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Construction of orthogonal multiscaling functions and multiwavelets with higher approximation order based on the matrix extension algorithm*

By

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Abstract

An algorithm is presented for constructing orthogonal multiscaling functions and multiwavelets with higher approximation order in terms of any given orthogonal multiscaling functions. That is, let $\Phi(x) = [\phi_1(x),\ \phi_2(x), \ldots, \phi_r(x)]^T \in (L^2(R))^r$ be an orthogonal multiscaling function with multiplicity r and approximation order m. We can construct a new orthogonal multiscaling function $\Phi^{new}(x) = [\Phi^T(x), \phi_{r+1}(x), \phi_{r+2}(x), \ldots, \phi_{r+s}(x)]^T$ with approximation order $n (n > m)$. Namely, we raise approximation order of a given multiscaling function by increasing its multiplicity. Corresponding to the new orthogonal multiscaling function $\Phi^{new}(x)$, orthogonal multiwavelet $\Psi^{new}(x)$ is constructed. In particular, the spacial case that $r = s$ is discussed. Finally, we give an example illustrating how to use our method to construct an orthogonal multiscaling function with higher approximation order and its corresponding multiwavelet.

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1. Introduction

Several types of uniwavelets are constructed based on multiresolution analysis, such as Daubechies' orthogonal wavelets [1], [2] and semi-orthogonal spline wavelets by Chui and Wang [3] et al. However, multiwavelets can have some features that uniwavelets cannot. Thus, multiwavelets provide interesting applications in signal processing and some other fields (See [4], [5]). In recent years, multiscaling functions and multiwavelets have been studied extensively

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[6]–[13]. Goodman, Lee and Tang [6] established a characterization of multiscaling functions and their corresponding multiwavelets. Chui and Wang [7] introduced semi-orthogonal spline multiwavelets. Examples of cubic and quintic finite elements and their corresponding multiwavelets were studied by Strang and Strela [8]. Geronimo, Hardin and Massopust [9] used fractal interpolation to construct orthogonal multiscaling functions, and their corresponding multiwavelets were given in [10]. In [11], Donovan, Geronimo, and Hardin showed that there exist compactly supported orthogonal polynomial spline multiscaling functions with arbitrarily high regularity. In applications of multiwavelets, the properties of multiscaling function and multiwavelet are also desired, such as orthogonality, symmetry, approximation order and regularity and so on. The properties of multiscaling functions and multiwavelets are discussed in many papers. Ashino, Nagase and Vaillancourt [14], Cohen, Daubechies and Plonka [15] Plonka and Strela [16], Strela [17], Shen [18], Keinert [19], Chui and Lian [20], Lian [21] and many others, have obtained important results on the existence, regularity, orthogonality, approximation order and symmetry of multiwavelets. One of the properties of a multiscaling function which has great practical interest is the approximation order (See [16], [17], [19]–[21]). One known ways to raise approximation order are through the use of two-scale similarity transforms (TSTs) (See [16], [17]) and lifting scheme (See [19]).

Similar to the construction of the uniscaling functions, multiscaling functions with multiplicity r also can be constructed based on multiresolution analysis. But the main difficulties in construction of multiwavelets are verification the convergence of the infinite product of two-scale matrix symbol (See [22], [23]). Are there any easier methods to construct orthogonal multiwavelets? Can multiwavelets be constructed based on uniwavelets? Generally, can multiwavelets with multiplicity $r+s$ be constructed based on multiwavelets with multiplicity r? Our main motivation is to raise approximation orders of orthogonal multiscaling functions by increasing their multiplicities. To answer this question, in this paper, we will give a general scheme to construct orthogonal multiwavelets with arbitrary approximation order from any given orthogonal multiscaling functions. Corresponding orthogonal multiwavelets are constructed.

Let $\Phi(x) = [\phi_1(x), \phi_2(x), \ldots, \phi_r(x)]^T \in (L^2(R))^r$, satisfying the following two-scale matrix equation:

$$\Phi(x) = \sum_k P_k \Phi(2x-k), \tag{1}$$

for some $r \times r$ matrices sequence $\{P_k\}_{k\in Z}$ called the two-scale matrix sequence, $\Phi(x)$ is called multiscaling functions and r is called multiplicity.

By taking Fourier transform the both sides of (1), we have

$$\hat{\Phi}(w) = P(e^{-iw/2})\hat{\Phi}\left(\frac{w}{2}\right). \tag{2}$$

Here $P(z) = \frac{1}{2}\sum_{k\in Z} P_k z^k$ called the two-scale matrix symbol of the two-scale matrix sequence $\{P_k\}_{k\in Z}$ of $\Phi(x)$.

By repeated applications of (2), we have

$$\hat{\Phi}(w) = \left(\prod_{j=1}^{\infty} P(e^{-iw/2^j}) \right) \hat{\Phi}(0).$$

If the infinite product $\prod_{j=1}^{\infty} P(e^{-iw/2^j})$ converges, then the $\hat{\Phi}(w)$ is well-defined and we will say that $\hat{\Phi}(w)$ is generated by $P(w)$.

In [24], Cabrelli, et al gave the following criteria to ensure the convergence of the above infinite product.

The infinite matrix product $\left(\prod_{j=1}^{\infty} P(e^{-iw/2^j}) \right)$ converges uniformly on compact sets to a continuous matrix-valued function if and only if the eigenvalues $\lambda_i, i = 1, 2, \ldots, r$ of the matrix $P(1)$ satisfy $\lambda_1 = 1, |\lambda_i| < 1, i = 2, 3, \ldots, r$ (See [4], [24]).

Let $\Psi(x) = [\psi_1(x), \psi_2(x), \ldots, \psi_r(x)]^T$ be an orthogonal multiwavelet corresponding to $\Phi(x)$, satisfying the following equation:

$$\Psi(x) = \sum_{k \in Z} Q_k \Phi(2x - k), \tag{3}$$

for some $r \times r$ matrices sequence $\{Q_k\}_{k \in Z}$. (3) can be rewritten as $\hat{\Psi}(w) = Q(e^{-iw/2})\hat{\Phi}(\frac{w}{2})$, where $Q(z) = \frac{1}{2} \sum_{k \in Z} Q_k z^k$.

