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Hypoellipticity for a class of kinetic equations

By

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Abstract

In this work, we study a class of operators coming from the linearization of some kinetic equations such as Boltzmann equations and Vlasov-Fokker-Planck equations. Since it is not a standard class of pseudo-differential operators, we obtain hypoelliptic estimates in some weight functions space and show the regularity of weak solutions for linear and semilinear equations.

1. Introduction

Recently the mathematical study of Boltzmann equation without Grad's angular cut-off has been developed from a new point of view in [1], [5], [6], [7], where it is stressed that the nonlinear collision term $Q(f, f)$ behaves essentially as a fractional power of the Laplacian $(-\triangle)^{2\alpha} f$ if the collision kernel has a singularity $\theta^{1-N-2\alpha}$ at the angular $\theta = 0$, where $0 < \alpha < 1$ and N is the number of space dimension (physically equal to 3). The smoothness of the solution for the spatially homogeneous case was fairly well discussed ([6], [7] for example), on the other hand, there seems to be no result in the spatially inhomogeneous case. As an attempt linking to the way to the complete research in the smoothness of solutions to the Cauchy problem for Boltzmann equation, we consider the following kinetic equations

$$Pu = \partial_t u + x \cdot \nabla_y u + \sigma(-\widetilde{\triangle}_x)^\alpha u = f, \tag{1.1}$$

where $(x, y) \in \mathbb{R}^{2n}$ and $0 < \sigma_0 \leq \sigma, \sigma \in C_b^\infty$ Here $(-\widetilde{\triangle}_x)^\alpha = |\widetilde{D}_x|^{2\alpha}$ is a Fourier multiplier with a smooth symbol $|\widetilde{\xi}|^{2\alpha}$, which is equal to $|\xi|^{2\alpha}$ if $|\xi| \geq 2$ and to $|\xi|^2$ if $|\xi| \leq 1$. If $\alpha = 1$, this is a linear Vlasov-Fokker-Planck equation(see [8], [9], cf.[2], [11]), and F. Bouchut [4] has proved the maximal hypoellipticity of operators P with a gain of $2/3$ (see Theorem 1.5 of [4]). When $0 < \alpha < 1$, as stated above, the equation (1.1) is a linearlized model of Boltzmann equation without angular cutoff. Some regularity results (which are restrictive comparing to the case $\alpha = 1$) are also given in [4], for the weak

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solution of equation (1.1) with a supplemental partial regularity with respect to x variable (see Proposition 1.1 and Corollary 1.2 of [4]).

In the present paper, we study the equation (1.1) from the pure analysis point of view, noting that the equation (1.1) is not a classical (pseudo-) partial differential equation because the coefficient x is unbounded and $(-\widetilde{\triangle}_x)$ is not pseudodifferential operator with respect to variables t, y. We first state the existence of the weak solution to the Cauchy problem for equation (1.1) with initial data $u|_{t=0} = u_0$, before considering the hypoellipticity which means the smoothness of a weak solution.

Theorem 1.1. *Assume that $f \in L^1(]0, T[; H^s(\mathbb{R}^{2n}))$ for some $0 < T < \infty, s \geq 0$ and $u_0 \in H^s(\mathbb{R}^{2n})$. If $0 < \alpha < 1$, then the Cauchy problem of equation (1.1) with initial data $u|_{t=0} = u_0$ admits a unique weak solution*

$$u \in L^\infty(]0, T[; H^s(\mathbb{R}^{2n})), \ \ (-\widetilde{\triangle}_x)^{\alpha/2} u \in L^2(]0, T[; H^s(\mathbb{R}^{2n})).$$

For the regularity of weak solution, we have a gain of order $\frac{1}{4}(\alpha - \frac{1}{3})$ with a weight $\langle x \rangle = (1 + |x|^2)^{1/2}$ as follows:

Theorem 1.2. *Let $1/3 < \alpha < 1$ and $f \in H^s(]a, b[\times\mathbb{R}^{2n}))$ for $s \geq 0$. If $u \in L^2(]a, b[\times\mathbb{R}^{2n})$ is a weak solution of equation $Pu = f$ on $]a, b[\times\mathbb{R}^{2n}$, then there exists $k_0 \in \mathbb{N}$ such that*

$$\langle x \rangle^{-k_0 - 1} u \in H^{s + \frac{\alpha}{4} - \frac{1}{12}}(]a', b'[\times\mathbb{R}^{2n}),$$

for any $a < a' < b' < b$. In particular, if $f \in H^\infty(]a, b[\times\mathbb{R}^{2n}))$, then $u \in C^\infty(]a, b[\times\mathbb{R}^{2n}))$.

