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Borcherds products for higher level modular forms

By

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Abstract

We generalize Borcherds' construction of infinite products to higher level vector valued modular forms of Nebentypus. Then we obtain meromorphic modular functions for the orthogonal group whose zeros or poles lie on the Heegner divisors. The construction involves twisted Siegel theta functions and the singular theta integral.

1. Introduction

In his paper [2], R. E. Borcherds constructed a multiplicative lifting from vector valued modular forms which are modular with respect to the Weil representation ρ_L of $\mathrm{Mp}_2(\mathbb{Z})$ attached to a non-degenerate even integral lattice L of signature $(2, b^-)$ to modular forms on the orthogonal group $\Gamma(L)$ of L. Here a vector valued modular form F may have poles at the cusp ∞ and must satisfy some integrality condition on the Fourier coefficients. Its lifting image Ψ_F, which has Borcherds infinite product expansion, is a meromorphic function on the Hermitian symmetric space attached to L and its zeros or poles are located on the Heegner divisors. The Heegner divisors are a generalization of imaginary quadratic irrationals in the one-dimensional complex upper half plane case. Their orders are explicitly determined by the singularity of F, that is, the Fourier coefficients of terms with negative exponents. When L splits two orthogonal hyperbolic planes over $\mathbb{Z}$, the images of this lifting are characterized by these conditions on their divisors ([6]).

Now, as Borcherds himself pointed out, the lifting works well only for full modular forms. Some more considerations are needed if one wants to lift higher level modular forms. In [2], he overcame this difficulty by embedding scalar valued higher level modular forms into vector valued full modular forms and obtained some important examples. Also he gave an application of this in another direction, see [3].

On the other hand, Zagier [15] and Kim [10], [11] treats Hauptmoduls for congruence subgroups of genus 0 and studies twisted traces of their singular

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moduli. In the one-dimensional full modular case, Borcherds' theorem and Zagier's result are equivalent to each other. However, the connection between Borcherds' construction and traces of singular moduli is not clearly understood in the higher level case. Recently, Bruinier and Yang ([7]) constructed twisted Borcherds products in the Hilbert modular case. Their construction is based on the argument of [5]. Specifically the automorphic Green function twisted by a genus character is crucial.

In the present paper, we extend Borcherds lifting to higher level modular forms of Nebentypus and give twisted Borcherds products. The resulting infinite product is quite similar to that in [7]. Our way to construct the twisted infinite products is different from the above ones and the same as in [2] for full modular forms, but needs a suitably modified Siegel theta function in considering the singular theta correspondence. For this purpose, we use the Siegel theta function twisted by some Dirichlet character. Then the construction proceeds in the same way as for the full modular case. For example, we can follow the argument of the partial Fourier transform of the Siegel theta function and unfolding trick for the singular theta integral. As a result, we obtain meromorphic modular functions for the orthogonal group whose zeros and poles lie on the Heegner divisors. Their multiplicities are described by singularities of higher level modular forms. Not as for the full modular case, the singularity occurs from each cusp.

More specifically, let K be a Lorentzian lattice of signature $(1, b^- - 1)$ and $M = \mathbb{Z}z + \mathbb{Z}z'$ with $z^2 = {z'}^2 = 0$, $(z, z') = N$. We put $L = K \oplus M$. We let χ be a real even primitive Dirichlet character modulo f with $f|N$ and assume that $\{p; p|f\} = \{p; p|N\}$. Then our main result Theorem 6.1 is that if a higher level modular form F of weight $1 - b^-/2$ with respect to ρ_K and $\Gamma_0(N)$ of Nebentypus χ satisfies some integrality condition on its Fourier coefficients at each cusp then the infinite product

$$\prod_{k\ (f)} \prod_{\substack{\lambda \in K' \\ (\lambda, W) > 0}} \left(1 - \zeta^k \mathbf{e}((\lambda, Z))\right)^{\chi(k) c_\lambda(\lambda^2/2)} \tag{1.1}$$

where $c_\lambda(\lambda^2/2)$ are Fourier coefficients of F at the cusp ∞ and $\zeta = \mathbf{e}(1/f)$ defines a meromorphic function on the Hermitian symmetric domain attached to L which is of weight 0 for the orthogonal group of L with some unitary character. Moreover we find that its zeros or poles lie on the Heegner divisors and that their orders are explicitly described by the singularity of F at each cusp.

To find such a higher level vector valued modular form F as above, we may twist a vector valued full modular form by some Dirichlet character. If we take various characters, then by our theorem we obtain a family of meromorphic modular functions from a full modular form. We are interested in the nature of their CM values or the subgroup of meromorphic modular functions generated by these functions. However this would require some amount of work as in [5]–[7].

Notations. For $z \in \mathbb{C}$ we write $\mathbf{e}(z) = \exp(2\pi i z)$. We choose a branch of $\sqrt{z} = z^{\frac{1}{2}}$ so that $\arg\left(\sqrt{z}\,\right) \in (-\frac{\pi}{2}, \frac{\pi}{2}\,]$ and put $z^{\frac{k}{2}} = (z^{\frac{1}{2}})^k$ for each integer k. As usual, $\mathfrak{H}$ denotes the complex upper half plane.

In sums of the form $\sum_{k\,(N)}$, the summation runs over complete representatives of integers modulo N. We understand products of the form $\prod_{k\,(N)}$ similarly.

For a lattice L and its associated symmetric bilinear form $(\ ,\ )$, we denote the norm of $\gamma \in L\otimes\mathbb{C}$ by $\gamma^2 = (\gamma, \gamma)$ where the form $(\ ,\ )$ is bilinearly extended to $L\otimes\mathbb{C}$. We also put $|\gamma| = \sqrt{|\gamma^2|}$ for $\gamma \in L\otimes\mathbb{C}$. For $N \in \mathbb{Z}$, we mean $(2N)$ to be an even integral lattice of rank one generated by a vector of norm $2N$. We denote by $II_{1,1}$ the even unimodular lattice of rank two generated by vectors z and z' with $z^2 = z'^{\,2} = 0$, $(z, z') = 1$. If L is a lattice, we denote by $L(N)$ the lattice whose underlying $\mathbb{Z}$-module is L and norm is N multiple of that of L. For lattices L and M, we let $L \oplus M$ be the orthogonal direct sum of L and M. Hereafter lattices are always assumed to be non-degenerate and even integral.

A sublattice M of a lattice L is called primitive if L/M is torsion free and a vector λ of L is primitive if it generates a primitive sublattice.

A lattice is Lorentzian if it has signature $(1, b^-)$ or $(b^+, 1)$.

2. Review of Borcherds product

We first review the Borcherds' construction of infinite product. Some results are needed in the later sections. For details and proofs, see [2] and also [6].

The metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$ is a double covering group of $\mathrm{SL}_2(\mathbb{Z})$ and is defined as $\mathrm{Mp}_2(\mathbb{Z}) = \{\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \pm\sqrt{c\tau + d}\,\right) \mid \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})\}$. We have $(\alpha, \phi(\tau))\,(\beta, \varphi(\tau)) = (\alpha\beta, \phi(\beta\tau)\varphi(\tau))$ for $(\alpha, \phi(\tau)), (\beta, \varphi(\tau)) \in \mathrm{Mp}_2(\mathbb{Z})$, where $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathfrak{H}$ by usual fractional linear transformations.

It is known that $T = \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1\right)$ and $S = \left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau}\,\right)$ generate $\mathrm{Mp}_2(\mathbb{Z})$. Their relation is given by $S^2 = (ST)^3 = Z$, where $Z = \left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), i\right)$ is a generator of the center of $\mathrm{Mp}_2(\mathbb{Z})$. The order of Z is four.

For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we put $\widetilde{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} = \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \sqrt{c\tau + d}\,\right)$.

Let L be a lattice of signature (b^+, b^-) with the associated symmetric bilinear form $(\ ,\ )$. We write $L' = \{\lambda \in L \otimes \mathbb{Q} \,|\, (\lambda, \mu) \in \mathbb{Z} \text{ for all } \mu \in L\,\}$ for the dual lattice of L. Then L'/L is a finite Abelian group and is called the discriminant group of L. Its order equals the absolute value of the determinant of the Gram matrix of L. Since L is non-degenerate, the mod 1 reduction of $\gamma^2/2$ defines a $\mathbb{Q}/\mathbb{Z}$-valued non-degenerate quadratic form on L'/L. The associated symmetric bilinear form is the mod 1 reduction of that of L'. We write also $(\ ,\ )$ for it.

Let $\mathbb{C}[L'/L]$ be the group algebra of L'/L. We denote its standard basis by $(\mathfrak{e}_\gamma)_{\gamma\in L'/L}$. The standard Hermitian inner product on $\mathbb{C}[L'/L]$ is given by

$$\left\langle \sum_{\gamma\in L'/L} c_\gamma \mathfrak{e}_\gamma\,,\ \sum_{\gamma\in L'/L} d_\gamma \mathfrak{e}_\gamma \right\rangle = \sum_{\gamma\in L'/L} c_\gamma \overline{d_\gamma}. \tag{2.1}$$

The Weil representation ρ_L attached to L is a unitary representation of $\mathrm{Mp}_2(\mathbb{Z})$ on $\mathbb{C}[L'/L]$ and is defined by the action of generators:

$$\rho_L(T)\mathfrak{e}_\gamma = \mathbf{e}(\gamma^2/2)\mathfrak{e}_\gamma, \tag{2.2}$$

$$\rho_L(S)\mathfrak{e}_\gamma = \frac{\sqrt{i}^{\,b^- - b^+}}{\sqrt{|L'/L|}} \sum_{\delta \in L'/L} \mathbf{e}(-(\gamma,\delta))\mathfrak{e}_\delta. \tag{2.3}$$

Then we readily verify $\rho_L(Z)\mathfrak{e}_\gamma = i^{b^- - b^+}\mathfrak{e}_{-\gamma}$.

We denote the matrix coefficients of ρ_L by $\rho_{\gamma\delta}(\alpha,\phi) = \langle \rho_L(\alpha,\phi)\mathfrak{e}_\delta,\, \mathfrak{e}_\gamma\rangle$. Shintani computed this coefficient $\rho_{\gamma\delta}(\alpha,\phi)$ explicitly ([14, Proposition 1.6]).

Proposition 2.1. *Let γ, $\delta \in L'/L$ and $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. Then we have*

$$\rho_{\gamma\delta}(\widetilde{\alpha}) = \begin{cases} \sqrt{i}^{\,(b^- - b^+)(1 - sgn(d))}\mathbf{e}(ab\gamma^2/2)\,\delta_{\gamma,a\delta} & \text{if } c = 0, \\ \frac{\sqrt{i}^{\,(b^- - b^+)sgn(c)}}{|c|^{(b^+ + b^-)/2}\sqrt{|L'/L|}} \sum_{\varepsilon \in L/cL} \mathbf{e}\left(\frac{a(\gamma+\varepsilon,\,\gamma+\varepsilon) - 2(\gamma+\varepsilon,\,\delta) + d(\delta,\delta)}{2c}\right) & \text{if } c \neq 0 \end{cases}$$

where $\delta_{*,*}$ is the Kronecker's delta.

As a result of this formula, we can immediately verify that ρ_L factors through the finite group $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ if $b^+ + b^-$ is even, and through a double covering group of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ if $b^+ + b^-$ is odd, where N is the smallest positive integer such that $L'(N)$ is even integral. The integer N is called the level of L.

