \neq \varphi(u)-u$ for all u.

Therefore, the rotation surface can be parametrized as

$$R_1(u,v) = (-2uv, \varphi + u - uv^2, \varphi - u - uv^2). \tag{1}$$

Now, if the axis of rotating is lightlike line which is spanned by the vector $(0,1,-1)$, parametrization of the rotation surface is

$$R_3(u,v) = (-2v\varphi, \varphi + u - v^2\varphi, \varphi - u + v^2\varphi).$$

Similarly, we assume that the axes of rotating are lightlike lines which are spanned by $(1,0,1)$ and $(1,0,-1)$. Since the surface is non degenerate, we can assume that the profile curve γ_2 lies in the x_1x_3-plane without loss of generality, and its parametrization is given by $\gamma_2(u) = (\varphi(u)+u, 0, \varphi(u)-u)$.

Therefore, the other rotation surfaces is parametrized as

$$R_2(u,v) = (\varphi + u - uv^2, -2uv, \varphi - u - uv^2),$$
$$R_4(u,v) = (\varphi + u - v^2\varphi, -2v\varphi, \varphi - u + v^2\varphi).$$

Hence, we have $\langle \gamma_1', \gamma_1' \rangle = \langle \gamma_2', \gamma_2' \rangle = 4\varphi'$.

For a moment, we assume that $\varphi' \neq 0$. If,

i) $\varphi' > 0 \Rightarrow \gamma_1$ and γ_2 profile curves are spacelike, then it is (L,S)-type,

ii) $\varphi' < 0 \Rightarrow \gamma_1$ and γ_2 profile curves are timelike, then it is (L,T)-type.

A helicoidal surface in Minkowski 3-space with the lightlike axis which is spanned by $(0,1,1)$, and which has pitch $a \in \mathbb{R}$ is as follows

$$H_1(u,v) = \begin{pmatrix} 1 & -v & v \\ v & 1-\frac{v^2}{2} & \frac{v^2}{2} \\ v & -\frac{v^2}{2} & 1+\frac{v^2}{2} \end{pmatrix} \begin{pmatrix} 0 \\ \varphi+u \\ \varphi-u \end{pmatrix} + a \begin{pmatrix} 0 \\ v \\ v \end{pmatrix}$$
$$= \begin{pmatrix} -2uv \\ \varphi + u - uv^2 + av \\ \varphi - u - uv^2 + av \end{pmatrix}.$$

$H_1(u,v)$ reduces to a rotation surface when $a = 0$.

Similarly, parametrizations of the helicoidal surfaces with the lightlike axes which are spanned by $(1,0,1)$, $(0,1,-1)$ and the $(1,0,-1)$ respectively, and which has pitch $a \in \mathbb{R}$ are as follows respectively

$$\begin{aligned}
H_2(u,v) &= (\varphi + u - uv^2 + av, -2uv, \varphi - u - uv^2 + av),\\
H_3(u,v) &= (-2v\varphi, \varphi + u - v^2\varphi + av, \varphi - u + v^2\varphi - av),\\
H_4(u,v) &= (\varphi + u - v^2\varphi + av, -2v\varphi, \varphi - u + v^2\varphi - av).
\end{aligned}$$

3. Bour's theorem on surfaces with lightlike axis

In this section, we study an isometric relation between a spacelike (timelike) generalized helicoid and a spacelike (timelike) rotation surface of lightlike axis. We classified spacelike (timelike) generalized helicoid (rotation surfaces) with null axis as follow

(Axis, Profile Curve)-Type	*Profile Curve*
Case 1. $I.(L,S)$	spacelike
Case 2. $II.(L,T)$	timelike
Case 3. $I.(L,S)$	spacelike

Table 1. Types of axis and profile curve

Lightlike Axis Spanned by	*Surface-Type*	$H=0$	$e_H = e_R$ $(H \neq 0)$
$(0,1,1)$	spacelike	$\varphi = c_1\frac{u^3}{3} - \frac{a^2}{4u} + c_2$, $c_1 > 0$	$\varphi = \sqrt{2}u - \frac{a^2}{4u} + c$
$(0,1,1)$	timelike	$\varphi = -c_1\frac{u^3}{3} - \frac{a^2}{4u} + c_2$, $c_1 > 0$	$\varphi = -\sqrt{2}u - \frac{a^2}{4u} + c$
$(1,0,-1)$	spacelike	$u = c_1\frac{\varphi^3}{3} - \frac{a^2}{4\varphi} + c_2$ or $\varphi = c_1$	$e_H \neq e_R$

Table 2. Minimal and the same Gauss map surfaces

In this study, it can be showed for the other case 4. $II.(L,T)$-type surfaces. The techniques of proofs are same for the A_2 (A_4) semi-orthogonal matrix and the lightlike axis which is spanned by $(1,0,1)$ $((1,0,-1))$.

Case 1. *$I.(L,S)$-type.*
First of all, we consider the (L,S)-type surfaces, namely, the axis is lightlike and the profile curve is spacelike.

