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A note on anisotropic first-passage percolation

By

Masato Takei

Abstract

We consider a first-passage percolation problem on the square lattice, where the distribution function of time coordinates of horizontal edges may be different from that of vertical edges. Some basic limit theorems for first-passage times and minimal lengths of optimal paths are obtained. Especially, we show that as long as the system is in the supercritical phase, the expectation of first-passage time from the origin to a point with distance n converges to a finite constant, which is independent of the directions, as $n \to \infty$.

1. Introduction and results

First-passage percolation was introduced by Hammersley and Welsh [5]. We consider a generalized first-passage percolation problem on the square lattice $\mathbb{L}^2 = (\mathbb{Z}^2, \mathbb{E}^2)$. We write the edge between $u, v \in \mathbb{Z}^2$ for $\langle u, v\rangle \in \mathbb{E}^2$. An edge e is called *horizontal* if $e = \langle (x, y), (x+1, y)\rangle$ for some $(x, y) \in \mathbb{Z}^2$ and *vertical* if $e = \langle (x, y), (x, y+1)\rangle$ for some $(x, y) \in \mathbb{Z}^2$. Let $\{t(e); e \in \mathbb{E}^2\}$ be a family of nonnegative random variables. We regard $t(e)$ as the time needed for a particle to traverse the edge e. The setting of *anisotropic first-passage percolation* ([2]) is as follows: The random variable $t(e)$ is independent of each other, and its distribution function is $F_h(s)$ if e is horizontal, and $F_v(s)$ if e is vertical. In this paper, we are interested in the first-passage time between two regions of $\mathbb{L}^2$.

1.1. Anisotropic Bernoulli percolation

First we consider a special case, *anisotropic Bernoulli percolation*. Each edge has two possible states, *open* and *closed*. A horizontal edge is open with probability p_h, while a vertical edge is open with probability p_v. The corresponding probability measure is denoted by P_{p_h,p_v}, and the expectation with respect to P_{p_h,p_v} is denoted by E_{p_h,p_v}. Let C_0 be the open cluster of the origin and $\theta(p_h, p_v) = P_{p_h,p_v}\{|C_0| = \infty\}$, where $|C|$ denotes the number of edges in a subgraph C. Moreover, The critical line of this model is $p_h + p_v = 1$ (see [4,

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§11.9]):

$$\theta(p_h, p_v) \begin{cases} = 0 & \text{if } p_h \neq 1,\ p_v \neq 1,\ p_h + p_v \leq 1, \\ > 0 & \text{otherwise.} \end{cases}$$

In the appendix, we give some formulae about the ratio of boundary to volume of the open cluster of the origin.

1.2. Limit theorems for first-passage times

We return to the general anisotropic first-passage percolation problem with distribution functions $F_h(s)$ and $F_v(s)$. We denote the probability measure by P, and the expectation with respect to P by E.

Let $\pi : e_1 \to \cdots \to e_p$ be a self-avoiding path on $\mathbb{L}^2$, where $e_1, \ldots, e_p \in \mathbb{E}^2$. The *passage time* of π is defined by

$$T(\pi) = \sum_{i=1}^{p} t(e_i).$$

The *first-passage time* between $u, v \in \mathbb{Z}^2$ is defined by

$$T(u, v) = \inf\{T(\pi); \pi \text{ is a path from } u \text{ to } v\}.$$

More generally, let $T(A, B) = \inf\{T(u, v); u \in A,\ v \in B\}$ for $A, B \subset \mathbb{Z}^2$. Let $\mathbf{0} = (0, 0)$, $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$. We study the following first-passage times:

• *Point-to-point first-passage times*:
For any unit vector $\mathbf{u} \in \mathbb{R}^2$, we define $T(\mathbf{0}, n\mathbf{u})$ by

$$T(\mathbf{0}, \text{the set of lattice points nearest to } n\mathbf{u}).$$

In particular, the following are called a-processes.

$$a^r_{0,n} = T(\mathbf{0}, n\mathbf{e}_1),\ a^l_{0,n} = T(\mathbf{0}, -n\mathbf{e}_1),$$
$$a^u_{0,n} = T(\mathbf{0}, n\mathbf{e}_2),\ a^d_{0,n} = T(\mathbf{0}, -n\mathbf{e}_2).$$