Then vector valued modular forms are defined as follows.

Definition 2.1. Let L be a lattice of signature $(b^+,\, b^-)$ and $k \in \frac{1}{2}\mathbb{Z}$. A holomorphic function $F : \mathfrak{H} \to \mathbb{C}[L'/L]$ is called a nearly holomorphic modular form of weight k with respect to ρ_L and $\mathrm{Mp}_2(\mathbb{Z})$ if it satisfies the following conditions.

(i) $F(\alpha\tau) = \phi(\tau)^{2k}\rho_L(\alpha,\,\phi)F(\tau)$ for all $(\alpha,\,\phi) \in \mathrm{Mp}_2(\mathbb{Z})$, $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

(ii) F is meromorphic at the cusp ∞. This means that F can be expanded as $F(\tau) = \sum_{\gamma \in L'/L} \mathfrak{e}_\gamma \sum_{\substack{n \in \mathbb{Q} \\ n \gg -\infty}} c_\gamma(n)\,\mathbf{e}(n\tau)$.

Then the Fourier polynomial $\sum_{\gamma \in L'/L} \mathfrak{e}_\gamma \sum_{\substack{n \in \mathbb{Q} \\ n < 0}} c_\gamma(n)\,\mathbf{e}(n\tau)$ is called the principal part or the singularity of F.

We put $V = L \otimes \mathbb{R}$. We write $Gr(L)$ for the Grassmannian of L. This is a real analytic manifold whose underlying set consists of b^+-dimensional maximal positive definite subspaces of V. If $v \in Gr(L)$, we denote by $v^\perp$ the orthogonal complement of v in V. For $x \in V$ and $v \in Gr(L)$, we write x_v (resp. $x_{v^\perp}$) for the v-component (resp. the $v^\perp$-component) of x in $V = v \oplus v^\perp$.

We let $O(V) = \{\, g \in SL(V) \,|\, (gx, gy) = (x, y) \text{ for all } x, y \in V \,\}$ be the special orthogonal group of V and $O(L) = \{\, g \in O(V) \,|\, g(L) \subset L \,\}$ be the orthogonal group of L. The group $O_d(L) = \{\, g \in O(L) \,|\, g \text{ acts trivially on } L'/L \,\}$ is called the discriminant kernel of $O(L)$. Moreover we put $O^+(V)$ the connected component of $O(V)$ and $\Gamma(L) = O^+(V) \cap O_d(L)$.

Let $\tau \in \mathfrak{H}$, $v \in Gr(L)$ and $r, t \in V$. The Siegel theta function attached to L is defined by

$$\Theta_L(\tau, v\,; r, t) = \sum_{\gamma \in L'/L} \mathfrak{e}_\gamma \theta_{L+\gamma}(\tau, v\,; r, t), \tag{2.4}$$

$$\theta_{L+\gamma}(\tau, v\,; r, t) = \sum_{\lambda \in L+\gamma} \mathbf{e}\left(\tau(\lambda+t)_v^2/2 + \overline{\tau}(\lambda+t)_{v^\perp}^2/2 - (\lambda + t/2,\, r)\right). \tag{2.5}$$

When $r = t = 0$, we write simply $\Theta_L(\tau, v)$ and $\theta_{L+\gamma}(\tau, v)$ for them.

By the standard argument using the Poisson summation formula, Borcherds showed the following transformation behavior ([2, Theorem 4.1]).

Theorem 2.1. *If* $\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \sqrt{c\tau+d}\, \right) \in Mp_2(\mathbb{Z})$, *we have*

$$\begin{aligned}&\Theta_L(\,(a\tau+b)/(c\tau+d), v\,;\, ar+bt, cr+dt\,)\\ &\quad = \sqrt{c\tau+d}^{\,b^+}\, \overline{\sqrt{c\tau+d}}^{\,b^-}\, \rho_L(\, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \sqrt{c\tau+d}\,)\, \Theta_L(\tau, v\,; r, t).\end{aligned}$$

Moreover, if $g \in O_d(L)$ *then* $\Theta_L(\tau, gv) = \Theta_L(\tau, v)$.

Now we can define the singular theta integral. We recall the basic ideas in [2]. If one wants to define the integral $\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}} \langle F(\tau), \Theta_L(\tau, v)\rangle\, y^{\frac{b^+}{2}} \frac{dxdy}{y^2}$ which is not absolutely convergent, we need to regularize it in two steps. First, if we denote by $\mathcal{F} = \{\tau \in \mathfrak{H} \,|\, |\tau| \geq 1, |\mathrm{Re}(\tau)| \leq \frac{1}{2}\}$ the standard fundamental domain of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}$ and put $\mathcal{F}_u = \{\tau \in \mathcal{F} \,|\, \mathrm{Im}(\tau) \leq u\}$, then we try to define it by $\lim_{u\to\infty} \int_{\mathcal{F}_u} \langle F(\tau), \Theta_L(\tau, v)\rangle\, y^{\frac{b^+}{2}} \frac{dxdy}{y^2}$. However, the limit may diverge and needs one more regularization. If we multiply the integrand by y^{-s}, we can show that it converges for $\mathrm{Re}(s) \gg 0$ and can be continued meromorphically in s. Then we take the constant term in the Laurent expansion at $s = 0$ and define the theta integral by its value. (Another definition of the singular theta integral using non-holomorphic Poincare series is given in [6]. It is more natural and gives precise information but requires a bit of work. The holomorphic infinite product constructed by the two methods are the same. Thus we adopt the above definition to avoid computational complications.)

Definition 2.2. If F is a nearly holomorphic modular form of weight $\frac{b^+-b^-}{2}$ with respect to ρ_L and $\mathrm{Mp}_2(\mathbb{Z})$, then the singular theta integral Φ_F of F is defined by

$$\Phi_F(v) = \mathop{\mathcal{C}}_{s=0} \left[\lim_{u\to\infty} \int_{\mathcal{F}_u} \langle F(\tau), \Theta_L(\tau, v)\rangle\; y^{\frac{b^+}{2}}\, \frac{dxdy}{y^{2+s}} \right]$$

for $v \in Gr(L)$, where $\mathop{\mathcal{C}}_{s=0}$ means the constant term of the Laurent expansion at $s = 0$.

We can calculate the singularity of the singular theta integral and find that it has the logarithmic singularity along the Heegner divisors which corresponds to the singularity of the nearly holomorphic modular form. This gives the information on the zeros or poles of Borcherds product.

The Heegner divisors are defined as follows. We put $\lambda^\perp = \{\, v \in Gr(L) \,|\, v \perp \lambda \,\}$ for $\lambda \in L'$ with negative norm. The subset $\lambda^\perp$ is isomorphic to a Grassmannian manifold attached to a quadratic space of signature $(b^+,\, b^- - 1)$. For $\beta \in L'$ and $m \in \mathbb{Q}_{<0}$, the Heegner divisor $H(\beta, m)$ of index (β, m) in $Gr(L)$ is defined by

$$H(\beta, m) = \bigcup_{\substack{\lambda \in L + \beta \\ \lambda^2/2 = m}} \lambda^\perp \,. \tag{2.6}$$

We easily verify that $H(\beta, m)$ is a locally finite union of the sub-Grassmannians.

For $b^+ = 2$, the Grassmannian $Gr(L)$ has a complex structure. We see this in some detail. Let $P(V_\mathbb{C})$ be the projective space associated to $V_\mathbb{C} = V \otimes_\mathbb{R} \mathbb{C}$. We denote by $[Z_L]$ the canonical image in $P(V_\mathbb{C})$ of $Z_L \in V_\mathbb{C} \setminus \{0\}$. We put $\mathcal{K} = \{\, [Z_L] \in P(V_\mathbb{C}) \,|\, (Z_L, Z_L) = 0, (Z_L, \overline{Z_L}) > 0 \,\}$. Then it is easily seen that $\mathcal{K}$ is a complex manifold and consists of two connected components which are preserved by the transitive action of the connected component $O^+(V)$ of the identity of the special orthogonal group $O(V)$. The action of $O(V) \backslash O^+(V)$ interchanges them. We denote by $\mathcal{K}^+$ one fixed connected component of $\mathcal{K}$. For $v \in Gr(L)$, we choose an oriented base X_L, Y_L of v that satisfies $X_L \perp Y_L$ and ${X_L}^2 = {Y_L}^2 > 0$. The image of $Z_L = X_L + iY_L$ in $P(V_\mathbb{C})$ is an element of $\mathcal{K}$. If the orientation is suitably chosen, the mapping $v \mapsto [Z_L]$ is a bijection from $Gr(L)$ onto $\mathcal{K}^+$ and defines a complex structure on $Gr(L)$.

If L has a norm 0 vector, a tube domain realization of $Gr(L)$ is also defined. Suppose that $z \in L$ is a primitive norm 0 vector. We take $z' \in L'$ with $(z, z') = 1$. Then we put $K = L \cap z^\perp \cap z'^\perp$. The Lorentzian lattice K is isomorphic to $(L \cap z^\perp)/\mathbb{Z}z$ and introduces coordinates on V as $V = (K \otimes \mathbb{R}) \oplus \mathbb{R}z' \oplus \mathbb{R}z$. If $Z_L = Z + az' + bz \in V_\mathbb{C}$ with $Z \in K \otimes \mathbb{C}$ and $a, b \in \mathbb{C}$, then we write $Z_L = (Z, a, b)$ for it. If a vector $Z \in K \otimes \mathbb{C}$ has positive imaginary part, the image of $(Z, 1, -Z^2/2 - {z'}^2/2)$ in $P(V_\mathbb{C})$ is an element of $\mathcal{K}$. We easily see that this mapping is a biholomorphic isomorphism from vectors in $K \otimes \mathbb{C}$ with positive imaginary part onto $\mathcal{K}$. Corresponding to $\mathcal{K}^+$, we choose one of the two connected components of $\{\, Y \in K \otimes \mathbb{R} \,|\, Y^2 > 0 \,\}$ and denote it by $\mathcal{C}$. Thus we have a tube domain realization $\mathfrak{H}_{b^-} = \{\, Z \in K \otimes \mathbb{C} \mid \mathrm{Im}(Z) \in \mathcal{C} \,\}$ of $Gr(L)$ associated to z and z'.

The main theorem in [2] is described in the tube domain realization $\mathfrak{H}_{b^-}$. For example, the sub-Grassmannian $\lambda^\perp$ for $\lambda = \lambda_K + az' + bz \in L'$ is written as $\lambda^\perp = \{\, Z \in \mathfrak{H}_{b^-} \mid aZ^2/2 - (\lambda_K, Z) - a{z'}^2/2 - b = 0 \,\}$.

As we have seen above, the sub-Grassmannians in $Gr(K)$ are also defined. In contrast to the case of L, the Grassmannian $Gr(K)$ which is isomorphic to $\{\, Y/|Y| \,|\, Y \in \mathcal{C} \,\}$ is real hyperbolic space so that $\lambda^\perp$ for $\lambda \in K'$ with $\lambda^2 < 0$ is a hyperplane of codimension 1. Thus the complement of all of them in $Gr(K)$ is not connected. We call its connected components the Weyl chambers of $Gr(K)$. If W is a Weyl chamber, then we will also call the subset $\{\, Z \in \mathfrak{H}_{b^-} \,|\, \mathrm{Im}(Z) \in W \,\}$ of $\mathfrak{H}_{b^-}$ a Weyl chamber. (In [6], Weyl chambers are defined in more sharpened form. We could not reach it in our approach.) If $\lambda \in K'$, we easily verify that $(\lambda, Z) > 0$ for any $Z \in W$ is equivalent to $(\lambda, Z) > 0$ for some $Z \in W$. If this condition holds, we write $(\lambda, W) > 0$.