Theorem 3.1. *A spacelike generalized helicoid*

$$H_1(u,v) = (-2uv, \varphi + u - uv^2 + av, \varphi - u - uv^2 + av)$$

is isometric to a spacelike rotation surface

$$R_1(u_R, v_R) = \Big(-4uv + 2a, \varphi - \frac{a^2}{4u} - 2uv^2 + 2av \\ +2u, \varphi - \frac{a^2}{4u} - 2uv^2 + 2av - 2u\Big)$$

so that helices on the generalized helicoid correspond to parallel circles on the rotation surface.

Proof. We assume that the profile curve is on the x_2x_3-plane. Since a generalized helicoid is given by rotating the profile curve around the axis and simultaneously displacing parallel to the axis, so that the speed of displacement is proportional to the speed of rotation, from (1), we have the following representation of a generalized helicoid

$$(2)\quad H_1(u_H, v_H) = (-2u_Hv_H, \varphi_H + u_H - u_Hv_H^2 + av_H, \varphi_H - u_H - u_Hv_H^2 + av_H)$$

where a is a constant.

The coefficients of the first fundamental form and the line element of the generalized helicoid (2) are given by

$$E_H = 4\varphi'_H, F_H = 2a, G_H = 4u_H^2,$$
$$ds_H^2 = 4\varphi'_H du_H^2 + 4a du_H dv_H + 4u_H^2 dv_H^2.$$

Because of

$$Q_H = E_HG_H - F_H^2 = 16u_H^2\varphi'_H - 4a^2,$$
$$\text{if, } u_H^2 > \frac{a^2}{4\varphi'_H} \text{ then } H_1(u_H, v_H) \text{ is spacelike,}$$
$$\text{if, } u_H^2 < \frac{a^2}{4\varphi'_H} \text{ then } H_1(u_H, v_H) \text{ is timelike.}$$

Since $Q_H = 16u_H^2\varphi'_H - 4a^2 > 0$ and $Q_R = 8(16u_R^2\varphi'_R - 4a^2) > 0$ in case 1, both two surfaces are spacelike. Helices in $H_1(u_H, v_H)$ are curves defined by $u_H = const.$, so curves in $H_1(u_H, v_H)$ that are orthogonal to helices supply the orthogonal condition as follow

$$2a du_H + 4u_H^2 dv_H = 0.$$

Thus we obtain

$$v_H = -\int \frac{a}{2u_H^2} du_H + c$$

where c is constant. Hence if we put

$$\bar{v}_H = v_H - \frac{a}{2u_H}$$

then curves that are orthogonal to helices are given by $\bar{v}_H = const.$. Substituting the equation

$$dv_H = d\bar{v}_H - \frac{a}{2u_H^2} du_H$$

into the line element, we have

$$ds_H^2 = (4\varphi'_H - \frac{a^2}{u_H^2}) du_H^2 + 4u_H^2 d\bar{v}_H^2. \tag{3}$$

By putting

$$\bar{u}_H = \int \sqrt{4\varphi'_H - \frac{a^2}{u_H^2}} du_H, \; f_H(\bar{u}_H) = 2u_H,$$

(3) reduces to

$$ds_H^2 = d\bar{u}_H^2 + f_H^2(\bar{u}_H) d\bar{v}_H^2. \tag{4}$$

On the other hand, an (L, S)-type rotation surface

$$R_1(u_R, v_R) = (-2u_R v_R, \varphi_R + u_R - u_R v_R^2, \varphi_R - u_R - u_R v_R^2) \tag{5}$$

has the line element

$$ds_R^2 = 4\varphi'_R du_R^2 + 4u_R^2 dv_R^2. \tag{6}$$

Hence, if we put

$$\bar{u}_R = \int \sqrt{4\varphi'_R} du_R, \quad f_R(\bar{u}_R) = 2u_R, \quad \bar{v}_R = v_R,$$

then (6) reduces to

$$ds_R^2 = d\bar{u}_R^2 + f_R^2(\bar{u}_R) d\bar{v}_R^2. \tag{7}$$

Comparing (4) with (7), if

$$\bar{u}_H = \bar{u}_R, \quad \bar{v}_H = \bar{v}_R, \quad f_H(\bar{u}_H) = f_R(\bar{u}_R),$$

then we have an isometry between $H_1(u_H, v_H)$ and $R_1(u_R, v_R)$. Therefore it follows that

$$\int \sqrt{4\varphi'_H - \frac{a^2}{u_H^2}} du_H = \int \sqrt{4\varphi'_R} du_R$$

and we have

$$\varphi_R = \varphi_H + \frac{a^2}{4u_H}.$$

□

Theorem 3.2. *If two surfaces in Theorem* 3.1 *are maximal, then*

$$\varphi = c_1 \frac{u^3}{3} - \frac{a^2}{4u} + c_2$$

where c_1, c_2 are constants, and c_1 is positive.

