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# On the annihilation of local cohomology modules

By

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**Abstract**

Let $R$ be a (not necessary finite dimensional) commutative noetherian ring and let $C$ be a semi-dualizing module over $R$. There is a generalized Gorenstein dimension with respect to $C$, namely $\mathrm{G}_C$-dimension, sharing the nice properties of Auslander's Gorenstein dimension. In this paper, we establish the Faltings' Annihilator Theorem and it's uniform version (in the sense of Raghavan) for local cohomology modules over the class of finitely generated $R$-modules of finite $\mathrm{G}_C$-dimension, provided $R$ is Cohen-Macaulay. Our version contains variations of results already known on the Annihilator Theorem.

## 1. Introduction

In this paper, we contribute to the study of the annihilation theorem of local cohomology modules. If $R$ is noetherian and $M$ is a finitely generated (henceforth finite) R-module, then the 0th local cohomology module of $M$ with support in an ideal $\mathfrak{a}$, $H^0_{\mathfrak{a}}(M)$, is always finite, simply because it is a submodule of $M$. But what about the following question: what is the largest integer $n$ such that all the modules $H^i_{\mathfrak{a}}(M)$ $(i < n)$ are finite ? This question is closely related to the question of which ideals annihilate the local cohomology modules, and the classical theorem on this subject is Faltings' Annihilator Theorem [F]. The original Annihilator Theorem holds when the underlying ring $R$ is regular or, by independence of base, a homomorphic image of such a ring. The present paper relies on the observation that the following is key in Faltings' proof of the Annihilator Theorem: A finite module $M$ over a regular ring has finite projective dimension, $\mathrm{pd}_R M < \infty$, and so satisfies the Auslander-Buchsbaum equality, $\operatorname{depth} R = \operatorname{depth} M + \mathrm{pd}_R M$. With this observation the second author and K. Khashyarmanesh (in [KS1]) extended the Annihilator Theorem to Gorenstein rings (where all finite modules have finite Gorenstein

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dimension in the sense of [AB]), and the present paper takes yet another step: Over a Cohen-Macaulay ring the Annihilator Theorem is established for finite modules of finite (generalized) Gorenstein dimension with respect to a semi-dualizing module $C$ (in the sense of [Gol]).

Let us explain the results of the paper more precisely. Let $C$ be a finite $R$-module. We say that $C$ is a *semi-dualizing module* for $R$ if $\operatorname{Hom}_R(C, C)$ is canonically isomorphic to $R$ and $\operatorname{Ext}^i_R(C, C) = 0$ for all $i > 0$. There is a generalized Gorenstein dimension with respect to $C$, denoted $\mathrm{G}_C$-dim, sharing the nice properties of Auslander's classical G-dimension [C]. It is finer than projective dimension, i.e. $\mathrm{G}_C\text{-}\dim_R M \le \operatorname{pd}_R M$ for all finite $R$-modules $M$, and equality holds if $\operatorname{pd}_R M < \infty$. Moreover, over local rings it satisfies a version of the Auslander-Buchsbaum depth formula, that is for any finite module $M$ with $\mathrm{G}_C\text{-}\dim_R M < \infty$, $\operatorname{depth} R = \operatorname{depth} M + \mathrm{G}_C\text{-}\dim_R M$.

Using this fact we establish Faltings' Theorem over the class of modules of finite generalized Gorenstein dimension over a Cohen-Macaulay ring. Next, we extends a theorem due to Raghavan [R] known as the 'Uniform annihilation of local cohomology' to the finite $\mathrm{G}_C$-dimension arena.

Our results, as we will see in some corollaries, contains variations of results from [BRS], [BS], [KS1], [KS2] and [R] and provide a generalization of the previously known results related to the Faltings' famous theorem.

## 2. Main Theorem

Throughout the paper $R$ denotes a commutative noetherian ring (with non-zero identity), $C$ is a semi-dualizing module for $R$, and $M$ is a finite $R$-module. $R$ is not assumed to be of finite dimensional, unless it is specified. We begin by a brief review of the notions we use throughout the paper.