If L has a primitive norm 0 vector z, we can calculate the Fourier expansion of the singular theta integral Φ_F in terms of a similar theta integral for K using the partial Fourier transform of the Siegel theta function and the unfolding argument. In fact, the singular theta integral for the Lorentzian lattice K is a piecewise linear function which is linear on each Weyl chamber. Then we define the Weyl vector attached to a Weyl chamber by the vector in $K \otimes \mathbb{R}$ representing the theta integral restricted on it.

From the Fourier expansion of the theta integral, Borcherds' infinite product is constructed. The convergence of the product is proved by the Hardy-Ramanujan-Rademacher asymptotics (see [1, Lemma 5.3]). Moreover, its various properties can be shown through the relation to the theta integral.

To describe the results of Borcherds precisely, we have to prepare one more notation. If we put $L'_0 = \{ \lambda \in L' \mid (\lambda, z) \equiv 0 \bmod N \}$ where N is the unique positive integer with $(z, L) = N\mathbb{Z}$, we have the projection $p : L'_0/L \to K'/K$ defined as follows. Let $\zeta \in L$ be a vector with $(z, \zeta) = N$. Then the lattice L can be written as $L = K \oplus \mathbb{Z}\zeta \oplus \mathbb{Z}z$ ([6, Proposition 2.2]). We have a mapping $p : L'_0 \to K'$ defined by $p(\lambda) = \lambda_K - \frac{(\lambda,z)}{N}\zeta_K$. Here λ_K for $\lambda \in L \otimes \mathbb{Q}$ is the orthogonal projection to $K \otimes \mathbb{R}$ with respect to the coordinates introduced by z and z'. Since p maps L onto K, we have the induced mapping $p : L'_0/L \to K'/K$.

We now state Theorem 13.3 of [2].

Theorem 2.2. *Let L be a non-degenerate even integral lattice of signature $(2, b^-)$. Assume that L contains a primitive norm 0 vector z. We take $z' \in L'$ with $(z, z') = 1$ and put $K = L \cap z^\perp \cap z'^\perp$. We assume moreover that K also contains a primitive norm 0 vector.*

Let $F(\tau) = \sum_{\gamma \in L'/L} \mathfrak{e}_\gamma \sum_{\substack{n \in \mathbb{Q} \\ n \gg -\infty}} c_\gamma(n)\, \mathbf{e}(n\tau)$ be a nearly holomorphic modular form of weight $1 - \frac{b^-}{2}$ with respect to ρ_L and $Mp_2(\mathbb{Z})$. We suppose that the coefficients $c_\gamma(n)$ are integral for $n < 0$. Then there exists a meromorphic function Ψ_F on $\mathfrak{H}_{b^-}$ which satisfies the following properties.

(i) *It is a meromorphic modular form on $\mathfrak{H}_{b^-}$ of weight $\frac{c_0(0)}{2}$ for the orthogonal group $\Gamma(L)$ with some multiplier system of finite order.*

(ii) *The divisor of Ψ_F is given by*

$$div(\Psi_F) = \frac{1}{2} \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z} + \gamma^2/2 \\ n < 0}} c_\gamma(n) H(\gamma, n).$$

Here the multiplicities of $H(\gamma, n)$ are 2 if $2\gamma = 0$ in L'/L and 1 if $2\gamma \neq 0$ in L'/L.

(iii) *The relation $\Phi_F(Z) = -4 \log |\Psi_F(Z)| - c_0(0) \left(2 \log |Y| + \Gamma'(1) + \log(2\pi)\right)$ holds.*

(iv) *If Z, which satisfies $|Im(Z)| \gg 0$, is in a Weyl chamber $W \subset \mathfrak{H}_{b^-}$ and* *outside the poles of* Ψ_F*, then* Ψ_F *can be expanded to an infinite product which converges absolutely and uniformly on any compact subset of that domain as follows*:

$$\Psi_F(Z) = C\mathbf{e}\left((\rho_F(W), Z)\right) \prod_{\substack{\lambda \in K' \\ (\lambda, W) > 0}} \prod_{\substack{\delta \in L_0'/L \\ p(\delta) = \lambda + K}} (1 - \mathbf{e}((\lambda, Z) + (\delta, z')))^{c_\delta(\lambda^2/2)}.$$

Here C *is a constant of absolute value*

$$\prod_{\substack{k \in \mathbb{Z}/N\mathbb{Z} \\ k \neq 0}} (1 - \mathbf{e}(k/N))^{c_{kz/N}(0)/2},$$

and $\rho_F(W) \in K \otimes \mathbb{R}$ *denotes the Weyl vector attached to* F *and* W*.*

The Weyl vector attached to F and W is defined be means of the singular theta integral for the Lorentzian lattice K. For the precise definition, see [2, Section 10] or [5, Section 3.1 and Section 3.4].

3. The higher level vector valued modular forms

In this section, we define higher level vector valued modular forms and relate them to some other scalar valued modular forms.

For an integer $N \neq 0$, we put $\widetilde{\Gamma}_0(N) = \{(\alpha, \phi(\tau)) \in \mathrm{Mp}_2(\mathbb{Z}) \mid \alpha \in \Gamma_0(N)\}$ which is the inverse image of $\Gamma_0(N)$ under the covering map $\mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z})$.

Definition 3.1. Let L be a lattice of signature (b^+, b^-) and $k \in \frac{1}{2}\mathbb{Z}$. For an integer $N \neq 0$, we let χ be a Dirichlet character modulo N. Then a holomorphic function $F : \mathfrak{H} \to \mathbb{C}[L'/L]$ is called a nearly holomorphic modular form of weight k with respect to ρ_L and $\widetilde{\Gamma}_0(N)$ with character χ if it satisfies the following conditions.

(i) $F(\alpha\tau) = \chi(d)\,\phi(\tau)^{2k}\rho_L(\alpha, \phi)F(\tau)$ for all $(\alpha, \phi) \in \widetilde{\Gamma}_0(N)$, $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

(ii) F is meromorphic at every cusps. This means if we put $(F|_k(\alpha, \phi))(\tau) = \phi(\tau)^{-2k}\rho_L(\alpha, \phi)^{-1}F(\alpha\tau)$ for any $(\alpha, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$ then we have the Fourier expansion of the form $(F|_k(\alpha, \phi))(\tau) = \sum_{\gamma \in L'/L} \mathfrak{e}_\gamma \sum_{\substack{n \in \mathbb{Q} \\ n \gg -\infty}} c_\gamma(n)\,\mathbf{e}(n\tau)$.

The $\mathbb{C}$-vector space of these functions is denoted by $M_k^!(\widetilde{\Gamma}_0(N), \chi\,\rho_L)$. When $N = 1$, we simply write $M_k^!(\rho_L)$ for it.

When the rank of a lattice is one, we can define the twist of vector valued modular forms as follows.

Proposition 3.1. *Let* N *be a non-zero integer and* $L = (2N)$*. For a Dirichlet character* χ *modulo* f *with* $f|N$*, we view it as a function on* $L'/L \simeq \mathbb{Z}/2N\mathbb{Z}$ *via a fixed generator of* L*. If* $F(\tau) = \sum_{\gamma \in L'/L} \mathfrak{e}_\gamma f_\gamma(\tau) \in M_k^!(\rho_L)$ *then* $F_\chi(\tau) = \sum_{\gamma \in L'/L} \mathfrak{e}_\gamma \chi(\gamma) f_\gamma(\tau)$ *is a modular form in* $M_k^!(\widetilde{\Gamma}_0(N), \chi\,\rho_L)$*.*

Proof. It suffices to check the transformation behavior under the action of $\widetilde{\alpha}$, $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$.

We denote a generator of L by λ_0 and put $\gamma = \frac{s}{2N}\lambda_0$, $\delta = \frac{t}{2N}\lambda_0 \in L'/L$ for $s, t \in \mathbb{Z}$. Then we write $\rho_{s,t}$ instead of $\rho_{\gamma\delta}$.

If $c \neq 0$, by virtue of Proposition 2.1 we can compute directly and find

$$\rho_{s,t}(\widetilde{\alpha}) = \frac{\sqrt{i}^{-\mathrm{sgn}(Nc)}}{\sqrt{2|Nc|}}\, \mathbf{e}\left(\frac{as^2 - 2st + dt^2}{4Nc}\right) \sum_{k=1}^{c} \mathbf{e}\left(\frac{Na}{c}k^2 + \frac{as - t}{c}k\right).$$

If we set $c = Nc'$, we have

$$\begin{aligned}
\sum_{k=1}^{c} \mathbf{e}\left(\frac{Na}{c}k^2 + \frac{as-t}{c}k\right) &= \sum_{l=0}^{c'-1} \sum_{m=1}^{N} \mathbf{e}\left(\frac{a}{c'}(l + c'm)^2 + \frac{as-t}{c}(l + c'm)\right) \\
&= \sum_{l=0}^{c'-1} \mathbf{e}\left(\frac{a}{c'}l^2 + \frac{as-t}{c}l\right) \sum_{m=1}^{N} \mathbf{e}\left(\frac{as-t}{N}m\right).
\end{aligned}$$

Therefore $\rho_{s,t}(\widetilde{\alpha}) \neq 0$ only for $t \equiv as \ (N)$. Using this we see that

$$\begin{aligned}
F_\chi(\alpha\tau)\sqrt{c\tau + d}^{-2k} &= \sum_{s\ (2N)} \mathfrak{e}_s \chi(s) \sum_{\substack{t\ (2N) \\ t \equiv as\ (N)}} \rho_{s,t}(\widetilde{\alpha}) f_t(\tau) \\
&= \chi(a)^{-1} \sum_{s\ (2N)} \mathfrak{e}_s \sum_{\substack{t\ (2N) \\ t \equiv as\ (N)}} \rho_{s,t}(\widetilde{\alpha}) \chi(t) f_t(\tau) \\
&= \chi(d) \rho_L(\widetilde{\alpha}) F_\chi(\tau).
\end{aligned}$$

Clearly this holds also when $c = 0$. □

4. Twisted Siegel theta functions

In this section, we define twisted Siegel theta functions for a lattice L of signature (b^+, b^-). This is a vector valued interpretation of that of [13].

For this purpose, we assume that L is written as $L = K \oplus M$ where K is a lattice of signature $(b^+ - 1, b^- - 1)$ and M is a lattice generated by vectors z and z' with $z^2 = z'^2 = 0$, $(z, z') = N$. The lattice M is isomorphic to $II_{1,1}(N)$.

Definition 4.1. We let $\tau \in \mathfrak{H}$, $v \in Gr(L)$ and $r, t \in V$. If χ is a Dirichlet character modulo f with $f|N$, the χ-twisted Siegel theta function attached to L is defined by

$$\begin{aligned}
\Theta_{L,\chi}(\tau, v\,; r, t) &= \sum_{\gamma \in K'/K} \mathfrak{e}_\gamma\, \theta_{K+\gamma,\chi}(\tau, v\,; r, t), \\
\theta_{K+\gamma,\chi}(\tau, v\,; r, t) &= \sum_{k\ (f)} \chi(k)\, \theta_{L+\gamma+\frac{k}{f}z}(\tau, v\,; r, t).
\end{aligned}$$

As in the full modular case, we write simply $\Theta_{L,\chi}(\tau, v)$ and $\theta_{K+\gamma,\chi}(\tau, v)$ when $r = t = 0$.

