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On traversable length inside semi-cylinder in 2d supercritical bond percolation

By

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Abstract

We investigate a limit theorem on traversable length inside semi-cylinder in the 2-dimensional supercritical Bernoulli bond percolation, which gives an extension of Theorem 2 in [5]. This type of limit theorems was originally studied for the extinction time for the 1-dimensional contact process on a finite interval in [10]. Actually, our main result Theorem 2.1 is stated under a rather general 2-dimensional bond percolation setting.

1. Introduction

Grimmett [5] proved that traversable length by open paths has logarithmic scale in the 2-dimensional subcritical Bernoulli bond percolaiton. By the self-duality, this assertion is equivalent to that exponential scale length is traversable by open paths in the 2-dimensional supercritical Bernoulli bond percolation. More precisely, the supercritical version of the assertion is the following limit theorem: For $p > 1/2$,

$$\begin{aligned}\lim_{N\to\infty} \mathcal{P}_p \left(\begin{array}{c} \text{there exists some crossing open path} \\ \text{from the bottom to the top in } T(e^{aN}, N) \end{array} \right) \\ = \begin{cases} 1 & \text{if } a < \alpha(1-p), \\ 0 & \text{if } a > \alpha(1-p), \end{cases}\end{aligned} \tag{1.1}$$

where $T(M,N) = \{(x_1,x_2) \in \mathbb{Z}^2 : 1 \le x_1 \le N,\ 0 \le x_2 \le M\}$ for $M, N \in (1,\infty)$ and

$$\alpha(r) = -\lim_{N\to\infty} \frac{1}{N} \log \mathcal{P}_r \left(\begin{array}{c} (0,0) \text{ is connected to} \\ \text{the vertical line } x_1 = N \text{ by open paths} \end{array} \right) \tag{1.2}$$

for $r \in (0,1]$. A similar result as in the subcritical case was obtained by Higuchi [8] for a class of site percolation in (strongly) mixing random fields on the d-dimensional lattice. One of typical examples is the 2-dimensional Ising percolation in high temperature phase without external magnetic fields.

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In the 1-dimensional contact process, same types of limit theorems for the extinction time of the process on a finite interval were proved in subcritical region (see [3]) and supercritical region (see [4]). A planar graph duality in the graphical representation for the contact process plays a central role in [4]. For this reason, the method in [4] was applied to the Bernoulli bond percolaiton to give another proof of (1.1). Further, Durrett and Schonmann [4] obtained that the traversable length $\sigma_N/\mathcal{P}_p[\sigma_N]$ scaled by its mean converges to a mean one exponential distribution in the sense of weak convergence, where

$$\sigma_N = \sup\left\{x_2 \in \mathbb{N} : \begin{array}{c}\text{there exists some open path}\\ \text{from } I_N \text{ to } (x_1, x_2) \text{ in } [1, N] \times [0, \infty)\end{array}\right\}$$

and $I_N = \{1, \ldots, N\} \times \{0\}$. As for higher dimensional versions of these types of limit theorems, we refer to Part I of [9]. Chen, Liu, and Zhang [2] carried out similar analysis of reversible nearest neighbor particle systems.

On the basis of the argument in [4], Wagner and Anantharam [10] studied the extinction time σ_N^{CP} for the 1-dimensional contact process with piecewise homogeneous birth rates and an identical death rate on a finite interval. The precise definition is as follows: Let the death rates for all vertices be identically equal to the normalized rate 1. Divide the interval $[1, N]$ into K intervals $I_{N,i}$'s with length $k_i N$'s. For every interval $I_{N,i}$, the birth rates for all vertices in $I_{N,i} \cap \mathbb{Z}$ are assumed to be equal to λ_i. One of results in [10] affirms that if all λ_i's are larger than the critical point λ_c of the (original) 1-dimensional contact process,

$$\lim_{N\to\infty} \mathbb{P}_N^{\mathrm{CP}}\left(\left|\frac{\log \sigma_N^{\mathrm{CP}}}{N} - \sum_{i=1}^{K} k_i \gamma^{\mathrm{CP}}(\lambda_i)\right| > \delta\right) = 0 \tag{1.3}$$

holds for any $\delta > 0$, where

$$\gamma^{\mathrm{CP}}(\lambda) = -\lim_{L\to\infty} \frac{1}{L} \log \mathcal{P}_\lambda^{\mathrm{CP}}\left(\begin{array}{c}\text{the 1-dimensional contact process with } L \text{ initial}\\ \text{particles on } \{1, \ldots, L\} \text{ eventually extincts}\end{array}\right).$$

In this paper, we consider a similar type of limit theorems as (1.3) for a class of 2-dimensional bond percolation models with the exponential decay of dual connectivity (DC) and the ratio weak mixing (RWM). Especially, (DC) is a more important notion since dual models of 2-dimensional bond percolation are also 2-dimensional bond percolation. The self-duality holds for the (infinite volume) random-cluster model with parameters (q, p) (which is the Bernoulli bond percolation when $q = 1$) in 2-dimensions. Using this property, (DC) is proved for the random-cluster model with $q = 1, 2$ and large enough q in whole subcritical region. It is also believed that the random-cluster model with $q \geq 1$ has (DC) in whole subcritical region. In addition, for the random-cluster model with $q \geq 1$, (RWM) also follows from (DC) (see Theorem 3.4 and Remark 3.5 in [1]).

The rest of this paper is structured as follows. Our results and some definitions are described in Section 2. Section 3 is devoted to the proof of Theorem 2.1 which goes along the same way as in [10] except for using (DC) and (RWM) instead of the independence property. Section 4 is devoted to the proofs of Theorems 2.2, 2.3, and 2.4.