Let $\Phi(x)$ be an orthogonal multiscaling function, and $\Psi(x)$ be an orthogonal multiwavelet corresponding to $\Phi(x)$, with two scale matrix symbol $P(z)$ and $Q(z)$, respectively. Then $P(z)$, and $Q(z)$ satisfy the following equations (See [12], [13]):

$$\begin{cases} P(z)P(z)^* + P(-z)P(-z)^* = I_{r \times r}, \\ P(z)Q(z)^* + P(-z)Q(-z)^* = O_{r \times r}, \\ Q(z)Q(z)^* + Q(-z)Q(-z)^* = I_{r \times r}, \end{cases} \tag{4}$$

where O and I_r denote the zero matrix and unity matrix, respectively. Here and throughout, the asterisk denotes conjugate transpose of matrix.

2. Construction of orthogonal multiscaling functions

In this section, we will introduce a procedure of constructing orthogonal multiscaling functions with multiplicity $r + s$ in terms of any given orthogonal multiscaling functions with multiplicity r.

To construct orthogonal multiscaling functions, we need the following lemma.

Lemma 1. *Let $\Phi(x)$ be an orthogonal multiscaling function, and $\Psi(x)$ be an orthogonal multiwavelet corresponding to $\Phi(x)$, with two-scale matrix* *symbol $P(z)$ and $Q(z)$, respectively. Suppose $Q^k(z), k = 1, 2, \ldots, r$ is the kth row of $Q(z)$. Then*

$$\left\{ \begin{array}{l} P(z)Q^k(z)^* + P(-z)Q^k(-z)^* = O_{r\times 1}, k = 1, 2, \ldots, r, \\ Q^j(z)Q^k(z)^* + Q^j(-z)Q^k(-z)^* = \delta_{j,k}, j, k = 1, 2, \ldots, r. \end{array} \right. \tag{5}$$

Proof. In terms of orthogonality of $\Phi(x)$ and $\Psi(x)$, then $P(z)$, and $Q(z)$ satisfy (4). Substituting $Q(z) = [Q^1(z)^*, Q^2(z)^*, \ldots, Q^r(z)^*]^*$ into (4), we obtain

$$\left\{ \begin{array}{l} P(z)\left[Q^1(z)^*, \ldots, Q^r(z)^*\right] + P(-z)\left[Q^1(-z)^*, \ldots, Q^r(-z)^*\right] = O_{r\times r}, \\ \left[Q^1(z)^*, \ldots, Q^r(z)^*\right]^* \left[Q^1(z)^*, \ldots, Q^r(z)^*\right] \\ + \left[Q^1(-z)^*, \ldots, Q^r(-z)^*\right]^* \left[Q^1(-z)^*, \ldots, Q^r(-z)^*\right] = I_{r\times r}. \end{array} \right.$$

This means that (5) holds. □

Let $h_{i,j}(z), i = 1, 2, \ldots, r; j = 1, 2, \ldots, s; |z| = 1$ satisfy the following conditions:

(C1): $h_{i,j}(z) = h_{i,j}(-z), i = 1, 2, \ldots, s; j = 1, 2, \ldots, r;$

(C2): For any integer $i, 1 \le i \le s$, $\sum_{j=1}^{r} |h_{i,j}(z)|^2 = a$, where $0 < a < 1$;

(C3): For any integer $i, k, 1 \le i < k \le s$, $\sum_{j=1}^{r} h_{i,j}(z)h_{k,j}(z)^* = 0$.

Construct $s \times r$ matrix $A(z)$ as follow

$$A(z) = \begin{bmatrix} h_{1,1}(z) & h_{1,2}(z) & \cdots & h_{1,r}(z) \\ h_{2,1}(z) & h_{2,2}(z) & \cdots & h_{2,r}(z) \\ \cdots & \cdots & \cdots & \cdots \\ h_{s,1}(z) & h_{s,2}(z) & \cdots & h_{s,r}(z) \end{bmatrix} \begin{bmatrix} Q^1(z) \\ Q^2(z) \\ \cdots \\ Q^r(z) \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{r} h_{1,j}(z)Q^j(z) \\ \sum_{j=1}^{r} h_{2,j}(z)Q^j(z) \\ \cdots \\ \sum_{j=1}^{r} h_{s,j}(z)Q^j(z) \end{bmatrix}. \tag{6}$$

Lemma 2. *Let $A(z)$ defined in (6) be $s \times r$ matrix. Then $A(z)A(z)^* + A(-z)A(-z)^* = aI_{s\times s}$.*

Proof. Since $A(z)A(z)^*$

$$= \begin{bmatrix} \sum_{j=1}^{r} h_{1,j}(z)Q^j(z) \\ \vdots \\ \sum_{j=1}^{r} h_{s,j}(z)Q^j(z) \end{bmatrix} \left[\sum_{k=1}^{r} h_{1,k}(z)^*Q^k(z)^*, \ldots, \sum_{k=1}^{r} h_{s,k}(z)^*Q^k(z)^*\right]$$

$$= \begin{bmatrix} \sum_{j,k=1}^{r} h_{1,j}(z)h_{1,k}(z)^*Q^j(z)Q^k(z)^* & \cdots & \sum_{j,k=1}^{r} h_{1,j}(z)h_{s,k}(z)^*Q^j(z)Q^k(z)^* \\ \cdots & \cdots & \cdots \\ \sum_{j,k=1}^{r} h_{s,j}(z)h_{1,k}(z)^*Q^j(z)Q^k(z)^* & \cdots & \sum_{j,k=1}^{r} h_{s,j}(z)h_{s,k}(z)^*Q^j(z)Q^k(z)^* \end{bmatrix}.$$

Consider (5) and $h_{i,j}(z)$ satisfying the conditions. We have $A(z)A(z)^* + A(-z)A(-z)^* =$

$$\begin{bmatrix} \sum_{j=1}^{r} |h_{1,j}(z)|^2 & 0 & \cdots & 0 \\ 0 & \sum_{j=1}^{r} |h_{2,j}(z)|^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \sum_{j=1}^{r} |h_{s,j}(z)|^2 \end{bmatrix} = aI_{s\times s}.$$

□

Remark 1. In definition of $A(z)$, $h_{i,j}(z), i = 1, 2, \ldots, r; j = 1, 2, \ldots, s$ are required satisfying conditions (C1),(C2),(C3). There exist a lot of functions satisfying these conditions. For example, $h_{i,i}(z) = \sqrt{a}, i = \min\{r, s\}; h_{i,j}(z) = 0, i \neq j$.