Theorem 4.1. *If* $\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \sqrt{c\tau+d}\,\right) \in \widetilde{\Gamma}_0(N)$, *we have*

$$\Theta_{L,\chi}(\,(a\tau+b)/(c\tau+d), v\,;\, ar+bt, cr+dt\,)$$
$$= \chi(d)\sqrt{c\tau+d}^{\,b^+}\overline{\sqrt{c\tau+d}}^{\,b^-}\,\rho_K(\,\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \sqrt{c\tau+d}\,)\,\Theta_{L,\chi}(\tau, v\,; r, t).$$

Moreover $\Theta_{L,\chi}(\tau, v)$ *is invariance under the action of* $O_d(L)$.

Proof. We note $\rho_L = \rho_K \otimes \rho_M$. By Theorem 2.1 we get

$$\Theta_{L,\chi}(\alpha\tau, v\,;\, ar+bt, cr+dt)\sqrt{c\tau+d}^{\,-b^+}\overline{\sqrt{c\tau+d}}^{\,-b^-}$$
$$= \sum_{\gamma\in K'/K}\mathfrak{e}_\gamma \sum_{k\,(f)}\chi(k)\,\theta_{L+\gamma+\frac{k}{f}z}(\alpha\tau, v\,; ar+bt, cr+dt)\sqrt{c\tau+d}^{\,-b^+}\overline{\sqrt{c\tau+d}}^{\,-b^-}$$
$$= \sum_{\gamma\in K'/K}\mathfrak{e}_\gamma \sum_{k\,(f)}\chi(k)\sum_{\delta\in K'/K}\sum_{\varepsilon\in M'/M}\rho_{\gamma,\delta}(\widetilde{\alpha})\rho_{\frac{k}{f}z,\varepsilon}(\widetilde{\alpha})\,\theta_{L+\delta+\varepsilon}(\tau, v\,; r, t).$$

As in the proof of Proposition 3.1, we find $\rho_{\frac{k}{f}z,\varepsilon}(\widetilde{\alpha}) = \delta_{\varepsilon,\frac{ak}{f}z}$. Therefore we obtain

$$\sum_{\gamma\in K'/K}\mathfrak{e}_\gamma \sum_{k\,(f)}\chi(k)\sum_{\delta\in K'/K}\rho_{\gamma,\delta}(\widetilde{\alpha})\,\theta_{L+\delta+\frac{ak}{f}z}(\tau, v\,; r, t)$$
$$= \sum_{\gamma\in K'/K}\mathfrak{e}_\gamma\,\chi(d)^{-1}\sum_{\delta\in K'/K}\rho_{\gamma,\delta}(\widetilde{\alpha})\sum_{k\,(f)}\chi(k)\,\theta_{L+\delta+\frac{k}{f}z}(\tau, v\,; r, t)$$
$$= \chi(d)^{-1}\sum_{\gamma\in K'/K}\mathfrak{e}_\gamma\sum_{\delta\in K'/K}\rho_{\gamma,\delta}(\widetilde{\alpha})\,\theta_{K+\delta,\chi}(\tau, v\,; r, t)$$
$$= \chi(d)\rho_K(\widetilde{\alpha})\,\Theta_{L,\chi}(\tau, v\,; r, t).$$

This proves the assertion. □

To calculate the Fourier expansion of the singular theta integral, we need to express the theta functions as the infinite sum involving theta functions for smaller lattices. For this, we just rewrite in our case Theorem 5.2 of [2] or Theorem 2.4 of [6] which is proved by means of the partial Fourier transformation. We note that z is a primitive norm 0 vector in L, $\frac{z'}{N} \in L'$ with $(z, \frac{z'}{N}) = 1$ and $(z, L) = N\mathbb{Z}$. Then we obtain the following.

Lemma 4.1. *Let* w *be the orthogonal complement of* z_v *in* v. *We put*

$$\mu = -\frac{z'}{N} + \frac{z_v}{2{z_v}^2} + \frac{z_{v^\perp}}{2z_{v^\perp}^2}$$

which is a vector of $V \cap z^\perp = (K\otimes\mathbb{R}) \oplus \mathbb{R}z$. *Then*

$$\theta_{L+\gamma+\frac{k}{f}z}(\tau, v) = \frac{1}{\sqrt{2y}\,|z_v|}\sum_{\substack{c,\,d\in\mathbb{Z}\\ c\equiv 0\,(N)}}\mathbf{e}\left(-\frac{d}{f}k\right)\exp\left(-\frac{\pi|c\tau+d|^2}{2{z_v}^2 y}\right)$$
$$\times\,\theta_{K+\gamma}(\tau, w\,; d\mu, -c\mu).$$

We denote by $G(\chi) = \sum_{k\,(f)} \chi(k)\,\mathbf{e}(\frac{k}{f})$ the Gaussian sum of a Dirichlet character χ modulo f.

Proposition 4.1. *If χ is primitive, then*

$$\theta_{K+\gamma,\,\chi}(\tau, v) = \frac{\chi(-1)\,G(\chi)}{\sqrt{2y}\,|z_v|} \sum_{\substack{c\equiv 0\ (N)\\ (d,\,\frac{fc}{N})=1}} \overline{\chi}(d) \sum_{n\geq 1} \overline{\chi}(n) \exp\left(-\frac{\pi n^2 |c\tau+d|^2}{2{z_v}^2 y}\right) \times \theta_{K+\gamma}(\tau, w\,;\, dn\mu, -cn\mu).$$

Proof. We note $\sum_{k\,(f)} \chi(k)\,\mathbf{e}(-\frac{d}{f}k) = G(\chi)\,\overline{\chi}(-d)$ since χ is primitive. From this formula and lemma 4.1, we obtain the sum over $c \equiv 0\ (N), d \in \mathbb{Z}$. Let $c = Nc'$, $c' \in \mathbb{Z}$ and $n = (c', d)$. If we replace $c'/n, d/n$ by c', d, we can take the summation over $n \geq 1, c', d \in \mathbb{Z}$ with $(c', d) = 1$. Again we put $c = Nc'$ and obtain the assertion. □

For the later use, we see how $\Theta_{L,\,\chi}(\tau, v)$ behaves under the action of $\mathrm{Mp}_2(\mathbb{Z})$. If $\alpha = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$\begin{aligned}(\Theta_{L,\,\chi}|\,\widetilde{\alpha})\,(\tau, v) =& \rho_K(\widetilde{\alpha})^{-1}\,\Theta_{L,\,\chi}(\alpha\tau, v)\sqrt{c\tau+d}^{\,-b^+}\overline{\sqrt{c\tau+d}}^{\,-b^-}\\ =& \sum_{\gamma\in K'/K} \mathfrak{e}_\gamma \sum_{k\,(f)} \chi(k) \sum_{\varepsilon\in M'/M} \rho_{\frac{k}{f}z,\varepsilon}(\widetilde{\alpha})\,\theta_{L+\gamma+\varepsilon}(\tau, v)\\ =& \sum_{\gamma\in K'/K} \mathfrak{e}_\gamma \sum_{\varepsilon\in M'/M} \left(\sum_{k\,(f)} \chi(k)\,\rho_{\frac{k}{f}z,\varepsilon}(\widetilde{\alpha})\right) \theta_{L+\gamma+\varepsilon}(\tau, v).\end{aligned}$$

In the rest of this section, we compute the constant $\sum_{k\,(f)} \chi(k)\,\rho_{\frac{k}{f}z,\,\varepsilon}(\widetilde{\alpha})$ when χ is primitive. If $c = 0$, it can be easily observed by Proposition 2.1. In the case of $c \neq 0$ we also have the expression of it as a finite product of roots of unity, Gaussian sums, values of Dirichlet characters and rational numbers. As we will not need this result in the subsequent sections, the reader may skip the following proposition.

Proposition 4.2. *For a non-zero integer N, we put $M = \mathbb{Z}z + \mathbb{Z}z'$ with $z^2 = {z'}^2 = 0$, $(z, z') = N$. Let χ be a primitive Dirichlet character modulo f with $f|N$ and $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ with $c \neq 0$. If P is the set of prime divisors of f, we write $f = \prod_{p\in P} f_p$, $\chi = \prod_{p\in P} \chi_p$ where f_p (resp. χ_p) is the p-component of f (resp. χ). In this case, χ_p is a primitive Dirichlet character modulo f_p.*

We fix integers s and t. For an integer n, we put $n' = n/(N, c)$, $n'' = n/(N/f, (N, c))$. We take integers β_0 and γ_0 satisfying $N'a\beta_0 + c'\gamma_0 = 1$.

Let P_1 (resp. P_2) be the set of primes consisting of $p \in P$ which satisfies $(N, c)''_p \geq f_p^{1/2}$ and $\chi_p(1 + (N, c)''_p) = \mathbf{e}\left((N/f)''^{-1}(ds)''\gamma_0 t'(N, c)''_p/f_p\right)$ (resp. $(N, c)''_p < f_p^{1/2}$ and $\chi_p(1 + f_p/(N, c)''_p) = \mathbf{e}\left((N/f)''^{-1}(ds)''\gamma_0 t'/(N, c)''_p\right)$). The expression $(N, c)''_p$ means the p-component of the integer $(N, c)''$ and $(N/f)''^{-1}$ means some representative in $\mathbb{Z}$ of $(N/f)''$ mod $(N, c)''$.

If $s \in \left(\frac{N}{f},\,(N,c)\right)\mathbb{Z}$, $t \in (N,c)\mathbb{Z}$ *and* $P = P_1 \cup P_2$*, then on the constant involving the matrix coefficients of* ρ_M *we find*

$$\begin{aligned}
&\sum_{k\,(f)} \chi(k)\,\rho_{\frac{k}{f}z,\frac{s}{N}z+\frac{t}{N}z'}\left(\widetilde{\begin{pmatrix} a & b \\ c & d\end{pmatrix}}\right) \\
&\qquad = \frac{1}{|N'|}\,\mathbf{e}\left(-\frac{N'\beta_0 - d}{N'c}st'\right)\prod_{p\neq q}\chi_p(f_q) \\
&\qquad\qquad \times \prod_{p\in P_1}\chi_p((N/f)''^{-1}(ds)'')\,\mathbf{e}\left(-\frac{(N/f)''^{-1}(ds)''\gamma_0 t'}{f_p}\right)\frac{f_p}{(N,c)''_p} \\
&\qquad\qquad \times \prod_{p\in P_2} G(\chi_p)\,\overline{\chi}_p(-\gamma_0 t'),
\end{aligned}$$

and otherwise 0*. Here the product* $\prod_{p\neq q}$ *is taken over all pairs of different primes in* P

Proof. We compute matrix coefficients of ρ_M by Proposition 2.1. For $c \neq 0$, we have

$$\begin{aligned}
&\sum_{k\,(f)} \chi(k)\,\rho_{\frac{k}{f}z,\frac{s}{N}z+\frac{t}{N}z'}(\widetilde{\alpha}) \\
&\quad = \frac{1}{N|c|}\sum_{k\,(f)}\chi(k)\sum_{\alpha,\,\beta\,(c)}\mathbf{e}\left(\frac{a}{2c}\left(\left(\frac{k}{f}+\alpha\right)z+\beta z'\right)^2\right. \\
&\qquad\qquad\qquad \left. -\frac{1}{c}\left(\left(\frac{k}{f}+\alpha\right)z+\beta z',\,\frac{s}{N}z+\frac{t}{N}z'\right)+\frac{d}{2c}\left(\frac{s}{N}z+\frac{t}{N}z'\right)^2\right) \\
&\quad = \frac{1}{N|c|}\,\mathbf{e}\left(\frac{dst}{Nc}\right)\sum_{k\,(f)}\chi(k)\,\mathbf{e}\left(-\frac{t}{c}\frac{k}{f}\right)\sum_{\beta\,(c)}\mathbf{e}\left(\left(\frac{Na}{f}k - s\right)\frac{\beta}{c}\right) \\
&\qquad\qquad \times \sum_{\alpha\,(c)}\mathbf{e}\left((Na\beta - t)\frac{\alpha}{c}\right).
\end{aligned}$$

The sum over α is not 0 only if $Na\beta \equiv t\ (c)$. In that case, t satisfies $t \in (N,c)\mathbb{Z}$ so that the summation over β can be replaced by β with $N'a\beta \equiv t'$ (c') where primes have been defined in Proposition 4.2.