Proof. First we consider a helicoid (2). Differentiating H_u and H_v, we obtain

$$H_{uu} = (0, \varphi'', \varphi''), \quad H_{uv} = (-2, -2v, -2v), \quad H_{vv} = (0, -2u, -2u).$$

The Gauss map e_H of the generalized helicoid is

$$e_H = \frac{1}{\sqrt{4a^2 - 16u^2\varphi'}} \begin{pmatrix} -4uv + 2a \\ -2uv^2 + 2av - 2u\varphi' + 2u \\ -2uv^2 + 2av - 2u\varphi' - 2u \end{pmatrix}. \tag{8}$$

Hence, the mean curvature H_H is given by

$$H_H = \frac{-8u^3\varphi'' + 16u^2\varphi' - 8a^2}{(4a^2 - 16u^2\varphi')^{3/2}} \tag{9}$$

by virtue of the first and second fundamental forms

$$E_H = 4\varphi', \quad F_H = 2a, \quad G_H = 4u^2,$$

$$L_H = \frac{4u\varphi''}{\sqrt{4a^2 - 16u^2\varphi'}}, \quad M_H = \frac{-4a}{\sqrt{4a^2 - 16u^2\varphi'}}, \quad N_H = \frac{-8u^2}{\sqrt{4a^2 - 16u^2\varphi'}}.$$

Next we calculate the Gauss map e_R and the mean curvature H_R of the rotation surface. Since

$$R_u = \left(-4v, \varphi' + \frac{a^2}{4u^2} - 2v^2 + 2, \varphi' + \frac{a^2}{4u^2} - 2v^2 - 2\right),$$

$$R_v = (-4u, -4uv + 2a, -4uv + 2a)$$

the Gauss map e_R of the rotation surface is given by

$$e_R = \frac{1}{\sqrt{4a^2 - 16u^2\varphi'}} \begin{pmatrix} -4\sqrt{2}uv + 2\sqrt{2}a \\ -2\sqrt{2}uv^2 + 2\sqrt{2}av - \sqrt{2}(\varphi' + \frac{a^2}{4u}) + 2\sqrt{2}u \\ -2\sqrt{2}uv^2 + 2\sqrt{2}av - \sqrt{2}(\varphi' + \frac{a^2}{4u}) - 2\sqrt{2}u \end{pmatrix}. \tag{10}$$

By the straight calculation, we have the coefficients of the second fundamental form as follows

$$L_R = \frac{4\sqrt{2}u\varphi'' - \frac{2\sqrt{2}a^2}{u^2}}{\sqrt{4a^2 - 16u^2\varphi'}}, \quad M_R = \frac{-8\sqrt{2}a}{\sqrt{4a^2 - 16u^2\varphi'}}, \quad N_R = \frac{-16\sqrt{2}u^2}{\sqrt{4a^2 - 16u^2\varphi'}}.$$

Hence the mean curvature H_R is

$$H_R = \frac{-4\sqrt{2}u^3\varphi'' + 8\sqrt{2}u^2\varphi' - 4\sqrt{2}a^2}{(4a^2 - 16u^2\varphi')^{3/2}}. \tag{11}$$

Thus we obtain

$$u^3\varphi'' - 2u^2\varphi' + a^2 = 0.$$

This equation means that the generalized helicoid and the rotation surface have zero mean curvature from (9) and (11). Therefore, if we solve this differential equation, we can see function φ easily. □

Example 3.1. A maximal spacelike helicoidal surface with lightlike axis (Figure 1) is isometric to a maximal spacelike rotation surface with lightlike axis (Figure 2). Moreover, $\varphi = c_1\frac{u^3}{3} - \frac{a^2}{4u} + c_2$ where $c_1 = 1$, $c_2 = 0$.

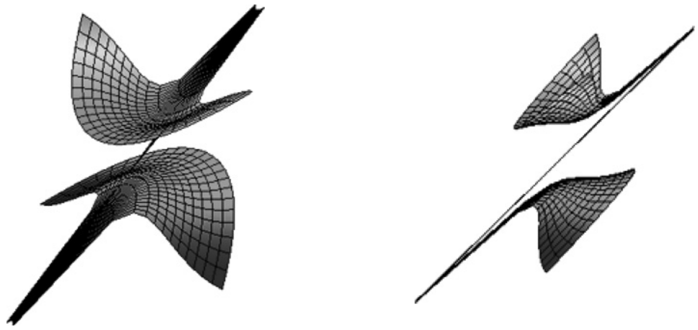

Figure 1. a-b. Maximal spacelike helicoidal surface with lightlike axis

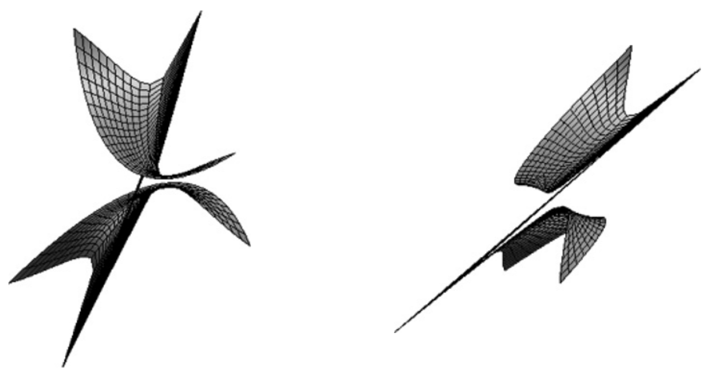

Figure 2. a-b. Maximal spacelike rotation surface with lightlike axis

Theorem 3.3. *If two surfaces in Theorem* 3.1 *have the same Gauss map, then*

$$\varphi = \sqrt{2}u - \frac{a^2}{4u} + c$$

where c is constant.

