J. Math. Kyoto Univ. (JMKYAZ)
46-2 (2006), 235–247

On the modulus of extremal Beltrami coefficients

By

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Abstract

Let R be a hyperbolic Riemann surface. Suppose the Teichmüller space $T(R)$ of R is infinite-dimensional. Let μ be an extremal Beltrami coefficient on R and let $[\mu]$ be the point in $T(R)$. In this note, it is shown that if μ is not uniquely extremal, then there exists an extremal Beltrami coefficient ν in $[\mu]$ with non-constant modulus. As an application, it follows, as is well known, that there exist infinitely many geodesics between $[\mu]$ and the base point $[0]$ in $T(R)$ if μ is non-uniquely extremal.

1. Introduction

Let R be a hyperbolic Riemann surface and let $QC(R)$ be the space of all quasiconformal mappings f from R to a variable Riemann surface $f(R)$. The Teichmüller spac $T(R)$ is the space of these mappings factored by an equivalence relation. Two mappings, f and g, are equivalent if there is a conformal mapping c from $f(R)$ onto $g(R)$ and a homotopy through quasiconformal mappings h_t mapping R onto $g(R)$ such that $h_0 = c \circ f$, $h_1 = g$ and $h_t(p) = c \circ f(p) = g(p)$ for every p in the ideal boundary of R. Let $[f]$ or $[\mu]$ denote the equivalence class of a quasiconformal mapping f in $QC(R)$, where μ is the Beltrami coefficient of f. Since the Beltrami coefficient μ uniquely determines the mapping f up to postcomposition by a conformal mapping, the Teichmüller space $T(R)$ may be represented as the space of equivalence classes of Beltrami coefficients μ in the unit ball $M(R)$ of the space $L^\infty(R)$. The equivalence class of the Beltrami coefficient zero is the basepoint of $T(R)$.

Given $f \in QC(R)$, let $\mu \in M(R)$ be the Beltrami coefficient of f. Let $K[f] = \frac{1+\|\mu\|_\infty}{1-\|\mu\|_\infty}$ denote the maximal dilation of f. We define

$$k_0([\mu]) = \inf\{\|\nu\|_\infty : \nu \in [\mu]\},$$

2000 *Mathematics Subject Classification(s)*. Primary 30C75; Secondary 30C62.

Received June 28, 2004
Revised December 27, 2005

*The authors were supported by the National Natural Science Foundation of China (Grant No. 10401036, 10271063, 10571009) and a Foundation for the Author of National Excellent Doctoral Dissertation (Grant No. 200518) of PR China.

and

$$K_0[f] = K_0([\mu]) = \frac{1 + k_0([\mu])}{1 - k_0([\mu])}.$$

We say that μ is extremal in $[\mu]$ (f is extremal in $[f]$) if $\|\mu\|_\infty = k_0([\mu])$, and uniquely extremal if $\|\nu\|_\infty > k_0([\mu])$ for any other $\nu \in [\mu]$. We call that a Beltrami coefficient μ is of constant modulus if $|\mu|$ is a constant almost everywhere on R.

For any μ, let $h^*(\mu)$ be the infimum over all compact subsets E contained in R of the essential supremum norm of the Beltrami coefficient $\mu(z)$ as z varies over $R \backslash E$. Define $h([\mu])$ to be the infimum of $h^*(\mu)$ taken over all representatives μ of the class $[\mu]$. The number

$$H([\mu]) = \frac{1 + h([\mu])}{1 - h([\mu])}$$

is called the boundary dilatation of the class $[\mu]$. Obviously $h([\mu]) \leq k_0([\mu])$ and following [3], [5], we call a point $[\mu]$ in $T(R)$ a Strebel point if $h([\mu]) < k_0([\mu])$.

Let $A(R)$ be the Banach space of all holomorphic functions φ on R with L^1-norm

$$\int_R |\varphi(z)| < \infty,$$

and let $A_1(R)$ be the unit sphere of $A(R)$. By Strebel's frame mapping theorem, every Strebel point $[\mu]$ is represented by the unique Beltrami differential of the form $k|\varphi|/\varphi$, where $k = k_0([\mu]) \in (0, 1)$ and φ is a unit vector in $A_1(R)$.

Two elements μ and ν in $L^\infty(R)$ are infinitesimally equivalent, which is denoted by $\mu \approx \nu$, if $\iint_R \mu\phi dxdy = \iint_R \nu\phi dxdy$ for all $\phi \in A(R)$. Denote by $N(R)$ the set of all the elements in $L^\infty(R)$ which are infinitesimally equivalent to zero. Then $B(R) = L^\infty(R)/N(R)$ is the tangent space of the Teichmüller space $T(R)$ at the basepoint.