2. Results

2.1. Main result

Let $\mathbb{E}_\Lambda = \{\{x,y\} : x, y \in \Lambda \text{ such that } |x-y|_1 = 1\}$ for every $\Lambda \subset \mathbb{Z}^2$ and $\mathbb{E} = \mathbb{E}_{\mathbb{Z}^2}$, where $|x-y|_1$ means the l_1-distance between x and y. We take as state space the set $\{0,1\}^{\mathbb{E}}$ and denote $(\omega_b)_{b\in\mathbb{E}} \in \{0,1\}^{\mathbb{E}}$ by ω. For $\omega \in \{0,1\}^{\mathbb{E}}$, we declare a bond $b \in \mathbb{E}$ to be *open* (resp. *closed*) (*in* ω) if $\omega_b = 1$ (resp. $\omega_b = 0$). Let (b_i) be a finite or an infinite sequence of bonds such that $b_i \neq b_j$ if $i \neq j$. We call such (b_i) a *path* if $b_1 \cap b_2 \neq \emptyset$ and $(b_i \backslash b_{i-1}) \cap b_{i+1} \neq \emptyset$ for all $i \geq 2$. We call a path (b_i) an *open path* (*in* ω) if all bonds b_i's are open. For $x, y \in \mathbb{Z}^2$ and $n \in \mathbb{N}$, we call a path $(b_i)_{i=1}^n$ an *open path from* x *to* y if $(b_i)_{i=1}^n$ is an open path such that $x \in b_1 \backslash b_2$ and $y \in b_n \backslash b_{n-1}$. For $\Delta, \Lambda \subset \mathbb{Z}^2$ and $n \in \mathbb{N}$, we call a path $(b_i)_{i=1}^n$ an *open path from* Δ *to* Λ if $(b_i)_{i=1}^n$ is an open path from x to y for some $x \in \Delta$ and $y \in \Lambda$. We denote by $\{\Delta \longleftrightarrow \Lambda\}$ the event where such a path exists. For $\Delta \subset \mathbb{Z}^2$, we define $\{\Delta \longleftrightarrow \infty\}$ as the event that there exists some infinite open path $(b_i)_{i=1}^\infty$ with $x \in b_1 \backslash b_2$ for some $x \in \Delta$. In notation below, we often replace $\{x\}$ with x. For $\Lambda \subset \mathbb{R}^2$, we call a path (b_i) a *path in* Λ when $b_i \in \mathbb{E}_{\Lambda\cap\mathbb{Z}^2}$ for all i. Similarly, we add '*in* Λ' to the other terminologies above. Let b^* denote the dual bond of $b \in \mathbb{E}$. We declare the dual bond b^* to be open if and only if b is closed. We define a *dual open path* and some related notions in a similar way as in the case of an open path. For sets Δ and Λ of the dual lattice $(\mathbb{Z}^2)^*$, we denote by $\{\Delta \overset{*}{\longleftrightarrow} \Lambda\}$ the event that there exists some dual open path from x to y for some $x \in \Delta$ and $y \in \Lambda$.

Let Φ be a 2-dimensional bond percolation model, which is a probability measure on $\{0,1\}^{\mathbb{E}}$. For every $\Lambda \subset \mathbb{Z}^2$, let $\mathcal{F}_\Lambda$ denote the σ-field generated by $\{\omega_b : b \in \mathbb{E}_\Lambda\}$. We say that Φ possesses the *bounded energy property* (BE) if there exists some $r \in (0,1)$ such that for any $b \in \mathbb{E}$,

$$r \leq \Phi(\omega_b = 1 \mid \mathcal{F}_{b^c}) \leq 1 - r.$$

We say that Φ satisfies the *exponential decay of dual connectivity property* (DC) if for some $\zeta, C \in (0,\infty)$ and any $x, y \in (\mathbb{Z}^2)^*$,

$$\Phi(x \overset{*}{\longleftrightarrow} y) \leq Ce^{-\zeta|x-y|_1}.$$

We say that Φ satisfies the *ratio weak mixing property* (RWM) if there exist some $c, C \in (0,\infty)$ such that for any $\Delta, \Lambda \subset \mathbb{Z}^2$ with $\Delta \cap \Lambda = \emptyset$,

$$\sup\left\{\left|\frac{\Phi(A\cap B)}{\Phi(A)\Phi(B)} - 1\right| : A \in \mathcal{F}_\Delta,\ B \in \mathcal{F}_\Lambda,\ \text{and}\ \Phi(A)\Phi(B) > 0\right\}$$
$$\leq C \sum_{x\in\Delta,\, y\in\Lambda} e^{-c|x-y|_1}.$$

For $r \in \mathbb{R}$, we denote by $\lceil r \rceil$ and $\lfloor r \rfloor$ the smallest integer larger than r and the largest integer smaller than or equal to r, respectively. Let $\mathbb{R}_+ = [0, \infty)$ and $I_N = \{1, \dots, N\} \times \{0\}$. We consider

$$\sigma_N = \sup\{n \in \mathbb{N} : I_N \longleftrightarrow \{1, \dots, N\} \times \{n\} \text{ in } [1, N] \times \mathbb{R}_+\} \tag{2.1}$$

in the following bond percolation model $\mathbb{P}_N$: Let $k_1, \dots, k_K > 0$ with $k_1 + \cdots + k_K = 1$ for a fixed $K \in \mathbb{N}$. Define $l_0 = 0$ and $l_i = k_1 + \cdots + k_i$ for every $1 \le i \le K$.

(P1) For every $1 \le i \le K$,

$$\mathbb{P}_N(\cdot \mid \mathcal{F}_{\Lambda^c}) = \Phi_i(\cdot \mid \mathcal{F}_{\Lambda^c}) \quad \mathbb{P}_N\text{-a.s.}$$

for any finite $\Lambda \subset \big([\lceil l_{i-1}N \rceil, \lfloor l_i N \rfloor] \times \mathbb{R}\big) \cap \mathbb{Z}^2$.

(P2) $\mathbb{P}_N$ satisfies the FKG inequality.

(P3) $\mathbb{P}_N$ satisfies (DC).