### 2.1. $\mathrm{G}_C$-dimension

Let $C$ be a semi-dualizing module over $R$. A finite module $M$ is said to belong to the $\mathrm{G}_C$-class, $\mathrm{G}_C(R)$, of $R$ if the canonical map $\delta_M : M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, C), C)$ is isomorphism and

$$\operatorname{Ext}^i_R(M, C) = 0 = \operatorname{Ext}^i_R(\operatorname{Hom}_R(M, C), C)$$

for all $i > 0$. It is clear that always $R \in \mathrm{G}_C(R)$.

Now let $M$ be a non-zero finite $R$-module. We say that $M$ is of finite $\mathrm{G}_C$-dimension, if there exists an exact sequence (which we call it $\mathrm{G}_C$-resolution of $M$)

$$0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0,$$

in which for any integer $i$, $G_i \in \mathrm{G}_C(R)$. We denote this by $\mathrm{G}_C\text{-}\dim_R M < \infty$. Otherwise we set $\mathrm{G}_C\text{-}\dim_R M = \infty$. By convention, $\mathrm{G}_C\text{-}\dim_R 0 = -\infty$. When $M$ is of finite $\mathrm{G}_C$-dimension, we define it to be the infimum of the length of its $\mathrm{G}_C$-resolutions. It is clear that with $C = R$, we get the classical Auslander G-dimension. For more details see [C, Sec. 3].

We need the following properties of $G_C$-dimension.

**Proposition 2.1.** *With the above notations, we have the following.*

(i) *In any exact sequence* $0 \to L \to G \to M \to 0$, *with* $G \in \mathrm{G}_C(R)$, *if* $\mathrm{G}_C\text{-}\dim_R M$ *is finite, then so is* $\mathrm{G}_C\text{-}\dim_R L$. *Moreover, if* $\mathrm{G}_C\text{-}\dim_R M > 0$, *then* $\mathrm{G}_C\text{-}\dim_R L = \mathrm{G}_C\text{-}\dim_R M - 1$.

(ii) *If* $M \in \mathrm{G}_C(R)$, *then there exists a short exact sequence* $0 \to M \to C^k \to N \to 0$, *for some integer* $k$, *such that* $N \in \mathrm{G}_C(R)$.

*Proof.* (i) This statement, can be obtained by a slight modification of the proof of similar result for G-dimension, see for example [AM, 2.3 and 2.5].

(ii) Apply the functor $(\ )^* = \mathrm{Hom}_R(\ , C)$, on a finite presentation of $M^*$,

$$0 \to K \to R^k \to M^* \to 0,$$

to deduce the exact sequence

$$0 \to M \to C^k \to K^* \to 0.$$

We show that $\mathrm{G}_C\text{-}\dim_R K^* = 0$. It follows from the above exact sequence that $\mathrm{G}_C\text{-}\dim_R K^* \leq 1$. Moreover, the exact sequence of 'Ext' modules which results from application of the functor $\mathrm{Hom}_R(\ , C)$ to this exact sequence, in view of the facts that $\mathrm{Hom}_R(C^k, C)$ is canonically isomorphic to $R^k$ and $\mathrm{Ext}^1_R(C^k, C) = 0$, implies that $\mathrm{Ext}^1_R(K^*, C) = 0$. The result now follows from the fact that when $\mathrm{G}_C\text{-}\dim_R K^*$ is finite, it is equal to the number $\mathrm{G}_C\text{-}\dim_R K^* = \sup\{i : \mathrm{Ext}^i_R(K^*, C) \neq 0\}$. □

### 2.2. Annihilator Theorem

Using the terminology of [BS, Sec.9], we define the $\mathfrak{b}$-finiteness dimension $f^{\mathfrak{b}}_{\mathfrak{a}}(M)$ of $M$ relative to $\mathfrak{a}$ by

$$f^{\mathfrak{b}}_{\mathfrak{a}}(M) = \inf\{i \in \mathbb{N}_0 \ : \ \mathfrak{b}^n H^i_{\mathfrak{a}}(M) \neq 0 \text{ for all } n \in \mathbb{N}\},$$

where, by convention, the infimum of the empty set of integers is interpreted as $\infty$. Moreover the $\mathfrak{b}$-minimum $\mathfrak{a}$-adjusted depth $\lambda^{\mathfrak{b}}_{\mathfrak{a}}(M)$ of $M$ is defined by Faltings by the formula

$$\lambda^{\mathfrak{b}}_{\mathfrak{a}}(M) = \inf\{\mathrm{depth} M_{\mathfrak{p}} + \mathrm{ht}((\mathfrak{a}+\mathfrak{p})/\mathfrak{p}) : \mathfrak{p} \in \mathrm{Spec}(A) \backslash V(\mathfrak{b})\},$$

where $V(\mathfrak{b})$ denotes the set of prime ideals containing $\mathfrak{b}$.