We remark that k_0 is in order of $[4s(\alpha - \frac{1}{3})^{-1}] + 1$.

Using this linear theorem, we can get the following results for semi-linear Cauchy problems

$$\begin{cases} Pu = F(u) \\ u|_{t=0} = u_0 \end{cases} \tag{1.2}$$

with $F \in C^\infty(\mathbb{R})$ and $F(0) = 0$.

Theorem 1.3. *If $s > n, 1/3 < \alpha < 1$ and $u_0 \in H^s(\mathbb{R}^{2n})$, then there exists a $T > 0$ such that the Cauchy problem (1.2) has a solution.*

$$u \in C^0([0, T[; H^s(\mathbb{R}^{2n})), \ \ (-\widetilde{\triangle}_x)^{\alpha/2} u \in L^2(]0, T[; H^s(\mathbb{R}^{2n})),$$

and

$$u \in H^{+\infty}_{loc}(]0, T[\times\mathbb{R}^{2n}) \subset C^\infty(]0, T[\times\mathbb{R}^{2n}).$$

More precisely, for the regularity we see that for any $m \in \mathbb{N}$ there exists an $m_0 \in \mathbb{N}$ such that

$$\langle x \rangle^{-m_0} u \in H^m(]a, b[\times\mathbb{R}^{2n}),$$

for any $0 < a < b < T$.

This paper is organized as follows: In Section 2, we prove the existence of weak solutions for linear and non linear Cauchy problems. In Section 3, we study the subellipticity of operators P, and get the sub-elliptic regularity for weak solution in Section 4. Finally in Section 5, we prove the smoothness of weak solutions for linear and non linear Cauchy problems.

2. Existence for Cauchy problems

2.1. Linear Cauchy problems

We consider now the linear Cauchy problem (1.1). For the existence of weak solution for linear equation, we follow the idea of proof of Theorem 23.1.2 in [10].

We give now the precise definition of the operator $(-\widetilde{\triangle}_x)^\alpha = |\widetilde{D_x}|^{2\alpha}$ for $\alpha \in \mathbb{R}$, where $|\widetilde{D_x}|^\alpha$ is a Fourier multiplier of symbol $|\xi|^\alpha\chi(\xi) + |\xi|(1-\chi(\xi))$, with $\chi \in C^\infty(\mathbb{R}^n), 0 \le \chi \le 1, \chi(\xi) = 1$ if $|\xi| \ge 2$ and $\chi(\xi) = 0$ if $|\xi| \le 1$. We first study the commutators of this operator with functions in C_b^∞ and unbounded function (the coefficients of our operators P) $x_k, k = 1, \cdots, n$. We give the following technical lemma.

Lemma 2.1. *Let Ω be an open (unbounded) domain of $]T_1, T_2[\times\mathbb{R}^{2n}$, $a \in C_b^\infty(\Omega), \beta \in \mathbb{R}$. Then there exists $C > 0$ depending only on the boundedness of a and their derivation such that*

$$\|[a,\, |\widetilde{D_x}|^\beta]v\|_{L^2(\Omega)} \le C\{\||\widetilde{D_x}|^{(\beta-1)}v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\}, \tag{2.1}$$

for any $v \in C_0^\infty(\Omega)$.

Moreover, we have that $[x_k,\, |\widetilde{D_x}|^\beta]$ is a Fourier multiplier and

$$\|[x_k,\, |\widetilde{D_x}|^\beta]v\|_{L^2(\Omega)} \le |\beta|\||\widetilde{D_x}|^{(\beta-1)}v\|_{L^2(\Omega)} + C\|v\|_{L^2}, \tag{2.2}$$

for any $v \in C_0^\infty(\Omega), k = 1, \cdots, n$.