Here, for every $1 \le i \le K$, bond percolation model Φ_i is assumed to possess the translation invariance, the FKG inequality, (BE), (DC) and (RWM). Notice that for a fixed $K \in \mathbb{N}$, the constants in (BE), (DC), and (RWM) for Φ_i's can be uniformly chosen, respectively.

Define

$$\gamma_i = -\lim_{N \to \infty} \frac{1}{N} \log \Phi_i \left(\{I_N \longleftrightarrow \infty \text{ in } \mathbb{R} \times \mathbb{R}_+\}^c\right). \tag{2.2}$$

Existence of the above limit follows from the subadditive argument together with the FKG inequality. Further, by (BE) and (DC),

$$\gamma_i \in (0, \infty).$$

Theorem 2.1. *For any $k_1, \dots, k_K > 0$ with $k_1 + \cdots + k_K = 1$ and $\delta > 0$,*

$$\mathbb{P}_N\left(\left|\frac{\log \sigma_N}{N} - \sum_{i=1}^{K} k_i \gamma_i\right| > \delta\right) \to 0 \tag{2.3}$$

as N goes to infinity.

2.2. Independent bond percolation

A probability measure $\mathbb{P}$ on $\{0,1\}^{\mathbb{E}}$ is said to be *independent bond percolation* if every bond becomes open independently of all the other bonds. Theorem 2.1 immediately leads the following corollary:

Corollary 2.1. *Consider the Bernoulli bond percolation $\mathcal{P}_p$. Suppose that $\Phi_i = \mathcal{P}_{p_i}$ with $p_i \in (1/2, 1)$ for every $1 \le i \le K$. Then, for any $k_1, \dots, k_K > 0$ with $k_1 + \cdots + k_K = 1$ and $\delta > 0$,* (2.3) *holds for independent bond percolation $\mathbb{P}_N$'s with* (P1).

Remark 1. Let

$$\gamma(p) = -\lim_{N\to\infty} \frac{1}{N} \log \mathcal{P}_p\left(\{I_N \longleftrightarrow \infty \text{ in } \mathbb{R}\times\mathbb{R}_+\}^c\right)$$

for $p \in [0,1)$. Comparing the case where $K = 1$ in Corollary 2.1 with (1.1) (obtained by Grimmett in [5]), we can see that $\gamma(p) = \alpha(1-p)$ as Durrett and Schonmann [4] pointed out. Here, $\alpha(\cdot)$ is the function in (1.2). According to Theorems 6.10 and 6.14 in [6], $\alpha(\cdot)$ is continuous on $(0,1]$.

For $b \in \mathbb{E}$ with $b = \{(x_1,x_2),(y_1,y_2)\}$, let $X(b) = \min\{x_1,y_1\}$. Consider a sequence $\{K_N\}_{N\in\mathbb{N}}$ of positive integers and let $l_i^{(N)} = i/K_N$ for every $1 \le i \le K_N$. Define $\{\mathrm{Cyl}_N(i)\}_{i=1}^{K_N}$ as follows: $\mathrm{Cyl}_N(1) = \left(-\infty, \lfloor l_1^{(N)} N\rfloor\right] \times \mathbb{R}$, $\mathrm{Cyl}_N(K_N) = \left[\lceil l_{K-1}^{(N)} N\rceil, \infty\right) \times \mathbb{R}$, and $\mathrm{Cyl}_N(i) = \left[\lceil l_{i-1}^{(N)} N\rceil, \lfloor l_i^{(N)} N\rfloor\right] \times \mathbb{R}$ for every $2 \le i \le K_N - 1$.

Theorem 2.2. *Consider the Bernoulli bond percolation $\mathcal{P}_p$ and a continuous function $\rho : [0,1] \to (1/2,1)$. Take a sequence $\{K_N\}_{N\in\mathbb{N}}$ of positive integers such that as N goes to infinity, $K_N \to \infty$, and $L_N^m/N \to \infty$ for some $m > 1$, where $L_N = N/K_N$. Let $p_i^{(N)} = \rho(l_i^{(N)})$ for every $1 \le i \le K_N$. Define $\mathbb{P}_N$ as independent bond percolation such that density of edge b is $p_i^{(N)}$ if $X(b) \in \mathrm{Cyl}_N(i)$. Then, for any $\delta > 0$,*

$$\mathbb{P}_N\left(\left|\frac{\log \sigma_N}{N} - \int_0^1 \gamma(\rho(u))du\right| > \delta\right) \to 0 \tag{2.4}$$

as N goes to infinity. Here, $\gamma(\cdot)$ is the function in Remark 1.

2.3. Random-cluster models

Let $q \ge 1$ throughout this paper. Let $\omega, \xi \in \{0,1\}^{\mathbb{E}}$ and $\Lambda \subset \mathbb{Z}^2$. A connected component of the graph $(\mathbb{Z}^2, \{b \in \mathbb{E} : \omega_b = 1\})$ is called a *cluster* (*in* ω). The number of clusters intersecting Λ is denoted by $k(\omega,\Lambda)$. Let $\omega_\Lambda\xi$ denote the bond configuration such that $(\omega_\Lambda\xi)_b = \omega_b$ if $b \in \mathbb{E}_\Lambda$ and $(\omega_\Lambda\xi)_b = \xi_b$ otherwise. For a finite set $\Lambda \subset \mathbb{Z}^2$ and $p \in [0,1]$, the finite volume random-cluster measure $\Phi^\xi_{\Lambda,p,q}$ on $\{0,1\}^{\mathbb{E}_\Lambda}$ with the boundary condition ξ is given by

$$\Phi^\xi_{\Lambda,p,q}(\omega) = \frac{1}{Z^\xi_\Lambda(p,q)}\left(\prod_{b\in\mathbb{E}_\Lambda} p^{\omega_b}(1-p)^{1-\omega_b}\right) q^{k(\omega_\Lambda\xi,\Lambda)} \quad \text{for } \omega \in \{0,1\}^{\mathbb{E}_\Lambda},$$

where $Z^\xi_\Lambda(p,q)$ is the normalizing constant.

