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Magnetic Schrödinger operators and the $\overline{\partial}$-equation

By

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Abstract

In this paper we characterize compactness of the canonical solution operator to $\overline{\partial}$ on weigthed L^2 spaces on $\mathbb{C}$. For this purpose we consider certain Schrödinger operators with magnetic fields and use a condition which is equivalent to the property that these operators have compact resolvents. We also point out what are the obstructions in the case of several complex variables.

1. Introduction

Let $\varphi : \mathbb{C} \longrightarrow \mathbb{R}$ be a $\mathcal{C}^2$-weight function and consider the Hilbert spaces

$$L^2_\varphi = \left\{ f : \mathbb{C} \longrightarrow \mathbb{C} \text{ measureable } : \|f\|^2_\varphi := \int_{\mathbb{C}} |f(z)|^2 \, e^{-2\varphi(z)} \, d\lambda(z) < \infty \right\}.$$

It is essentially due to L. Hörmander [H] that for a suitable weight function φ and for every $f \in L^2_\varphi$ there exists $u \in L^2_\varphi$ satisfying $\overline{\partial} u = f$. In fact there exists a continuous solution operator $\tilde{S} : L^2_\varphi \longrightarrow L^2_\varphi$ for $\overline{\partial}$, i.e. $\|\tilde{S}(f)\|_\varphi \leq C \|f\|_\varphi$ and $\overline{\partial}\tilde{S}(f) = f$, see also [Ch].

Let A^2_φ denote the space of entire functions belonging to L^2_φ and let

$$P_\varphi : L^2_\varphi \longrightarrow A^2_\varphi$$

denote the Bergman projection. Then $S = (I - P_\varphi)\tilde{S}$ is the uniquely determined canonical solution operator to $\overline{\partial}$, i.e. $\overline{\partial} S(f) = f$ and $S(f) \perp A^2_\varphi$.

In this paper we discuss the compactness of the canonical solution operator to $\overline{\partial}$ on weighted L^2-spaces. The question of compactness of the solution operator to $\overline{\partial}$ is of interest for various reasons—see [FS1] and [FS2] for an excellent survey and [C], [CD], [K], [L].

A similar situation appears in [SSU] where the Toeplitz C^* -algebra $\mathcal{T}(\Omega)$ is considered and the relation between the structure of $\mathcal{T}(\Omega)$ and the $\overline{\partial}$-Neumann problem is discussed (see [SSU]).

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The connection of $\overline{\partial}$ with the theory of Schrödinger operators with magnetic fields appears in [Ch], [B] and [FS3].

For the case of one complex variable we use results of Iwatsuka ([I]) to discuss compactness of the canonical solution operator to $\overline{\partial}$ (see also [HeMo]).

Multiple difficulties arise in the case of several complex variables, mainly because the geometric structures underlying the analysis become much more complicated. We try to point out the different situation and the obstructions which appear in the case of several complex variables.

2. Schrödinger operators with magnetic fields in one complex variable

A nonnegative Borel measure ν defined on $\mathbb{C}$ is said to be doubling if there exists a constant C such that for all $z \in \mathbb{C}$ and $r \in \mathbb{R}^+$,

$$\nu(B(z,2r)) \leq C\nu(B(z,r)).$$

$\mathcal{D}$ denotes the set of all doubling measures ν for which there exists a constant δ such that for all $z \in \mathbb{C}$,

$$\nu(B(z,1)) \geq \delta.$$

Let $\varphi : \mathbb{C} \longrightarrow \mathbb{R}$ be a subharmonic function. Then $\Delta\varphi$ defines a nonnegative Borel measure, which is finite on compact sets.

Let $\mathcal{W}$ denote the set of all subharmonic $\mathcal{C}^2$ functions $\varphi : \mathbb{C} \longrightarrow \mathbb{R}$ such that $\Delta\varphi \in \mathcal{D}$.