Given $\mu \in L^\infty(R)$, we denote by $[\mu]_B$ the set of all elements $\nu \in L^\infty(R)$ infinitesimally equivalent to μ, and set

$$\|\mu\| = \inf\{\|\nu\|_\infty : \nu \in [\mu]_B\}. \tag{1.1}$$

We say that μ is infinitesimally extremal (in $[\mu]_B$) if $\|\mu\|_\infty = \|\mu\|$, and we say it is infinitesimally uniquely extremal if $\|\nu\|_\infty > \|\mu\|$ for any other $\nu \in [\mu]_B$.

In a parallel manner we can define the boundary dilatation for the infinitesimal Teichmüller class $[\mu]_B$. The boundary dilatation $b([\mu]_B)$ is the infimum over all elements in the equivalence class $[\mu]_B$ of the quantity $b^*(\nu)$. Here $b^*(\nu)$ is the infimum over all compact subsets E contained in R of the essential supremum of the Beltrami coefficient ν as z varies over $R - E$.

An infinitesimally equivalent class $[\mu]_B$ is called an infinitesimal Strebel point if $\|\mu\| > b([\mu]_B)$. It follows from the infinitesimal frame mapping theorem (see Theorem 2.4 in [7]) that if $[\mu]_B$ is an infinitesimal Strebel point, then there exists a unique vector φ in $A_1(R)$ such that μ and $\|\mu\||\varphi|/\varphi$ are infinitesimally equivalent.

In [1], Božin, Lakic et al. gave a series of characteristic conditions for a Beltrami coefficient μ to be (infinitesimally) uniquely extremal. For simplicity, we state parts of characteristic conditions in the special case.

Theorem A. *Let μ be a Beltrami coefficient in $M(R)$ with constant modulus. Then the following conditions are equivalent*:
(a) *μ is uniquely extremal in its class $[\mu]$ in $T(R)$;*
(b) *μ is infinitesimally uniquely extremal in its class $[\mu]_B$ in $B(R)$;*
(c) *for every measurable subset E of R with nonzero measure, there exists a sequence of unit vectors φ_n in $A_1(R)$ such that*

$$\frac{1}{\int_E |\varphi_n|}\left(\|\mu\|_\infty - Re\int_R \mu\varphi_n\right) \to 0, \text{ as } n \to \infty;$$

(d) *μ is extremal in $[\mu]$ and, for every compact subset E of R with nonzero measure and every $r > 0$, $[\mu\chi_E + \frac{1}{1+r}\mu\chi_{R-E}]$ is a Strebel point in $T(R)$;*
(e) *μ is infinitesimally extremal in $[\mu]_B$ and, for every compact subset E of R with nonzero measure and every $r > 0$, $[\mu\chi_E + \frac{1}{1+r}\mu\chi_{R-E}]_B$ is an infinitesimal Strebel point in $B(R)$.*

When $[\mu]$ in $T(R)$ contains more than one extremal Beltrami coefficient, the situation is very complicated. It is of interest to consider the problem as follows.

Problem 1. *If $[\mu]$ in $T(R)$ admits more than one extremal Beltrami coefficient, can we say that there always exists an extremal Beltrami coefficient in $[\mu]$ with non-constant modulus?*

When R is the unit disk Δ, a positive answer to this problem is actually implied by Reich's proof of his theorem in [8] (also see [16]). His proof depends on the Polygon Inequality due to Reich and Strebel [10]. However, the Polygon Inequality is not generalized for general hyperbolic Riemann surfaces except for some special surfaces, for example, see [13]. And hence for more general hyperbolic Riemann surfaces, the solution requires a different technique. The main aim of this paper is to answer Problem 1 affirmatively. We avoid using the Polygon Inequality and our proof is self-contained.

Theorem 1.1. *Suppose μ in $M(R)$ is extremal with $\|\mu\|_\infty = k$ and is not uniquely extremal. Then there exists a compact subset E of R with nonzero measure and an extremal Beltrami coefficient $\nu \in [\mu]$ such that $|\nu| \le \frac{k}{1+r_0}$ on E for some $r_0 > 0$.*

Corollary 1.1. *Suppose μ in $M(R)$ is extremal with $\|\mu\|_\infty = k$. If for every extremal Beltrami coefficient ν in $[\mu]$, $|\nu| = k$ a.e in R, then μ is uniquely extremal with constant modulus.*

Corollary 1.1 shows that the case (2) of Theorem 1 in [11] really does not exist.