Taking the thermodynamic limit, there exist the infinite volume random-cluster measures $\Phi^w_{p,q}$ and $\Phi^f_{p,q}$ corresponding to the wired boundary condition $\xi \equiv 1$ and the free one $\xi \equiv 0$, respectively. The percolation threshold $p_c(q)$ is defined by

$$\begin{aligned} p_c(q) &= \inf\{p \in [0,1] : \Phi^w_{p,q}(O \longleftrightarrow \infty) > 0\} \\ &= \inf\{p \in [0,1] : \Phi^f_{p,q}(O \longleftrightarrow \infty) > 0\}, \end{aligned}$$

where O indicates the origin of $\mathbb{Z}^2$ (see Sections 4 and 5 in [7]).

Remark 2. (i) Let $p_{sd}(q) = \sqrt{q}/(1+\sqrt{q})$. It holds that $p_c(q) \geq p_{sd}(q)$ and there exists a unique infinite volume random-cluster measure $\Phi_{p,q}$ for $p \neq p_{sd}(q)$. Further, $p_c(q) = p_{sd}(q)$ when $q = 1, 2$ and $q \geq 25.72$ (see Sections 6.2 and 6.4 in [7]).

(ii) If $q = 1, 2$ or $q \geq 25.72$ and $p > p_c(q)$, (DC) holds for the infinite volume random-cluster measure $\Phi_{p,q}$. For sufficiently large $p > p_c(q)$, (DC) holds for the infinite volume random-cluster measure $\Phi_{p,q}$ (see Sections 6.2 and 6.4 in [7]).

(iii) In the infinite volume random-cluster measure $\Phi_{p,q}$ with $p \neq p_{sd}(q)$, (DC) implies (RWM) (see Theorem 3.4 and Remark 3.5 in [1]).

Recall that $X(b) = \min\{x_1, y_1\}$ for $b \in \mathbb{E}$ with $b = \{(x_1, x_2), (y_1, y_2)\}$. Let us fix $K \in \mathbb{N}$ and $k_1, \ldots, k_K > 0$ with $k_1 + \cdots + k_K = 1$. For $p_1, \ldots, p_K \in [0, 1]$ and $N \in \mathbb{N}$, let $\mathcal{R}_N(p_1, \ldots, p_K; q)$ denote the set of all infinite volume random-cluster measures defined by the DLR equation which possess a cluster-weight q and an edge-weight p_i for every edge b with $X(b) \in \mathrm{Cyl}(i)$, where $\mathrm{Cyl}(1) = \big(-\infty, \lfloor l_1 N \rfloor\big] \times \mathbb{R}$, $\mathrm{Cyl}(K) = \big[\lceil l_{K-1} N \rceil, \infty\big) \times \mathbb{R}$, and $\mathrm{Cyl}(i) = \big[\lceil l_{i-1} N \rceil, \lfloor l_i N \rfloor\big] \times \mathbb{R}$ for every $2 \leq i \leq K-1$.

Theorem 2.3. *Consider the infinite volume random-cluster measure $\Phi_{p,q}$ for $p \in (p_c(q), 1)$. For every $1 \leq i \leq K$, suppose that $p_i > p_c(q)$ and $\Phi_i = \Phi_{p_i,q}$ satisfies* (DC). *Then, the set $\mathcal{R}_N(p_1, \ldots, p_K; q)$ is nonempty for any $k_1, \ldots, k_K > 0$ with $k_1 + \cdots + k_K = 1$ and $N \in \mathbb{N}$. Moreover,* (2.3) *holds for any $\delta > 0$ if $\mathbb{P}_N \in \mathcal{R}_N(p_1, \ldots, p_K; q)$ for all $N \in \mathbb{N}$.*

Theorem 2.4. *Consider the infinite volume random-cluster measure $\Phi_{p,q}$ for $p \in (p_c(q), 1)$. Suppose that $p_i > p_c(q)$ and $\Phi_{p_i,q}$ satisfies* (DC) *for every $1 \leq i \leq K$. Consider the semi-cylindrical random-cluster measure $\mathbb{P}^w_{N,cyl}$ corresponding to the wired boundary condition such that its cluster-weight is q and for every $1 \leq i \leq K$, its edge-weight is p_i for every edge b with $X(b) \in \big[\lceil l_{i-1} N \rceil, \lfloor l_i N \rfloor\big] \times \mathbb{R}_+$. Then, for any $k_1, \ldots, k_K > 0$ with $k_1 + \cdots + k_K = 1$ and $\delta > 0$,* (2.3) *holds for $\mathbb{P}^w_{N,cyl}$'s.*

3. Proof of Theorem 2.1

Although we can prove Theorem 2.1 along the line in [10] by using (DC) and (RWM) instead of independency, for self-consistency we will give its full proof. We write $\bar{\gamma} = \sum_{i=1}^K k_i \gamma_i$. We sometimes omit the index i from the notation.

3.1. Upper bound

We will show that for any $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}_N \left(\frac{\log \sigma_N}{N} > \bar{\gamma} + \delta \right) = 0. \tag{3.1}$$

For $M \in \mathbb{N}$, define

$$A_N^M = \{I_N \longleftrightarrow \mathbb{Z} \times \{M\} \text{ in } \mathbb{R} \times [0, M]\}^c ,$$
$$B_N = \left\{ \left(\frac{1}{2}, \frac{1}{2}\right) \stackrel{*}{\longleftrightarrow} \left(N + \frac{1}{2}, \frac{1}{2}\right) \text{ in } R_N^+ \right\},$$

and

$$B_N^M = \left\{ \left(\frac{1}{2}, \frac{1}{2}\right) \stackrel{*}{\longleftrightarrow} \left(N + \frac{1}{2}, \frac{1}{2}\right) \text{ in } R_N^+(M) \right\},$$

where $R_N^+ = [1/2, N + (1/2)] \times [1/2, \infty)$ and $R_N^+(M) = [1/2, N + (1/2)] \times [1/2, M - (1/2)]$. For every $1 \leq i \leq K$, the following three limits exist as in the case of γ_i (see (2.2)):