We have taken integers β_0 and γ_0 which satisfy $N'a\beta_0 + c'\gamma_0 = 1$. Then the summation over β (c) with $N'a\beta \equiv t'$ (c') can be replaced by $\beta = \beta_0 t' + c'l$

with $l = 1, \ldots, (N,c)$. We immediately see

$$\begin{aligned}
&\sum_{\substack{\beta\ (c) \\ N'a\beta \equiv t'\ (c')}} \mathbf{e}\left(\left(\frac{Na}{f}k - s\right)\frac{\beta}{c}\right) \\
&= \mathbf{e}\left(-\frac{\beta_0 st'}{c}\right)\mathbf{e}\left(\frac{N'a\beta_0 t'}{fc'}k\right)\sum_{l=1}^{(c,N)} \mathbf{e}\left(\left(\frac{Na}{f}k - s\right)\frac{l}{(N,c)}\right)
\end{aligned}$$

The sum is not 0 only if $\frac{Na}{f}k \equiv s\ ((N,c))$.

Consequently we find that if $s \in \left(\frac{N}{f},\ (N,c)\right)\mathbb{Z}$ and $t \in (N,c)\mathbb{Z}$ then

$$\begin{aligned}
&\sum_{k\ (f)} \chi(k)\, \rho_{\frac{k}{f}z,\ \frac{s}{N}z + \frac{t}{N}z'}(\widetilde{\alpha}) \\
&= \frac{1}{|N'|}\mathbf{e}\left(\frac{dst}{Nc}\right) \sum_{\substack{k\ (f) \\ \frac{Na}{f}k \equiv s\ ((N,c))}} \chi(k)\,\mathbf{e}\left(-\frac{t}{c}\frac{k}{f}\right)\mathbf{e}\left(-\frac{\beta_0 st'}{c}\right)\mathbf{e}\left(\frac{N'a\beta_0 t'}{fc'}k\right) \\
&= \frac{1}{|N'|}\mathbf{e}\left(\frac{dst}{Nc}\right)\mathbf{e}\left(-\frac{\beta_0 st'}{c}\right) \sum_{\substack{k\ (f) \\ \frac{Na}{f}k \equiv s\ ((N,c))}} \chi(k)\,\mathbf{e}\left(\frac{(N'a\beta_0 - 1)\,t'}{fc'}k\right) \\
&= \frac{1}{|N'|}\mathbf{e}\left(-\frac{N'\beta_0 - d}{N'c}st'\right) \sum_{\substack{k\ (f) \\ \frac{Na}{f}k \equiv s\ ((N,c))}} \chi(k)\,\mathbf{e}\left(-\frac{\gamma_0 t'}{f}k\right).
\end{aligned}$$

Otherwise it equals 0.

We can replace the summation over β with $\frac{Na}{f}k \equiv s\ ((N,c))$ by β with $k \equiv \left(\frac{N}{f}\right)^{\prime\prime -1}(ds)''\ ((N,c)'')$. Here tilde has been defined in the Proposition. We easily find $(N,c)''$ divides f.

To calculate the sum, we concentrate on this Gaussian type sum and investigate its properties.

If χ is a primitive Dirichlet character modulo f, we put

$$G(k, \chi\,; \alpha, \beta) = \sum_{\substack{n\ (f) \\ n \equiv \alpha\ (k)}} \chi(n)\,\mathbf{e}\left(\frac{\beta}{f}n\right) \tag{4.1}$$

for $0 < k | f$ and $\alpha, \beta \in \mathbb{Z}$.

The next lemma is well known for the usual Gaussian sum.

Lemma 4.2. *We let χ_1, χ_2 be primitive Dirichlet characters modulo f_1, f_2 respectively and k_1, k_2 be integers such that $k_1 | f_1$, $k_2 | f_2$. If $(f_1, f_2) = 1$, then*

$$G(k_1k_2, \chi_1\chi_2\,; \alpha, \beta) = \chi_1(f_2)\,\chi_2(f_1)\,G(k_1, \chi_1\,; \alpha, \beta)\,G(k_2, \chi_2\,; \alpha, \beta).$$

Proof. For an integer n, we take integers n_1, n_2 such that $n = n_2 f_1 + n_1 f_2$. We easily find that $n \equiv \alpha \ (k_1 k_2)$ is equivalent to $n_1 \equiv \alpha \ (k_1)$, $n_2 \equiv \alpha \ (k_2)$. Then the assertion immediately follows from the usual argument. $\square$

By the preceding lemma, we can reduce the computation to the case of $f = p^e$, where p is a prime number. In this case, we write $G(r, \chi\,; \alpha, \beta)$ instead of $G(p^r, \chi\,; \alpha, \beta)$.

First, we show the following lemma.

Lemma 4.3. *For $0 \le r \le e$, we have*

$$G(r, \chi\,; \alpha, \beta) = \frac{p^{e-r}}{G(\overline{\chi})} \, \mathbf{e}\left(\frac{\alpha\beta}{p^e}\right) G(e-r, \overline{\chi}\,; -\beta, \alpha).$$

Proof. As in the preceding computations, we replace $\chi(n)$ by its Fourier expansion and find

$$\begin{aligned} G(r, \chi\,; \alpha, \beta) &= \sum_{\substack{n \ (p^e) \\ n \equiv \alpha \ (p^r)}} \left(\sum_{k \ (p^e)} \frac{1}{G(\overline{\chi})} \, \overline{\chi}(k) \, \mathbf{e}\left(\frac{n}{p^e} k\right) \right) \mathbf{e}\left(\frac{\beta}{p^e} n\right) \\ &= \frac{1}{G(\overline{\chi})} \sum_{k \ (p^e)} \overline{\chi}(k) \sum_{\substack{n \ (p^e) \\ n \equiv \alpha \ (p^r)}} \mathbf{e}\left(\frac{k+\beta}{p^e} n\right). \end{aligned}$$

If we write in the later sum $n = \alpha + p^r l$, $l = 1, \ldots, p^{e-r}$, then we have

$$\begin{aligned} &\frac{1}{G(\overline{\chi})} \, \mathbf{e}\left(\frac{\alpha\beta}{p^e}\right) \sum_{k \ (p^e)} \overline{\chi}(k) \, \mathbf{e}\left(\frac{\alpha}{p^e} k\right) \sum_{l=1}^{p^{e-r}} \mathbf{e}\left(\frac{k+\beta}{p^{e-r}} l\right) \\ &\quad = \frac{p^{e-r}}{G(\overline{\chi})} \, \mathbf{e}\left(\frac{\alpha\beta}{p^e}\right) \sum_{\substack{k \ (p^e) \\ k \equiv -\beta \ (p^{e-r})}} \overline{\chi}(k) \, \mathbf{e}\left(\frac{\alpha}{p^e} k\right) \\ &\quad = \frac{p^{e-r}}{G(\overline{\chi})} \, \mathbf{e}\left(\frac{\alpha\beta}{p^e}\right) G(e-r, \overline{\chi}\,; -\beta, \alpha). \end{aligned}$$

Thus we proved the assertion. $\square$

We readily find $G(r, \chi\,; \alpha, \beta) = \chi(\alpha) \, G(r, \chi\,; 1, \alpha\beta)$. If $r \ge e/2$, the mapping $l \mapsto \chi(1 + p^r l)$ is an additive character of $\mathbb{Z}/p^{e-r}\mathbb{Z}$. Therefore we may write $\chi(1 + p^r l) = \mathbf{e}\left(\frac{\gamma}{p^{e-r}} l\right)$ for some $\gamma \in \mathbb{Z}$, so that we obtain

$$\begin{aligned} G(r, \chi\,; 1, \alpha\beta) &= \mathbf{e}\left(\frac{\alpha\beta}{p^e}\right) \sum_{l=1}^{p^{e-r}} \mathbf{e}\left(\frac{\gamma + \alpha\beta}{p^{e-r}} l\right) \\ &= \begin{cases} \mathbf{e}\left(\frac{\alpha\beta}{p^e}\right) p^{e-r} & \text{if} \quad \chi(1+p^r) = \mathbf{e}\left(-\frac{\alpha\beta}{p^{e-r}}\right), \\ 0 & \text{if} \quad \chi(1+p^r) \ne \mathbf{e}\left(-\frac{\alpha\beta}{p^{e-r}}\right). \end{cases} \end{aligned}$$

If $r < e/2$, then we can compute $G(r, \chi\,; 1, \alpha\beta)$ by Lemma 4.3.

As a result, we obtain

$$G(r, \chi\,; \alpha, \beta) = \begin{cases} \chi(\alpha)\,\mathbf{e}\left(\frac{\alpha\beta}{p^e}\right) p^{e-r} & \text{if} \quad r \geq e/2,\ \chi(1+p^r) = \mathbf{e}\left(-\frac{\alpha\beta}{p^{e-r}}\right), \\ G(\chi)\,\overline{\chi}(\beta) & \text{if} \quad r < e/2,\ \chi(1+p^{e-r}) = \mathbf{e}\left(-\frac{\alpha\beta}{p^r}\right), \\ \quad 0 & \text{otherwise.} \end{cases}$$

This proves Proposition 4.2. □

5. The Singular theta integral

In this section, we define the singular theta integral of a higher level nearly holomorphic vector valued modular form.

In order to define the singular theta integral of a nearly holomorphic modular form with respect to $\widetilde{\Gamma}_0(N)$, we just consider the integration on $\Gamma_0(N)\backslash\mathfrak{H}$ and regularize it on each cusp by a similar way as in the full modular case.

Although we can not take the standard fundamental domain for $\Gamma_0(N)$, we choose one of them and truncate it. Then we multiply the integrand by y^{-s} corresponding to each cusp so that the invariance of it under the action of $\Gamma_0(N)$ does not hold. However if we take the limit to the whole domain, we can show that the integral converges when $\mathrm{Re}(s)$ is sufficiently large and can be meromorphically continued to the whole s-plane. Taking its constant term of the Laurent expansion at $s = 0$, we obtain the singular theta integral which is a function on $Gr(L)$ invariant under $\Gamma(L)$.