Lemma 3. *Let $P(z)$ and $Q(z)$ be two scale matrix symbols associated with $\Phi(x)$ and $\Psi(x)$, respectively. Suppose that $A(z)$ defined in (6) is the $s \times r$ matrix. Then*

$$P(z)A(z)^* + P(-z)A(-z)^* = O_{r\times s},$$

$$A(z)Q(z)^* + A(-z)Q(-z)^* = \begin{bmatrix} h_{1,1}(z) & h_{1,2}(z) & \cdots & h_{1,r}(z) \\ h_{2,1}(z) & h_{2,2}(z) & \cdots & h_{2,r}(z) \\ \cdots & \cdots & \cdots & \cdots \\ h_{s,1}(z) & h_{s,2}(z) & \cdots & h_{s,r}(z) \end{bmatrix}.$$

Theorem 1. *Let $P(z)$ and $Q(z)$ be two-scale matrix symbols associated with $\Phi(x)$ and $\Psi(x)$, respectively, $A(z)$ defined in (6) be $s \times r$ matrix, and $B(z)$ be $s \times s$ matrix, satisfying $B(z)B(z)^* + B(-z)B(-z)^* = (1 - a)I_{s\times s}$, where $a \in (0, 1)$. Define*

$$P^{new}(z) = \begin{bmatrix} P(z) & O \\ A(z) & B(z) \end{bmatrix}. \tag{7}$$

Then

$$P^{new}(z)P^{new}(z)^* + P^{new}(-z)P^{new}(-z)^* = I_{(r+s)\times(r+s)}. \tag{8}$$

Proof. Since $P(z)$ and $Q(z)$ are two scale matrix symbols associated with $\Phi(x)$ and $\Psi(x)$, respectively, then $|P(z)|^2 + |P(-z)|^2 = I_{r\times r}$. By Lemma 1 and

Lemma 2, we have

$$\begin{aligned}
&P^{new}(z)P^{new}(z)^* + P^{new}(-z)P^{new}(-z)^* \\
&\quad = \begin{bmatrix} P(z) & 0 \\ A(z) & B(z) \end{bmatrix} \begin{bmatrix} P(z)^* & A(z)^* \\ 0 & B(z)^* \end{bmatrix} \\
&\qquad + \begin{bmatrix} P(-z) & 0 \\ A(-z) & B(-z) \end{bmatrix} \begin{bmatrix} P(-z)^* & A(-z)^* \\ 0 & B(-z)^* \end{bmatrix} \\
&\quad = \begin{bmatrix} P(z)P(z)^* + P(-z)P(-z)^* & P(z)A(z)^* + P(-z)A(-z)^* \\ A(z)P(z)^* + A(-z)P(-z)^* & U(z) \end{bmatrix} \\
&\quad = \begin{bmatrix} I_{r\times r} & O_{r\times s} \\ O_{s\times r} & I_{s\times s} \end{bmatrix} = I_{(r+s)\times(r+s)}.
\end{aligned}$$

Here $U(z) = A(z)A(z)^* + A(-z)A(-z)^* + B(z)B(z)^* + B(-z)B(-z)^* = I_{s\times s}$. □

Remark 2. There exist a lot of $B(z)$ satisfying the condition $B(z)B(z)^* + B(-z)B(-z)^* = (1-a)I_{s\times s}$. For example, let $\mathcal{P}(z)$ be the two scale matrix symbol associated with an orthogonal multiscaling function with multiplicity s. Then $\mathcal{P}(z)\mathcal{P}(z)^* + \mathcal{P}(-z)\mathcal{P}(-z)^* = I_{s\times s}$. Take $B(z) = (1-a)^{\frac{1}{2}}\mathcal{P}(z)$. It is easy to verify that $B(z)$ satisfies the condition $B(z)B(z)^* + B(-z)B(-z)^* = (1-a)I_{s\times s}$. Additionally, let $p^i(z), i = 1, 2, \ldots, s$ be two scale symbols associated with orthogonal uniscaling function $\phi^i(x)$. Take $B(z) = (1-a)^{\frac{1}{2}}\mathrm{diag}[p^1(z), p^2(z), \ldots, p^s(z)]$. Then $B(z)$ satisfies the condition $B(z)B(z)^* + B(-z)B(-z)^* = (1-a)I_{s\times s}$.

Theorem 2. *Let $P^{new}(z)$ defined in (7) be a lower triangle matrix. Under the conditions of Theorem 1, if the eigenvalues $\lambda_i, i = 1, 2, \ldots, r$ of the matrix $P(1)$ satisfy $\lambda_1 = 1, |\lambda_i| < 1, i = 2, 3, \ldots, r$, and the eigenvalues $\mu_i, i = 1, 2, \ldots, s$ of the matrix $B(1)$ satisfy $|\mu_i| < 1, i = 1, 2, \ldots, s$, then 1 must be a simple eigenvalue of the matrix $P^{new}(1)$, and all other eigenvalues λ of $P^{new}(1)$ must satisfy $|\lambda| < 1$.*

Proof. Since $P^{new}(1) = \begin{bmatrix} P(1) & O \\ A(1) & B(1) \end{bmatrix}$, then $|\lambda E_{r+s} - P^{new}(1)| = |\lambda E_r - P(1)||\lambda E_s - B(1)|$. Obviously, all the eigenvalues of the matrices $P(1)$ and $B(1)$ must be the eigenvalues of the matrix $P^{new}(1)$. This completes the proof of Theorem 2. □

According to [24], the infinite matrix product $\prod_{j=1}^{\infty} P^{new}(e^{-iw/2^j})$ converges. Thus an orthogonal multiscaling function $\Phi^{new}(x)$ with multiplicity $r+s$ is well-defined, in terms of Fourier transform, by

$$\begin{aligned}
&\hat{\Phi}^{new}(w) = [\hat{\phi}_1(w), \ldots, \hat{\phi}_r(w), \hat{\phi}_{r+1}(w), \ldots, \hat{\phi}_{r+s}(w)]^T \\
&= \begin{bmatrix} P(e^{-iw/2}) & 0 \\ A(e^{-iw/2}) & B(e^{-iw/2}) \end{bmatrix} \left[\hat{\phi}_1\left(\frac{w}{2}\right), \ldots, \hat{\phi}_r\left(\frac{w}{2}\right), \hat{\phi}_{r+1}\left(\frac{w}{2}\right), \ldots, \hat{\phi}_{r+s}\left(\frac{w}{2}\right)\right]^T.
\end{aligned}$$