$$\gamma_i^M = -\lim_{N \to \infty} \frac{1}{N} \log \Phi_i(A_N^M),$$
$$\mu_i = -\lim_{N \to \infty} \frac{1}{N} \log \Phi_i(B_N),$$

and

$$\mu_i^M = -\lim_{N \to \infty} \frac{1}{N} \log \Phi_i(B_N^M).$$

Lemma 3.1. *For every* $1 \leq i \leq K$,

$$\gamma_i = \mu_i = \lim_{M \to \infty} \mu_i^M .$$

Proof. Note that γ^M and μ^M are decreasing in M. By the definitions of γ and γ^M,

$$\begin{aligned} \exp(-(\gamma + \varepsilon)N) &\leq \Phi\left(\{I_N \longleftrightarrow \infty \text{ in } \mathbb{R} \times \mathbb{R}_+\}^c\right) \\ &= \lim_{M \to \infty} \Phi(A_N^M) \\ &\leq \lim_{M \to \infty} \exp(-\gamma^M N) \end{aligned}$$

for any $\varepsilon > 0$ and sufficiently large N, which together with $\gamma \leq \gamma^M$ implies that $\gamma = \lim_{M \to \infty} \gamma^M$. Similarly, $\mu = \lim_{M \to \infty} \mu^M$. Further, $\gamma^M = \mu^M$ for any $M \in \mathbb{N}$ since

$$r^2 \Phi(B_N^M) \leq \Phi(A_N^M) \leq r^{-2M} \Phi(B_N^M)$$

follow from the FKG inequality and (BE). Thus, $\mu = \gamma$ holds. □

Proof of the upper bound (3.1). By Lemma 3.1, there exists some integer M such that for all $1 \leq i \leq K$,

$$\mu_i^M \leq \gamma_i + (\delta/6). \tag{3.2}$$

Take a positive η satisfying

$$\eta \le \min\left\{ \frac{\delta}{6K\log(1/r)},\ \frac{1}{3}\min\{k_1,\dots,k_K\} \right\}.$$

Let us fix such η and M. For every $1 \le i \le K$, consider

$$B_{N,i} = \left\{ \begin{array}{c} \left(\lceil (l_{i-1}+\eta)N\rceil - \frac{1}{2}, \frac{1}{2}\right) \overset{*}{\longleftrightarrow} \left(\lfloor (l_i-\eta)N\rfloor - \frac{1}{2}, \frac{1}{2}\right) \\ \text{in } \left[\lceil l_{i-1}N\rceil + \frac{1}{2}, \lfloor l_i N\rfloor - \frac{1}{2}\right] \times \left[\frac{1}{2}, M-\frac{1}{2}\right] \end{array} \right\}$$

and

$$F_N = \{\text{all dual bonds in } \mathbb{B}_N \text{ are open}\},$$

where

$$\begin{aligned} \mathbb{B}_N = &\left\{ \left\{ \left(j-\frac{1}{2},\frac{1}{2}\right), \left(j+\frac{1}{2},\frac{1}{2}\right) \right\} : 1 \le j \le \lfloor \eta N\rfloor \text{ or } \lceil (1-\eta)N\rceil \le j \le N \right\} \\ &\cup \bigcup_{i=1}^{K-1} \left\{ \left\{ \left(j-\frac{1}{2},\frac{1}{2}\right), \left(j+\frac{1}{2},\frac{1}{2}\right) \right\} : \lceil (l_i-\eta)N\rceil \le j \le \lfloor (l_i+\eta)N\rfloor \right\}. \end{aligned}$$

By the FKG inequality, (P1), (RWM) and (3.2),

$$\begin{aligned} \mathbb{P}_N\left(B_N^M\right) &\ge \mathbb{P}_N(F_N)\prod_{i=1}^K \mathbb{P}_N\left(B_{N,i}\right) \\ &\ge \frac{1}{2} r^{2K\eta N}\prod_{i=1}^K \Phi_i(B_{N,i}) \ge \frac{1}{2}\exp\left(-\left\{\bar{\gamma} + (2\delta/3)\right\}N\right) \end{aligned} \tag{3.3}$$

for sufficiently large N. Let $H = \{x \in \mathbb{Z}^2 : x_2 = iM \text{ for some } i \in \mathbb{N}\}$. By comparing σ_N with σ_N conditioned by the event that all bonds in $\mathbb{E}_H$ are open, it is not difficult to see that for any $l \in \mathbb{N}$,

$$\mathbb{P}_N(\sigma_N > l) \le \mathbb{P}_N(\sigma_N \ge M)^{\lfloor l/M\rfloor} \le \left(1 - \mathbb{P}_N\left(B_N^M\right)\right)^{\lfloor l/M\rfloor}. \tag{3.4}$$

From (3.3) and (3.4), we can conclude (3.1). □

3.2. Lower bound

Because of (3.1), we obtain Theorem 2.1 once we can prove that for any $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}_N\left(\frac{\log \sigma_N}{N} < \bar{\gamma} - \delta\right) = 0. \tag{3.5}$$

Let

$$C_N = \left\{ \text{for some } k \in \mathbb{Z},\ \left(\frac{1}{2},\frac{1}{2}\right) \overset{*}{\longleftrightarrow} \left(N+\frac{1}{2}, k+\frac{1}{2}\right) \text{ in } R_N \right\},$$

where $R_N = [1/2, N+(1/2)] \times \mathbb{R}$. By Proposition 3.1 mentioned below, we can see that for any $l \in \mathbb{N}$ and sufficiently large N,

$$\mathbb{P}_N(\sigma_N < l) \leq (l+1)\mathbb{P}_N(C_N) \leq (l+1)\exp\left(-\left\{\bar{\gamma} - \frac{\delta}{2}\right\} N\right),$$

which implies (3.5).

Proposition 3.1.