The analogous problem in the infinitesimal setting is considered in Section 4. Applying Theorem 1.1 and the result in [2], in Section 5 we give an alternative proof that there exist infinitely many geodesics between $[\mu]$ and the base point $[0]$ in $T(R)$ if μ is non-uniquely extremal.

2. Non-Strebel Points

The first lemma is inspired by the lemma in [8].

Lemma 2.1. *If $\mu \in M(R)$ is extremal with $\|\mu\|_\infty = k$, then for every measurable subset E of R with nonzero measure and every $r > 0$, the Beltrami coefficient $\mu_r = \mu\chi_E + \frac{1}{1+r}\mu\chi_{R-E}$ has the property $k_0([\mu_r]) \geq \frac{k}{1+r}$.*

Proof. Let η be an extremal Beltrami coefficient in $[\mu_r]$. Then there exist homotopic quasiconformal mappings g and h with Beltrami coefficient μ_r and η, respectively, such that $g(R) = h(R)$ and $g(p) = h(p)$ for every point on the ideal boundary of R. Let f be the quasiconformal mapping with the Beltrami coefficient μ. It follows that f and $f \circ g^{-1} \circ h$ are equivalent in $T(R)$. Since f is extremal by hypothesis, it follows that

$$\frac{1+k}{1-k} = K[f] \leq K[f \circ g^{-1} \circ h] \leq K[F]K[h], \tag{2.1}$$

where $F = f \circ g^{-1}$. Note that

$$|\mu_F(g(z))| = |\frac{\mu(z) - \mu_r(z)}{1 - \overline{\mu(z)}\mu_r(z)}| = \begin{cases} \frac{r|\mu(z)|}{1+r-|\mu(z)|^2}, & z \in R - E, \\ 0, & z \in E. \end{cases}$$

We have

$$|\mu_F(g(z))| \leq \frac{rk}{1+r-k^2}, \; z \in R.$$

Thus,

$$K[F] \leq \frac{1+k}{1-k}\frac{1+r-k}{1+r+k}. \tag{2.2}$$

Combining (2.1) and (2.2), we obtain

$$K[h] = K_0[h] \geq \frac{1+r+k}{1+r-k} = \frac{1+\frac{k}{1+r}}{1-\frac{k}{1+r}},$$

which proves the lemma. □

Theorem 2.1. *Suppose that $\mu \neq 0$ is extremal with $\|\mu\|_\infty = k$ and there exists a compact subset E of R such that*

$$\inf\left\{\frac{1}{\int_E |\varphi|}\left(k - Re\int_R \mu\varphi\right) : \varphi \in A_1(R)\right\} = \gamma > 0. \tag{2.3}$$

Then $[\mu_r] = [\mu\chi_E + \frac{1}{1+r}\mu\chi_{R-E}]$ is a non-Strebel point and $k_0([\mu_r]) = \frac{k}{1+r}$ for every $r \in [0, \frac{(1-k)\gamma}{k(1+k)})$.

Proof. Suppose $[\mu_r]$ is a Strebel point for some $r \geq 0$. By Lemma 2.1, we have $k_0([\mu_r]) \geq s = \frac{k}{1+r}$. Thus, by Strebel's frame mapping theorem, there exists $s_r = k_0([\mu_r]) \geq s$ and a unit vector φ in $A_1(R)$ such that μ_r and $s_r\frac{|\varphi|}{\varphi}$ are equivalent. Therefore, by the Main Inequality [9, 4], we have

$$\frac{1+s}{1-s} \leq \frac{1+s_r}{1-s_r} = K_0([\mu_r]) \leq \int_R |\varphi| \frac{|1+\mu_r\varphi/|\varphi||^2}{1-|\mu_r|^2}.$$

Let $\lambda = \frac{\mu}{1+r}$. We have

$$\frac{1+s}{1-s} \leq \int_{R-E} |\varphi| \frac{|1+\lambda\varphi/|\varphi||^2}{1-|\lambda|^2} + \int_E |\varphi| \frac{|1+\mu\varphi/|\varphi||^2}{1-|\mu|^2} = X + Y,$$

where

$$X = \int_R |\varphi| \frac{|1+\lambda\varphi/|\varphi||^2}{1-|\lambda|^2},\ Y = \int_E |\varphi| [\frac{|1+\mu\varphi/|\varphi||^2}{1-|\mu|^2} - \frac{|1+\lambda\varphi/|\varphi||^2}{1-|\lambda|^2}].$$

By a simple computation,

$$X \leq \frac{1+s^2+2Re\int_R \lambda\varphi}{1-s^2},$$
$$Y \leq \frac{2kr}{(1-k)(1+r-k)} \int_E |\varphi|.$$