• *Point-to-line first-passage times* (b-processes):

$$b^r_{0,n} = T(\mathbf{0}, \{x = n\}),\ b^l_{0,n} = T(\mathbf{0}, \{x = -n\}),$$
$$b^u_{0,n} = T(\mathbf{0}, \{y = n\}),\ b^d_{0,n} = T(\mathbf{0}, \{y = -n\}).$$

• *Point-to-box first-passage time* (c-process) : $c_{0,n} = T(\mathbf{0}, \partial B(n))$, where $\partial B(n) = \{(x, y) \in \mathbb{Z}^2; \max\{|x|, |y|\} = n\}$.
Note that $\theta^r_{0,n} = \theta^l_{0,n}$ and $\theta^u_{0,n} = \theta^d_{0,n}$ in distribution, where $\theta = a$ or b.

We assume that $m_h := \int s dF_h(s) < \infty$ and $m_v := \int s dF_v(s) < \infty$. By the subadditivity of the point-to-point first-passage time, there exists the *time constant* for the direction $\mathbf{u}$:

$$\mu(\mathbf{u}) = \mu(F, \mathbf{u}) := \lim_{n \to \infty} \frac{ET(\mathbf{0}, n\mathbf{u})}{n}.$$

By the subadditive ergodic theorem,

$$\lim_{n\to\infty} \frac{T(\mathbf{0}, n\mathbf{u})}{n} = \mu(\mathbf{u}) \text{ a.s. and in } L^1.$$

An edge e is called *open* if $t(e) = 0$, otherwise *closed*. Then, associated with our first-passage percolation process, the anisotropic Bernoulli percolation process with parameters $p_h = F_h(0)$ and $p_v = F_v(0)$ arises. The coupling measure governing these processes is denoted again by P.

On the positivity of $\mu_h := \mu(\mathbf{e}_1)$ and $\mu_v := \mu(\mathbf{e}_2)$, it is known that the critical line is $F_h(0) + F_v(0) = 1$, in the sense of the following proposition.

Proposition 1.1 ([2]). *Assume that m_h, $m_v < \infty$. Let $p_h = F_h(0)$ and $p_v = F_v(0)$.*

(1) *If $p_h + p_v < 1$, then $\mu_h > 0$ and $\mu_v > 0$.* (*In fact, their argument shows that $\mu(\mathbf{u}) > 0$ for any* $\mathbf{u}$.)

(2) *If $p_h \neq 1$, $p_v \neq 1$ and $p_h + p_v \geq 1$, then $\mu_h = \mu_v = 0$.* (*In fact, their argument shows that $\mu(\mathbf{u}) = 0$ for any* $\mathbf{u}$.)

(3) $\begin{cases} \textit{If } p_h = 1, \textit{ then } \mu_h = 0,\ \mu_v = \inf\{s \geq 0; F_v(s) > 0\}, \\ \textit{If } p_v = 1, \textit{ then } \mu_v = 0,\ \mu_h = \inf\{s \geq 0; F_h(s) > 0\}. \end{cases}$

As for other passage times,

Proposition 1.2. *Assume that $m_h, m_v < \infty$.*

(1) *For the b-processes, the following hold almost surely and in L^1;*

$$\lim_{n\to\infty} \frac{b^r_{0,n}}{n} = \lim_{n\to\infty} \frac{b^l_{0,n}}{n} = \mu_h, \ \lim_{n\to\infty} \frac{b^u_{0,n}}{n} = \lim_{n\to\infty} \frac{b^d_{0,n}}{n} = \mu_v.$$

(2) $\lim_{n\to\infty} \frac{c_{0,n}}{n} = \min\{\mu_h, \mu_v\}$ *a.s. and in L^1.*

(1) is a consequence of the shape theorem (see [6, p. 166]). This method can be applied to (2) also (as in [8, (3.16)]).

Since $c_{0,n}$ is increasing in n, the limit $\rho = \lim_{n\to\infty} c_{0,n}$ exists (possibly $+\infty$). If there exists an infinite open cluster, then $\rho < \infty$ a.s. by Kolmogorov 0-1 law.