Faltings' Annihilator Theorem [F] states that, if $R$ is a homomorphic image of a regular ring or if $R$ has a dualizing complex, then for every choice of the ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $R$, $\lambda^{\mathfrak{b}}_{\mathfrak{a}}(M) = f^{\mathfrak{b}}_{\mathfrak{a}}(M)$. There are some refinements of the conditions of the theorem, see [B], [BRS] and [KS1].

### 2.3. Main Theorem

Our main aim in this paper is to prove the following extension of the Faltings' Annihilator Theorem.

**Theorem.** *Let* $R$ *be a Cohen-Macaulay ring, let* $C$ *be a semi-dualizing module over* $R$ *and let* $M$ *be a finite* $R$*-module of finite* $\mathrm{G}_C$*-dimension.*

($\alpha$) *For any ideals* $\mathfrak{a}$ *and* $\mathfrak{b}$ *of* $R$*, there exists an integer* $k$ *such that*

$$\mathfrak{b}^k H^i_{\mathfrak{a}}(M) = 0 \quad \textit{for all} \;\; i < \lambda^{\mathfrak{b}}_{\mathfrak{a}}(M).$$

($\beta$) *If* $R$ *is of finite dimension, then there exists an integer* $k$*, such that for every ideals* $\mathfrak{a}$ *and* $\mathfrak{b}$ *of* $R$*,*

$$\mathfrak{b}^k H^i_{\mathfrak{a}}(M) = 0 \quad \textit{for all} \;\; i < \lambda^{\mathfrak{b}}_{\mathfrak{a}}(M).$$

Part ($\alpha$) of the theorem extends Faltings' Annihilator Theorem to the finite $\mathrm{G}_C$-dimension arena, while part ($\beta$) provides an extension of the main theorem of [R, 3.1].

By [R, 2.2], an ordered pair $(\mathfrak{a}, \mathfrak{b})$ of ideals of $R$ is called an ideal pair for $M$ if $0 :_R M \subseteq \mathfrak{a}$ and $\mathfrak{b} \nsubseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$. We shall need the following lemma.

**Lemma 2.1** ([R, 2.3])**.** *Let* $R$ *be a commutative ring,* $M$ *be a finitely generated* $R$*-module and* $(\mathfrak{a}, \mathfrak{b})$ *be an ideal pair for* $M$*. Then* $\lambda^{\mathfrak{b}}_{\mathfrak{a}}(M) \leq ht(\mathfrak{a})$*.*

## 3. Proof of the Theorem

The approach we shall take for the proof of the theorem is modelled very closely on [BS, Chapter 9] and [KS1] (for part ($\alpha$)) and [R] (for part ($\beta$)). So we try to give detailed references to those sources and explain only the new parts of the arguments.

In the following lemma, we adopt the convention that the intersection of an empty set of submodules of $M$ is to be taken as $M$.

**Lemma 3.1** (Compare [BS, 9.4.3])**.** *Let* $M$ *be a finite* $R$*-module and* $C$ *be a semi-dualizing module over* $R$*. Let* $n \in \mathbb{N}$ *and* $\mathfrak{p} \in \operatorname{Spec} R$ *be such that* $\mathrm{G}_{C_{\mathfrak{p}}}\text{-}\dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = 0$*. Then there exists* $s \in R \setminus \mathfrak{p}$ *such that for any proper ideal* $\mathfrak{a}$ *of* $R$*,* $sH^i_{\mathfrak{a}}(M) = 0$ *for all* $i < \min\{\mathrm{grade}(\mathfrak{a}, R), n\}$*.*