Proof. If the generalized helicoid and the rotation surface have the same Gauss map, comparing (9) and (11), we obtain

$$v(4\sqrt{2}u - 4u) = 2\sqrt{2}a - 2a,$$
$$2uv^2(\sqrt{2}-1) + 2av(1-\sqrt{2}) + \varphi'(-2u+\sqrt{2}) + \frac{\sqrt{2}a^2}{4u} \mp (1-\sqrt{2})2u = 0.$$

Hence we can see function φ easily. □

Example 3.2. A spacelike ($H \neq 0$) helicoidal surface with lightlike axis (Figure 3) is isometric to a timelike ($H \neq 0$) rotation surface with lightlike axis (Figure 4). In addition,

$$\varphi = \sqrt{2}u - \frac{a^2}{4u} + c$$

where $c = 0$.

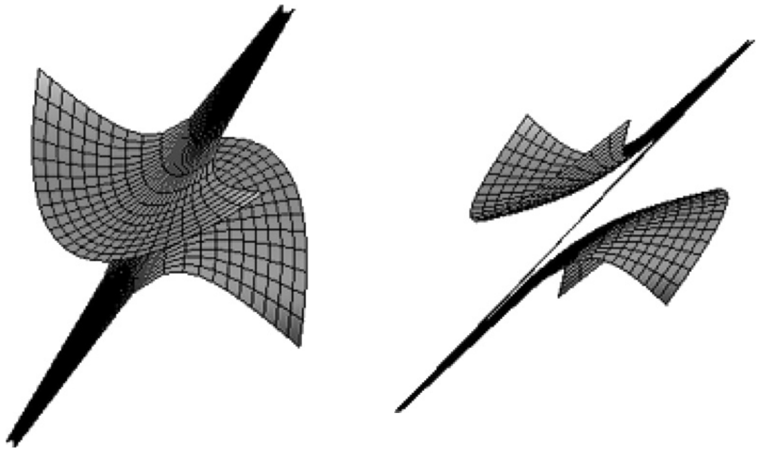

Figure 3. a-b. Spacelike ($H \neq 0$) helicoidal surface with lightlike axis

Case 2. *I.(L, T)-type.*

Theorem 3.4. *A timelike generalized helicoid*

$$H_1(u,v) = (-2uv, \varphi + u - uv^2 + av, \varphi - u - uv^2 + av)$$

is isometric to a timelike rotation surface

$$R_{1T}(u_R, v_R) = \left(4uv - 2a, \varphi + \frac{a^2}{4u} + 2uv^2 - 2av - 2u, \varphi + \frac{a^2}{4u} + 2uv^2 - 2av + 2u\right) \tag{12}$$

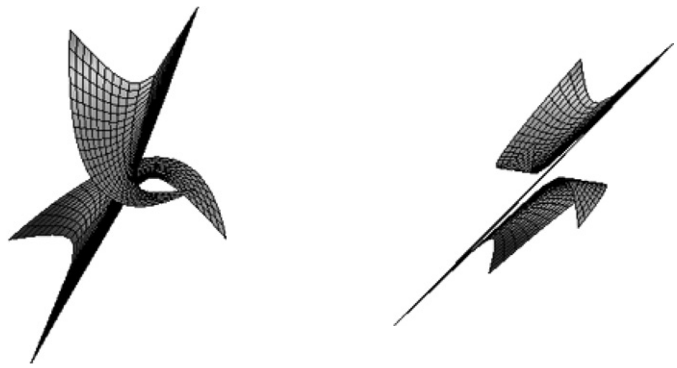

Figure 4. a-b. Spacelike ($H \neq 0$) rotation surface with lightlike axis

so that helices on the generalized helicoid correspond to parallel circles on the rotation surface.

Proof. Since $Q_H = 16u_H^2\varphi'_H - 4a^2 < 0$ and $Q_R = 8(16u_R^2\varphi'_R - 4a^2) < 0$ in case 2, both two surfaces are timelike. The line element of the generalized helicoid is

$$ds_H^2 = \left(4\varphi'_H - \frac{a^2}{u_H^2}\right) du_H^2 + 4u_H^2 d\bar{v}_H^2.$$

Because of $(4\varphi' - \frac{a^2}{u^2}) < 0$ and $u^2 < 0$, it reduces to $ds^2 = -du^2 - f^2(u)dv^2$. Hence, $\varphi_R = -\varphi_H - \frac{a^2}{4u_H}$, $u_R = -2u$, $v_R = v - \frac{a}{2u}$. Therefore, (5) reduces to (12). By the methods in Theorem 3.1, the generalized helicoid is isometric to the rotation surface. □