Theorem 2.1. *Let $\varphi \in \mathcal{W}$. The canonical solution operator $S : L^2_\varphi \longrightarrow L^2_\varphi$ to $\overline{\partial}$ is compact if and only if there exists a real valued continuous function μ on $\mathbb{C}$ such that $\mu(z) \to \infty$ as $|z| \to \infty$ and*

$$\int_{\mathbb{C}} \mathcal{S}\phi(z)\,\overline{\phi(z)}\,d\lambda(z) \geq \int_{\mathbb{C}} \mu(z)\,|\phi(z)|^2\,d\lambda(z)$$

for all $\phi \in \mathcal{C}_0^\infty(\mathbb{C})$, where

$$\mathcal{S} = -\frac{\partial^2}{\partial z \partial \overline{z}} - \frac{\partial\varphi}{\partial \overline{z}}\frac{\partial}{\partial z} + \frac{\partial\varphi}{\partial z}\frac{\partial}{\partial \overline{z}} + \left|\frac{\partial\varphi}{\partial z}\right|^2 + \frac{\partial^2\varphi}{\partial z \partial \overline{z}}.$$

Proof. Consider the equation $\overline{\partial}u = f$ for $f \in L^2_\varphi$. The canonical solution operator to $\overline{\partial}$ gives a solution with minimal L^2_φ-norm. We substitute $v = u\,e^{-\varphi}$ and $g = f\,e^{-\varphi}$ and the equation becomes

$$\overline{D}v = g\ ,\ \text{where } \overline{D} = e^{-\varphi}\,\frac{\partial}{\partial \overline{z}}\,e^{\varphi}.$$

u is the minimal solution to the $\overline{\partial}$-equation in L^2_φ if and only if v is the solution to $\overline{D}v = g$ which is minimal in $L^2(\mathbb{C})$.

The formal adjoint of $\overline{D}$ is $D = -e^{\varphi}\frac{\partial}{\partial z}e^{-\varphi}$. As in [Ch] we define $\text{Dom}(\overline{D}) = \{f \in L^2(\mathbb{C}) \ : \ \overline{D}f \in L^2(\mathbb{C})\}$ and likewise for D. Then $\overline{D}$ and D are closed unbounded linear operators from $L^2(\mathbb{C})$ to itself. Further we define $\text{Dom}(\overline{D}D) = \{u \in \text{Dom}(D) \ : \ Du \in \text{Dom}(\overline{D})\}$ and we define $\overline{D}D$ as $\overline{D} \circ D$ on this domain. Any function of the form $e^{\varphi}\, g$, with $g \in \mathcal{C}_0^2$ belongs to $\text{Dom}(\overline{D}D)$ and hence $\text{Dom}(\overline{D}D)$ is dense in $L^2(\mathbb{C})$. Since $\overline{D} = \frac{\partial}{\partial \overline{z}} + \frac{\partial \varphi}{\partial \overline{z}}$ and $D = -\frac{\partial}{\partial z} + \frac{\partial \varphi}{\partial z}$ we see that

$$\begin{aligned}\mathcal{S} = \overline{D}D &= -\frac{\partial^2}{\partial z \partial \overline{z}} - \frac{\partial \varphi}{\partial \overline{z}}\frac{\partial}{\partial z} + \frac{\partial \varphi}{\partial z}\frac{\partial}{\partial \overline{z}} + \left|\frac{\partial \varphi}{\partial z}\right|^2 + \frac{\partial^2 \varphi}{\partial z \partial \overline{z}} \\ &= -\frac{1}{4}\,((d - iA)^2 - \Delta\varphi),\end{aligned}$$

where $A = A_1\, dx + A_2\, dy = -\varphi_y\, dx + \varphi_x\, dy$. Hence $\mathcal{S} = \overline{D}D$ is a Schrödinger operator with electric potential $\Delta\varphi$ and with magnetic field $B = dA$, ([CFKS]).

Now let $\|u\|^2 = \int_{\mathbb{C}} |u(z)|^2\, d\lambda(z)$ for $u \in L^2(\mathbb{C})$ and

$$(u, v) = \int_{\mathbb{C}} u(z)\overline{v(z)}\, d\lambda(z)$$

denote the inner product of $L^2(\mathbb{C})$.