As before, let K be a lattice of signature $(b^+ - 1,\, b^- - 1)$ and $M = \mathbb{Z}z + \mathbb{Z}z'$ with $z^2 = {z'}^2 = 0$, $(z,\, z') = N$. We put $L = K \oplus M$. We let χ be a Dirichlet character modulo f with $f|N$.

Let $\{\, l\,\}$ be the complete representatives of the set of $\Gamma_0(N)$-equivalent classes of cusps and h_l be the width of a cusp l. We take $\alpha_l \in \mathrm{SL}_2(\mathbb{Z})$ such that $\alpha_l(\infty) = l$ for each l. We consider the integration over $\coprod_l \alpha_l \mathcal{F}_{l,u}$ where $\mathcal{F}_{l,u} = \{\, \tau \pm \frac{i}{2} \,|\, \tau \in \mathcal{F}_u,\, i = 1, \ldots, h_l \,\}$.

Definition 5.1. If $F(\tau) \in M^!_{\frac{b^+ - b^-}{2}}(\widetilde{\Gamma}_0(N),\, \chi\,\rho_K)$ and $v \in Gr(L)$, then the singular theta integral $\varPhi_F$ of F is defined by

$$\varPhi_F(v) = \underset{s=0}{\mathcal{C}} \left[\lim_{u \to \infty} \sum_l \int_{\mathcal{F}_{l,u}} \langle F(\alpha_l \tau),\, \Theta_{L,\,\chi}(\alpha_l \tau, v) \rangle \; \mathrm{Im}(\alpha_l \tau)^{\frac{b^+}{2}} \frac{dxdy}{y^{2+s}} \right].$$

Proposition 5.1. *The regularized integral of $\varPhi_F$ converges when $Re(s) > \frac{b^+}{2} - 1$ and can be meromorphically continued to the complex s-plane. It may have a simple pole at $s = \frac{b^+}{2} - 1$ and is holomorphic elsewhere. Therefore the singular theta integral $\varPhi_F(v)$ is well-defined for any $v \in Gr(L)$.*

Proof. The proof is identical to that of [2] Theorem 7.1. We follow its argument.

It suffices to prove the convergence and meromorphic continuation of the integral

$$\int_{y>1}\int_{0<x<h_l} \langle\, (F|\,\widetilde{\alpha}_l)\,(\tau),\ (\Theta_{L,\chi}|\,\widetilde{\alpha}_l)\,(\tau,v)\,\rangle\, y^{\frac{b^+}{2}-2-s}\, dxdy \tag{5.1}$$

for each l. We put $(F|\,\widetilde{\alpha}_l)\,(\tau) = \sum_{\gamma\in K'/K} \mathfrak{e}_\gamma \sum_{\substack{n\in\mathbb{Q}\\ n\gg-\infty}} c_\gamma(n)_l\, q^n$. The inner product of (5.1) is expanded as

$$\begin{aligned}
&\langle\, (F|\,\widetilde{\alpha}_l)\,(\tau),\ (\Theta_{L,\chi}|\,\widetilde{\alpha}_l)\,(\tau,v)\,\rangle\\
&= \sum_{\substack{\gamma\in K'/K\\ \varepsilon\in M'/M}} \sum_{k\,(f)} \overline{\chi}(k)\,\overline{\rho}_{\frac{k}{f}z,\varepsilon}(\widetilde{\alpha}_l) \sum_{\substack{n\in\mathbb{Q}\\ n\gg-\infty}} c_\gamma(n)_l\, q^n\, \overline{\theta_{L+\gamma+\varepsilon}(\tau,v)}
\end{aligned}$$

If we carry out the integral over x, we obtain a linear combination of

$$\sum_{\substack{\lambda\in L+\gamma+\varepsilon\\ n=\lambda^2/2}} c_\gamma(n)_l\, \exp\left(-2\pi\left(n+\frac{{\lambda_v}^2-\lambda_{v^\perp}^2}{2}\right)y\right)\, y^{\frac{b^+}{2}-2-s}. \tag{5.2}$$

We will verify that the integral over $y>1$ of (5.2) is absolutely convergent except for finitely many terms and remaining terms can be continued meromorphically.

Since $(F|\,\widetilde{\alpha}_l)\,(\tau)$ grows at most exponentially as $y\to\infty$, there exists a constant $A>0$ such that $|c_\gamma(n)_l|\exp(-2\pi ny)\le\exp(2\pi Ay)$ for any $y>1, \gamma\in K'/K$ and $n\in\mathbb{Q}$. If we put $\sigma=\mathrm{Re}(s)$, we have an estimate

$$\begin{aligned}
&\int_{y>1} \sum_{\substack{\lambda\in L+\gamma+\varepsilon\\ n=\lambda^2/2}} |c_\gamma(n)_l|\, \exp\left(-2\pi\left(n+\frac{{\lambda_v}^2-\lambda_{v^\perp}^2}{2}\right)y\right)\, y^{\frac{b^+}{2}-2-\sigma}\, dy\\
&\le \int_{y>1} \sum_{\lambda\in L+\gamma+\varepsilon} \exp\left(2\pi\left(A-\frac{{\lambda_v}^2-\lambda_{v^\perp}^2}{2}\right)y\right)\, y^{\frac{b^+}{2}-2-\sigma}\, dy.
\end{aligned}$$

Except for finitely many λ, we have $A-\frac{{\lambda_v}^2-\lambda_{v^\perp}^2}{2} < -\frac{1}{2\pi}\left(\frac{{\lambda_v}^2-\lambda_{v^\perp}^2}{2}\right)$. For these λ, the above integral and sum are estimated by

$$\int_{y>1} \sum_{\lambda\in L'\smallsetminus\{0\}} \exp\left(-\frac{{\lambda_v}^2-\lambda_{v^\perp}^2}{2}\,y\right)\, y^{\frac{b^+}{2}-2-\sigma}\, dy \tag{5.3}$$

Since the positive definite norm in the bracket attains the minimum for $\lambda\in L'\smallsetminus\{0\}$, it is easily seen that (5.3) is convergent for any s.

For remaining finitely many λ, we have to show that the integral

$$c_\gamma(\lambda^2/2)_l \int_{y>1} \exp(\,-2\pi\lambda_v^2 y\,)\, y^{\frac{b^+}{2}-2-s}\, dy$$

can be meromorphically continued to the whole s-plane. If $\lambda_v^2 > 0$ it converges absolutely. If $\lambda_v^2 = 0$ then we may compute it explicitly.

This completes the proof of Proposition 5.1. $\square$

Now we calculate the singularity of the singular theta integral in the sense of [2]. It lies on Heegner divisors as in the full modular case and in contrast to that case it occurs from each cusp of $\Gamma_0(N)$.

We first recall the definition of singularity. Let X be a real analytic manifold and f, g be functions on some open dense subset of X. We say that f has a singularity of type g at $x \in X$ if $f - g$ can be continued to a real analytic function near x.

Proposition 5.2. *Suppose $F(\tau) \in M^!_{\frac{b^+-b^-}{2}}(\widetilde{\Gamma}_0(N), \chi\,\rho_K)$ has its Fourier expansion $(F|\,\widetilde{\alpha}_l)(\tau) = \sum_{\gamma\in K'/K} \mathfrak{e}_\gamma \sum_{\substack{n\in\mathbb{Q} \\ n\gg-\infty}} c_\gamma(n)_l\, q^n$ for each l. Then the singularity of $\Phi_F(v)$ at $v_0 \in Gr(L)$ is given by*

$$\sum_l \sum_{\substack{\lambda\in L'\setminus\{0\} \\ \lambda_{v_0}=0}} \left(\sum_{k\,(f)} \overline{\chi}(k)\, \overline{\rho}_{\frac{k}{f}z,\lambda_M}(\widetilde{\alpha}_l) \right) h_l\, c_{\lambda_K}(\lambda^2/2)_l\, (-2\pi\lambda_v^2)^{1-\frac{b^+}{2}} \log(\lambda_v^2)$$

if $b^+ = 0, 2$ and

$$-\sum_l \sum_{\substack{\lambda\in L'\setminus\{0\} \\ \lambda_{v_0}=0}} \left(\sum_{k\,(f)} \overline{\chi}(k)\, \overline{\rho}_{\frac{k}{f}z,\lambda_M}(\widetilde{\alpha}_l) \right) h_l\, c_{\lambda_K}(\lambda^2/2)_l\, (2\pi\lambda_v^2)^{1-\frac{b^+}{2}}\, \Gamma\left(-1+\frac{b^+}{2}\right)$$

otherwise. Here we denote the orthogonal projection of $\lambda \in L'$ to $K\otimes\mathbb{R}$ (resp. $M\otimes\mathbb{R}$) by λ_K (resp. λ_M).

Proof. It suffices to compute the singularity of

$$\mathop{\mathcal{C}}_{s=0}\left[\int_{y>1}\int_{0<x<h_l} \langle\, (F|\,\widetilde{\alpha}_l)(\tau),\, (\Theta_{L,\chi}|\,\widetilde{\alpha}_l)(\tau, v)\,\rangle\; y^{\frac{b^+}{2}-2-s}\, dxdy\right].$$

As we have seen in the proof of Proposition 5.1, it is the constant term of

$$-\sum_{\lambda\in L'} \left(\sum_{k\,(f)} \overline{\chi}(k)\, \overline{\rho}_{\frac{k}{f}z,\lambda_M}(\widetilde{\alpha}_l) \right) h_l\, c_{\lambda_K}(\lambda^2/2)_l \int_{y>1} \exp(-2\pi\lambda_v^2 y)\, y^{\frac{b^+}{2}-2-s}\, dy.$$

Since $\lambda = 0$ and $\lambda \in L'$ with $\lambda_{v_0}^2 > 0$ give real analytic functions near v_0, it is sufficient to consider only $\lambda \in L'\setminus\{0\}$ with $\lambda_{v_0} = 0$. From [2, Lemma 6.1], the singularity from these λ can be worked out. Then we have the assertion. $\square$

We can compute the Fourier expansion of $\Phi_F(v)$ using the unfolding trick as in [2] or [6] under some conditions. Thus the proof of Theorem 7.1 in [2] works also in our case.

Before stating the theorem, we recall one more notation. For $v \in Gr(L)$ let w be the orthogonal complement of z_v in v. The orthogonal projection $L \otimes \mathbb{R} \to K \otimes \mathbb{R}$ induces an isometric embedding of w into $K \otimes \mathbb{R}$ (see [6, p.42]). Thus for $\lambda \in K'$ the notation λ_w makes sense.

Theorem 5.1. *Let K be a Lorentzian lattice and χ be a primitive Dirichlet character modulo f with $f|N$. We assume $\{p; p|f\} = \{p; p|N\}$. Let $F(\tau) = \sum_{\gamma \in K'/K} \mathfrak{e}_\gamma \sum_{\substack{n \in \mathbb{Q} \\ n \gg -\infty}} c_\gamma(n)\, q^n$ be in $M^{!}_{1-\frac{b^-}{2}}(\widetilde{\Gamma}_0(N), \chi\, \rho_K)$. If $v \in Gr(L)$ is outside the Heegner divisors and $|z_v| \ll 1$, then we have the Fourier expansion*

$$\begin{aligned}\Phi_F(v) &= 2\,\chi(-1)\,\overline{G(\chi)}\,L(1,\chi)\,c_0(0)\\ &\quad + 2\,\chi(-1)\,\overline{G(\chi)} \sum_{\lambda \in K' \smallsetminus \{0\}} \sum_{n \geq 1} \frac{\chi(n)}{n}\, c_\lambda(\lambda^2/2)\, \mathbf{e}(n(\lambda,\mu)) \exp\left(-2\pi n \frac{|\lambda_w|}{|z_v|}\right).\end{aligned}$$

Proof. We first calculate for arbitrary even integral lattice K, then we specialize the result to the case of Lorentzian lattice.