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}_N\left(C_N\right) = -\bar{\gamma}. \tag{3.6}$$

We prepare some notation and lemmas to prove (3.6). Let $R_N(M) = [1/2, N+(1/2)] \times [-M+(1/2), M-(1/2)]$ for $M \in \mathbb{N}$. Define

$$C_N^M = \left\{\text{for some } k \in \mathbb{Z}, \left(\frac{1}{2}, \frac{1}{2}\right) \overset{*}{\longleftrightarrow} \left(N+\frac{1}{2}, k+\frac{1}{2}\right) \text{ in } R_N(M)\right\},$$
$$D_N = \left\{\text{for some } j,k \in \mathbb{Z}, \left(\frac{1}{2}, j+\frac{1}{2}\right) \overset{*}{\longleftrightarrow} \left(N+\frac{1}{2}, k+\frac{1}{2}\right) \text{ in } R_N\right\},$$
$$D_N^M = \left\{\text{for some } j,k \in \mathbb{Z}, \left(\frac{1}{2}, j+\frac{1}{2}\right) \overset{*}{\longleftrightarrow} \left(N+\frac{1}{2}, k+\frac{1}{2}\right) \text{ in } R_N(M)\right\},$$
$$E_N = \left\{\left(\frac{1}{2}, \frac{1}{2}\right) \overset{*}{\longleftrightarrow} \left(N+\frac{1}{2}, \frac{1}{2}\right) \text{ in } R_N\right\}$$

and

$$E_N^M = \left\{\left(\frac{1}{2}, \frac{1}{2}\right) \overset{*}{\longleftrightarrow} \left(N+\frac{1}{2}, \frac{1}{2}\right) \text{ in } R_N(M)\right\}.$$

Lemma 3.2. *For every* $1 \leq i \leq K$,

$$\begin{aligned} \sup_{N\in\mathbb{N}} \frac{1}{N} \log \Phi_i(E_N) &= \lim_{N\to\infty} \frac{1}{N} \log \Phi_i(E_N) \\ &= \lim_{M\to\infty} \lim_{N\to\infty} \frac{1}{N} \log \Phi_i(E_N^M) = -\gamma_i. \end{aligned} \tag{3.7}$$

Proof. Note that

$$\begin{aligned} B_N^{2M} \supset &\left\{\left(\frac{1}{2}, M+\frac{1}{2}\right) \overset{*}{\longleftrightarrow} \left(N+\frac{1}{2}, M+\frac{1}{2}\right) \text{ in } R_N(M)+(0,M)\right\} \\ &\cap \{\text{all dual bonds in } \mathbb{B}_N \text{ are open}\}, \end{aligned}$$

where

$$\begin{aligned} \mathbb{B}_N = &\left\{\left\{\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{3}{2}, \frac{1}{2}\right)\right\}, \left\{\left(N-\frac{1}{2}, \frac{1}{2}\right), \left(N+\frac{1}{2}, \frac{1}{2}\right)\right\}\right\} \\ &\cup \left\{\left\{\left(\beta, j-\frac{1}{2}\right), \left(\beta, j+\frac{1}{2}\right)\right\} : 1 \leq j \leq M \text{ and } \beta = \frac{3}{2}, N-\frac{1}{2}\right\}. \end{aligned}$$

Then, by the FKG inequality and (BE),

$$\lim_{N\to\infty} \frac{1}{N} \log \Phi_i(B_N^{2M}) \geq \lim_{N\to\infty} \frac{1}{N} \log \Phi_i(E_N^M)$$

for any $M \in \mathbb{N}$. From this and the fact that $B_N^M \subset E_N^M$,

$$\lim_{N\to\infty} \frac{1}{N} \log \Phi_i(B_N^M) = \lim_{N\to\infty} \frac{1}{N} \log \Phi_i(E_N^M)$$

for any $M \in \mathbb{N}$. Thus, we can obtain (3.7) as in the proof of Lemma 3.1. □

Lemma 3.3. *For every* $1 \leq i \leq K$,

$$\lim_{N\to\infty} \frac{1}{N} \log \Phi_i(C_N) = -\gamma_i. \tag{3.8}$$

Proof. By Lemma 3.2 and the fact that $E_N \subset C_N$,

$$-\gamma \leq \liminf_{N\to\infty} \frac{1}{N} \log \Phi(C_N). \tag{3.9}$$

By (DC),

$$\Phi(C_N \backslash C_N^M) \leq 2CNe^{-\zeta M} \tag{3.10}$$

for any $M \in \mathbb{N}$. From (3.9) and (3.10),

$$\limsup_{N\to\infty} \frac{1}{N} \log \Phi(C_N) = \limsup_{N\to\infty} \frac{1}{N} \log \Phi(C_N^{aN}) \tag{3.11}$$

for $a = \lceil 6\gamma/\zeta \rceil$. Note that

$$\frac{1}{N} \log \Phi(C_N^{aN}) \leq \frac{1}{N} \log(2aN) + \frac{1}{N} \sup_{|k|\leq aN} \log \Phi(C_N(k)), \tag{3.12}$$

where

$$C_N(k) = \left\{ \left(\frac{1}{2}, \frac{1}{2}\right) \overset{*}{\longleftrightarrow} \left(N + \frac{1}{2}, k + \frac{1}{2}\right) \text{ in } R_N \right\}$$

for every $k \in \mathbb{Z}$. Further, Lemma 3.2 maintains

$$\sup_{k\in\mathbb{Z}} \Phi(C_N(k)) \leq e^{-\gamma N}, \tag{3.13}$$

since by the translation invariance and the FKG inequality,

$$\Phi(C_N(k))^2 \leq \Phi(E_{2N})$$

for any $k \in \mathbb{Z}$. From (3.12) and (3.13),

$$\limsup_{N\to\infty} \frac{1}{N} \log \Phi(C_N^{aN}) \leq -\gamma. \tag{3.14}$$

Therefore, (3.8) follows from (3.9), (3.11), and (3.14). □

Lemma 3.4. *For every $1 \le i \le K$,*

$$\limsup_{N\to\infty} \frac{1}{N} \log \Phi_i \left(D_N^{N^2} \right) \le -\gamma_i. \tag{3.15}$$

Proof. The fact that $\Phi(D_N^{N^2}) \le (2N^2+1)\Phi(C_N)$ and Lemma 3.3 immediately show (3.15). □

Remark 3. This lemma together with Lemma 3.2 means that in (3.15), the upper limit and the inequality can be replaced with limit and equality, respectively.