Let $\mathcal{D} = \{u, d, l, r\}$. We quote almost sure convergence results in [13]. The original proof works for our anisotropic case also. We remark that moment conditions required in [13] can be dropped by a simple modification of his own method.

Proposition 1.3 (cf. [13] Theorem 5). *Assume that $p_h \neq 1, p_v \neq 1$ and $p_h + p_v > 1$.*

(1) *For any $* \in \mathcal{D}$, the family $\{a^*_{0,n}\}$ is tight, i.e.*

$$\lim_{L\to\infty} \inf_{n\geq 1} P\{a^*_{0,n} \leq L\} = 1.$$

On the other hand,

$$\liminf_{n\to\infty} a^*_{0,n} = \rho \textit{ and } \limsup_{n\to\infty} a^*_{0,n} = +\infty \textit{ a.s.}$$

(2) *For any* $* \in \mathcal{D}$, $\lim_{n\to\infty} b^*_{0,n} = \rho$ *almost surely. In fact,* $b^*_{0,n} = c_{0,n}$ *eventually a.s.*

For the expectations, we can obtain the following result. This theorem says that even if F_h is quite different from F_v, the limit of the expectation of the passage time is independent of the directions whenever the system is on the critical line or in the supercritical regime.

Theorem 1.1. *Assume that* $p_h \neq 1$, $p_v \neq 1$ *and* $p_h + p_v \geq 1$. *If* $E\rho < \infty$, *then the following hold.*
(1) *For any unit vector* $\mathbf{u} \in \mathbb{R}^2$, $\lim_{n\to\infty} ET(\mathbf{0}, n\mathbf{u}) = 2E\rho$.
(2) *For any* $* \in \mathcal{D}$, $\lim_{n\to\infty} Eb^*_{0,n} = E\rho$.

When $p_h + p_v > 1$, $m_h, m_v < \infty$ is sufficient for $E\rho < \infty$ (cf. [13, Properties (a)]). In the supercritical case, ρ is the first passage time from the origin to the minimal open circuit surrounding it and connected to the infinite open cluster (see Lemma 2.3). Thus we can interpret ρ as the minimal cost to go far away from the origin. In a fixed configuration and for a large n, $T(\mathbf{0}, n\mathbf{u})$ depends on the situation around the terminal point $n\mathbf{u}$, while the first passage time from $n\mathbf{u}$ to the infinite open cluster is equal to ρ in law. This gives intuition for the above results. We note that our proof of Theorem 1.1 remains valid for the critical case, although there is no infinite open cluster. In this case, the situation is less clear and the finiteness of $E\rho$ becomes a delicate problem (see [14]).

1.3. Limit theorems for minimal route lengths

A path π from A to B is called a *route* for the passage time $T(A, B)$ if $T(\pi) = T(A, B)$. For example, we can always find a route for $c_{0,n}$. In two dimensions, for any F_h and F_v, there exist routes for $a_{0,n}$ and $b_{0,n}$ almost surely. A proof for the standard first-passage percolation (i.e. $F_h = F_v = F$) is found in §4.3 of [9]. It works for our anisotropic setting.

The *minimal route length* for $c_{0,n}$ is defined by

$$N^c_{0,n} = \inf\{|\pi|; \pi \text{ is a route for } c_{0,n}\}.$$

Similarly, the minimal route length for $\theta^*_{0,n}$ is denoted by $N^{\theta,*}_{0,n}$, where $\theta = a$ or b, and $* \in \mathcal{D}$. It remains difficult to obtain limit theorems for the minimal route lengths. The argument in [12] for the supercritical case works for our anisotropic model also:

Proposition 1.4 (cf. [12]). *If* $p_h \neq 1$, $p_v \neq 1$ *and* $p_h + p_v > 1$, *then there are constants* $\lambda_h, \lambda_v > 1$, *which depend only on* p_h *and* p_v, *such that*

$$\lim_{n\to\infty} \frac{N^{\theta,l}_{0,n}}{n} = \lim_{n\to\infty} \frac{N^{\theta,r}_{0,n}}{n} = \lambda_h, \ \lim_{n\to\infty} \frac{N^{\theta,u}_{0,n}}{n} = \lim_{n\to\infty} \frac{N^{\theta,d}_{0,n}}{n} = \lambda_v$$

almost surely and in L^1, *where* $\theta = a$ *or* b.