We can see easily that $\phi_1(x), \phi_2(x), \ldots, \phi_r(x)$, the first r components of the orthogonal multiscaling function $\Phi^{new}(x)$, are old functions generated by $P(z)$; $\phi_{r+1}(x), \phi_{r+2}(x), \ldots, \phi_{r+s}(x)$, the final s components of the orthogonal multiscaling function $\Phi^{new}(x)$, are new functions that we construct. That is

$$\begin{aligned}\hat{\Phi}(w) =&[\hat{\phi}_1(w), \ldots, \hat{\phi}_r(w)]^T = P(e^{-iw/2}) \left[\hat{\phi}_1\left(\frac{w}{2}\right), \ldots, \hat{\phi}_r\left(\frac{w}{2}\right)\right]^T, \\ &[\hat{\phi}_{r+1}(w), \ldots, \hat{\phi}_{r+s}(w)]^T = A(e^{-iw/2}) \left[\hat{\phi}_1\left(\frac{w}{2}\right), \ldots, \hat{\phi}_r\left(\frac{w}{2}\right)\right]^T \\ &+ B(e^{-iw/2}) \left[\hat{\phi}_{r+1}\left(\frac{w}{2}\right), \ldots, \hat{\phi}_{r+s}\left(\frac{w}{2}\right)\right]^T.\end{aligned}$$

According to the above discussion, we have the following construction theorem.

Theorem 3. *Let $\Phi(x) = [\phi_1(x), \phi_2(x), \ldots, \phi_r(x)]^T$ be an orthogonal multiscaling function, and $\Psi(x)$ be an orthogonal multiwavelet corresponding to $\Phi(x)$, with two-scale matrix symbols $P(z)$ and $Q(z)$, respectively. Under the condition of Theorem 1 and Theorem 2, there are $\phi_{r+1}, \phi_{r+2}, \ldots, \phi_{r+s}$ such that $\Phi^{new}(x) = [\Phi^T(x), \phi_{r+1}, \phi_{r+2}, \ldots, \phi_{r+s}]^T$ is an orthogonal multiscaling function with multiplicity $r+s$ and its two scale matrix symbol $P^{new}(z)$ is given by* (7).

3. Explicit formula for constructing orthogonal multiwavelets

In the above section, we give a method of constructing orthogonal multiscaling functions. In this section, we will discuss the construction of the corresponding multiwavelets.

Construct the matrices $Q^{new}(z), M(z)$ by

$$\left\{\begin{aligned} Q^{new}(z) &= \begin{bmatrix} XQ(z) & YB(z) \\ O & (1-a)^{-\frac{1}{2}} z^k B(-z)^* \end{bmatrix}, \\ M(z) &= \begin{bmatrix} P^{new}(z) & P^{new}(-z) \\ Q^{new}(z) & Q^{new}(-z) \end{bmatrix}, \end{aligned}\right. \tag{9}$$

where X is $r \times r$ matrix, Y is $r \times s$ matrix, and k is odd number.

Next we will give an explicit construction formula for orthogonal multiwavelets corresponding to $\Phi^{new}(x)$.

Theorem 4. *Let $\Phi(x) = [\phi_1(x), \phi_2(x), \ldots, \phi_r(x)]^T$ be an orthogonal multiscaling function, and $\Psi(x)$ be an orthogonal multiwavelet corresponding to $\Phi(x)$, with two-scale matrix symbols $P(z)$ and $Q(z)$, respectively, and $B(z)$ be $s \times s$ diagonal matrix satisfying $B(z)B(z)^* + B(-z)B(-z)^* = (1-a)I_{s\times s}$, where $0 < a < 1$. If matrices X, Y satisfy the following conditions*:

$$\left\{\begin{aligned} &\begin{bmatrix} h_{1,1} & h_{1,2} & \cdots & h_{1,r} \\ h_{2,1} & h_{2,2} & \cdots & h_{2,r} \\ \cdots & \cdots & \cdots & \cdots \\ h_{s,1} & h_{s,2} & \cdots & h_{s,r} \end{bmatrix} X^* + (1-a)Y^* = O_{s\times r}, \\ &XX^* + (1-a)YY^* = I_{r\times r}, \end{aligned}\right. \tag{10}$$

then under the condition of Theorem 1 and Theorem 3, $M(z)$ defined in (9) is a unitary matrix. Further, suppose that $\Phi^{new}(x) = [\Phi^T(x), \phi_{r+1}(x), \phi_{r+2}(x), \dots, \phi_{r+s}(x)]^T$ is the orthogonal multiscaling function generated by $P^{new}(z)$, $Q^{new}(z)$ defined in (9) is an upper triangle matrix. Then orthogonal multiwavelet corresponding to $\Phi^{new}(x)$ is given, in terms of Fourier transform, by

$$\hat{\Psi}^{new}(w) = Q^{new}(e^{-iw/2})\hat{\Phi}^{new}\left(\frac{w}{2}\right).$$

Proof. According to the wavelet construction theorem, we only need prove $M(z)$ is a unitary matrix. That is, $P^{new}(z)$ and $Q^{new}(z)$ must satisfy the following equations:

$$P^{new}(z)P^{new}(z)^* + P^{new}(-z)P^{new}(-z)^* = I_{(r+s)\times(r+s)}, \tag{11}$$

$$P^{new}(z)Q^{new}(z)^* + P^{new}(-z)Q^{new}(-z)^* = O_{(r+s)\times(r+s)}, \tag{12}$$

$$Q^{new}(z)Q^{new}(z)^* + Q^{new}(-z)Q^{new}(-z)^* = I_{(r+s)\times(r+s)}. \tag{13}$$

By Theorem 1, (11) holds. Next, we only need to prove (12) and (13) hold. In fact,

$$\begin{aligned} P^{new}(z)Q^{new}(z)^* &= \begin{bmatrix} P(z) & 0 \\ A(z) & B(z) \end{bmatrix} \begin{bmatrix} Q(z)^*X^* & O \\ B(z)^*Y^* & (1-a)^{-\frac{1}{2}}\overline{z}^k B(-z) \end{bmatrix} \\ &= \begin{bmatrix} P(z)Q(z)^*X^* & O \\ A(z)Q(z)^*X^* + B(z)B(z)^*Y^* & (1-a)^{-\frac{1}{2}}\overline{z}^k B(z)B(-z) \end{bmatrix}. \end{aligned}$$