From Proposition 4.1 and the assumption $\{p; p|f\} = \{p; p|N\}$, we write

$$\begin{aligned}&\langle F(\tau),\, \Theta_{L,\chi}(\tau, v)\rangle\, y^{\frac{b^+}{2}}\\ &\quad = \frac{\chi(-1)\,\overline{G(\chi)}}{\sqrt{2}\,|z_v|} \sum_{n \geq 1} \chi(n) \sum_{\substack{c \equiv 0\ (N) \\ (d,\, Nc)=1}} \chi(d) \exp\left(-\frac{\pi n^2 |c\tau + d|^2}{2{z_v}^2 y}\right)\\ &\qquad\qquad \times \langle F(\tau),\, \Theta_K(\tau, w\,;\, dn\mu, -cn\mu)\rangle\, y^{\frac{b^+-1}{2}}.\end{aligned}$$

If we put $\Gamma_\infty = \{\, \left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right) \mid h \in \mathbb{Z}\,\}$, then we obtain a bijection between $\{\,(c,\, d) \in \mathbb{Z}^2 \smallsetminus \{0\} \mid c \equiv 0\,(N),\, (d,\, Nc) = 1\,\}$ and $\Gamma_\infty \backslash \Gamma_0(N)$. Since the sum of $c = d = 0$ vanishes, we find

$$\begin{aligned}&\sum_{\substack{c \equiv 0\ (N) \\ (d,\, Nc)=1}} \chi(d) \exp\left(-\frac{\pi n^2 |c\tau + d|^2}{2{z_v}^2 y}\right) \langle F(\tau),\, \Theta_K(\tau, w\,;\, dn\mu, -cn\mu)\rangle\, y^{\frac{b^+-1}{2}}\\ &= \sum_{\alpha \in \Gamma_\infty \backslash \Gamma_0(N)} \exp\left(-\frac{\pi n^2}{2{z_v}^2\, \mathrm{Im}(\alpha\tau)}\right) \langle F(\alpha\tau),\, \Theta_K(\alpha\tau, w\,;\, n\mu, 0)\rangle\, \mathrm{Im}(\alpha\tau)^{\frac{b^+-1}{2}}.\end{aligned}$$

Therefore we can use the unfolding trick as in the full modular case.

Then we have

$$\begin{aligned}
&\sum_{l} \int\limits_{\mathcal{F}_{l,u}} \langle F(\alpha_l\tau),\, \Theta_{L,\chi}(\alpha_l\tau, v)\rangle \operatorname{Im}(\alpha_l\tau)^{\frac{b^+}{2}} \frac{dxdy}{y^{2+s}} \\
&= \frac{\chi(-1)\,\overline{G(\chi)}}{\sqrt{2}\,|z_v|} \sum_{l} \int\limits_{\mathcal{F}_{l,u}} \sum_{n\geq 1} \chi(n) \sum_{\alpha\in\Gamma_\infty\backslash\Gamma_0(N)} \exp\left(-\frac{\pi n^2}{2{z_v}^2 \operatorname{Im}(\alpha\alpha_l\tau)}\right) \\
&\qquad\qquad \times \langle F(\alpha\alpha_l\tau),\, \Theta_K(\alpha\alpha_l\tau, w\,;\, n\mu, 0)\rangle \operatorname{Im}(\alpha\alpha_l\tau)^{\frac{b^+-1}{2}} \frac{dxdy}{y^{2+s}} \\
&= \frac{\chi(-1)\,\overline{G(\chi)}}{\sqrt{2}\,|z_v|} \int\limits_{\mathcal{F}_u} \sum_{n\geq 1} \chi(n) \sum_{\alpha\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})} \exp\left(-\frac{\pi n^2}{2{z_v}^2 \operatorname{Im}(\alpha\tau)}\right) \\
&\qquad\qquad \times \langle F(\alpha\tau),\, \Theta_K(\alpha\tau, w\,;\, n\mu, 0)\rangle \operatorname{Im}(\alpha\tau)^{\frac{b^+-1}{2}} \frac{dxdy}{y^{2+s}}.
\end{aligned}$$

We can then exchange the integral and sums as shown in [2, Theorem 7.1]. For $\alpha \notin \Gamma_\infty$, $\exp\left(-\pi n^2/2{z_v}^2 \operatorname{Im}(\alpha\tau)\right)$ decreases rapidly if $|z_v|$ is sufficiently small, so that it kills off the growth of $F(\alpha\tau)$. Therefore the integral and sums for these α are absolutely convergent. We may replace y^{-s} by $\operatorname{Im}(\alpha\tau)^{-s}$ in considering to take the constant term. For $\alpha \in \Gamma_\infty$ the regularization process justifies the exchange. Considering the contribution of $\pm 1 \in \Gamma_0(N)$, we have

$$\begin{aligned}
\Phi_F(v) = \frac{\sqrt{2}\,\chi(-1)\,\overline{G(\chi)}}{|z_v|} \mathop{\mathcal{C}}_{s=0} \Bigg[&\sum_{n\geq 1} \chi(n) \int\limits_{y>0} \int\limits_{0<x<1} \exp\left(-\frac{\pi n^2}{2{z_v}^2\, y}\right) \\
&\times \langle F(\tau),\, \Theta_K(\tau, w\,;\, n\mu, 0)\rangle\, y^{\frac{b^+-5}{2}-s}\, dxdy \Bigg].
\end{aligned}$$

The Fourier expansions of F and Θ_K give us the following expansion of the sum in the bracket:

$$\sum_{n\geq 1} \chi(n) \int\limits_{y>0} \exp\left(-\frac{\pi n^2}{2{z_v}^2\, y}\right) \sum_{\lambda\in K'} c_\lambda(\lambda^2/2)\, \mathbf{e}(n(\lambda,\mu)) \exp(-2\pi\lambda_w^2 y)\, y^{\frac{b^+-5}{2}-s}\, dy.$$

From the estimate on $c_\lambda(\lambda^2/2)$, we can exchange the integral and sums if $\operatorname{Re}(s) \gg 0$. Consequently it becomes

$$\sum_{\lambda\in K'} \sum_{n\geq 1} \chi(n)\, c_\lambda(\lambda^2/2)\, \mathbf{e}(n(\lambda,\mu)) \int\limits_{y>0} \exp\left(-\frac{\pi n^2}{2{z_v}^2\, y} - 2\pi\lambda_w^2 y\right) y^{\frac{b^+-5}{2}-s}\, dy.$$

From [2, Lemma 7.2 and Lemma 7.3], the integral over $y > 0$ can be written as

$$2\left(\frac{n}{2|z_v||\lambda_w|}\right)^{\frac{b^+-3}{2}-s} K_{\frac{b^+-3}{2}-s}\left(2\pi n \frac{|\lambda_w|}{|z_v|}\right)$$

if $\lambda_w \neq 0$ and as

$$\left(\frac{\pi n^2}{2z_v^2}\right)^{\frac{b^+-3}{2}-s}\Gamma\left(s-\frac{b^+-3}{2}\right)$$

if $\lambda_w = 0$, where $K_\mu(t)$ denotes the modified Bessel function of the third kind.

Now we specialize the above results to the case $b^+ = 2$. Note that $K_{1/2}(t) = \sqrt{\pi/(2t)}\, e^{-t}$. If we use an estimate of the Fourier coefficients of F by Hardy-Ramanujan circle method given in [1, Lemma 5.3] and the assumption that v is outside the Heegner divisors, we can show that we may simply put $s = 0$ in the sum to take the constant term of the Laurent expansion at $s = 0$. Putting $b^+ = 0$ and $s = 0$, we find that the integral over $y > 0$ for $\lambda_w \neq 0$ and $\lambda_w = 0$ are the same. Then it is easily seen that the vector $\lambda = 0$ contributes

$$2\,\chi(-1)\,\overline{G(\chi)}\,L(1,\chi)\,c_0(0)$$

and all the other vectors contribute

$$2\,\chi(-1)\,\overline{G(\chi)}\sum_{\lambda\in K'\setminus\{0\}}\sum_{n\geq 1}\frac{\chi(n)}{n}\,c_\lambda(\lambda^2/2)\,\mathbf{e}(n(\lambda,\mu))\exp\left(-2\pi n\frac{|\lambda_w|}{|z_v|}\right).$$

$\square$

6. Borcherds products

In this section, we construct Borcherds products for higher level vector valued modular forms. We let the signature of a lattice L be $(2, b^-)$. Results are described in the tube domain realization $\mathfrak{H}_{b^-}$ of $Gr(L)$ introduced in Section 2. In that coordinates, we have $|z_v| = 1/|Y|$, $|\lambda_w| = |(\lambda, Y)|/|Y|$ and $(\lambda, \mu) = (\lambda, X)$ for $Z = X + iY \in \mathfrak{H}_{b^-}$ corresponding to $v \in Gr(L)$.

Theorem 6.1. *Let K be a Lorentzian lattice of signature $(1, b^- - 1)$ and $M = \mathbb{Z}z + \mathbb{Z}z'$ with $z^2 = {z'}^2 = 0, (z, z') = N$. We put $L = K \oplus M$. Suppose that χ is a real even primitive Dirichlet character modulo f with $f|N$. We assume $\{p; p|f\} = \{p; p|N\}$.*

We let $F(\tau)$ be in $M^!_{1-\frac{b^-}{2}}(\widetilde{\Gamma}_0(N), \chi\,\rho_K)$. At a cusp l, we have the Fourier expansion $(F|\,\widetilde{\alpha}_l)(\tau) = \sum_{\gamma\in K'/K}\mathfrak{e}_\gamma\sum_{\substack{n\in\mathbb{Q}\\ n\gg-\infty}} c_\gamma(n)_l q^n$ for $\alpha_l \in SL_2(\mathbb{Z})$ such that $\alpha_l(\infty) = l$. Suppose that $\left(\sum_{k\,(f)}\chi(k)\,\overline{\rho}_{\frac{k}{f}z,\varepsilon}(\widetilde{\alpha}_l)\right) h_l\, c_\gamma(n)_l$ is an integer for any cusp l, $\gamma \in K'/K$, $\varepsilon \in M'/M$ and $n < 0$. Then there exists a meromorphic function Ψ_F on $\mathfrak{H}_{b^-}$ which satisfies the following properties.