As for $N^c_{0,n}$, we have the following

Theorem 1.2. *If $p_h \neq 1$, $p_v \neq 1$ and $p_h + p_v > 1$, then*

$$\lim_{n\to\infty} \frac{N^c_{0,n}}{n} = \min\{\lambda_h, \lambda_v\} \text{ a.s. and in } L^1.$$

A claim in [7] is that for the standard first-passage percolation, the existence of the limit of $N^b_{0,n}/n$ implies the existence of the limit of $N^c_{0,n}/n$ in the whole parameter region. Unfortunately, the argument in [7] is not correct. Here we justify the above claim at least for the supercritical phase, and extend it to the anisotropic case. Although the statement of the theorem looks like Proposition 1.2 (2), we need a separate proof.

2. Proofs

Proof of Theorem 1.1. Almost the same arguments in [13], [14] work. Here we give a simpler proof of (1) than in [13], which can be applied to some dependent models, such as the two-dimensional Ising first-passage percolation.

Since each route for $T(\mathbf{0}, n\mathbf{u})$ intersects both $\partial B(n/3)$ and $n\mathbf{u} + \partial B(n/3)$, we have

$$T(\mathbf{0}, n\mathbf{u}) \geq T(\mathbf{0}, \partial B(n/3)) + T(n\mathbf{u}, n\mathbf{u} + \partial B(n/3)).$$

By the translation-invariance, $ET(\mathbf{0}, n\mathbf{u}) \geq 2Ec_{0,n/3}$. Using the monotone convergence theorem, we can conclude that $\liminf_{n\to\infty} ET(\mathbf{0}, n\mathbf{u}) \geq 2E\rho$.

Let us consider the event

$$G_m := \{\text{there is an open circuit } C_m \text{ surrounding } \mathbf{0} \text{ in } B(m) \setminus B(n)\},$$

where $B(n) = [-n, n]^2 \cap \mathbb{Z}^2$. For fixed $n \in \mathbb{N}$, $\lim_{m\to\infty} P(G_m) = 1$. We take a route r_1 for $c_{0,m}$ and a route r_2 for $T(n\mathbf{u}, \partial B(m))$ arbitrarily. On G_m, both r_1 and r_2 must intersect C_m, which implies that

$$T(\mathbf{0}, n\mathbf{u}) \leq t(r_1) + 0 + t(r_2) = c_{0,m} + T(n\mathbf{u}, \partial B(m)).$$

Noting that $T(n\mathbf{u}, \partial B(m)) \leq T(n\mathbf{u}, n\mathbf{u} + \partial B(m + 2n))$, and the distribution of $T(n\mathbf{u}, n\mathbf{u} + \partial B(m + 2n))$ equals that of $c_{0,m+2n}$,

$$\begin{aligned} ET(\mathbf{0}, n\mathbf{u}) &= E[T(\mathbf{0}, n\mathbf{u}) : G_m] + E[T(\mathbf{0}, n\mathbf{u}) : G^c_m] \\ &\leq E[c_{0,m} + c_{0,m+2n}] + n\max\{m_h, m_v\} \cdot P(G^c_m). \end{aligned}$$

Letting $m \to \infty$ and then, $n \to \infty$, we have $\limsup_{n\to\infty} ET(\mathbf{0}, n\mathbf{u}) \leq 2E\rho$. This completes the proof. □

Before proving Theorem 1.2, we prepare some lemmata. The first one is about some fundamental results on the anisotropic Bernoulli percolation.

Lemma 2.1. (1) *The radius distribution of* C_0 *decays exponentially if* $p_h + p_v < 1$. *In particular, we have* $E_{p_h,p_v}[|C_0|^m] < +\infty$ *for all* $m \in \mathbb{N}$. *By the duality, the radius of each dual closed cluster decays exponentially if* $p_h + p_v > 1$.

