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Analyticity and quasianalyticity of positive definite distributions

By

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Abstract

We show that if f is a positive definite distribution which is analytic on some neighborhood of the origin, then f is analytic everywhere in $\mathbb{R}^n$.

1. Introduction

A continuous function $f(x)$ on $\mathbb{R}^n$ is said to be *positive definite* if

$$\sum_{j,k=1}^{m} f(x_j - x_k)\zeta_j \bar{\zeta}_k \geq 0 \tag{1.1}$$

for any $x_1, \ldots, x_m \in \mathbb{R}^n$ and $\zeta_1, \ldots, \zeta_m \in \mathbb{C}$.

A distribution f is said to be *positive definite* if

$$\langle f, \varphi * \varphi^* \rangle \geq 0 \quad \text{for any } \varphi \in \mathcal{D}, \tag{1.2}$$

where $\varphi^*(x) = \overline{\varphi(-x)}$ and $\mathcal{D}$ is the space of all C^∞ functions on $\mathbb{R}^n$ having compact support.

In view of [10], for a continuous function f, the above two definitions are equivalent, where $\langle f, \varphi * \varphi^* \rangle = \int \int \overline{f(x-y)}\varphi(x)\overline{\varphi(y)}dxdy$.

By the Bochner-Schwarz theorem ([10, p. 141]) every positive definite distribution f is the Fourier transform of a positive tempered measure μ. Thus if f is a positive definite distribution, then f belongs to $\mathcal{S}'$ and (1.2) holds for $\varphi \in \mathcal{S}$, so f is a positive definite tempered distribution, where $\mathcal{S}$ is the Schwarz space and $\mathcal{S}'$ is its dual space. Hence it is well known that the class of positive definite distributions and the class of positive definite tempered distributions are identical.

The regularity of positive definite continuous functions in $\mathbb{R}^n$ depends on the regularity of those functions on a neighborhood of the origin. The following theorems show that a positive definite continuous function which behaves well on some neighborhood of the origin also behaves well in $\mathbb{R}^n$.

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Theorem 1.1 ([8, p. 186]). *For $1 \le k \le \infty$, let a continuous function $f(x)$ be of positive definite and belong to the class C^{2k} in some neighborhood of the origin; then $f(x)$ is everywhere C^{2k}.*

Theorem 1.2 ([5]). *Let a continuous function $f(x)$ be of positive definite and analytic (quasianalytic) in some neighborhood of the origin; then $f(x)$ is analytic (quasianalytic, respectively) everywhere in $\mathbb{R}^n$.*

A similar argument for the analyticity is easily derived from the well-known theorems for general positive semidefinite analytic kernels ([3], [9], [12]).

In this paper, we extend the above results to distributions. We show that if f is a positive definite distribution and is analytic (quasianalytic) in some neighborhood of the origin, then f is analytic (quasianalytic, respecitvely) everywhere in $\mathbb{R}^n$.

To prove the main theorem, we consider the convolution of the heat kernel function and the tempered distribution f, which is called the defining function of f. The defining function of a positive definite tempered distribution f is a positive definite continuous function and a solution of the heat equation with the initial value f. We represent f as the difference of $(-\Delta)^m g$ and h for some $m \in \mathbb{N}$ where g and h are positive definite, g is continuous and h is real analytic. Applying hypoellipticity of the Laplacian operator and Theorem 1.1, we show that g is a C^∞ function in $\mathbb{R}^n$, so f is a positive definite C^∞ function in $\mathbb{R}^n$ which is analytic (quasianalytic) on some neighborhood of the origin. Due to Theorem 1.2, f is an analytic (quasianalytic) in $\mathbb{R}^n$.

2. Preliminary

Throughout this paper we use conventional multi-index notations such as for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\partial^\alpha = \partial_1{}^{\alpha_1} \cdots \partial_n{}^{\alpha_n}$, and $\partial_j = \frac{\partial}{\partial x_j}$, where $\mathbb{N}_0$ denotes the set of all nonnegative integers.