Proof of Proposition 3.1. From (3.3),

$$\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}_N \left(C_N \right) \ge \bar{\gamma}. \tag{3.16}$$

For every $\eta > 0$ as in the proof of (3.1) and $1 \le i \le K$, let

$$N_i = \lceil (l_i - \eta) N \rceil - \lfloor (l_{i-1} + \eta) N \rfloor.$$

Define

$$D_{N,i} = \left\{ \begin{array}{c} \text{for some } j,k \in \mathbb{Z}, \\ \left(\lceil (l_{i-1}+\eta)N \rceil - \frac{1}{2}, j + \frac{1}{2} \right) \overset{*}{\longleftrightarrow} \left(\lceil (l_i - \eta)N \rceil - \frac{1}{2}, k + \frac{1}{2} \right) \\ \text{in } \left[\lceil (l_{i-1}+\eta)N \rceil - \frac{1}{2}, \lceil (l_i - \eta)N \rceil - \frac{1}{2} \right] \times \left[0, N_i^2 - \frac{1}{2} \right] \end{array} \right\}.$$

By (P1), (RWM), and Lemma 3.4,

$$\begin{aligned} &\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}_N \left(C_N^{aN} \right) \\ &\quad \le \limsup_{\eta \searrow 0} \limsup_{N\to\infty} \frac{1}{N} \log \left(2\Phi_1(D_{N,1}) \mathbb{P}_N(D_{N,2} \cap \cdots \cap D_{N,K}) \right) \\ &\quad \le \limsup_{\eta \searrow 0} \sum_{1\le i\le K} \limsup_{N\to\infty} \frac{1}{N} \log \Phi_i(D_{N,i}) \\ &\quad = -\bar{\gamma} \end{aligned} \tag{3.17}$$

for any $a > 0$. Let $a = \lceil 6\gamma/\zeta \rceil$. Then, by (DC) and (3.16),

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}_N \left(C_N \right) = \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}_N \left(C_N^{aN} \right),$$

which together with (3.17) implies that

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}_N \left(C_N \right) \le -\bar{\gamma}. \tag{3.18}$$

This and (3.16) can lead (3.6). □

4. Proofs of Theorems 2.2, 2.3, and 2.4

Proof of Theorem 2.2. Let us fix the integer m as in Theorem 2.2 and an integer M. Let $p_- = \min\{\rho(u) : u \in [0,1]\}$ and $p_+ = \max\{\rho(u) : u \in [0,1]\}$. Note that $1/2 < p_- \le p_+ < 1$. We will show that

$$\frac{1}{N}\log \mathcal{P}_p(D_N^{N^m}) \quad \text{and} \quad \frac{1}{N}\log \mathcal{P}_p(B_N^M) \tag{4.1}$$

are Lipschitz continuous functions in p on $[p_-, p_+]$ uniformly in $N \in \mathbb{N}$, which implies that

$$\lim_{N\to\infty}\sum_{i=1}^{K_N}\gamma(p_i)k_i = \int_0^1 \gamma(\rho(u))du$$

and both terms in (4.1) converge uniformly in $p \in [p_-, p_+]$ as N goes to infinity. Using these facts, we can obtain (2.4) in a similar way as in the proof of Theorem 2.1.

For simplicity, we consider

$$\frac{1}{N}\log \mathcal{P}_p\left(\widetilde{D}_N\right) \quad \text{for } p \in [r_-, r_+],$$

where $r_- = 1 - p_+$, $r_+ = 1 - p_-$ and

$$\widetilde{D}_N = \{\text{for some } j,k \in \mathbb{Z}, (0,j) \longleftrightarrow (N,k) \text{ in } [0,N]\times[0,2N^m-1]\}.$$

Note that $0 < r_- \le r_+ < 1/2$ and $\widetilde{D}_N$ is a local event. Let $\Delta_N = ([0,N] \times [0,2N^m-1]) \cap \mathbb{Z}^2$ and $\mathbb{E}_N = \mathbb{E}_{\Delta_N}$. By abusing notation, $[0,N]\times[0,2N^m-1]$ is also denoted by Δ_N. Let Ω_N indicate the number of open bonds in $\mathbb{E}_N$. By Russo's formula (see Section 2.5 in [6] or Section 2.4 in [7]),

$$\frac{d}{dp}\mathcal{P}_p\left(\widetilde{D}_N\right) = \frac{1}{p(1-p)}\mathrm{cov}_p(\Omega_N, \mathbf{1}_{\widetilde{D}_N}), \tag{4.2}$$

where cov_p means the covariance with respect to $\mathcal{P}_p$ and $\mathbf{1}_{\widetilde{D}_N}$ denotes the indicator function of $\widetilde{D}_N$.