Proof. Now $|D_x|^\beta\chi(D_x) \in Op(S^\beta_{1,0}(\mathbb{R}^n_x))$, then $[a, |D_x|^\beta\chi(D_x)]$ is a pseudo-differential operators of order $(\beta - 1)$, its principal symbol is

$$\sum_{k=1}^n (\partial_{x_k} a)\,(\beta|\xi|^{\beta-2}\,(i\xi_k)\,\chi(\xi) + |\xi|^\beta i\partial_{\xi_k}\chi(\xi)),$$

which deduces that

$$\|[a, |D_x|^\beta\chi(D_x)]v\|_{L^2} \le C\||\widetilde{D_x}|^{(\beta-1)}v\|_{L^2} + \||D_x|(1-\chi(D_x))(v)\|_{L^2}.$$

For the terms $|D_x|^\beta(1-\chi(D_x))$, we just use the boundedness of a and $|\xi|^\beta(1-\chi(\xi))$ to get the L^2 boundedness. This is the reason why we give (2.2) for unbounded function x_k. Direct calculation gives

$$\begin{aligned} x_k(|\widetilde{D_x}|^\beta v)(x) &= \mathcal{F}^{-1}\Big(D_{\xi_k}\Big(\big(|\xi|^\beta\chi(\xi)+|\xi|(1-\chi(\xi))\big)\hat{v}(\xi)\Big)\Big) \\ &= \mathcal{F}^{-1}\Big(\Big(D_{\xi_k}\big(|\xi|^\alpha\chi(\xi)+|\xi|(1-\chi(\xi))\big)\Big)\,\hat{v}(\xi) \\ &\quad + \big(|\xi|^\alpha\chi(\xi)+|\xi|(1-\chi(\xi))\big)D_{\xi_k}\hat{v}(\xi)\Big) \\ &= \mathcal{F}^{-1}\Big(\Big(D_{\xi_k}\big(|\xi|^\alpha\chi(\xi)+|\xi|(1-\chi(\xi))\big)\Big)\,\hat{v}(\xi)\Big) \\ &\quad + \Big(|\widetilde{D_x}|^\beta(x_k\, v)\Big)(x). \end{aligned}$$

But, for $|\xi|\geq 1$,

$$|D_{\xi_k}|\xi|^\beta| \leq |\beta||\xi|^{\beta-1},$$

and for $|\xi|\leq 2$, $|D_{\xi_k}|\xi||\leq C$. We get that

$$\|[x_k,\, |\widetilde{D_x}|^\beta]v\|_{L^2} \leq |\beta|\||\widetilde{D_x}|^{(\beta-1)}v\|_{L^2} + C\|v\|_{L^2}.$$

We have proved Lemma 2.1. □

Proposition 2.1. *Assume that $s\geq 0$, $f\in L^1(]0,T[;H^s(\mathbb{R}^{2n}))$ for some $0<T<\infty$ and $u_0\in H^s(\mathbb{R}^{2n})$. Suppose that u is a regular solution of Cauchy problem of equation* (1.1) *with initial data $u|_{t=0}=u_0$. Then there exists a $C>0$ such that*

$$\begin{aligned} &\|\Lambda^s_{x,y}u\|^2_{L^\infty(]0,T[;L^2(\mathbb{R}^{2n}))} + \sigma_0\|\Lambda^\alpha_x\Lambda^s_{x,y}u\|^2_{L^2(]0,T[;L^2(\mathbb{R}^{2n}))} \\ &\qquad \leq C\left\{\|\Lambda^s_{x,y}f\|^2_{L^1(]0,T[;L^2(\mathbb{R}^{2n}))} + \|\Lambda^s_{x,y}u_0\|^2_{L^2(\mathbb{R}^{2n})}\right\}, \end{aligned} \tag{2.3}$$

where

$$\Lambda_x = (1+|D_x|^2)^{1/2},\ \Lambda_y = (1+|D_y|^2)^{1/2},\ \Lambda_{x.y} = (1+|D_x|^2+|D_y|^2)^{1/2}.$$

Now as in the proof of Theorem 23.1.2 in [10], The "energy estimate" (2.3) and Hahn-Banach theorem give the existence of solution for the Cauchy problem (1.1):

$$u\in L^\infty(]0,T[;H^s(\mathbb{R}^{2n})),\ \ \Lambda^\alpha_x u\in L^2(]0,T[;H^s(\mathbb{R}^{2n})),$$

we have proved Theorem 1.1.