In [Ch] the following results are proved : If $u \in \text{Dom}(D)$ and $Du \in \text{Dom}(\overline{D})$, then

$$\|Du\|^2 = (\overline{D}(Du), u).$$

$\overline{D}D$ is a closed operator and

$$\|u\| \leq C\|\overline{D}Du\|$$

for all $u \in \text{Dom}(\overline{D}D)$. Moreover, for any $f \in L^2(\mathbb{C})$ there exists a unique $u \in \text{Dom}(\overline{D}D)$ satisfying $\overline{D}Du = f$. Hence $\mathcal{S}^{-1} = (\overline{D}D)^{-1}$ is a bounded operator on $L^2(\mathbb{C})$.

Now we claim that the canonical solution operator $S : L^2_{\varphi} \longrightarrow L^2_{\varphi}$ to $\overline{\partial}$ is compact if and only if $\mathcal{S}^{-1} : L^2(\mathbb{C}) \longrightarrow L^2(\mathbb{C})$ is compact.

For this we remark that v is the minimal solution to $\overline{\partial}v = g$ in L^2_{φ} if and only if $u = v\, e^{-\varphi}$ is the minimal solution to $\overline{D}u = g\, e^{-\varphi}$ in $L^2(\mathbb{C})$. Hence the canonical solution operator S to $\overline{\partial}$ is compact if and only if the canonical solution operator to $\overline{D}u = f$ is compact. By the above properties of the operators D and $\overline{D}$ we have

$$\|D\mathcal{S}^{-1}f\|^2 = (\overline{D}D(\overline{D}D)^{-1}f, (\overline{D}D)^{-1}f) = (f, (\overline{D}D)^{-1}f) \leq \|\mathcal{S}^{-1}\|\|f\|^2,$$

hence

$$\|D\mathcal{S}^{-1}f\| \leq \|\mathcal{S}^{-1}\|^{1/2}\, \|f\|$$

and $T = D\mathcal{S}^{-1}$ is a bounded operator on $L^2(\mathbb{C})$ with $\overline{D}Tf = f$ and $Tf \perp \ker\overline{D}$, which means that T is the canonical solution operator to $\overline{D}u = f$. Since $\mathcal{S}^{-1}$ is a selfadjoint operator (see for instance [I]) it follows that

$$\mathcal{S}^{-1} = T^*T.$$

Since T is compact if and only if T^*T is compact (see [W]), our claim is proved.

To prove the theorem we use Iwatsuka's result ([I]) that the operator $\mathcal{S}$ has compact resolvent if and only if the condition in Theorem 2.1 holds. □

Theorem 2.2. *If $\varphi(z) = |z|^2$, then the canonical solution operator $S : L^2_\varphi \longrightarrow L^2_\varphi$ to $\overline{\partial}$ fails to be compact.*

Proof. In our case the magnetic field B is the form $B = dA = B(x,y)dx \wedge dy = \Delta\varphi dx \wedge dy$. Hence for $\varphi(z) = |z|^2$ we have $\Delta\varphi(z) = 4$ for each $z \in \mathbb{C}$. Let Q_w be the ball centered at w with radius 1. Then

$$\int_{Q_w} (|B(x,y)|^2 + \Delta\varphi(z))\, d\lambda(z)$$

is a constant as $|w| \to \infty$, so the assertion follows from [I] Theorem 5.2. □

Theorem 2.3. *Let $\varphi \in \mathcal{W}$ and suppose that $\Delta\varphi(z) \to \infty$ as $|z| \to \infty$. Then the canonical solution operator $S : L^2_\varphi \longrightarrow L^2_\varphi$ to $\overline{\partial}$ is compact.*

Proof. Since in our case $|B(x,y)| = \Delta\varphi(z) \to \infty$, as $|z| \to \infty$ the conclusion follows from the proof of Theorem 2.1 and [AHS], [D] or [I]. □

Remark 2.1. In [Has2] it shown that for $\varphi(z) = |z|^2$ even the restriction of the canonical solution operator S to the Fock space A^2_φ fails to be compact and that for $\varphi(z) = |z|^m,\ m > 2$ the restriction of S to A^2_φ fails to be Hilbert Schmidt.