Theorem 3.5. *If two surfaces in Theorem* 3.4 *are minimal then*

$$\varphi = -c_1\frac{u^3}{3} - \frac{a^2}{4u} + c_2$$

where c_1, c_2 *are constants, and* c_1 *is positive.*

Proof. Similarly Theorem 3.2, we obtain $u^3\varphi'' - 2u^2\varphi' + a^2 = 0$. If we solve this differential equation, we can see function φ easily. □

Example 3.3. A minimal timelike helicoidal surface with lightlike axis (Figure 5) is isometric to a minimal timelike rotation surface with lightlike axis (Figure 6). Moreover, $\varphi = -c_1\frac{u^3}{3} - \frac{a^2}{4u} + c_2$ where $c_1 = 1$, $c_2 = 0$.

Theorem 3.6. *If two surfaces in Theorem* 3.4 *have the same Gauss map*

$$\varphi = -\sqrt{2}u - \frac{a^2}{4u} + c$$

where c *is constant.*

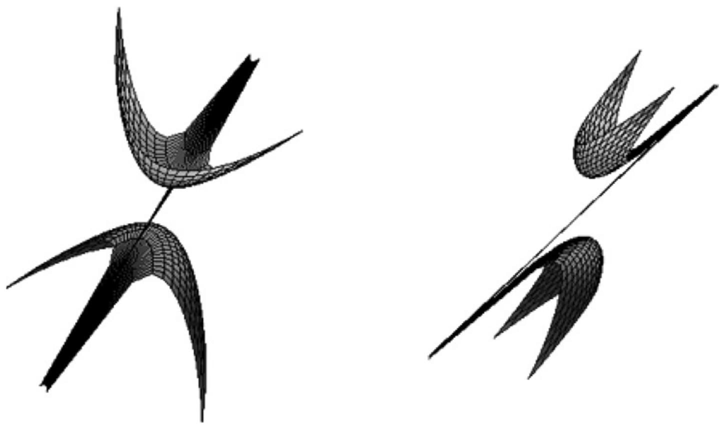

Figure 5. a-b. Minimal timelike helicoidal surface with lightlike axis

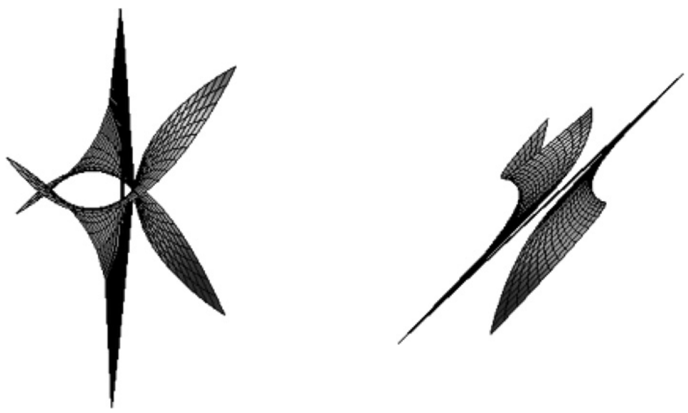

Figure 6. a-b. Minimal timelike rotation surface with lightlike axis

Proof. From the methods in Theorem 3.4,

$$e_H = \frac{1}{\sqrt{4a^2 - 16u^2\varphi'}} \begin{pmatrix} -4uv + 2a \\ -2uv^2 + 2av - 2u\varphi' + 2u \\ -2uv^2 + 2av - 2u\varphi' - 2u \end{pmatrix},$$

$$e_R = \frac{1}{\sqrt{4a^2 - 16u^2\varphi'}} \begin{pmatrix} 4\sqrt{2}uv - 2\sqrt{2}a \\ 2\sqrt{2}uv^2 - 2\sqrt{2}av + \sqrt{2}(\varphi' + \frac{a^2}{4y}) - 2\sqrt{2}u \\ 2\sqrt{2}uv^2 - 2\sqrt{2}av + \sqrt{2}(\varphi' + \frac{a^2}{4u}) + 2\sqrt{2}u \end{pmatrix},$$

comparing e_H and e_R, we obtain

$$v(2\sqrt{2}u + 2u) = \sqrt{2}a + a,$$

$$2uv^2(\sqrt{2}+1) - 2av(\sqrt{2}+1) + \varphi'(\sqrt{2}+2u) + \frac{\sqrt{2}a^2}{4u} \pm (\sqrt{2}+1)2u = 0.$$

Thus, we can see function φ easily. $\square$

Example 3.4. The timelike ($H \neq 0$) helicoidal surface with lightlike axis (Figure 7) is isometric to the timelike ($H \neq 0$) rotation surface with lightlike axis (Figure 8). Then, $\varphi = -\sqrt{2}u - \frac{a^2}{4u} + c$ where $c = 0$.