By [13], we have $P(z)Q(z)^* + P(-z)Q(-z)^* = O_{r\times r}$. Hence $[P(z)Q(z)^* + P(-z)Q(-z)^*]X^* = O_{r\times r}$. Using the condition $B(z)B(z)^* + B(-z)B(-z)^* = (1-a)I_{s\times s}$ and Lemma 3, we obtain

$$\begin{aligned} &[A(z)Q(z)^* + A(-z)Q(-z)^*]X^* + [B(z)B(z)^* + B(-z)B(-z)^*]Y^* \\ &= \begin{bmatrix} h_{1,1} & h_{1,2} & \cdots & h_{1,r} \\ h_{2,1} & h_{2,2} & \cdots & h_{2,r} \\ \cdots & \cdots & \cdots & \cdots \\ h_{s,1} & h_{s,2} & \cdots & h_{s,r} \end{bmatrix} X^* + (1-a)Y^* = O_{s\times r}. \end{aligned}$$

Therefore (12) holds. Again

$$\begin{aligned} &Q^{new}(z)Q^{new}(z)^* \\ &= \begin{bmatrix} XQ(z) & YB(z) \\ O & (1-a)^{-\frac{1}{2}}z^k B(-z)^* \end{bmatrix} \begin{bmatrix} Q(z)^*X^* & O \\ B(z)^*Y^* & (1-a)^{-\frac{1}{2}}\overline{z}^k B(-z) \end{bmatrix} \\ &= \begin{bmatrix} XQ(z)Q(z)^*X^* + YB(z)B(z)^*Y^* & (1-a)^{-\frac{1}{2}}\overline{z}^k YB(z)B(-z) \\ (1-a)^{-\frac{1}{2}}z^k B(-z)^*B(z)^*Y^* & (1-a)^{-1}z^k\overline{z}^k B(-z)^*B(-z) \end{bmatrix}. \end{aligned}$$

By (10), we have $Q^{new}(z)Q^{new}(z)^* + Q^{new}(-z)Q^{new}(-z)^*$

$$\begin{aligned} &= \begin{bmatrix} XX^* + (1-a)YY^* & O \\ O & (1-a)^{-1}[B(z)^*B(z) + B(-z)^*B(-z)] \end{bmatrix} \\ &= \begin{bmatrix} I_{r\times r} & O \\ O & I_{s\times s} \end{bmatrix} = I_{(r+s)\times(r+s)}, \end{aligned}$$

which implies that $M(z)$ is a unitary matrix. This completes the proof of Theorem 4. $\square$

Next, we will discuss a special setting: $r = s$. Similar to Theorem 3, we also have the following corollary.

Corollary 1. *Let $\Phi(x) = [\phi_1(x), \phi_2(x), \ldots, \phi_r(x)]^T$ be an orthogonal multiscaling function, and $\Psi(x)$ be an orthogonal multiwavelet corresponding to $\Phi(x)$, with two scale matrix symbols $P(z)$ and $Q(z)$, respectively, and $B(z)$ be $r \times r$ matrix, satisfying $B(z)B(z)^* + B(-z)B(-z)^* = (1-a)I_{r\times r}$, where $0 < a < 1$. Suppose that H is any unitary matrix. Define*

$$P^{\#}(z) = \begin{bmatrix} P(z) & O \\ \sqrt{a}HQ(z) & B(z) \end{bmatrix}. \tag{14}$$

Then there are $\phi_{r+1}(x), \ldots, \phi_{2r}(x)$ such that $\Phi^{\#}(x) = [\Phi^T(x), \phi_{r+1}(x), \phi_{r+2}(x), \ldots, \phi_{2r}(x)]^T$ is an orthogonal multiscaling function with multiplicity $2r$ and its two scale matrix symbol $P^{\#}(z)$ is given by (14).

To give an explicit formula of constructing an orthogonal multiwavelet corresponding to $\Phi^{\#}(x)$, we suppose that matrix $B(z)$ satisfies $B(z)B(-z) = B(-z)B(z)$. Extraordinarily, we suppose that $B(z)$ is $r \times r$ diagonal matrix.

Corollary 2. *Suppose that $B(z)$ is $r \times r$ diagonal matrix, satisfying*

$$B(z)B(z)^* + B(-z)B(-z)^* = (1-a)I_{r\times r}, 0 < a < 1.$$

Let k be an odd number. Define

$$Q^{\#}(z) = \begin{bmatrix} \sqrt{1-a}Q(z) & -\sqrt{\frac{a}{1-a}}H^*B(z) \\ O & \sqrt{\frac{1}{1-a}}z^kB(-z)^* \end{bmatrix}. \tag{15}$$

Then under the condition of Corollary 1*, the orthogonal multiwavelet $\Psi^{\#}(x)$ corresponding to $\Phi^{\#}(x)$ is given, in terms of Fourier transform, by*

$$\hat{\Psi}^{\#}(w) = Q^{\#}(e^{-iw/2})\hat{\Phi}^{\#}\left(\frac{w}{2}\right). \tag{16}$$

Proof. It is easy to verify that $\begin{bmatrix} P^{\#}(z) & P^{\#}(-z) \\ Q^{\#}(z) & Q^{\#}(-z) \end{bmatrix}$ is a unitary matrix. Hence, according to the wavelet construction theorem, we are able to define the orthogonal multiwavelet $\Psi^{\#}(x)$ by (16). $\square$

4. Approximation order

One of the properties of a multiscaling function which has great practical interest is the approximation order. A multiscaling function $\Phi(x)$ has approximation order $m \geq 1$ if all powers of x up to $m-1$ can be locally written as linear combinations of its integer translates. Namely, there exist vectors $\mathbf{y}_k^{(j)} \in \mathbf{R}^r$ such that for $j = 0, 1, \ldots, m-1$

$$x^j = \sum_k [\mathbf{y}_k^{(j)}]^* \Phi(x-k).$$

Equivalently, a multiscaling functions $\Phi(x)$ has approximation order $m \geq 1$ if m is the largest integer for which there is a set of row vectors $\{\mathbf{u}^\ell\}_{\ell=0}^{m-1} \subset R^{1\times r}$, with $\mathbf{u}^0 \neq O_{1\times r}$ that satisfy

$$\sum_{k=0}^{\ell} \binom{\ell}{k} (-i)^{\ell-k} 2^k \mathbf{u}^k D^{\ell-k} P\left(e^{-\pi h i}\right) = \delta_{0,h} \mathbf{u}^\ell,$$

for $\ell = 0, 1, \ldots, m-1$ and $h = 0, 1$. D denotes the differentiation operator. See [16], [19] and [21] for details. As is known, if a multiscaling function has approximation order m, this implies that the corresponding multiwavelet has m vanishing moments.

In this section, we discuss the approximation orders of new orthogonal multiscaling functions constructed in Section 2.