3. Several complex variables.

In [Sch] it is shown that the restriction of the canonical solution operator to the Fock space A^2_φ fails to be compact, where

$$\varphi(z) = |z_1|^m + \cdots + |z_n|^m,$$

for $m \geq 2$ and $n \geq 2$. Hence the canonical solution operator cannot be compact on the corresponding L^2-spaces.

Here we investigate the solution operator on L^2-spaces and try to generalize the method from above for several complex variables.

Let $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}$ be a $\mathcal{C}^2$-weight function and consider the space

$$L^2(\mathbb{C}^n, \varphi) = \left\{ f : \mathbb{C}^n \longrightarrow \mathbb{C} \ : \ \int_{\mathbb{C}^n} |f|^2\, e^{-2\varphi}\, d\lambda < \infty \right\}$$

and the space $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ of $(0,1)$-forms with coefficients in $L^2(\mathbb{C}^n, \varphi)$.

For $v \in L^2(\mathbb{C}^n)$ let

$$\overline{D}v = \sum_{k=1}^{n} \left(\frac{\partial v}{\partial \overline{z}_k} + \frac{\partial \varphi}{\partial \overline{z}_k}\, v \right) d\overline{z}_k$$

and for $g = \sum_{j=1}^n g_j \, d\overline{z}_j \in L^2_{(0,1)}(\mathbb{C}^n)$ let

$$\overline{D}^* g = \sum_{j=1}^{n} \left(\frac{\partial \varphi}{\partial z_j}\, g_j - \frac{\partial g_j}{\partial z_j} \right),$$

where the derivatives are taken in the sense of distributions. It is easy to see that $\overline{\partial} u = f$ for $u \in L^2(\mathbb{C}^n, \varphi)$ and $f \in L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ if and only if $\overline{D}v = g$, where $v = u\, e^{-\varphi}$ and $g = f\, e^{-\varphi}$. It is also clear that the necessary condition $\overline{\partial} f = 0$ for solvability holds if and only if $\overline{D} g = 0$ holds. Here

$$\overline{D} g = \sum_{j,k=1}^{n} \left(\frac{\partial g_j}{\partial \overline{z}_k} + \frac{\partial \varphi}{\partial \overline{z}_k}\, g_j \right) d\overline{z}_k \wedge d\overline{z}_j.$$

Then

$$\overline{D}\,\overline{D}^* g = \overline{D}\left(\sum_{j=1}^{n} \left(\frac{\partial \varphi}{\partial z_j}\, g_j - \frac{\partial g_j}{\partial z_j} \right) \right)$$

$$= \sum_{k=1}^{n} \left[\sum_{j=1}^{n} \left(\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k}\, g_j - \frac{\partial^2 g_j}{\partial z_j \partial \overline{z}_k} + \frac{\partial g_j}{\partial \overline{z}_k}\, \frac{\partial \varphi}{\partial z_j} - \frac{\partial g_j}{\partial z_j}\, \frac{\partial \varphi}{\partial \overline{z}_k} + \frac{\partial \varphi}{\partial z_j}\, \frac{\partial \varphi}{\partial \overline{z}_k}\, g_j \right) \right] d\overline{z}_k.$$

Proposition 3.1. *The operator $\overline{D}\,\overline{D}^*$ defined on $Dom\overline{D}^* \cap ker\overline{D}$ has the form*

$$\sum_{k=1}^{n} \left[\sum_{j=1}^{n} \left(2\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k}\, g_j - \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_j}\, g_k - \frac{\partial^2 g_k}{\partial z_j \partial \overline{z}_j} \right.\right.$$

$$\left.\left. + \frac{\partial g_k}{\partial \overline{z}_j}\, \frac{\partial \varphi}{\partial z_j} - \frac{\partial g_k}{\partial z_j}\, \frac{\partial \varphi}{\partial \overline{z}_j} + \frac{\partial \varphi}{\partial z_j}\, \frac{\partial \varphi}{\partial \overline{z}_j}\, g_k \right) \right] d\overline{z}_k.$$