(2) *The probability that there is a left-right open crossing in the rectangle* $[-M, M] \times [-L, L]$ *is denoted by* $R^{\leftrightarrow}_{L,M} = R^{\leftrightarrow}_{L,M}(p_h, p_v)$. *Similarly,* $R^{\updownarrow}_{L,M} = R^{\updownarrow}_{L,M}(p_h, p_v)$ *denotes the top-bottom open crossing probability. Then,*

$$R^{\leftrightarrow}_{L,3M} \geq \left(R^{\updownarrow}_{L,M}\right)^3 \cdot \left(1 - \sqrt{1 - R^{\updownarrow}_{L,M}}\right)^4 \cdot \left(1 - \sqrt{1 - R^{\leftrightarrow}_{L,M}}\right)^8,$$
$$R^{\updownarrow}_{3L,M} \geq \left(R^{\leftrightarrow}_{L,M}\right)^3 \cdot \left(1 - \sqrt{1 - R^{\leftrightarrow}_{L,M}}\right)^4 \cdot \left(1 - \sqrt{1 - R^{\updownarrow}_{L,M}}\right)^8.$$

The exponential decay result (1) can be proved by the Menshikov argument (see [4, §5.2]). A version of the Russo-Seymour-Welsh lemma (2) is found in [10, p. 85].

Suppose that $p_h + p_v > 1$. Let I be the unique infinite open cluster. The minimal open circuit surrounding $n\mathbf{e}_1$ and connected to I is denoted by D_n. The next lemma corresponds to Lemma 2 in [12]. This is proved by the method in section 2 of [1] together with Lemma 2.1.

Lemma 2.2. *If* $p_h \neq 1$, $p_v \neq 1$ *and* $p_h + p_v > 1$, *then for any* $m \geq 1$,

$$E_{p_h,p_v}[|D_n|^m] < +\infty \text{ and } \lim_{n\to\infty} \frac{|D_n|}{n} = 0 \text{ a.s.}$$

The following lemma is a variant of Lemma 1 in [12].

Lemma 2.3. *Assume that* D_0 *lies in the box* $B(n)$. *Then a path* r *from* $\mathbf{0}$ *to* $\partial B(n)$ *is a route for* $c_{0,n}$ *if and only if* r *consists of two pieces* r_1 *and* r_2, *say* $r = r_1 * r_2$, *of the following nature:* r_1 *connects* $\mathbf{0}$ *to* D_0 *inside* D_0 (*except for its endpoint on* D_0) *and has minimal passage time among such paths.* r_2 *is contained in* I *and connects* D_0 *to* $\partial B(n)$.

We put

$$\begin{aligned}
\partial B(n)_u &= \{(x,y) \in \mathbb{Z}^2; -n \leq x \leq n,\ y = n\},\\
\partial B(n)_d &= \{(x,y) \in \mathbb{Z}^2; -n \leq x \leq n,\ y = -n\},\\
\partial B(n)_l &= \{(x,y) \in \mathbb{Z}^2; x = -n,\ -n \leq y \leq n\},\\
\partial B(n)_r &= \{(x,y) \in \mathbb{Z}^2; x = n,\ -n \leq y \leq n\}.
\end{aligned}$$

The set of routes for $c_{0,n}$ is denoted by $\mathcal{R}^c_{0,n}$. For any $* \in \mathcal{D}$, we define $\mathcal{R}^{c,*}_{0,n}$ is the set of paths in $\mathcal{R}^c_{0,n}$ whose endpoints are in $\partial B(n)_*$.

Lemma 2.4. *With probability one, there exists a random number* N *such that all the above four sets are non-empty and* $D_0 \subset B(n)$ *for all* $n \geq N$.

Proof. If there exists an infinite open cluster in $Q_n^r := \{(x,y) \in \mathbb{Z}^2; x \geq 0, |y| \leq x\}$ almost surely, then by Lemma 2.3, $\mathcal{R}_{0,n}^{c,r} \neq \emptyset$ for large n. Using Lemma 2.1, such an infinite cluster can be constructed by a standard sponge-crossing argument. □

The monotonicity of $N_{0,n}^b$ (for large n) is used in the proof of Lemma 4 in [12]. We can see that $N_{0,n}^c$ is also monotone for large n.