Let

$$b_u(z) = \sum_{j\in Z} b_j^u z^j = \frac{1}{2^{m-1}} \left(\frac{1+z}{2}\right)^{n_u} h_u(z), u = 1, 2, \ldots, s, \tag{17}$$

where $z = e^{-iw}$, n_u are positive integers, $h_u(z)$ are trigonometric polynomials satisfying $h_u(1) = 1$.

Define $s \times s$ diagonal matrix $B(z)$ by

$$B(z) = \mathrm{diag}[b_1(z), b_2(z), \ldots, b_s(z)]. \tag{18}$$

Then we have the following lemma.

Lemma 4. *Let $b_u(z)$ be trigonometric polynomials defined in* (7) *and b_j^u be the corresponding coefficients. Then*

$$2^m \sum_{j\in Z} b_{2j}^u = 2^m \sum_{j\in Z} b_{2j+1}^u = 1, u = 1, 2, \ldots, s,$$

$$\sum_{j\in Z} (2j)^k b_{2j}^u = \sum_{j\in Z} (2j+1)^k b_{2j+1}^u, k = 1, 2, \ldots, n_u - 1.$$

Further, suppose that $B(z) = \sum_{j\in Z} B_j z^j$, and $L = \min\{n_1, n_2, \ldots, n_s\}$. Then

$$\sum_{j\in Z} (2j)^k B_{2j} = \sum_{j\in Z} (2j+1)^k B_{2j+1}, k = 1, 2, \ldots, L.$$

Lemma 5. *If all $b_u(z), u = 1, 2, \ldots, s$ satisfy $|b_u(z)|^2 + |b_u(-z)|^2 = \frac{1}{2^{2m-2}}$, then*

$$B(z)B(z)^* + B(-z)B(-z)^* = \left[1 - \frac{2^{2m-2}-1}{2^{2m-2}}\right] I_{s\times s}. \tag{19}$$

Theorem 5. *Let $\Phi(x) = [\phi_1(x), \phi_2(x), \dots, \phi_r(x)]^T$ be an orthogonal multiscaling function, $\Psi(x)$ be the corresponding orthogonal multiwavelet, with two-scale matrix symbols $P(z)$, $Q(z)$ and approximation orders m, respectively. Suppose that $B(z)$ given by* (18) *satisfies* (19)*, and $A(z)$ defined in* (6) *satisfies $A(z)A(z)^* + A(-z)A(-z)^* = \frac{2^{2m-2}-1}{2^{2m-2}}$. Then the two scale matrix symbol $P^{new}(z)$ given by (7) can generate a new orthogonal multiscaling functions $\Phi^{new}(x) = [\Phi^T(x), \phi_{r+1}(x), \dots, \phi_{r+s}(x)]^T$ with approximation order $m + L$.*

Proof. By Theorem 3, $P^{new}(z)$ can generate a new orthogonal multiscaling function $\Phi^{new}(x)$. Next, we will prove that this new orthogonal multiscaling function has approximation order $m + L$.

Since the approximation order of $\Phi(x)$ is m, there are $\mathbf{a}^\ell \in R^r, \ell = 0, 1, \dots, m-1$, with $\mathbf{a}^0 \neq O_{1\times r}$, such that

$$\mathbf{a}^\ell \left(\sum_{j\in Z} P_{2j} - \frac{1}{2^\ell} I_{r\times r} \right) = -\sum_{k=0}^{\ell-1} (-1)^{\ell-k} \frac{1}{2^{\ell-k}} \binom{\ell}{k} \mathbf{a}^k \sum_{j\in Z} (2j)^{\ell-k} P_{2j}, \tag{20}$$

$$\mathbf{a}^\ell \left(\sum_{j\in Z} P_{2j+1} - \frac{1}{2^\ell} I_{r\times r} \right) = -\sum_{k=0}^{\ell-1} (-1)^{\ell-k} \frac{1}{2^{\ell-k}} \binom{\ell}{k} \mathbf{a}^k \sum_{j\in Z} (2j+1)^{\ell-k} P_{2j+1}. \tag{21}$$

Next, we will prove the approximation order of $\Phi^{new}(x)$ is $m + L$. That is, we will find a set of row vectors $\mathbf{w}^\ell \in R^{r+s}, \ell = 0, 1, \dots, m+L-1$, with $\mathbf{w}^0 \neq O_{1\times(r+s)}$ such that

$$\begin{aligned} &\mathbf{w}^\ell \left(\begin{bmatrix} \sum_{j\in Z} P_{2j} & O_{r\times s} \\ \sum_{j\in Z} A_{2j} & \sum_{j\in Z} B_{2j} \end{bmatrix} - \frac{1}{2^\ell} I_{(r+s)\times(r+s)} \right) \\ &= -\sum_{k=0}^{\ell-1} \frac{(-1)^{\ell-k}}{2^{\ell-k}} \binom{\ell}{k} \mathbf{w}^k \begin{bmatrix} \sum_{j\in Z} (2j)^{\ell-k} P_{2j} & O_{r\times s} \\ \sum_{j\in Z} (2j)^{\ell-k} A_{2j} & \sum_{j\in Z} (2j)^{\ell-k} B_{2j} \end{bmatrix}, \end{aligned} \tag{22}$$

$$\begin{aligned} &\mathbf{w}^\ell \left(\begin{bmatrix} \sum_{j\in Z} P_{2j+1} & O_{r\times s} \\ \sum_{j\in Z} A_{2j+1} & \sum_{j\in Z} B_{2j+1} \end{bmatrix} - \frac{1}{2^\ell} I_{(r+s)\times(r+s)} \right) \\ &= -\sum_{k=0}^{\ell-1} \frac{(-1)^{\ell-k}}{2^{\ell-k}} \binom{\ell}{k} \mathbf{w}^k \begin{bmatrix} \sum_{j\in Z} (2j+1)^{\ell-k} P_{2j+1} & O_{r\times s} \\ \sum_{j\in Z} (2j+1)^{\ell-k} A_{2j+1} & \sum_{j\in Z} (2j+1)^{\ell-k} B_{2j+1}. \end{bmatrix}. \end{aligned} \tag{23}$$