Proof. The condition $\overline{D} g = 0$ means that

$$\frac{\partial g_j}{\partial \overline{z}_k} + \frac{\partial \varphi}{\partial \overline{z}_k}\, g_j = \frac{\partial g_k}{\partial \overline{z}_j} + \frac{\partial \varphi}{\partial \overline{z}_j}\, g_k,$$

for $j, k = 1, \dots, n$. Now we apply the differentiation $\frac{\partial}{\partial z_j}$ on both sides and obtain

$$\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k}\, g_j + \frac{\partial^2 g_j}{\partial z_j \partial \overline{z}_k} + \frac{\partial g_j}{\partial z_j}\, \frac{\partial \varphi}{\partial \overline{z}_k} = \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_j}\, g_k + \frac{\partial^2 g_k}{\partial z_j \partial \overline{z}_j} + \frac{\partial g_k}{\partial z_j}\, \frac{\partial \varphi}{\partial \overline{z}_j}$$

Using this for the formula for $\overline{D}\,\overline{D}^*$ we get

$$\begin{aligned}\overline{D}\,\overline{D}^* g &= \overline{D}\left(\sum_{j=1}^n \left(\frac{\partial\varphi}{\partial z_j}\, g_j - \frac{\partial g_j}{\partial z_j}\right)\right)\\ &= \sum_{k=1}^n \left[\sum_{j=1}^n \left(2\frac{\partial^2\varphi}{\partial z_j\partial\overline{z}_k}\, g_j - \frac{\partial^2\varphi}{\partial z_j\partial\overline{z}_j}\, g_k - \frac{\partial^2 g_k}{\partial z_j\partial\overline{z}_j}\right.\right.\\ &\qquad \left.\left. + \frac{\partial g_k}{\partial\overline{z}_j}\,\frac{\partial\varphi}{\partial z_j} - \frac{\partial g_k}{\partial z_j}\,\frac{\partial\varphi}{\partial\overline{z}_j} + \frac{\partial\varphi}{\partial z_j}\,\frac{\partial\varphi}{\partial\overline{z}_j}\, g_k\right)\right] d\overline{z}_k.\end{aligned}$$

□

Remark 3.1. The only term where g_j appears in the last line is

$$2\frac{\partial^2\varphi}{\partial z_j\partial\overline{z}_k}\, g_j,$$

and we will get a diagonal system if we restrict to weight functions of a special form, for instance $\varphi(z) = |z_1|^2 + \cdots + |z_n|^2$, the case of the Fock space.

Proposition 3.2. *Suppose that the weight function φ is of the form*

$$\varphi(z_1, \ldots, z_n) = \varphi_1(z_1) + \cdots + \varphi_n(z_n),$$

where $\varphi_j : \mathbb{C} \longrightarrow \mathbb{R}$ are $\mathcal{C}^2$-functions for $j = 1, \ldots, n$.

Then the equation $\overline{D}\,\overline{D}^* g = h$, for $h = \sum_{k=1}^n h_k\, d\overline{z}_k$, splits into the n-equations

$$\begin{aligned}&2\,\frac{\partial^2\varphi}{\partial z_k\partial\overline{z}_k}\, g_k + \sum_{j=1}^n \left(-\frac{\partial^2\varphi}{\partial z_j\partial\overline{z}_j}\, g_k - \frac{\partial^2 g_k}{\partial z_j\partial\overline{z}_j}\right.\\ &\qquad\qquad \left. + \frac{\partial g_k}{\partial\overline{z}_j}\,\frac{\partial\varphi}{\partial z_j} - \frac{\partial g_k}{\partial z_j}\,\frac{\partial\varphi}{\partial\overline{z}_j} + \frac{\partial\varphi}{\partial z_j}\,\frac{\partial\varphi}{\partial\overline{z}_j}\, g_k\right) = h_k,\end{aligned}$$

for $k = 1, \ldots, n$. These equations can be represented as Schrödinger operators $\mathcal{S}_k$ with magnetic fields, where