Thus,

$$\frac{1+s}{1-s} \leq \frac{1+s^2+2Re\int_R \lambda\varphi}{1-s^2} + \frac{2kr}{(1-k)(1+r-k)} \int_E |\varphi|,$$

namely,

$$2\left(s - Re\int_R \lambda\varphi\right) \leq \frac{2kr(1-s^2)}{(1-k)(1+r-k)} \int_E |\varphi|.$$

Therefore, we get

$$k - Re\int_R \mu\varphi \leq \frac{(1+r+k)kr}{(1-k)(1+r)} \int_E |\varphi| \leq \frac{k(1+k)r}{1-k} \int_E |\varphi|.$$

Hence,

$$r \geq \frac{1-k}{k(1+k)\int_E |\varphi|} \left(k - Re\int_R \mu\varphi\right) \geq \frac{(1-k)\gamma}{k(1+k)}.$$

Thus, $[\mu_r]$ is a non-Strebel point for every $r \in [0, \frac{(1-k)\gamma}{k(1+k)})$. Hence, $k_0([\mu_r]) = H([\mu_r]) \leq \frac{k}{1+r}$. Again by Lemma 2.1, we must have $k_0([\mu_r]) = \frac{k}{1+r}$. □

Lemma 2.2. *Suppose that μ is extremal but not (infinitesimally) uniquely extremal with $\|\mu\|_\infty = k$. Then there exists a compact subset E of R with nonzero measure such that*

$$\inf\left\{\frac{1}{\int_E |\varphi|}\left(k - Re\int_R \mu\varphi\right) : \varphi \in A_1(R)\right\} = \gamma > 0. \tag{2.4}$$

Proof. If μ is of constant modulus, then the lemma is an immediate corollary of Theorem A.

If μ is not of constant modulus even if μ is (infinitesimally) uniquely extremal, then there exists a compact subset E of R such that $|\mu| < s < k$ on E. Thus, for any unit vector φ in $A_1(R)$,

$$\frac{1}{\int_E |\varphi|}\left(k - Re\int_R \mu\varphi\right) \geq \frac{1}{\int_E |\varphi|}\left(k\int_E |\varphi| - Re\int_E \mu\varphi\right) \geq k - s > 0.$$

□

3. Extremal Beltrami coefficients with non-constant modulus

By Lemma 2.2, Theorem 1.1 is a direct corollary of the following theorem.

Theorem 3.1. *Suppose μ in $M(R)$ is extremal with $\|\mu\|_\infty = k$. If there exists a compact subset G of R with nonzero measure such that*

$$\inf\left\{\frac{1}{\int_G |\varphi|}\left(k - Re\int_R \mu\varphi\right) : \varphi \in A_1(R)\right\} = \gamma > 0, \tag{3.1}$$

then there exists a compact subset E of R with nonzero measure and an extremal Beltrami coefficient $\nu \in [\mu]$ such that $|\nu| \leq \frac{k}{1+r_0}$ on E for some $r_0 > 0$.

Proof. Since μ satisfies (3.1), applying Theorem 2.1 to G, we can find some $r_0 > 0$ such that $[\mu_r] = [\mu\chi_G + \frac{1}{1+r}\mu\chi_{R-G}]$ is a non-Strebel point and $k_0([\mu_r]) = \frac{k}{1+r}$ for every $r \in [0, r_0]$.

Let η be an extremal Beltrami coefficient in $[\mu_r]$. Then there exist homotopic quasiconformal mappings g and h with Beltrami coefficient μ_r and η, respectively, such that $g(R) = h(R)$ and g is homotopic to h by a homotopy which fixes every point on the ideal boundary of R. Let f be the quasiconformal mapping with the Beltrami coefficient μ. By the same computation as in the proof of Lemma 2.1, we have

$$|\mu_F(g(z))| = |\frac{\mu(z) - \mu_r(z)}{1 - \overline{\mu(z)}\mu_r(z)}| = \begin{cases} \frac{r|\mu(z)|}{1+r-|\mu(z)|^2}, & z \in R - G, \\ 0, & z \in G, \end{cases}$$

and

$$K[F] \leq \frac{1+k}{1-k}\frac{1+r-k}{1+r+k},$$

where $F = f \circ g^{-1}$. Since $K[h] = \frac{1+k_0([\mu_r])}{1-k_0([\mu_r])}$, we obtain

$$K[f \circ g^{-1} \circ h] \leq K[F]K[h] \leq \frac{1+k}{1-k}\frac{1+r-k}{1+r+k}\frac{1+\frac{k}{1+r}}{1-\frac{k}{1+r}} = \frac{1+k}{1-k}.$$

Let ν denote the Beltrami coefficient of $f \circ g^{-1} \circ h$. Then ν is extremal in $[\mu]$.