*Proof.* Let $0 = \bigcap_{i=1}^{s} N_i$ be a primary decomposition of 0 in $M$. So for any ideal $\mathfrak{b}$ of $R$,

$$H^0_{\mathfrak{b}}(M) = \bigcap \{N_i : H^0_{\mathfrak{b}}(M/N_i) = 0\}.$$

Hence $\{H^0_{\mathfrak{b}}(M) : \mathfrak{b} \text{ is an ideal of } R\}$ is a finite set, say $\{L_1, L_2, \ldots, L_u\}$. Let $L_i$, $i = 1, \ldots, k$ be such that $(L_i)_{\mathfrak{p}} = 0$. So there exists $s' \in R \setminus \mathfrak{p}$ such that $s'L_i = 0$, for $i = 1, 2, \ldots, k$. But since $\mathrm{G}_{C_{\mathfrak{p}}}\text{-}\dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = 0$, by 2.2 (ii), there exists an exact sequence

$$0 \to M_{\mathfrak{p}} \to C^t_{\mathfrak{p}} \quad (*).$$

So for any ideal $\mathfrak{a}$ with $\mathrm{grade}(\mathfrak{a}, R) > 0$, by applying the functor $\Gamma_{\mathfrak{a}R_{\mathfrak{p}}}$ on $(*)$, we get that there exists $s'' \in R \setminus \mathfrak{p}$ such that $s''L_i = 0$ for $i = k+1, \ldots, u$. This implies that there exists $s \in R \setminus \mathfrak{p}$, such that $sH^0_{\mathfrak{a}}(M) = 0$, for all ideal $\mathfrak{a}$ with $\min\{\mathrm{grade}(\mathfrak{a}, R), 1\} > 0$. Now suppose inductively that $n > 1$ and the result has been proved for smaller values of $n$. By this assumption, it is only remains for us to prove that there exists $s \in R \setminus \mathfrak{p}$ such that $sH^{n-1}_{\mathfrak{a}}(M) = 0$, for every ideal $\mathfrak{a}$ with $\operatorname{grade}(\mathfrak{a}, R) > n-1$. Consider the exact sequence $0 \to K \to M \to M^{**} \to L \to 0$, where as usual $(\ )^*$ denotes the functor $\operatorname{Hom}_R(\ , C)$. The proof can be completed by a slight modification of the proof of [KS1, 2.9] and so we omit it. □

**Notation** (Compare [BS, 9.4.8]). Let $M$ be a finite $R$-module. Set

$$U(M) := \{\mathfrak{p} \in \operatorname{Spec}(R) : \mathrm{G}_{C_\mathfrak{p}}\text{-}\dim_{R_\mathfrak{p}} M_\mathfrak{p} \leq 0\}.$$

Let $C(M) = \operatorname{Spec}(R) \setminus U(M)$, and let $c(M) = \bigcap_{\mathfrak{p} \in C(M)} \mathfrak{p}$.

Let $\mathrm{G}_C\text{-}\dim_R M$ be finite. It is of important to decide whether $U(M)$ is an open subset of $\operatorname{Spec}(R)$ (in the Zariski topology). We here mention that, by modifying the argument of the proof of [BS, 9.4.7], one can see that the set $U(M) = \{\mathfrak{p} \in \operatorname{Spec}(R) : \mathrm{G}_{C_\mathfrak{p}}\text{-}\dim_{R_\mathfrak{p}} M_\mathfrak{p} \leq 0\}$ is an open subset of $\operatorname{Spec}(R)$.

The proof of the following proposition parallels that of [BS, 9.4.10] and so is omitted.

**Proposition 3.1.** *Let $n \in \mathbb{N}$. Then there exists $t \in \mathbb{N}$ such that for any proper ideal $\mathfrak{a}$ of $R$ and any integer $i < \min\{\operatorname{grade}(\mathfrak{a}, R),\ n\}$, $c(M)^t H^i_{\mathfrak{a}}(M) = 0$.*

Next result is the first place that we need the Cohen-Macaulayness of the underlying ring.