$$\begin{aligned}\mathcal{S}_k v &= 2\,\frac{\partial^2\varphi}{\partial z_k\partial\overline{z}_k}\, v\\ &\quad + \sum_{j=1}^n \left(-\frac{\partial^2\varphi}{\partial z_j\partial\overline{z}_j}\, v - \frac{\partial^2 v}{\partial z_j\partial\overline{z}_j} + \frac{\partial v}{\partial\overline{z}_j}\,\frac{\partial\varphi}{\partial z_j} - \frac{\partial v}{\partial z_j}\,\frac{\partial\varphi}{\partial\overline{z}_j} + \frac{\partial\varphi}{\partial z_j}\,\frac{\partial\varphi}{\partial\overline{z}_j}\, v\right)\end{aligned}$$

and v is a $\mathcal{C}^2$-function. The operators $\mathcal{S}_k$ can be written in the form

$$\mathcal{S}_k = \frac{1}{4}\left[-\sum_{j=1}^n \left(\frac{\partial}{\partial x_j} - i a_j\right)^2 - \sum_{j=1}^n \left(\frac{\partial}{\partial y_j} - i b_j\right)^2\right] + V_k,$$

where $z_j = x_j + iy_j$ *and* $a_j = -\frac{\partial\varphi}{\partial y_j}$, $b_j = \frac{\partial\varphi}{\partial x_j}$, *for* $j = 1, \ldots, n$ *and*

$$V_k = 2\frac{\partial^2\varphi}{\partial z_k \partial\overline{z}_k} - \sum_{j=1}^n \frac{\partial^2\varphi}{\partial z_j \partial\overline{z}_j},$$

for $k = 1, \ldots, n$.

Remark 3.2. (a) If the weight function φ is of the form

$$\varphi(z_1, \ldots, z_n) = \varphi_1(z_1) + \cdots + \varphi_n(z_n),$$

where $\varphi_j : \mathbb{C} \longrightarrow \mathbb{R}$ are $\mathcal{C}^2$-functions for $j = 1, \ldots, n$, then the magnetic field of the Schrödinger operators $\mathcal{S}_k$ is the 2-form

$$B = \sum_{j<l} B_{jl}\, d\tilde{x}_j \wedge d\tilde{x}_l,$$

where $\tilde{x}_{2j-1} = x_j$, $\tilde{x}_{2j} = y_j$, $\tilde{a}_{2j-1} = a_j$, $\tilde{a}_{2j} = b_j$ for $j = 1, \ldots, n$ and

$$B_{jl} = \frac{1}{4}\left(\frac{\partial\tilde{a}_l}{\partial\tilde{x}_j} - \frac{\partial\tilde{a}_j}{\partial\tilde{x}_l}\right).$$

If we write

$$|B| = \left(\sum_{j<l} |B_{jl}|^2\right)^{1/2},$$

then the assumptions on the weight function φ imply that

$$|B| = \frac{1}{4}\left[\sum_{j=1}^n \left(\frac{\partial^2\varphi}{\partial x_j^2} + \frac{\partial^2\varphi}{\partial y_j^2}\right)^2\right]^{1/2}.$$

The electric potentials V_k have the form

$$\begin{aligned} V_k &= 2\,\frac{\partial^2\varphi}{\partial z_k \partial\overline{z}_k} - \sum_{j=1}^n \frac{\partial^2\varphi}{\partial z_j \partial\overline{z}_j} \\ &= \frac{1}{2}\left(\frac{\partial^2\varphi}{\partial x_k^2} + \frac{\partial^2\varphi}{\partial y_k^2}\right) - \frac{1}{4}\sum_{j=1}^n \left(\frac{\partial^2\varphi}{\partial x_j^2} + \frac{\partial^2\varphi}{\partial y_j^2}\right). \end{aligned}$$

Hence the the socalled effective potentials (see [KS] , Corollary 1.14)

$$V^\delta_{k,\text{eff}} = V_k + \frac{\delta}{n-1}\,|B|\ , \quad \delta \in [0, 1)$$

do not tend to infinity as $|z|$ tends to infinity for weight functions like

$$\varphi(z) = \sum_{j=1}^n |z_j|^2,$$

causing the obstructions for the Schrödinger operators $\mathcal{S}_k$ to have compact resolvents.

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