Let $E = h^{-1} \circ g(G)$. Note that $f \circ g^{-1}$ is conformal on $g(G)$, we have $\nu(z) = \eta(z)$ for almost every $z \in E$, and hence $|\nu| \leq \frac{k}{1+r}$ on E.

This completes the proof of Theorem 1.1. □

We end the section with the following open problem.

Problem 2. *If $[\mu]$ in $T(R)$ contains more than one extremal Beltrami coefficient, can we say that there always exists an extremal Beltrami coefficient ν in $[\mu]$ and a measurable subset E of R with non-empty interior such that $|\nu| \leq \frac{k_0([\mu])}{1+r_0}$ a.e. on E for some $r_0 > 0$?*

4. Infinitesimally extremal Beltrami differentials with non-constant modulus

Lemma 4.1. *If $\mu \in L^\infty(R)$ is infinitesimally extremal with $\|\mu\|_\infty = k$, then for every measurable subset E of R with nonzero measure and every $r > 0$, the Beltrami coefficient $\mu_r = \mu\chi_E + \frac{1}{1+r}\mu\chi_{R-E}$ has the property $\|\mu_r\| \geq \frac{k}{1+r}$.*

Proof. Let η be an extremal in $[\mu_r]_B$. Then μ is infinitesimally equivalent to $\mu + \eta - \mu_r$, and

$$\mu - \mu_r = \begin{cases} \frac{r\mu(z)}{1+r}, & z \in R - E, \\ 0, & z \in E. \end{cases}$$

So, $\|\mu - \mu_r\|_\infty \leq \frac{rk}{1+r}$. Then we have

$$k = \|\mu\|_\infty \leq \|\mu + \eta - \mu_r\|_\infty \leq \|\eta\|_\infty + \|\mu - \mu_r\|_\infty. \tag{4.1}$$

Therefore,

$$\|\eta\|_\infty \geq k - \frac{rk}{1+r} = \frac{k}{1+r},$$

proving the lemma. □

Theorem 4.1. *Suppose that $\mu \neq 0$ is infinitesimally extremal with $\|\mu\|_\infty = k$ and there exists a compact subset E of R such that*

$$\inf\left\{\frac{1}{\int_E |\varphi|}\left(k - Re\int_R \mu\varphi\right) : \varphi \in A_1(R)\right\} = \gamma > 0. \tag{4.2}$$

Then $[\mu_r]_B = [\mu\chi_E + \frac{1}{1+r}\mu\chi_{R-E}]_B$ is an infinitesimal non-Strebel point and $\|\mu_r\| = \frac{k}{1+r}$ for every $r \in [0, \frac{\gamma}{k})$.

Proof. Suppose $[\mu_r]_B$ is an infinitesimal Strebel point for some $r \geq 0$. Then by the infinitesimal frame mapping theorem, there exists a unit vector φ in $A_1(R)$ such that μ_r and $\|\mu_r\|\frac{|\varphi|}{\varphi}$ are infinitesimally equivalent. By Lemma 4.1, we have $\|\mu_r\| \geq \frac{k}{1+r}$. Therefore, we have

$$\frac{k}{1+r} \leq \int_R \|\mu_r\|\frac{|\varphi|}{\varphi}\varphi = \int_R \mu_r\varphi = \int_E \mu\varphi + \int_{R-E}\frac{\mu}{1+r}\varphi.$$

Thus,

$$k - Re\int_R \mu\varphi \leq kr\int_E |\varphi|.$$

Hence,

$$r \geq \frac{1}{k\int_E |\varphi|}\left(k - Re\int_R \mu\varphi\right) \geq \frac{\gamma}{k}.$$

Thus, $[\mu_r]_B$ is an infinitesimal non-Strebel point for every $r \in [0, \frac{\gamma}{k})$. Hence, $\|\mu_r\| = b([\mu_r]_B) \leq \frac{k}{1+r}$. Again by Lemma 4.1, we must have $\|\mu_r\| = \frac{k}{1+r}$. □

Lemma 4.2. *Suppose μ in $L^\infty(R)$ is infinitesimally extremal with $\|\mu\|_\infty = k$. If there exists a compact subset E of R with nonzero measure such that*

$$\inf\left\{\frac{1}{\int_E |\varphi|}\left(k - Re\int_R \mu\varphi\right) : \varphi \in A_1(R)\right\} = \gamma > 0, \tag{4.3}$$

then there exists an extremal Beltrami coefficient $\nu \in [\mu]_B$ such that $|\nu| \leq \frac{k}{1+r_0}$ on E for some $r_0 > 0$.