Lemma 2.5. *If $N \leq n < n'$, then $N_{0,n}^c < N_{0,n'}^c$.*

Proof. Let r' be a path from $\mathbf{0}$ to $\partial B(n')$ with $T(r') = c_{0,n'}$ and $|r'| = N_{0,n'}^c$. We decompose $r' = r_1' * r_2'$ as in Lemma 2.3. Let r_2 be the piece of r_2' from its starting point to its first intersection with $\partial B(n)$. Then, again by Lemma 2.3, $r := r_1' * r_2$ is route for $c_{0,n}$. Thus we have

$$N_{0,n}^c \leq |r| = |r_1'| + |r_2| < |r_1'| + |r_2'| = |r'| = N_{0,n'}^c.$$

□

Hereafter we assume that $n \geq N$. For any $* \in \mathcal{D}$, we define

$$N_{0,n}^{c,*} = \inf\{|r|; r \in \mathcal{R}_{0,n}^{c,*},\, T(r) = c_{0,n}\}.$$

Clearly, $N_{0,n}^{c,*} \geq N_{0,n}^c$ and $N_{0,n}^{c,*} < N_{0,n+1}^{c,*}$ for any $* \in \mathcal{D}$. By a simple argument (among others, as in [7]), we have $N_{0,n}^c \leq N_{0,n}^{b,*} \leq N_{0,n}^{c,*}$ for any $* \in \mathcal{D}$. Note that for each n, $N_{0,n}^{c,*} = N_{0,n}^c$ for at least one direction $* \in \mathcal{D}$.

Proof of Theorem 1.2. As in the proof of Lemma 4 in [12], we can see that

$$N_{0,n}^c \leq N_{0,n}^{a,h} + |D_n| \text{ and } N_{0,n}^c \leq N_{0,n}^{a,v} + |D_n'|,$$

where D_n' denotes the minimal open circuit surrounding $n\mathbf{e}_2$ and connected to I. By Proposition 1.4 and Lemma 2.2, we have

$$\limsup_{n\to\infty} \frac{N_{0,n}^c}{n} \leq \min\{\lambda_h, \lambda_v\} \text{ a.s.}$$

For $\theta = b$ or c, let $N_{0,n}^{\theta,h} = \min\{N_{0,n}^{\theta,l}, N_{0,n}^{\theta,r}\}$ and $N_{0,n}^{\theta,v} = \min\{N_{0,n}^{\theta,u}, N_{0,n}^{\theta,d}\}$. If $N_{0,n}^c < N_{0,n}^{c,h}$ eventually, then $N_{0,n}^c = N_{0,n}^{c,v} = N_{0,n}^{b,v}$ for large n and

$$\lim_{n\to\infty} \frac{N_{0,n}^c}{n} = \lim_{n\to\infty} \frac{N_{0,n}^{b,v}}{n} = \lambda_v = \min\{\lambda_h, \lambda_v\}$$

by Proposition 1.4. Similarly, if $N_{0,n}^c < N_{0,n}^{c,v}$ eventually, then we have $\lim_{n\to\infty} \frac{N_{0,n}^c}{n} = \lambda_h = \min\{\lambda_h, \lambda_v\}$.

Suppose that we can find two subsequences $\{n_k\}$ and $\{m_k\}$ such that $N_{0,n_k}^c = N_{0,n_k}^{c,v} = N_{0,n_k}^{b,v}$ and $N_{0,m_k}^c = N_{0,m_k}^{c,h} = N_{0,m_k}^{b,h}$. Note that $N_{0,n}^c = N_{0,n}^{c,h} =$

$N^{b,h}_{0,n}$ and $N^c_{0,n+1} = N^{c,v}_{0,n+1} = N^{b,v}_{0,n+1}$ for infinitely many n's. To establish the existence of the limit of $N^c_{0,n}/n$, we shall prove that $\lambda_h = \lambda_v$ in such a case. Suppose that $\lambda_h > \lambda_v$. By Proposition 1.4, for given $\varepsilon > 0$, we can choose n so large that

$$\left| \frac{N^{b,h}_{0,n}}{n} - \lambda_h \right| < \varepsilon \text{ and } \left| \frac{N^{b,v}_{0,n+1}}{n+1} - \lambda_v \right| < \varepsilon.$$