For a set $E \subset \mathbb{E}$, let $|E|$ and ∂E mean the cardinality of E and the set of all boundary bonds of E, respectively. More precisely, ∂E is defined by

$$\partial E = \{e \in \mathbb{E} : e \notin E \text{ and } e \cap b \neq \emptyset \text{ for some } b \in E\}.$$

A set E is said to be *connected* if for any $b, b' \in E$, there exists some path in E which includes both b and b'. Define the open bond cluster $\widetilde{C}_{N,x}$ in Δ_N (containing $x \in \mathbb{Z}^2$) as follows:

$$\widetilde{C}_{N,x} = \{b \in \mathbb{E} : b \text{ is included in some open path from } x \text{ in } \Delta_N\}.$$

On the event $\widetilde{D}_N$, there exists some $\widetilde{C}_{N,(0,j)}$ crossing from the left to the right in Δ_N. Define Γ_N as $\widetilde{C}_{N,(0,j)}$ with the minimal $j \in \{0, \dots, 2N^m - 1\}$ among such $\widetilde{C}_{N,(0,j)}$'s. Then, by the FKG inequality,

$$\begin{aligned}
&\mathrm{cov}_p(\Omega_N, \mathbf{1}_{\widetilde{D}_N}) \\
&\quad = \sum_C \mathcal{P}_p(\Gamma_N = C)\Big(\mathcal{P}_p[\Omega_N \mid \Gamma_N = C] - \mathcal{P}_p[\Omega_N]\Big) \\
&\quad \le \sum_C \mathcal{P}_p(\Gamma_N = C)\Big((|\mathbb{E}_N| - |C| - |\partial C \cap \mathbb{E}_N|)p + |C| - |\mathbb{E}_N|p\Big) \\
&\quad \le (1-p)\sum_{n \ge N} \mathcal{P}_p(|\Gamma_N| \ge n),
\end{aligned} \tag{4.3}$$

where $\sum_C$ stands for the summation over all connected subsets of $\mathbb{E}_N$ crossing from the left to the right in Δ_N. Note that $\mathcal{P}_p(|\Gamma_N| \ge N) \ge p^N \ge r_-^N$ for all $p \in [r_-, r_+]$. Let C_O be the open cluster containing the origin O of $\mathbb{Z}^2$. In the subcritical regime, the cluster size distribution decays exponentially (see Section 6.3 in [6]). This fact together with the FKG inequality implies that for some $A \in (0, \infty)$ and all $p \in [r_-, r_+]$,

$$\begin{aligned}
\sum_{n \ge AN} \mathcal{P}_p(|\Gamma_N| \ge n) &\le 2N^m \sum_{n \ge AN} \mathcal{P}_p(|C_O| \ge \lfloor n/4 \rfloor) \\
&\le 8N^m \sum_{n \ge AN/4} \mathcal{P}_{r_+}(|C_O| \ge n) \\
&\le \mathcal{P}_p(|\Gamma_N| \ge N),
\end{aligned} \tag{4.4}$$

where $|C_O|$ means the cardinality of C_O. From (4.2)–(4.4),

$$\frac{d}{dp}\mathcal{P}_p\left(\widetilde{D}_N\right) \le \frac{2AN}{r_-}\mathcal{P}_p\left(\widetilde{D}_N\right), \tag{4.5}$$

which implies the first term in (4.1) is uniformly Lipschitz continuous in $N \in \mathbb{N}$. As for the second term in (4.1), the proof is similar as above and easier. □

Proof of Theorem 2.3. Let us fix $p_1, \dots, p_K$ and q as in Theorem 2.3 and write $\mathcal{R}_N = \mathcal{R}_N(p_1, \dots, p_K; q)$. By the definition of $\mathcal{R}_N$, (P1) holds for any element $\mathbb{P}_N$ of $\mathcal{R}_N$.

The set of all limit random-cluster measures which possess a cluster-weight q and an edge-weight p_i for every edge b with $X(b) \in \mathrm{Cyl}(i)$ is denoted by $\mathcal{W}_N$. The element of $\mathcal{W}_N$ corresponding to the wired (resp. free) boundary condition is denoted by $\mathbb{P}_N^w$ (resp. $\mathbb{P}_N^f$). Both measures $\mathbb{P}_N^w$ and $\mathbb{P}_N^f$ satisfy the FKG inequality. Further, comparing them with $\Phi_{p_0,q}$ in the FKG sense leads their (DC) property, where $p_0 = \min\{p_1, \dots, p_K\}$. Therefore, there exists a unique infinite cluster almost surely under both $\mathbb{P}_N^w$ and $\mathbb{P}_N^f$, which implies that $\mathbb{P}_N^w, \mathbb{P}_N^f \in \mathcal{R}_N$ (see Section 4.4 in [7]). Thus, $\mathbb{P}_N^w$ and $\mathbb{P}_N^f$ satisfy (P1), (P2), and (P3). This together with Theorem 2.1 maintains that for a certain $\bar{\gamma}$ independent of w and f and any $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}_N^w \left(\left| \frac{\log \sigma_N}{N} - \bar{\gamma} \right| > \delta \right) = \lim_{N\to\infty} \mathbb{P}_N^f \left(\left| \frac{\log \sigma_N}{N} - \bar{\gamma} \right| > \delta \right) = 0.$$

Then, by the FKG inequality,

$$\lim_{N\to\infty} \mathbb{P}_N \left(\frac{\log \sigma_N}{N} > \bar{\gamma} + \delta \right) \leq \lim_{N\to\infty} \mathbb{P}_N^w \left(\frac{\log \sigma_N}{N} > \bar{\gamma} + \delta \right) = 0$$

and

$$\lim_{N\to\infty} \mathbb{P}_N \left(\frac{\log \sigma_N}{N} < \bar{\gamma} - \delta \right) \leq \lim_{N\to\infty} \mathbb{P}_N^f \left(\frac{\log \sigma_N}{N} < \bar{\gamma} - \delta \right) = 0.$$

□

Proof of Theorem 2.4. By the FKG inequality, it is sufficient to prove that for any $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}_{N,cyl}^w \left(\frac{\log \sigma_N}{N} > \bar{\gamma} + \delta \right) = 0. \tag{4.6}$$

In the same way as in (3.3) and (3.4),

$$\begin{aligned} \mathbb{P}_{N,cyl}^w(\sigma_N > l) &\leq \left(1 - \mathbb{P}_{N,cyl}^w \left(B_N^M\right)\right)^{\lfloor l/M \rfloor} \\ &\leq \left(1 - \frac{1}{2}\exp\left(-\left\{\bar{\gamma} + (2\delta/3)\right\} N\right)\right)^{\lfloor l/M \rfloor} \end{aligned}$$

for some $M \in \mathbb{N}$, which implies (4.6). □

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