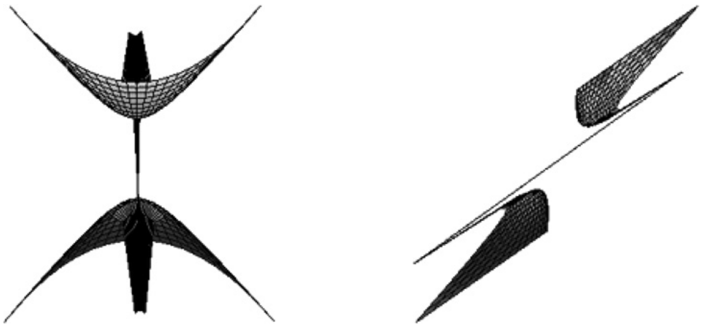

Figure 7. a-b. Timelike ($H \neq 0$) helicoidal surface with lightlike axis

Figure 8. a-b. Timelike ($H \neq 0$) rotation surface with lightlike axis

The techniques of proofs are same for the A_2 semi-orthogonal matrix and the lightlike axis which is spanned by $(1, 0, 1)$.

Now we give only sketch proofs for A_3 semi-orthogonal matrix and the lightlike axis which are spanned by $(0, 1, -1)$ as follow theorems.

Case 3. *$II.(L, S)$-type.*

Theorem 3.7. *A spacelike generalized helicoid*

$$H_3(u, v) = (-2v\varphi, \varphi + u - v^2\varphi + av, \varphi - u + v^2\varphi - av),$$

is isometric to a spacelike rotation surface

$$R_3(u_R, v_R) = \begin{pmatrix} -2\left(v + \int \frac{a\varphi'}{2\varphi}du\right)\left(\varphi - \int \frac{a^2\varphi'^2}{4\varphi^2}du\right) \\ \left(\varphi - \int \frac{a^2\varphi'^2}{4\varphi^2}du\right) + 2\varphi - \left(v + \int \frac{a\varphi'}{2\varphi}du\right)^2\left(\varphi - \int \frac{a^2\varphi'^2}{4\varphi^2}du\right) \\ \left(\varphi - \int \frac{a^2\varphi'^2}{4\varphi^2}du\right) - 2\varphi + \left(v + \int \frac{a\varphi'}{2\varphi}du\right)^2\left(\varphi - \int \frac{a^2\varphi'^2}{4\varphi^2}du\right) \end{pmatrix}$$

so that helices on the generalized helicoid correspond to parallel circles on the rotation surface.

Proof. The coefficients of the first fundamental form and the line element of the generalized helicoid $H_3(u,v)$ are given by

$$E_H = 4\varphi'_H, \quad F_H = 2a\varphi'_H, \quad G_H = 4\varphi_H^2,$$
$$ds_H^2 = 4\varphi'_H du_H^2 + 4a\varphi'_H du_H dv_H + 4\varphi_H^2 dv_H^2.$$

Because of

$$Q_H = E_H G_H - F_H^2 = 16\varphi_H^2\varphi'_H - 4a^2\varphi'^2_H,$$
$$\text{if, } \varphi_H^2 > \frac{a^2}{4\varphi'_H} \text{then } H_3(u_H, v_H) \text{ is spacelike,}$$
$$\text{if, } \varphi_H^2 < \frac{a^2}{4\varphi'_H} \text{then } H_3(u_H, v_H) \text{ is timelike.}$$

Since $Q_H > 0$ and $Q_R > 0$ in case 3, both two surfaces are spacelike. If

$$\bar{u}_H = \bar{u}_R, \quad \bar{v}_H = \bar{v}_R, \quad f_H(\bar{u}_H) = f_R(\bar{u}_R),$$

then we have an isometry between $H_3(u_H, v_H)$ and $R_3(u_R, v_R)$. Therefore it follows that

$$\int \sqrt{4\varphi'_H - \frac{a^2\varphi'^2_H}{\varphi_H^2}} du_H = \int \sqrt{4\varphi'_R} du_R$$

and we have

$$\varphi_R = \varphi_H - \int \frac{a^2\varphi'^2_H}{4\varphi_H^2} du_H.$$

Hence the rotation surface is

(13)

$$R_3(u_R, v_R) = \begin{pmatrix} -2\left(v - \frac{a}{2\varphi}\right)\left(\varphi - \int \frac{a^2\varphi'^2}{4\varphi^2} du\right) \\ \left(\varphi - \int \frac{a^2\varphi'^2}{4\varphi^2} du\right) + 2\varphi - \left(v - \frac{a}{2\varphi}\right)^2 \left(\varphi - \int \frac{a^2\varphi'^2}{4\varphi^2} du\right) \\ \left(\varphi - \int \frac{a^2\varphi'^2}{4\varphi^2} du\right) - 2\varphi + \left(v - \frac{a}{2\varphi}\right)^2 \left(\varphi - \int \frac{a^2\varphi'^2}{4\varphi^2} du\right) \end{pmatrix}.$$