Proof. Since μ satisfies (4.3), applying Theorem 4.1 to E, we can find some $r_0 >$ such that $[\mu_r]_B = [\mu\chi_G + \frac{1}{1+r}\mu\chi_{R-E}]_B$ is an infinitesimal non-Strebel point and $\|\mu_r\| = \frac{k}{1+r}$ for every $r \in [0, r_0]$.

Let η be an extremal element in $[\mu_r]_B$. Then $\|\eta\|_\infty = \frac{k}{1+r}$ and

$$\|\mu + \eta - \mu_r\|_\infty \leq \|\eta\|_\infty + \|\mu - \mu_r\|_\infty = k.$$

Since μ is infinitesimally equivalent to $\nu = \mu + \eta - \mu_r$, ν is infinitesimally extremal in $[\mu]_B$. In addition, $\nu = \eta$ on E and hence $|\nu| \leq \frac{k}{1+r}$ on E.

The proof of Lemma 4.2 is completed. □

Lemma 2.2 and Lemma 4.2 give

Corollary 4.1. *Suppose μ in $L^\infty(R)$ is infinitesimally extremal with $\|\mu\|_\infty = k$. If for every extremal element ν in $[\mu]_B$, $|\nu| = k$ a.e in R, then μ is uniquely extremal with constant modulus k.*

Here, we give a stronger result than the above corollary in a simple way.

Theorem 4.2. *Suppose μ in $L^\infty(R)$ is infinitesimally extremal. If for every extremal element ν in $[\mu]_B$, $|\nu| = |\mu|$ a.e in R, then μ is uniquely extremal.*

Proof. Suppose ν is an extremal element in $[\mu]_B$. Put $\mu_t = t\mu + (1-t)\nu$ for $t \in (0, 1)$. Then by hypothesis, $\mu_t \in [\mu]_B$ and for almost all $z \in R$,

$$|\mu(z)| = |t\mu(z) + (1-t)\nu(z)| \leq t|\mu(z)| + (1-t)|\nu(z)| = |\mu(z)|.$$

This happens if and only if $\mu(z) = \nu(z)$ a.e. in R, which implies that μ is uniquely extremal in $[\mu]_B$. □

We note that we cannot prove a parallel global result corresponding to Theorem 4.2 for $[\mu]$, that is, the following problem is still unsettled.

Problem 3. *Suppose μ in $M(R)$ is an extremal Beltrami coefficient in $[\mu]$. If for every extremal Beltrami coefficient ν in $[\mu]$, $|\nu| = |\mu|$ a.e in R, can we say that μ is uniquely extremal?*

Our main result of the paper is actually to solve Problem 3 in the special case that μ is of constant modulus.

Remark 1. Problem 3 cannot be reduced to Problem 1. The first author recently showed [15] that there exists a point $[\mu]$ in $T(R)$ admitting infinitely many extremal Beltrami coefficients such that every extremal Beltrami coefficient in $[\mu]$ is not of constant modulus, and so is its infinitesimal version.

It is easy to see from the proof of Theorem 4.2 that there exist infinitely many extremal elements in $[\mu]_B$ with non-constant modulus if μ is non-uniquely extremal. Is it also true for $[\mu]$?

Combining Lemma 2.2, 4.2, Theorem 4.1 with Theorem A, the following theorem is proved.

Theorem 4.3. *Suppose $\mu \neq 0$ in $L^\infty(R)$ is infinitesimally extremal with $\|\mu\|_\infty = k$. Then the following three conditions are equivalent*:
(1) *there exists an extremal element in $[\mu]_B$ with non-constant modulus*;
(2) *for any given extremal element $\nu \in [\mu]_B$, there exists a compact subset E of R with nonzero measure such that*

$$\inf\left\{\frac{1}{\int_E|\varphi|}\left(k - Re\int_R \nu\varphi\right) : \varphi \in A_1(R)\right\} = \gamma > 0;$$

(3) *for any given extremal element $\nu \in [\mu]_B$, there exists a compact subset E of R with nonzero measure such that $[\nu\chi_E + \frac{1}{1+r}\nu\chi_{R-E}]_B$ is an infinitesimal non-Strebel point for every $r \in [0, r_0)$ for some $r_0 > 0$.*

5. Geodesics in Teichmüller spaces

A hyperbolic Riemann surface can be viewed as a quotient space Δ/Γ in certain sense, where Γ is a Fuchsian group acting on the unit disk Δ. $M(R)$ is canonically identified with the set of Beltrami coefficients μ in $M(\Delta)$ compatible with Γ, that is, those μ for which

$$(\mu \circ \gamma)\overline{\gamma'}/\gamma' = \mu, \ \text{ for all } \gamma \in \Gamma.$$

Let $f^\mu : \Delta \to \Delta$ be the quasiconformal mapping with complex dilatation μ keeping 1, -1 and i fixed. It is well known that μ and ν in $M(R)$ are equivalent if and only if f^μ and f^ν coincide on $\partial\Delta$.