The Schwarz space $\mathcal{S}$ is the set of all C^∞ functions φ on $\mathbb{R}^n$ satisfying

$$\sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta \varphi(x)| < \infty \quad \text{for any multi-indices } \alpha, \beta, \tag{2.1}$$

$\mathcal{S}'$ denotes the strong dual space of $\mathcal{S}$ and an element of $\mathcal{S}'$ is called a tempered distribution.

We introduce the ultradifferentiable class including the analytic class, quasianalytic class and non-quasianalytic class.

Let $(M_p)_{p=0}^\infty$ be a sequence of positive numbers satisfying the following conditions:

(M.0) There exist positive constants C and H such that

$$p! \le CH^p M_p, \quad p = 0, 1, \ldots.$$

(M.1) $M_p^2 \le M_{p-1} M_{p+1}, \quad p = 1, 2, \ldots.$

(M.2)$'$ There exist positive constant A and B such that

$$M_{p+1} \le AB^p M_p, \quad p = 0, 1, \ldots.$$

For an open subset Ω of $\mathbb{R}^n$ and a sequence (M_p) as above, we denote by $\mathcal{E}^{\{M_p\}}(\Omega)$ ($\mathcal{E}^{(M_p)}(\Omega)$, respectively) the set of all $\varphi \in C^\infty(\Omega)$ such that for any compact subset K of Ω there exist $h > 0$ and $C > 0$ (for any $h > 0$ there exists a constant $C > 0$) such that

$$\sup_{x\in K} |\partial^\alpha \varphi(x)| \le C h^{|\alpha|} M_{|\alpha|}, \qquad \text{for any } \alpha \in \mathbb{N}_0^n.$$

An element of $\mathcal{E}^{\{M_p\}}(\Omega)$ ($\mathcal{E}^{(M_p)}(\Omega)$) is called $\{M_p\}$-*ultradifferentiable* ((M_p)-*ultradifferentiable*, respectively) in Ω.

For an instance, $\mathcal{E}^{\{p!^s\}}(\Omega)$ is known as the set of Gevrey functions of index s. In particular, if $s = 1$, then $\mathcal{E}^{\{p!\}}(\Omega)$ is the set of all (real) analytic functions in Ω and $\mathcal{E}^{(p!)}(\Omega)$ is the set of all analytic functions which extend to $\mathbb{C}^n$ as entire functions.

Remark 1. (i) The conditions (M.0), (M.1) and (M.2)$'$ are the most fundamental and essential in a sense that the sequence (M_p) can be rearranged without any change of $\mathcal{E}^*(\Omega)$ so that (M.1) should be satisfied and (M.2)$'$ makes $\mathcal{E}^*(\Omega)$ stable under the differentiation ([11]). Here * denote $\{M_p\}$ or (M_p). Moreover, (M.0) means that the analytic class is the smallest class to be considered here.

(ii) If the sequence (M_p) satisfies $\sum_{p=1}^\infty \frac{M_{p-1}}{M_p} = \infty$, then an element φ in $\mathcal{E}^*(\Omega)$ with $\partial^\alpha\varphi(x_0) = 0$ for all $\alpha \in \mathbb{N}_0^n$ is identically zero in a connected open set containing x_0. In this case every φ in $\mathcal{E}^*(\Omega)$ is said to be quasianalytic in Ω. But if $\sum_{p=1}^\infty \frac{M_{p-1}}{M_p} < \infty$, then $\mathcal{E}^*(\Omega)$ contains the cut-off functions and the partitions of unity.

To prove the main theorem we consider the convolution of a tempered distribution f and the heat kernel function.