(i) *It is a meromorphic modular form on $\mathfrak{H}_{b^-}$ of weight 0 for $\Gamma(L)$ with some unitary character of finite order.*

(ii) *The divisor of Ψ_F is given by*

$$div(\Psi_F) = \frac{1}{2}\sum_l\sum_{\substack{\gamma\in K'/K\\ \varepsilon\in M'/M}}\sum_{\substack{n\in\mathbb{Z}+\gamma^2/2\\ n<0}}\left(\sum_{k\,(f)}\chi(k)\,\overline{\rho}_{\frac{k}{f}z,\varepsilon}(\widetilde{\alpha}_l)\right)h_l\,c_\gamma(n)_l H(\gamma+\varepsilon, n)\,.$$

(iii) *The relation* $\Phi_F(Z) = -4\log|\Psi_F(Z)| + 2\,G(\chi)L(1,\chi)\,c_0(0)$ *holds.*

(iv) *If* Z, *which satisfies* $|Im(Z)| \gg 0$, *is in a Weyl chamber* $W \subset \mathfrak{H}_{b^-}$ *and outside the poles of* Ψ_F, *then* Ψ_F *can be expanded to an infinite product which converges absolutely and uniformly on any compact subset of that domain as follows*:

$$\Psi_F(Z) = C \prod_{k\ (f)} \prod_{\substack{\lambda\in K' \\ (\lambda,W)>0}} \left(1 - \zeta^k \mathbf{e}((\lambda,Z))\right)^{\chi(k)c_\lambda(\lambda^2/2)}.$$

Here C *is a constant of absolute value* 1 *and* $\zeta = \mathbf{e}(1/f)$.

Proof. The theorem is proved in the same way as [2, Theorem 13.3].

First, we define a function Ψ_F on some open subset of $\mathfrak{H}_{b^-}$ by

$$\Psi_F(Z) = \prod_{k\ (f)} \prod_{\substack{\lambda\in K'\setminus\{0\} \\ (\lambda,W)>0}} \left(1 - \zeta^k \mathbf{e}((\lambda,Z))\right)^{\chi(k)c_\lambda(\lambda^2/2)}. \tag{6.1}$$

If Z is in a Weyl chamber W and outside the poles of Ψ_F, we have to show that the infinite product (6.1) converges for $|\mathrm{Im}(Z)| \gg 0$. We need again an estimate on the Fourier coefficients of F given in [1, Lemma 5.3]. This gives the fourth assertion of Theorem 6.1.

Next, we prove the third assertion. We let $Z \in W$ and $Y^2 \gg 1$. From Theorem 5.1 we have

$$\begin{aligned}\Phi_F(Z) = {} & 2\,\overline{G(\chi)}L(1,\chi)\,c_0(0) \\ & + 2\,\overline{G(\chi)} \sum_{\lambda\in K'\setminus\{0\}} \sum_{n\ge1} \frac{\chi(n)}{n} c_\lambda(\lambda^2/2)\,\mathbf{e}(n(\lambda,X)) \exp\left(-2\pi n|(\lambda,Y)|\right).\end{aligned}$$

Since $\chi(n) = \frac{1}{G(\overline{\chi})}\sum_{k\ (f)} \overline{\chi}(k)\,\zeta^{nk}$ and χ is a real even character, we find

$$\begin{aligned}
& 2\,\overline{G(\chi)} \sum_{\lambda\in K'\setminus\{0\}} \sum_{n\ge1} \frac{\chi(n)}{n} c_\lambda(\lambda^2/2)\,\mathbf{e}(n(\lambda,X)) \exp\left(-2\pi n|(\lambda,Y)|\right) \\
&= 2 \sum_{k\ (f)} \chi(k) \sum_{\lambda\in K'\setminus\{0\}} c_\lambda(\lambda^2/2) \sum_{n\ge1} \frac{1}{n} \left(\zeta^k \mathbf{e}((\lambda,X) + i|(\lambda,Y)|)\right)^n \\
&= -2 \sum_{k\ (f)} \chi(k) \sum_{\lambda\in K'\setminus\{0\}} c_\lambda(\lambda^2/2) \log\left(1 - \zeta^k \mathbf{e}((\lambda,X) + i|(\lambda,Y)|)\right) \\
&= -4 \sum_{k\ (f)} \chi(k) \sum_{\substack{\lambda\in K'\setminus\{0\} \\ (\lambda,W)>0}} c_\lambda(\lambda^2/2) \log\left|1 - \zeta^k \mathbf{e}((\lambda,Z))\right| \\
&= -4 \log \left| \prod_{k\ (f)} \prod_{\substack{\lambda\in K'\setminus\{0\} \\ (\lambda,W)>0}} \left(1 - \zeta^k \mathbf{e}((\lambda,Z))\right)^{\chi(k)c_\lambda(\lambda^2/2)} \right|.
\end{aligned}$$

This proves the third assertion.

From this relation, the type of singularity of Φ_F, the integrality condition on the Fourier coefficients and pluriharmonicity of Φ_F, we can continue Ψ_F to a single valued meromorphic function on $\mathfrak{H}_{b^-}$ as in the proof of [6, Theorem 3.16]. In that process, we find the divisor of Ψ_F.

Finally we derive the first assertion from the relation in the third assertion and [2, Lemma 13.1],

This completes the proof. □

6.1. Examples

We focus on the one-dimensional case and present a few modular forms constructed from our Borcherds products.

Example 6.1. We take a real even primitive Dirichlet character χ modulo f. We put $L = (2f) \oplus II_{1,1}(f)$ and denote the generators of $(2f)$ and $II_{1,1}(f)$ by λ_0 with ${\lambda_0}^2 = 2f$ and z, z' with $z^2 = {z'}^2 = 0, (z, z') = f$. In this case, we write $\mathfrak{e}_j$ for $\mathfrak{e}_\gamma$, $\gamma = \frac{j}{2f}\lambda_0$, $j \in \mathbb{Z}$.

If we apply Proposition 3.1 to the theta function $\Theta_{(2f)}(\tau)$ of a lattice $(2f)$, then we get the function $\Theta_{(2f),\,\chi}(\tau) = \sum_{j\ (2f)} \mathfrak{e}_j \chi(j) \sum_{n \equiv j\ (2f)} \mathbf{e}\left(\frac{n^2}{4f}\tau\right)$ in $M_{\frac{1}{2}}(\widetilde{\Gamma}_0(f),\, \chi\, \rho_{(2f)})$. Since $\Theta_{(2f),\,\chi}(\tau)$ has no singularity at any cusps, we find from Theorem 6.1 that $\eta_\chi(\tau) = \prod_{k\ (f)} \prod_{n \geq 1} \left(1 - \zeta^k q^n\right)^{\chi(kn)}$ defines a modular function which is holomorphic on $\mathfrak{H}$ with some character of finite order. In fact, the function $\eta_\chi(\tau)$ is a modular function with respect to $\Gamma_0(f)$ and satisfies $\eta_\chi | W_{f^2} = \eta_\chi$ for the Fricke involution $W_{f^2} = \begin{pmatrix} 0 & -1 \\ f^2 & 0 \end{pmatrix}$. (This can be verified by the argument of Weil. See [12, Theorem 4.4.1] or [9, Lemma 1.3.]) Moreover we have $\Phi_{\Theta_{(2f),\chi}}(\tau) = -4 \log |\eta_\chi(\tau)|$. We would call $\eta_\chi(\tau)$ as a twisted Dedekind eta function. (T. Horie and N. Kanou [9] studies the arithmetic of another type of twisted function.)

Example 6.2. In the second example, we take an odd prime number p congruent to 1 modulo 4. We put $L = (2p) \oplus II_{1,1}(p)$ and denote generators of $(2p)$ and $II_{1,1}(p)$ by λ_0 with ${\lambda_0}^2 = 2p$ and z, z' with $z^2 = {z'}^2 = 0$, $(z, z') = p$ respectively. Let χ be the real even primitive Dirichlet character modulo p. We know that $\Gamma_0(p)$ has only two cusps ∞ and 0. Their widths are 1 and p respectively.

Let $F(\tau) = \sum_{j\ (2p)} \mathfrak{e}_j \sum_{n \in \mathbb{Q}} c_j(n)\, q^n$ be a modular form in $M^!_{\frac{1}{2}}(\rho_{(2p)})$. We calculate the Fourier coefficients of the twist $F_\chi(\tau)$ and rewrite the integrality condition in Theorem 6.1 explicitly.

We find at the cusp ∞ that $\sum_{k\ (p)} \chi(k)\, \rho_{\frac{k}{p}z,\,\varepsilon}(1) = \chi(m)$ if $\varepsilon \equiv \frac{m}{p} z \bmod II_{1,1}(p)$ and $= 0$ otherwise, and at the cusp 0 that $\sum_{k\ (p)} \chi(k)\, \rho_{\frac{k}{p}z,\,\varepsilon}(S) = \frac{\chi((z,\varepsilon))}{\sqrt{p}}$. We have to express $c_j(n)_0$ in terms of $c_j(n)$. We take $\widetilde{\alpha_0} = S$. By the transformation law of the full modular form $F(\tau)$, we have $(F_\chi|S)(\tau) = -\frac{\chi(2)}{2\sqrt{p}} \sum_{j\ (2p)} \mathfrak{e}_j \sum_{n \in \mathbb{Q}} \sum_{i\ (2p)} (1 + \mathbf{e}(\frac{j-i}{2}))\chi(j-i)\, c_i(n)\, q^n$ and therefore we obtain $c_j(n)_0 = -\frac{\chi(2)}{2\sqrt{p}} \sum_{i\ (2p)} (1 + \mathbf{e}(\frac{j-i}{2}))\chi(j-i)\, c_i(n)$.

Consequently if $c_j(n) \in \mathbb{Z}$ for any $n < 0$ and j modulo $2p$, the integrality condition for $F_\chi(\tau)$ is satisfied. We can always find such a modular form $F(\tau)$, for example the Siegel theta function $\Theta_{(2p)}(\tau)$ times any modular function in $\mathbb{Z}[j(\tau)]$.

Now we assume that the space of holomorphic Jacobi forms $J_{2,p}$ of weight 2 and index p is zero. For example, by [8, Theorem 9.1] we readily verify that when $p = 5$ this assumption is satisfied. Since $J_{2,p}$ is the obstruction space to find singularities and constant terms of modular forms in $M^!_{\frac{1}{2}}(\rho_{(2p)})$ by [3, Theorem 3.1] and [8, Theorem 5.1], there exists a unique modular form $F_d(\tau) = \mathfrak{e}_{j_0} q^{-d} + \sum_{j\ (2p)} \mathfrak{e}_j \sum_{n>0} c_j(n)\, q^n$ in $M^!_{\frac{1}{2}}(\rho_{(2p)})$ for an integer j_0 and a positive integer d with $-d \in \mathbb{Z} + j_0^2/4p$.

For simplicity, let $j_0 = 0$. As a result of the above discussion and Theorem 6.1, we conclude that there exists a meromorphic function $\Psi_d(\tau)$ on $\mathfrak{H}$ which is a modular function for $\Gamma_0(p)$ with some character with divisor $-\frac{1}{2} \sum_{a,b,c\ (p)} \chi(a)\chi(c) H(\frac{a}{p}\lambda_0 + \frac{b}{p}z + \frac{c}{p}z', -d)$. For $a, b, c \in \mathbb{Z}$, the Heegner divisor $H(\frac{a}{p}\lambda_0 + \frac{b}{p}z + \frac{c}{p}z', -d)$ is a union of quadratic irrationals on $\mathfrak{H}$ and given by

$$H\left(\frac{a}{p}\lambda_0 + \frac{b}{p}z + \frac{c}{p}z',\, -d\right) = \left\{ \tau \in \mathfrak{H} \;\middle|\; \begin{array}{c} pc'\tau^2 - 2a'\tau - \dfrac{b'}{p} = 0, \\ a' \equiv a\ (p),\ b' \equiv b\ (p),\ c' \equiv c\ (p), \\ {a'}^2 + b'c' = -dp \end{array} \right\}.$$

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