**Proposition 3.2.** *Let $C$ be a semi-dualizing module over a Cohen-Macaulay ring $R$, let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of $R$ and let $M$ be a non-zero finite module of finite $\mathrm{G}_C$-dimension. Consider the short exact sequence*

$$0 \to L \to R^n \to M \to 0.$$

*If $\lambda^{\mathfrak{b}}_{\mathfrak{a}}(M) < ht\mathfrak{a}$, then $\lambda^{\mathfrak{b}}_{\mathfrak{a}}(L) = \lambda^{\mathfrak{b}}_{\mathfrak{a}}(M) + 1$.*

*Proof.* Let $\lambda^{\mathfrak{b}}_{\mathfrak{a}}(M) = t$. So there exists a prime $\mathfrak{p} \in \operatorname{Spec}(R) \setminus V(\mathfrak{b})$ such that $\operatorname{depth} M_\mathfrak{p} + \operatorname{ht}\frac{\mathfrak{a}+\mathfrak{p}}{\mathfrak{p}} = t$. Note that $\mathrm{G}_{C_\mathfrak{p}}\text{-}\dim_{R_\mathfrak{p}} M_\mathfrak{p} > 0$, because otherwise $\operatorname{depth} R_\mathfrak{p} = \operatorname{depth} M_\mathfrak{p}$ and hence since $R$ is Cohen-Macaulay, $\operatorname{depth} R_\mathfrak{p} = \operatorname{ht}\mathfrak{p}$, and so we have

$$\lambda^{\mathfrak{b}}_{\mathfrak{a}}(M) = \operatorname{ht}\mathfrak{p} + \operatorname{ht}\frac{\mathfrak{a}+\mathfrak{p}}{\mathfrak{p}} \geq \operatorname{ht}\mathfrak{a},$$

which contradicts with our assumption. Therefore by 2.2 $(i)$, $\mathrm{G}_{C_\mathfrak{p}}\text{-}\dim_{R_\mathfrak{p}} L_\mathfrak{p} = \mathrm{G}_{C_\mathfrak{p}}\text{-}\dim_{R_\mathfrak{p}} M_\mathfrak{p} - 1$. Hence Auslander-Buchsbaum formula for $\mathrm{G}_C$-dimension, in view of the definition of $\lambda^{\mathfrak{b}}_{\mathfrak{a}}$, implies that

$$\lambda^{\mathfrak{b}}_{\mathfrak{a}}(L) \leq \lambda^{\mathfrak{b}}_{\mathfrak{a}}(M) + 1.$$

Moreover, since $\operatorname{depth} L_\mathfrak{p} \geq \operatorname{depth} M_\mathfrak{p}$, we always have $\lambda^{\mathfrak{b}}_{\mathfrak{a}}(M) \leq \lambda^{\mathfrak{b}}_{\mathfrak{a}}(L)$. So $\lambda^{\mathfrak{b}}_{\mathfrak{a}}(M) \leq \lambda^{\mathfrak{b}}_{\mathfrak{a}}(L) \leq \lambda^{\mathfrak{b}}_{\mathfrak{a}}(M)+1$. Now if $\lambda^{\mathfrak{b}}_{\mathfrak{a}}(L) = \operatorname{ht}\mathfrak{a}$, the result follows. Otherwise pick $\mathfrak{p} \in \operatorname{Spec}(R) \setminus V(\mathfrak{b})$ such that $\lambda^{\mathfrak{b}}_{\mathfrak{a}}(L) = \operatorname{depth} L_\mathfrak{p} + \operatorname{ht}\frac{\mathfrak{a}+\mathfrak{p}}{\mathfrak{p}}$ and follow the above procedure, to deduce the inequality $\lambda^{\mathfrak{b}}_{\mathfrak{a}}(L) \geq \lambda^{\mathfrak{b}}_{\mathfrak{a}}(M)+1$. The result hence follows. □