It is clear that $\mathbf{w}^\ell = [\mathbf{a}^\ell, 0, 0, \dots, 0] \in R^{r+s}, \ell = 0, 1, \dots, m-1$, satisfy (22) and (23). Hence we choose $\mathbf{w}^\ell = [\mathbf{a}^\ell, 0, 0, \dots, 0] \in R^{r+s}, \ell = 0, 1, \dots, m-1$, is the first m vectors in (22) and (23). The rest L row vectors are denoted by $\mathbf{w}^{m+\ell} = [\mathbf{a}^{m+\ell}, c^1_{m+\ell}, c^2_{m+\ell}, \dots, c^s_{m+\ell}], \ell = 0, 1, \dots, L-1$. Obviously, $\mathbf{w}^m$ must satisfy $\sum\limits_{j=1}^{s} |c_m^j| \neq 0$. In fact, if all $c_m^j = 0$, then $\mathbf{w}^m = [\mathbf{a}^m, 0, 0, \ldots, 0]$. This means that the approximation order of $\Phi(x)$ is $m+1$. If we use the notation $\mathbf{w}^\ell = [\mathbf{a}^\ell, c_\ell^1, c_\ell^2, \ldots, c_\ell^s]$, then $c_\ell^j = 0$ for $j = 1, 2, \ldots, s; \ell = 0, 1, \ldots, m-1$. Hence (22) is equivalent to

$$\begin{aligned}
&\mathbf{a}^\ell \left(\sum_{j\in Z} P_{2j} - \frac{1}{2^\ell} I_{r\times r} \right) + [c_\ell^1, c_\ell^2, \ldots, c_\ell^s] \sum_{j\in Z} A_{2j} \\
&\quad = -\sum_{k=0}^{\ell-1} \frac{(-1)^{\ell-k}}{2^{\ell-k}} \binom{\ell}{k} \left[\mathbf{a}^k \sum_{j\in Z} (2j)^{\ell-k} P_{2j} + [c_k^1, c_k^2, \ldots, c_k^s] \sum_{j\in Z} (2j)^{\ell-k} A_{2j} \right],
\end{aligned} \tag{24}$$

$$\begin{aligned}
&[c_\ell^1, c_\ell^2, \ldots, c_\ell^s] \left[\sum_{j\in Z} B_{2j} - \frac{1}{2^\ell} I_{s\times s} \right] \\
&\quad = -\sum_{k=0}^{\ell-1} \frac{(-1)^{\ell-k}}{2^{\ell-k}} \binom{\ell}{k} [c_k^1, c_k^2, \ldots, c_k^s] \sum_{j\in Z} (2j)^{\ell-k} B_{2j}.
\end{aligned} \tag{25}$$

Since $c_\ell^j = 0$ for $j = 1, 2, \ldots, s; \ell = 0, 1, \ldots, m-1$, then (25) becomes the following two identities

$$[c_m^1, c_m^2, \ldots, c_m^s] \left[\sum_{j\in Z} B_{2j} - \frac{1}{2^m} I_{s\times s} \right] = O_{s\times s}, \tag{26}$$

$$\begin{aligned}
&[c_{m+\ell}^1, c_{m+\ell}^2, \ldots, c_{m+\ell}^s] \left[\sum_{j\in Z} B_{2j} - \frac{1}{2^{m+\ell}} I_{s\times s} \right] \\
&\quad = -\sum_{k=0}^{\ell-1} \frac{(-1)^{\ell-k}}{2^{\ell-k}} \binom{m+\ell}{\ell-k} [c_{m+k}^1, c_{m+k}^2, \ldots, c_{m+k}^s] \sum_{j\in Z} (2j)^{\ell-k} B_{2j}, \\
&\quad \ell = 1, 2, \ldots, L-1.
\end{aligned} \tag{27}$$

By Lemma 4, $\sum\limits_{j\in Z} B_{2j} = \frac{1}{2^m} I_{s\times s}$. Hence $\sum\limits_{j\in Z} B_{2j} - \frac{1}{2^{m+\ell}} I_{s\times s} = \frac{2^\ell - 1}{2^{m+\ell}} I_{s\times s}$. Therefore

$$\begin{aligned}
&[c_{m+\ell}^1, c_{m+\ell}^2, \ldots, c_{m+\ell}^s] \\
&\quad = -\frac{2^m}{2^\ell - 1} \sum_{k=0}^{\ell-1} (-1)^{\ell-k} 2^k \binom{m+\ell}{\ell-k} [c_{m+k}^1, c_{m+k}^2, \ldots, c_{m+k}^s] \sum_{j\in Z} (2j)^{\ell-k} B_{2j}, \\
&\quad \ell = 1, 2, \ldots, L-1.
\end{aligned} \tag{28}$$

Similarly, applying (23), we have

$$[c_m^1, c_m^2, \ldots, c_m^s]\left[\sum_{j\in Z} B_{2j+1} - \frac{1}{2^m} I_{s\times s}\right] = O_{s\times s}, \tag{29}$$

(30)

$$\begin{aligned}&[c_{m+\ell}^1, c_{m+\ell}^2, \ldots, c_{m+\ell}^s]\left[\sum_{j\in Z} B_{2j+1} - \frac{1}{2^{m+\ell}} I_{s\times s}\right]\\ &\quad = -\sum_{k=0}^{\ell-1}\frac{(-1)^{\ell-k}}{2^{\ell-k}}\binom{m+\ell}{\ell-k}[c_{m+k}^1, c_{m+k}^2, \ldots, c_{m+k}^s]\sum_{j\in Z}(2j+1)^{\ell-k}B_{2j+1},\\ &\quad \ell = 1, 2, \ldots, L-1.\end{aligned}$$

Hence, we have

(31)

$$\begin{aligned}&[c_{m+\ell}^1, c_{m+\ell}^2, \ldots, c_{m+\ell}^s]\\ &= -\frac{2^m}{2^\ell - 1}\sum_{k=0}^{\ell-1}(-1)^{\ell-k}2^k\binom{m+\ell}{\ell-k}[c_{m+k}^1, c_{m+k}^2, \ldots, c_{m+k}^s]\sum_{j\in Z}(2j+1)^{\ell-k}B_{2j+1},\\ &\ell = 1, 2, \ldots, L-1.\end{aligned}$$

By (28) or (31), taking any $[c_m^1, c_m^2, \ldots, c_m^s] \neq \mathbf{O}_{1\times s}$, we can obtain $[c_{m+\ell}^1, c_{m+\ell}^2, \ldots, c_{m+\ell}^s], \ell = 1, 2, \ldots, L-1$. And then applying (24), we can obtain $\mathbf{a}^{m+\ell}$. This means that the rest $L-1$ row vectors $\mathbf{w}^{m+\ell} = [\mathbf{a}^{m+\ell}, c_{m+\ell}^1, c_{m+\ell}^2, \ldots, c_{m+\ell}^s], \ell = 1, 2, \ldots, L-1$ are obtained. Thereby, we prove that $\Phi^{new}(x)$ has approximation order $m+L$. This completes the proof of Theorem 5. □

Remark 3. Lemma 4 can guarantee that vectors $[c_{m+\ell}^1, c_{m+\ell}^2, \ldots, c_{m+\ell}^s], \ell = 1, 2, \ldots, L-1$ obtained by (28) and (31) are the same.