We denote by $E(x,t)$ the $n-$dimensional heat kernel function

$$E(x,t) = \begin{cases} (4\pi t)^{-n/2} \exp\left(-\dfrac{|x|^2}{4t}\right), & t > 0, \\ 0, & t \le 0. \end{cases}$$

Since for each $t > 0$, $E(\cdot,t) \in \mathcal{S}$, for $f \in \mathcal{S}'$ the function

$$F(x,t) = \langle f_y, E(x-y,t)\rangle$$

is well-defined on $\mathbb{R}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n, t > 0\}$ and called the defining function of f. For each $t > 0$, the defining function $F(\cdot,t)$ is no more a distribution but belongs to the class of $C^\infty(\mathbb{R}^n)$. Due to [13], we have a bound of the defining function., i.e., there exists $C, M, N > 0$ such that

$$|F(x,t)| \le C t^{-M}(1+|x|)^N \quad \text{for any } (x,t) \in \mathbb{R}^{n+1}_+ \tag{2.2}$$

and $\lim_{t\to 0^+} F(x,t) = f(x)$ in the following sense; for any $\varphi \in \mathcal{S}$,

$$\lim_{t\to 0^+} \int_{\mathbb{R}^n} F(x,t)\varphi(x)dx = \langle f, \varphi\rangle.$$

The next theorem show the positive definiteness of a tempered distribution and its defining function.

Theorem 2.1 ([4]). *Let* $f \in \mathcal{S}'$ *and* $F(x,t) = \langle f_y, E(x-y,t)\rangle$. *Then the following conditions are equivalent*:

(i) f *is a positive definite tempered distribution.*

(ii) *For each* $t > 0$, *the defining function* $F(\cdot, t)$ *of* f *is a positive definite continuous function.*

3. Analyticity and quasianalyticity

In the next theorem we show that a positive definite distribution which is well behaved near the origin also behaves well everywhere in $\mathbb{R}^n$. For $M \in \mathbb{R}$ we denote by $[M]$ the greatest integer which is less than or equal to M.

Theorem 3.1. *Let* f *be a positive definite distribution such that* f *is* $\{M_p\}$*-ultradifferentiable on some neighborhood of the origin. Then* f *is* $\{M_p\}$*-ultradifferentiable in* $\mathbb{R}^n$.

Proof. Let f be a positive definite distribution which is $\{M_p\}$-ultradifferentiable on a neighborhood Ω of the origin. By the Bochner-Schwarz theorem ([10], p. 141) f is the Fourier transform of a positive tempered measure μ. Thus f belongs to $\mathcal{S}'$ and (1.2) holds for $\varphi \in \mathcal{S}$, so f is a positive definite tempered distribution, where $\mathcal{S}$ is the Schwarz space and $\mathcal{S}'$ is its dual space.

Define a function on $\mathbb{R}^{n+1}_+$

$$F(x,t) = \langle f_y, E(x-y,t)\rangle. \tag{3.1}$$

Since for each $t > 0$, $E(\cdot,t) \in \mathcal{S}$ and E is a solution of the heat equation, i.e., $(\partial_t - \Delta)E(x,t) = 0$, F is real analytic in x and also a solution of the heat equation. In view of Theorem 2.1, for each $t > 0$, $F(\cdot,t)$ is a positive definite continuous function. Moreover, there exist positive constants C, M, N such that

$$|F(x,t)| \le Ct^{-M}(1+|x|)^N \quad \text{for any } (x,t) \in \mathbb{R}^{n+1}_+, \tag{3.2}$$

and for any $\varphi \in \mathcal{S}$

$$\lim_{t\to 0^+} \int F(x,t)\varphi(x)dx = \langle f, \varphi\rangle. \tag{3.3}$$

Let

$$m = [M] + 2 \tag{3.4}$$

and define a function

$$g(t) = \begin{cases} \dfrac{t^{m-1}}{(m-1)!}, & t \ge 0, \\ 0, & t < 0. \end{cases} \tag{3.5}$$

Let χ be a C^∞ function such that $\chi \equiv 1$ on $(-\infty, 1]$, $\chi \equiv 0$ on $[2, \infty)$ and $\chi \geq 0$. Define a function on $\mathbb{R}$

$$v(t) = g(t) \cdot \chi(t). \tag{3.6}$$

Then $v(t) \geq 0$ for any $t \in \mathbb{R}$ and v has a compact support $[0, 2]$. Applying Leibniz formula, it follows that

$$\frac{d^m}{dt^m} v(t) = \delta(t) + w(t), \tag{3.7}$$

where δ is the Dirac delta function. We can easily see that w is a C^∞ function having support in $[1, 2]$. Define a function on $\mathbb{R}^{n+1}_+$