□

Theorem 3.8. *If the surface of $H_3(u,v)$ in Theorem 3.7. is minimal then*

$$u = c_1 \frac{\varphi^3}{3} - \frac{a^2}{4\varphi} + c_2 \quad or \quad \varphi = c_1$$

where c_1, c_2 are constants, c_1 is positive. If the rotation surface which is given by (13) *in theorem* 3.7 *is minimal then we have an differential equation as follow*

$$\begin{aligned} &2\varphi' - A^2 B_u \varphi'' - \frac{4B_u + 2A^2 B_u^2}{B} + A^2 A_u^2 B B_u + 2AA_{uu}B + A^2 B_{uu}\varphi' \\ &+ \Big(A^2 A_{uu} B - AA_u B_u + A^2 B_{uu} + A^2 A_u^2 B \\ &-A^3 A_{uu} B - \frac{1}{2} B_{uu} - \frac{1}{2} A^4 B_{uu} \Big) B_u = 0 \end{aligned}$$

where $A = v - \frac{a}{2\varphi}$, $B = \varphi - \int \frac{a^2\varphi'^2}{4\varphi^2} du$.

Proof. We consider a helicoid $H_3(u,v)$. Differentiating H_u and H_v, we obtain

$$\begin{aligned} H_{uu} &= (-2v\varphi'', \varphi'' - v^2\varphi'', \varphi'' + v^2\varphi''), \\ H_{uv} &= (-2\varphi', -2v\varphi', 2v\varphi'), \quad H_{vv} = (0, -2\varphi, 2\varphi). \end{aligned}$$

The Gauss map e_H of the generalized helicoid is

$$e_H = \frac{1}{\sqrt{4a^2\varphi'^2 - 16\varphi^2\varphi'}} \begin{pmatrix} 4v\varphi\varphi' - 2a\varphi' \\ -2\varphi\varphi' + 2v^2\varphi\varphi' - 2av\varphi' + 2\varphi \\ -2\varphi\varphi' - 2v^2\varphi\varphi' + 2av\varphi' - 2\varphi \end{pmatrix}.$$

Therefore, the mean curvature H_H is

$$H_H = \frac{-8\varphi^3\varphi'' - 8a^2\varphi'^3 - 16\varphi^2\varphi'^2}{(4a^2\varphi'^2 - 16\varphi^2\varphi')^{3/2}}$$

by virtue of the second fundamental forms

$$L_H = \frac{4\varphi\varphi''}{\sqrt{4a^2\varphi'^2 - 16\varphi^2\varphi'}}, \quad M_H = \frac{-4a\varphi'^2}{\sqrt{4a^2\varphi'^2 - 16\varphi^2\varphi'}},$$
$$N_H = \frac{8\varphi^2\varphi'}{\sqrt{4a^2\varphi'^2 - 16\varphi^2\varphi'}}.$$

Hence we obtain

$$\varphi^3\varphi'' + a^2\varphi'^3 + 2\varphi^2\varphi'^2 = 0.$$

If we solve this equation, we have $u = c_1\frac{\varphi^3}{3} - \frac{a^2}{4\varphi} + c_2$ *or* $\varphi = c_1$ where c_1, c_2 are constants, and c_1 is positive.

Now we calculate the Gauss map e_R and the mean curvature H_R of the rotation surface (13). Since

$$A := \left(v - \frac{a}{2\varphi}\right), \; B := \left(\varphi - \int \frac{a^2\varphi'^2}{4\varphi^2} du\right)$$

we obtain

$$R(u_R, v_R) = \begin{pmatrix} -2AB \\ B + 2\varphi - A^2B \\ B - 2\varphi + A^2B \end{pmatrix},$$

$$R_u = \begin{pmatrix} -2(A_uB + AB_u) \\ B_u + 2\varphi' - 2AA_uB - A^2B_u \\ B_u - 2\varphi' + 2AA_uB + A^2B_u \end{pmatrix}, \quad R_v = \begin{pmatrix} -2B \\ -2AB \\ 2AB \end{pmatrix},$$

then the first fundamental form and its coefficients are as follows

$$E_R = 4A_u^2B^2 + 8\varphi' B_u, \;\; F_R = 4A_uB^2, \;\; G_R = 4B^2,$$

$$E_RG_R - F_R^2 = 32\varphi' B^2 B_u = \frac{2\varphi'}{\varphi^2}\left(\varphi - \int \frac{a^2\varphi'^2}{4\varphi^2}du\right)^2 (16\varphi^2\varphi' - 4a^2\varphi'^2).$$

The Gauss map e_R of the rotation surface is given by

$$e_R = \frac{\frac{\varphi\sqrt{2}}{\sqrt{\varphi'}}}{\sqrt{4a^2\varphi'^2 - 16\varphi^2\varphi'}} \begin{pmatrix} 2AB_u \\ B_u - 2\varphi' - A^2B_u \\ -B_u - 2\varphi' - A^2B_u \end{pmatrix}.$$