5. Example

We will illustrate by an example how to construct orthogonal multiwaveltet with higher approximation order in terms of any given orthogonal uniwavelet or multiwavelet based on our method.

Example. Let $\Phi(x) = (\phi_1, \phi_2)^T$, supp$\Phi(x) \subset [0, 2]$, be an orthogonal multiscaling function, satisfying the following equation [25]:

$$\Phi(x) = P_0\Phi(2x) + P_1\Phi(2x-1) + P_2\Phi(2x-2),$$

where

$$P_0 = \begin{bmatrix} 0 & \frac{2+\sqrt{7}}{4} \\ 0 & \frac{2-\sqrt{7}}{4}\end{bmatrix}, \qquad P_1 = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4}\end{bmatrix}, \qquad P_2 = \begin{bmatrix} \frac{2-\sqrt{7}}{4} & 0 \\ \frac{2+\sqrt{7}}{4} & 0\end{bmatrix}.$$

The corresponding orthogonal multiwavelet $\Psi(x)$ satisfies the following equation

$$\Psi(x) = Q_0\Phi(2x) + Q_1\Phi(2x-1) + Q_2\Phi(2x-2),$$

where

$$Q_0 = \begin{bmatrix} 0 & \frac{3}{4} \\ 0 & \frac{1}{4} \end{bmatrix}, \qquad Q_1 = \begin{bmatrix} -\frac{2+\sqrt{7}}{4} & -\frac{2-\sqrt{7}}{4} \\ -\frac{2-\sqrt{7}}{4} & -\frac{2+\sqrt{7}}{4} \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{3}{4} & 0 \end{bmatrix}.$$

Hence,

$$P(z) = \frac{1}{2}\begin{bmatrix} \frac{3}{4}z + \frac{2-\sqrt{7}}{4}z^2 & \frac{2+\sqrt{7}}{4} + \frac{1}{4}z \\ \frac{1}{4}z + \frac{2+\sqrt{7}}{4}z^2 & \frac{2-\sqrt{7}}{4} + \frac{3}{4}z \end{bmatrix},$$

$$Q(z) = \frac{1}{2}\begin{bmatrix} -\frac{2+\sqrt{7}}{4}z + \frac{1}{4}z^2 & \frac{3}{4} - \frac{2-\sqrt{7}}{4}z \\ -\frac{2-\sqrt{7}}{4}z + \frac{3}{4}z^2 & \frac{1}{4} - \frac{2+\sqrt{7}}{4}z \end{bmatrix}.$$

Take $B(z) = \frac{1}{2}\left[\frac{1+z}{2}\right]^2 \frac{(1+\sqrt{3})+(1-\sqrt{3})z}{2}$, $A(z) = \frac{\sqrt{3}}{2}Q^2(z) = \frac{1}{4}[-\frac{2-\sqrt{7}}{4}z+\frac{3}{4}z^2, \frac{1}{4}-\frac{2+\sqrt{7}}{4}z]$. It is easy to verify that $A(z)A(z)^* + A(-z)A(-z)^* = \frac{3}{4}, B(z)B(z)^* + B(-z)B(-z)^* = 1-\frac{3}{4}$. Hence, $a = \frac{3}{4}$. By (7), we obtain

$$P^{new}(z) = \begin{bmatrix} \frac{3}{8}z + \frac{2-\sqrt{7}}{8}z^2 & \frac{2+\sqrt{7}}{8} + \frac{1}{8}z & 0 \\ \frac{1}{8}z + \frac{2+\sqrt{7}}{8}z^2 & \frac{2-\sqrt{7}}{8} + \frac{3}{8}z & 0 \\ -\frac{2-\sqrt{7}}{16}z + \frac{3}{16}z^2 & \frac{1}{16} - \frac{2+\sqrt{7}}{16}z & \frac{1}{2}\left[\frac{1+z}{2}\right]^2 \frac{(1+\sqrt{3})+(1-\sqrt{3})z}{2} \end{bmatrix}. \tag{32}$$

Again applying Theorem 3, we obtain a new orthogonal multiscaling function $\Phi^{new}(x) = [\phi_1(x), \phi_2(x), \phi_3(x)]^T$, with two scale matrix symbol $P^{new}(z)$ given by (32). By Lemm 2.2 from [13], we get supp$\Phi^{new}(x) \subset [0,3]$.

Let $X = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{bmatrix}, Y = [-\sqrt{3}, 0]^T$. It is easy to verify that matrices X, Y satisfy (10). Thus, by (9), and taking $k = 3$, we construct the matrix $Q^{new}(z)$ by $Q^{new}(z) =$

$$\begin{bmatrix} -\frac{2\sqrt{3}-\sqrt{21}}{16}z + \frac{3\sqrt{3}}{16}z^2 & \frac{\sqrt{3}}{16} - \frac{2\sqrt{3}+\sqrt{21}}{16}z & -\frac{1+\sqrt{3}}{12} - \frac{3+\sqrt{3}}{12}z - \frac{3-\sqrt{3}}{12}z^2 - \frac{1-\sqrt{3}}{12}z^3 \\ -\frac{2+\sqrt{7}}{8}z + \frac{3}{8}z^2 & \frac{1}{8} - \frac{2-\sqrt{7}}{8}z & 0 \\ 0 & 0 & z^3\left[\frac{1-z}{2}\right]^2 \frac{(1+\sqrt{3})-(1-\sqrt{3})z}{2} \end{bmatrix}. \tag{33}$$

Hence, applying Theorem 4, an orthogonal multiwavelet $\Psi^{new}(x) = [\psi_1(x), \psi_2(x), \psi_3(x)]^T$ corresponding to $\Phi^{new}(x)$ can be constructed by two scale matrix symbol $Q^{new}(z)$ defined in (18).

Further, by Theorem 5, the approximation order of the new orthogonal multiscaling function $\Phi^{new}(x)$ which we constructed is 4. That is, we raise the approximation order of $\Phi(x)$ from 2 to 4.

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