**Remark 1.** Let $R$ be Cohen-Macaulay, $C$ be a semi-dualizing module over $R$, $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of $R$ and $M$ be a non-zero finite module of finite $\mathrm{G}_C$-dimension, say $s$. Let $\Omega^i M$ be the $i$th syzygy module of $M$ in a minimal free resolution. It follows from the above proposition that when $\lambda_{\mathfrak{a}}^{\mathfrak{b}}(M) < \mathrm{ht}\mathfrak{a}$, there exists a non-negative integer $j$, less than or equal to $s$ such that $\lambda_{\mathfrak{a}}^{\mathfrak{b}}(\Omega^j M) = \mathrm{ht}\mathfrak{a}$. Let $j$ be the smallest one. Then it is easy to see that $\lambda_{\mathfrak{a}}^{\mathfrak{b}}(\Omega^j M) = f_{\mathfrak{a}}^{\mathfrak{b}}(\Omega^j M)$ if and only if $\lambda_{\mathfrak{a}}^{\mathfrak{b}}(M) = f_{\mathfrak{a}}^{\mathfrak{b}}(M)$. So in the study of Faltings' Annihilator Theorem we may assume that $\lambda_{\mathfrak{a}}^{\mathfrak{b}}(M) = \mathrm{ht}\mathfrak{a}$. Moreover, with the above notations, suppose that for some integer $t$, with $0 \le t \le s$, there exists a fixed integer $k$, such that for any choice of ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $R$, $\mathfrak{b}^k H_{\mathfrak{a}}^i(\Omega^t M) = 0$ for all $i < \lambda_{\mathfrak{a}}^{\mathfrak{b}}(\Omega^t M)$. Then in view of the above proposition, it follows that for all ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $R$, such that $\lambda_{\mathfrak{a}}^{\mathfrak{b}}(\Omega^j M) < \mathrm{ht}\mathfrak{a}$, for all $j < t$, we have $\mathfrak{b}^k H_{\mathfrak{a}}^i(M) = 0$ for all $i < \lambda_{\mathfrak{a}}^{\mathfrak{b}}(M)$.

Now we are able to prove part ($\alpha$) of the Theorem.

*Proof of Theorem* ($\alpha$). Suppose contrary that there exist ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $R$ such that $\lambda_{\mathfrak{a}}^{\mathfrak{b}}(M) > f_{\mathfrak{a}}^{\mathfrak{b}}(M)$. By [BS, 9.2.6], we can assume, in our search for a contradiction, that $0 :_R M \subseteq \mathfrak{a}$. Fix $\mathfrak{b}$. Using noetherian property of $R$, choose $\mathfrak{a}$ to be maximal among such counterexamples. Similar to the proof of [BRS, 3.2] one see that in this situation $\mathfrak{a}$ will be a prime ideal. By replacing $M$ by $M/H_{\mathfrak{b}}^0(M)$, we may assume that $(\mathfrak{a}, \mathfrak{b})$ is an ideal pair for $M$. So by Lemma 2.5, $\lambda_{\mathfrak{a}}^{\mathfrak{b}}(M) \le \mathrm{ht}(\mathfrak{a})$. In view of Remark 3.4, we can (and do) assume that $\lambda_{\mathfrak{a}}^{\mathfrak{b}}(M) = \mathrm{ht}(\mathfrak{a})$. If $\mathfrak{b} \subseteq c(M)$, by Proposition 3.2, there is nothing to prove. So assume that $\mathfrak{b} \nsubseteq c(M)$ and let $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_t$ be the minimal primes over $c(M)$ not containing $\mathfrak{b}$. But now, by the aid of an argument similar to that used in the proof of the case 1 of [KS1, 2.10], one can deduce that $\lambda_{\mathfrak{a}}^{\mathfrak{b}}(M) < \mathrm{ht}\mathfrak{a}$. This contradiction completes the proof. □

The following lemma is needed for the proof of part ($\beta$) of the Theorem.

**Lemma 3.2.** *Let $R$ be a Cohen-Macaulay ring, $C$ be a semi-dualizing module over $R$, $\mathfrak{m}$ be a maximal ideal of $R$ and $M$ be a finite $R$-module of finite $\mathrm{G}_C$-dimension. Assume that $t$ is the integer obtained in* Proposition 3.2, *such that $c(M)^t H_{\mathfrak{a}}^i(M) = 0$, for all ideals $\mathfrak{a}$ of $R$ and all $i < \min\{\mathrm{grade}(\mathfrak{a}, R), \mathrm{ht}\mathfrak{m}\}$. Then, for any ideal $\mathfrak{b}$ of $R$ with $\lambda_{\mathfrak{m}}^{\mathfrak{b}}(M) = \mathrm{ht}\mathfrak{m}$, $\mathfrak{b}^t H_{\mathfrak{m}}^i(M) = 0$, for all $i < \mathrm{ht}\mathfrak{m}$.*