For any Beltrami coefficient $\mu \in M(\Delta)$, let H be the usual Hilbert transform defined by

$$H\mu(z) = -\frac{1}{\pi}\iint_\Delta \frac{\mu(\zeta)}{(\zeta - z)^2}d\xi d\eta.$$

Put

$$h_\mu = \mu + \mu H\mu + \mu H(\mu H\mu) + \cdots .$$

The following useful lemma can be found in [12].

Lemma 5.1. *Let μ and ν be two Beltrami coefficients in $M(\Delta)$, Then μ and ν are equivalent in $T(\Delta)$ if and only if $h_\mu - h_\nu \in N(\Delta)$.*

For two given points $[\mu]$ and $[\nu]$ in $T(R)$, the Teichmüller distance between them is defined as

$$d([\mu],[\nu]) = \frac{1}{2}\log\frac{1+\|\eta\|_\infty}{1-\|\eta\|_\infty},$$

where η is an extremal Beltrami coefficient in the equivalence class of the Beltrami coefficient of $f^\mu \circ (f^\nu)^{-1}$.

A geodesic α in $T(R)$ is defined to be the image of an injective continuous map Φ from a non-trivial compact real interval $[a,b]$ into $T(R)$ such that

$$d(\Phi(x),\Phi(z)) = d(\Phi(x),\Phi(y)) + d(\Phi(y),\Phi(z)),$$

whenever $a \le x \le y \le z \le b$. The points $\Phi(a)$ and $\Phi(b)$ are called the endpoints of α. In particular, if μ is extremal, then the image of the $\Phi:\ [0,\|\mu\|_\infty] \to T(R)$ determined by $\Phi(t) = [t\mu/\|\mu\|_\infty]$ is a geodesic joining $[0]$ and $[\mu]$.

Geodesic plays an important role in the geometry of Teichmüller spaces. If μ is uniquely extremal with constant modulus, then there exists a unique geodesic between two points $[0]$ and $[\mu]$. This was proved by Li Zhong [6] when the group Γ is trivial and by Tanigawa [14] in the general case. Earle et al. [2] proved that the converse is also true. Now, as an application of Theorem 1.1, we give a somewhat different proof from that of Earle et al.

Suppose that μ is extremal with non-constant modulus. Then the set $E = \{z \in R :\ |\mu(z)| \le r\|\mu\|_\infty\}$ has nonzero measure for some $r \in (0,1)$. For $t \in \Delta$, put

$$\Phi(t) = [t\mu/\|\mu\|_\infty]$$

and

$$\Phi_\varphi(t) = [\mu(t,\varphi)],$$

where $\mu(t,\varphi) = t\mu/\|\mu\|_\infty + \frac{1-r}{2}t(t-\|\mu\|_\infty)\chi_E|\varphi|/\varphi)$ and $\varphi \in A_1(R)$. These functions are holomorphic maps from Δ to $T(R)$ sending 0 to 0 and $\|\mu\|_\infty$ to $[\mu]$. So, by Theorem 5 in [2], they are holomorphic isometries with respect to the Poincaré metric on Δ and the Teichmüller metric on $T(R)$. Thus, $\Phi_\varphi([0,\|\mu\|_\infty])$ is a geodesic joining $[0]$ and $[\mu]$.

It remains to show that, the holomorphic isometries Φ_φ are different from each other when φ varies in $A_1(R)$. Suppose to the contrary, there would exist two different elements φ and ψ in $A_1(R)$ such that $[\mu(t,\varphi)] = [\mu(t,\psi)]$ for all $t \in \Delta$.

Let $p:\ \Delta \to R = \Delta/\Gamma$ be the canonical projection. Let $\widetilde{\mu}_\varphi$, $\widetilde{\mu}_\psi$, $\widetilde{\varphi}$ and $\widetilde{\psi}$ denote the lifts of $\mu(t,\varphi)$, $\mu(t,\psi)$, φ and ψ to Δ, respectively. Then $\widetilde{\mu}_\varphi$ and $\widetilde{\mu}_\psi$

are equivalent in $T(\Delta)$. By Lemma 5.1, $h_{\widetilde{\mu}_\varphi} - h_{\widetilde{\mu}_\psi} \in N(\Delta)$ for all $|t|<1$. By a simple computation, we have

$$h_{\widetilde{\mu}_\varphi} - h_{\widetilde{\mu}_\psi} = \frac{1-r}{2}\|\mu\|_\infty \chi_{p^{-1}(E)}\left(\frac{|\widetilde{\psi}|}{\widetilde{\psi}} - \frac{|\widetilde{\varphi}|}{\widetilde{\varphi}}\right) t + o(t), \text{ as } t \to 0.$$