If $\varepsilon < (\lambda_h - \lambda_v)/2$ and $n \geq (\lambda_v + \varepsilon)/(\lambda_h - \lambda_v - 2\varepsilon)$, then

$$\begin{aligned} N^c_{0,n+1} = N^{b,v}_{0,n+1} &\leq (\lambda_v + \varepsilon)(n+1) \\ &\leq (\lambda_h - \varepsilon)n \leq N^{b,h}_{0,n} = N^c_{0,n}. \end{aligned}$$

This contradicts Lemma 2.5. □

3. Appendix: The ratio of boundary to volume of the open cluster

We give some formulae about the ratio of boundary to volume of the open cluster of the origin in anisotropic Bernoulli bond percolation, which extend the results in [3], [11]. For a subgraph C of $\mathbb{L}^2$, let $|C|_h$ be the number of horizontal edges in C and $|C|_v$ be the number of vertical edges in C. The boundary edges of C is denoted by ΔC.

Theorem 3.1. *For $0 < p_h$, $p_v \leq 1$ and $n \in \mathbb{N}$,*

$$\frac{E_{p_h,p_v}|\Delta C_0 \cap S(n)|_h}{E_{p_h,p_v}|C_0 \cap S(n)|_h} = \frac{1-p_h}{p_h}, \quad \frac{E_{p_h,p_v}|\Delta C_0 \cap S(n)|_v}{E_{p_h,p_v}|C_0 \cap S(n)|_v} = \frac{1-p_v}{p_v},$$

where $S(n) = [-n,n]^2$. When $0 < p_h + p_v < 1$, we have

$$\frac{E_{p_h,p_v}|\Delta C_0|_h}{E_{p_h,p_v}|C_0|_h} = \frac{1-p_h}{p_h}, \quad \frac{E_{p_h,p_v}|\Delta C_0|_v}{E_{p_h,p_v}|C_0|_v} = \frac{1-p_v}{p_v}.$$

Proof. We can see that

$$\begin{aligned} &E_{p_h,p_v}[|\Delta C_0 \cap S(n)|_h] \\ &= \sum_{e \subset S(n);\text{horizontal}} P_{p_h,p_v}\{\text{a horizontal edge } e \text{ belongs to } \Delta C_0\} \\ &= \sum_{e \subset S(n);\text{horizontal}} \frac{1-p_h}{p_h} P_{p_h,p_v}\{\text{a horizontal edge } e \text{ belongs to } C_0\} \\ &= \frac{1-p_h}{p_h} E_{p_h,p_v}[|C_0 \cap S(n)|_h]. \end{aligned}$$

For the subcritical case, we get the desired result by using Lemma 2.1 (1) and the monotone convergence theorem. □

For the supercritical case, we can obtain the following information about the geometry of the infinite open cluster, which is an extension of Theorem 8.99 in [4].

Theorem 3.2. *Suppose that $0 < p_h$, $p_v \leq 1$ and $p_h + p_v > 1$. Let I be the infinite cluster and I_e be the edges both of whose endpoints are in I. Then, as $n \to \infty$,*

$$\frac{|\Delta I \cap S(n)|_h}{|I_e \cap S(n)|_h} \to \frac{1-p_h}{p_h} \text{ and } \frac{|\Delta I \cap S(n)|_v}{|I_e \cap S(n)|_v} \to \frac{1-p_v}{p_v} \text{ a.s.}$$

This can be proved by the argument for Theorem 8.99 in [4] with similar modifications as in the proof of Theorem 3.1.

These theorems have a little weaker version for higher-dimensional cases, proved by a similar method.

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Department of Mathematics
Faculty of Science and Technology
Keio University
3-14-1, Hiyoshi
Kohoku-ku, Yokohama 223-8522
Japan
e-mail: takei@math.keio.ac.jp

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