*Proof.* First assume that $R$ is local and $\mathfrak{m}$ is the unique maximal ideal of $R$. We show that $\mathfrak{b} \subseteq c(M)$. Suppose contrary that $\mathfrak{b} \nsubseteq c(M)$. Since $c(M)$ is a radical ideal, there exists $\mathfrak{p} \in \min c(M)$ that does not contain $\mathfrak{b}$. Since $\mathfrak{p} \supseteq c(M)$, $G_{C_{\mathfrak{p}}}\text{-}\dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} > 0$, and so $\mathrm{ht}\mathfrak{p} = \operatorname{depth} R_{\mathfrak{p}} > \operatorname{depth} M_{\mathfrak{p}}$. Hence

$$\lambda_{\mathfrak{m}}^{\mathfrak{b}}(M) \le \operatorname{depth} M_{\mathfrak{p}} + \mathrm{ht}\frac{\mathfrak{m}+\mathfrak{p}}{\mathfrak{p}} < \operatorname{depth} R_{\mathfrak{p}} + \mathrm{ht}\frac{\mathfrak{m}+\mathfrak{p}}{\mathfrak{p}} \le \mathrm{ht}\mathfrak{m},$$

which is the desired contradiction. Hence the result is complete, when $R$ is assumed to be local. Now let $R$ be a (not necessarily local) Cohen-Macaulay ring and $\mathfrak{m}$ be a maximal ideal of $R$. If $\mathfrak{b} \nsubseteq \mathfrak{m}$, we deduce that $\mathrm{ht}\mathfrak{m} = \lambda_{\mathfrak{m}}^{\mathfrak{b}}(M) \le$

depth $M_{\mathfrak{m}}$ and hence for all $i < \operatorname{ht}\mathfrak{m}$, we have $H^i_{\mathfrak{m}}(M) = 0$. So we assume that $\mathfrak{b} \subseteq \mathfrak{m}$. By the previous paragraph, we have $\mathfrak{b}R_{\mathfrak{m}} \subseteq c(M)_{\mathfrak{m}}$. Using this fact, it is easy to see that for all $i < \operatorname{ht}\mathfrak{m}$, $\operatorname{Supp}_R(\mathfrak{b}^t H^i_{\mathfrak{m}}(M)) = \emptyset$. The proof hence is complete. □

*Proof of Theorem* ($\beta$). As it is showed in the second paragraph of the proof of [R, 3.1], it is enough to prove the result for all ideal pairs for $M$. This we do. Let $(\mathfrak{a}, \mathfrak{b})$ be an ideal pair for $M$. Set $\dim R := d$. For any integer $n$, set

$$\mathfrak{I}_n := \{\mathfrak{a} \mid \mathfrak{a} \text{ is an ideal of } R \text{ and } \operatorname{ht}\mathfrak{a} = n\}.$$

We argue by a descending induction on $n$. First assume that $n = d$. Let $\mathfrak{a} \in \mathfrak{I}_n$ and $\mathfrak{m}_1, \mathfrak{m}_2, \cdots, \mathfrak{m}_h$ be all the maximal ideals that contain $\mathfrak{a}$. The Mayer-Vietoris sequence on local cohomology modules, implies that