By the straight calculation, we have the coefficients of the second fundamental form as follows

$$\begin{aligned} L_R = {} & \frac{\varphi\sqrt{2}}{\sqrt{\varphi'}\sqrt{4a^2\varphi'^2 - 16\varphi^2\varphi'}}[-4A^2A_{uu}BB_u - 4AA_uB_u^2 - 4A^2B_uB_{uu} \\ & + 2B_uB_{uu} - 8\varphi'\varphi'' + 8A_u^2B\varphi' + 8AA_{uu}B\varphi' + 16AA_uB_u\varphi' + 4A^2B_{uu}\varphi' \\ & + 4A^2B_u\varphi'' + 4A^2A_u^2BB_u + 4A^3A_{uu}BB_u + 8A^3A_uB_u^2 + 2A^4B_uB_{uu}], \end{aligned}$$

$$\begin{aligned} M_R = {} & \frac{\varphi\sqrt{2}}{\sqrt{\varphi'}\sqrt{4a^2\varphi'^2 - 16\varphi^2\varphi'}}[-4AB_u^2 + 8A_uB\varphi' + 4A^2A_uBB_u \\ & + 8AB_u\varphi' + 4A^3B_u^2], \end{aligned}$$

$$N_R = \frac{\varphi\sqrt{2}}{\sqrt{\varphi'}\sqrt{4a^2\varphi'^2 - 16\varphi^2\varphi'}}[8B\varphi' + 4A^2BB_u].$$

Hence the mean curvature H_R is

$$\begin{aligned} H_R = {} & \frac{\frac{\sqrt{2}\varphi^3}{\varphi'\sqrt{\varphi'}}}{(4a^2\varphi'^2 - 16\varphi^2\varphi')^{3/2}} \Bigg[-\frac{16B_u\varphi'}{B} - 4A^2A_u^2BB_u\varphi' - \frac{8A^2B_u^2\varphi'}{B} \\ & + 4A^2A_{uu}BB_u - 4AA_uB_u^2 + 4A^2B_uB_{uu} - 2B_uB_{uu} + 8\varphi'\varphi'' - 8AA_{uu}B\varphi' \\ & -4A^2B_{uu}\varphi' - 4A^2B_u\varphi'' + 4A^2A_u^2BB_u - 4A^3A_{uu}BB_u - 2A^4B_uB_{uu}\Bigg]. \end{aligned}$$

If we put in $H_R = 0$ as follows equations

$$A_u = \frac{a\varphi'}{2\varphi^2}, \quad B_u = \left(\varphi' - \frac{a^2\varphi'^2}{4\varphi^2}\right), \quad A_{uu} = \frac{a}{2}\left(\frac{\varphi\varphi'' - 2\varphi'^2}{\varphi^3}\right),$$

$$B_{uu} = \varphi'' - \frac{a^2}{2}\left(\frac{\varphi\varphi'\varphi'' - \varphi'^3}{\varphi^3}\right),$$

then we have an interesting differential equation. The solution is also an attracted problem. □

Theorem 3.9. *If isometric spacelike generalized helicoid and spacelike rotation surface have non-zero mean curvature in Theorem* 3.7 *then these Gauss maps are definitely different.*

Proof. Comparing Gauss maps e_H and e_R, we obtain

$$2v\varphi\varphi' - a\varphi' = \frac{\varphi\sqrt{2}}{\sqrt{\varphi'}}AB_u \tag{14}$$

$$-2\varphi\varphi' + 2v^2\varphi\varphi' - 2av\varphi' + 2\varphi = \frac{\varphi\sqrt{2}}{\sqrt{\varphi'}}(B_u - 2\varphi' - A^2B_u) \tag{15}$$

$$-2\varphi\varphi' - 2v^2\varphi\varphi' + 2av\varphi' - 2\varphi = \frac{\varphi\sqrt{2}}{\sqrt{\varphi'}}(-B_u - 2\varphi' - A^2B_u). \tag{16}$$

This differential equations are the quadrature-type. Therefore, (14) reduces to

$$\left(\frac{a^4}{4\varphi^4}\right)\varphi'^2 - \left(\frac{a^2}{2\varphi^2} + 2\right)\varphi' + 1 = 0.$$

Hence the solutions are

$$u = -\frac{a^2}{4\varphi} + \varphi - \frac{\sqrt{2}}{2}\sqrt{2\varphi^2 + a^2} + \frac{a^2\sqrt{2}}{2\,|a|}\log\left(\frac{2a^2 + 2\,|a|\,\sqrt{2\varphi^2 + a^2}}{\varphi}\right) + c_1,$$

$$u = -\frac{a^2}{4\varphi} + \varphi + \frac{\sqrt{2}}{2}\sqrt{2\varphi^2 + a^2} - \frac{a^2\sqrt{2}}{2\,|a|}\log\left(\frac{2a^2 + 2\,|a|\,\sqrt{2\varphi^2 + a^2}}{\varphi}\right) + c_2$$

where c_1, c_2 are constants. If (15) and (16) differential equations are compared then we obtain different solutions. □

Corollary 3.1. *Two surfaces of Theorem* 3.9 *have the different Gauss map.*

From the methods in Theorem 3.7, Theorem 3.8 and Theorem 3.9, it can be showed for the other case 4. $II.(L, T)$-type surfaces in this study.

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