$$G(x,t) = \int_0^\infty F(x, t+s) v(s) ds$$

and on $\mathbb{R}^n \times [0, \infty)$

$$H(x,t) = \int_0^\infty F(x, t+s) w(s) dx.$$

Since F is a solution of the heat equation, so are G and H. In view of (3.2) and (3.4), there exists $C_1 > 0$ such that

$$|G(x,t)| \leq C_1(1+|x|)^N, \quad |H(x,t)| \leq C_1(1+|x|)^N \quad \text{for any } (x,t) \in \mathbb{R}^{n+1}_+. \tag{3.8}$$

Since w has a compact support in $[1, 2]$, $H(x,t)$ is a C^∞ function on $\mathbb{R}^n \times [0, \infty)$. Due to (3.4) and (3.8), G can be continuously extended to $\mathbb{R}^n \times [0, \infty)$. We write

$$g(x) = G(x, 0) \qquad \text{and} \qquad h(x) = H(x, 0).$$

Then g is a continuous function. Since w has a compact support in $[1, 2]$ and for each $t > 0$ $F(\cdot, t)$ is real analytic in x, h is a real analytic function. We next show that g is a positive definite continuous function. Since for each $t > 0$, $F(\cdot, t)$ is positive definite, by Lebesgue dominated convergence theorem

$$\begin{aligned}
&\int\int \overline{g(x-y)}\varphi(x)\overline{\varphi(y)}dxdy \\
&\quad = \int\int \lim_{t\to 0^+} \int \overline{F(x-y, t+s)} v(s) ds \ \varphi(x)\overline{\varphi(y)} dxdy \\
&\quad = \lim_{t\to 0^+} \int\int\int \overline{F(x-y, t+s)}\varphi(x)\overline{\varphi(y)} dxdy \ v(s) ds \geq 0
\end{aligned}$$

for any $\varphi \in \mathcal{S}$. Thus g is a positive definite continuous function.

Observing that G is a solution of heat equation, i.e., $(\partial_t - \Delta)G = 0$, by (3.7)

$$\begin{aligned}
(-\Delta)^m G(x,t) &= (-\partial_t)^m G(x,t) \\
&= F(x,t) + \int_0^\infty F(x, t+s) w(s) ds,
\end{aligned} \tag{3.9}$$

and taking limit $t \to 0^+$ we have

$$(-\Delta)^m g = f + h \qquad \text{in } \mathcal{S}'. \tag{3.10}$$

By the assumption of the theorem, f is a C^∞ function on Ω. Since h is a C^∞ function on Ω, $(-\Delta)^m g$ is a C^∞ function on Ω. By the hypoellipticity of the Laplacian operator ([1], p.187) g is a C^∞ function on Ω. Observing that g is a positive definite continuous function which belongs to $C^\infty(\Omega)$, by Theorem 1.1 g is a C^∞ function on $\mathbb{R}^n$. Thus by (3.10) f is a C^∞ function on $\mathbb{R}^n$ and f is a positive definite continuous function which is $\{M_p\}$−ultradifferentiable on Ω. Therefore by Theorem 1.2, $f \in \mathcal{E}^{\{M_p\}}(\mathbb{R}^n)$. □

Following the same proof of the above theorem, we can easily show (M_p)-ultradifferentiability of a positive definite distribution.

Corollary 3.1. *If f is a positive definite distribution which is (M_p)-ultradifferentiable on some neighborhood of the origin, then $f \in \mathcal{E}^{(M_p)}(\mathbb{R}^n)$.*

Taking $M_p = p!$ for each p, we have the analyticity of positive definite distributions.

Corollary 3.2. *If f is a positive definite distribution which is analytic on some neighborhood of the origin, then f is analytic everywhere on $\mathbb{R}^n$.*

By (3.10) we can express a positive definite distribution in the following way.

Corollary 3.3. *Let f be a positive definite distribution. Then there exist $m \in \mathbb{N}$, a positive definite continuous functions g and a real analytic function h such that*

$$f = (-\Delta)^m g - h.$$

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