Thus, we conclude $\chi_{p^{-1}(E)}(\frac{|\widetilde{\psi}|}{\widetilde{\psi}} - \frac{|\widetilde{\varphi}|}{\widetilde{\varphi}}) \in N(\Delta)$ and consequently $\chi_E(\frac{|\psi|}{\psi} - \frac{|\varphi|}{\varphi}) \in N(R)$. This implies that $\varphi = \psi$ which contradicts the hypothesis, and hence Φ_φ and Φ_ψ are different from each other.

Combing Theorem 6 in [2], Lemma 2.2, Theorem 2.1, Theorem 4.3 and the proof of Theorem 3.1 with the above discussion, one can easily prove the following theorem.

Theorem 5.1. *Suppose $\mu \neq 0$ is an extremal Beltrami coefficient in $M(R)$ with $\|\mu\|_\infty = k$. Then the following conditions are equivalent*:
(1) *there exists an extremal Beltrami coefficient in $[\mu]$ with non-constant modulus*;
(2) *there exists an extremal element in $[\mu]_B$ with non-constant modulus*;
(3) *for any given extremal Beltrami coefficient $\nu \in [\mu]$, there exists a compact subset E of R with nonzero measure such that*

$$\inf\left\{\frac{1}{\int_E|\varphi|}\left(k - Re\int_R \nu\varphi\right) : \varphi \in A_1(R)\right\} = \gamma > 0;$$

(4) *for any given extremal element $\nu \in [\mu]_B$, there exists a compact subset E of R with nonzero measure such that*

$$\inf\left\{\frac{1}{\int_E|\varphi|}\left(k - Re\int_R \nu\varphi\right) : \varphi \in A_1(R)\right\} = \gamma > 0;$$

(5) *for any given extremal Beltrami coefficient $\nu \in [\mu]$, there exists a compact subset E of R with nonzero measure such that $[\nu\chi_E + \frac{1}{1+r}\nu\chi_{R-E}]$ is a non-Strebel point for every $r \in [0, r_0)$ for some $r_0 > 0$*;
(6) *for any given extremal element $\nu \in [\mu]_B$, there exists a compact subset E of R with nonzero measure such that $[\nu\chi_E + \frac{1}{1+r}\nu\chi_{R-E}]_B$ is an infinitesimal non-Strebel point for every $r \in [0, r_0)$ for some $r_0 > 0$*;
(7) *there exist infinitely many geodesics joining $[0]$ and $[\mu]$*;
(8) *there exist infinitely many holomorphic isometries $\Phi : \Delta \to T(R)$ such that $\Phi(0) = 0$ and $\Phi(\|\mu\|_\infty) = [\mu]$.*

Obviously, we have

Corollary 5.1. *Suppose $\mu \neq 0$ is an extremal Beltrami coefficient in $M(R)$. Then the following conditions are equivalent*:
(*a*) *μ is uniquely extremal with constant modulus*;
(*b*) *μ is infinitesimally uniquely extremal with constant modulus*;
(*c*) *for any compact subset E of R with nonzero measure,*

$$\inf\left\{\frac{1}{\int_E|\varphi|}\left(\|\mu\|_\infty - Re\int_R \mu\varphi\right) : \varphi \in A_1(R)\right\} = 0;$$

(*d*) *for any compact subset* E *of* R *with nonzero measure,* $[\mu\chi_E + \frac{1}{1+r}\mu\chi_{R-E}]$ *is a Strebel point for every* $r > 0$;
(*e*) *for any compact subset* E *of* R *with nonzero measure,* $[\mu\chi_E + \frac{1}{1+r}\mu\chi_{R-E}]_B$ *is an infinitesimal Strebel point for every* $r > 0$;
(*f*) *there exists a unique geodesic joining* $[0]$ *and* $[\mu]$;
(*g*) *there exists only one holomorphic isometries* $\Phi : \Delta \to T(R)$ *such that* $\Phi(0) = 0$ *and* $\Phi(\|\mu\|_\infty) = [\mu]$.

Corollary 5.1 indicates that the above condition (c) or the condition (c) in Theorem A is actually also a sufficient condition for μ to be uniquely extremal with constant modulus.

Acknowledgements. The authors wish to thank Professor Edgar Reich for sending them his inspiring paper [8]. Much gratitude is also due to the referee for helpful suggestions and careful corrections.

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