$$H^i_{\mathfrak{a}}(M) \cong \bigoplus_{j=1}^{h} H^i_{\mathfrak{m}_j}(M).$$

Hence it is enough to prove the assertion just for maximal ideals of $\mathfrak{I}_n$. So let $\mathfrak{m} \in \mathfrak{I}_n$. Since we assumed that $(\mathfrak{m}, \mathfrak{b})$ is an ideal pair for $M$, we may assume that $\lambda^{\mathfrak{b}}_{\mathfrak{m}}(M) \leq \operatorname{ht}\mathfrak{m}$. But in view of Remark 3.4, we may assume that $\lambda^{\mathfrak{b}}_{\mathfrak{m}}(M) = \operatorname{ht}\mathfrak{m}$. Hence the result follows in this case from the above lemma. Now suppose inductively that $n < d$ and the result is proved for $n+1$. Let $\mathfrak{a} \in \mathfrak{I}_n$ and $(\mathfrak{a}, \mathfrak{b})$ be an be an ideal pair for $M$. If $\mathfrak{b} \subseteq c(M)$, there is nothing to do anymore. So let $\mathfrak{b} \nsubseteq c(M)$. Again by Remark 3.4, we can assume that $\lambda^{\mathfrak{b}}_{\mathfrak{a}}(M) = \operatorname{ht}\mathfrak{a}$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_v$ be the minimal primes over $c(M)$ not containing $\mathfrak{b}$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ be the minimal prime over $\mathfrak{a}$ of hight $n$. Now, by using the same argument as in the proof of the Case 1 of [KS1, 2.10], we may deduce that there exists an element $x \in \cap_{i=1}^{v}\mathfrak{p}_i \setminus \cup_{j=1}^{r}\mathfrak{q}_j$. Hence $\operatorname{ht}\mathfrak{a} + Rx \geq t+1$ and $\mathfrak{b}_x \subseteq (c(M))_x$. Now, by inductive hypothesis, there exists an integer $k'$ such that $\mathfrak{b}^{k'} H^i_{\mathfrak{a}'}(M) = 0$ for all ideal $\mathfrak{a}'$ with $\operatorname{ht}\mathfrak{a}' \geq t+1$ and all $i < \lambda^{\mathfrak{b}}_{\mathfrak{a}'}(M)$. In particular, for $\mathfrak{a}' = \mathfrak{a} + Rx$. (Note that $\lambda^{\mathfrak{b}}_{\mathfrak{a}+Rx}(M) > \lambda^{\mathfrak{b}}_{\mathfrak{a}}(M)$). Set $k = k' + l$, where $l$ is an integer such that for every ideal $\mathfrak{c}$ of $R$, $(c(M))^l H^i_{\mathfrak{c}}(M) = 0$ for all $i < \min\{\operatorname{ht}\mathfrak{c}, \dim R\}$. The exact sequence $\cdots \to H^i_{\mathfrak{a}+Rx}(M) \to H^i_{\mathfrak{a}}(M) \to H^i_{\mathfrak{a}R_x}(M_x) \to \cdots$, now implies the result. □

**Corollary 3.1.** *Let $S$ be a homomorphic image of a Cohen-Macaulay ring $R$, $C$ be a semi-dualizing module over $R$ and $M$ be a finite $S$-module such that* $\mathrm{G}_C$-$\dim_R M < \infty$. *Then for any ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $S$, $\lambda^{\mathfrak{b}}_{\mathfrak{a}}(M) = f^{\mathfrak{b}}_{\mathfrak{a}}(M)$.*

*Proof.* In view of [BS, 9.1.7] and [BS, 9.2.6], we may assume that $S$ itself is a Cohen-Macaulay ring with a semi-dualizing module. The result now follows from the part ($\alpha$). □

**Corollary 3.2.** *The Local-Global principle for the annihilation of local cohomology modules holds for all finite modules of finite $G_C$-dimension over a Cohen-Macaulay ring.*

*Proof.* The result follows using [BRS, 3.4] and part ($\alpha$) of the Theorem. □

**Corollary 3.3.** *Faltings' Annihilator Theorem holds over a homomorphic image of a* (*not necessarily finite dimensional*) *commutative noetherian Gorenstein ring.*

*Proof.* As before we may assume that ring itself is Gorenstein. So by [Got] Gorenstein dimension of all finite modules is finite. Hence if in part ($\alpha$) of the Theorem we let $C$ to be $R$ itself the result follows, because in this case $\mathrm{G}_C$-dimension will be equal to G-dimension and so all modules have finite $\mathrm{G}_C$-dimension. □

By putting $C = R$ in part ($\beta$) of the Theorem, we get the following version of the uniform annihilation of local cohomology modules.

**Corollary 3.4** (compare [KS2])**.** *Let $R$ be a finite dimensional Gorenstein ring and $M$ be a finite $R$-module. Then there exists an integer $k$, such that for every ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $R$, $\mathfrak{b}^k H^i_{\mathfrak{a}}(M) = 0$, for all $i < \lambda^{\mathfrak{b}}_{\mathfrak{a}}(M)$.*

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