Proof of Proposition 2.1. If u is a regular solution of Cauchy problem, we have

$$(\partial_t u, \Lambda^{2s}_{x,y}u) + (x\cdot\nabla_y u, \Lambda^{2s}_{x,y}u) + (\sigma(-\widetilde{\Delta}_x)^\alpha u, \Lambda^{2s}_{x,y}u) = (f, \Lambda^{2s}_{x,y}u), \tag{2.4}$$

where $(\cdot,\cdot) = (\cdot,\cdot)_{L^2(\mathbb{R}^n_x\times\mathbb{R}^n_y)}$. Noting that

$$\mathrm{Re}(x\cdot\nabla_y\Lambda^s_{x,y}u, \Lambda^s_{x,y}u)_{L^2(\mathbb{R}^{2n})} = 0$$

and

$$\frac{\sigma_0}{2}\||\widetilde{D}_x|^\alpha \Lambda^s_{x,y} u(t)\|^2_{L^2(\mathbb{R}^{2n})} \le (\sigma(-\widetilde{\Delta}_x)^\alpha \Lambda^s_{x,y} u, \Lambda^s_{x,y} u) + C\|\Lambda^s_{x,y} u(t)\|^2_{L^2(\mathbb{R}^{2n})},$$

we study the commutator terms

$$|([\Lambda^s_{x,y},\, x\cdot\nabla_y]u, \Lambda^s_{x,y} u)_{L^2(\mathbb{R}^{2n})}| \le C\|\Lambda^s_{x,y} u(t)\|^2_{L^2(\mathbb{R}^{2n})},$$

since

$$[\Lambda^s_{x,y},\, x\cdot\nabla_y] = s\Lambda^{s-2}_{x,y}\nabla_x\cdot\nabla_y \in Op(S^s(\mathbb{R}^{2n})).$$

Then, the integration by parts in (2.4) and Cauchy-Schwarz inequality deduce that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Lambda^s_{x,y} u(t)\|^2_{L^2(\mathbb{R}^{2n})} + \sigma_0\||\widetilde{D_x}|^\alpha \Lambda^s_{x,y} u(t)\|^2_{L^2(\mathbb{R}^{2n})}&\\ \le \|\Lambda^s_{x,y} f(t)\|_{L^2(\mathbb{R}^{2n})}\|\Lambda^s_{x,y} u(t)\|_{L^2(\mathbb{R}^{2n})} + C\|\Lambda^s_{x,y} u(t)\|^2_{L^2(\mathbb{R}^{2n})}&.\end{aligned}$$

We have also

$$\begin{aligned}\||\widetilde{D_x}|^\alpha \Lambda^s_{x,y} u(t)\|^2_{L^2(\mathbb{R}^{2n})} &\le \|\Lambda^\alpha_x \Lambda^s_{x,y} u(t)\|^2_{L^2(\mathbb{R}^{2n})}\\ &\le 2^{2\alpha}\left\{\||\widetilde{D_x}|^\alpha \Lambda^s_{x,y} u(t)\|^2_{L^2(\mathbb{R}^{2n})} + \|\Lambda^s_{x,y} u(t)\|^2_{L^2(\mathbb{R}^{2n})}\right\}.\end{aligned}\tag{2.5}$$

Integrating on $[0,t]$ for any $t\in]0,T]$, we have that,

$$\begin{aligned}\|\Lambda^s_{x,y} u(t)\|^2_{L^2(\mathbb{R}^{2n})} + \sigma_0\|\Lambda^\alpha_x\Lambda^s_{x,y} u\|^2_{L^2(]0,t[;L^2(\mathbb{R}^{2n}))}&\\ \le 2e^{4CT}\|\Lambda^s_{x,y} f\|^2_{L^1(]0,T[;L^2(\mathbb{R}^{2n}))}&\\ + \frac{1}{2}\|\Lambda^s_{x,y} u\|^2_{L^\infty(]0,T[;L^2(\mathbb{R}^{2n}))} + e^{2CT}\|\Lambda^s_{x,y} u_0\|^2_{L^2(\mathbb{R}^{2n})}&.\end{aligned}$$

We have proved Proposition 2.1. □

We can also prove the following results.

Proposition 2.2. *Assume that* $s\ge 0$, $\langle x\rangle f \in L^1(]0,T[;H^s(\mathbb{R}^{2n}))$ *for some* $0<T<\infty$ *and* $\langle x\rangle u_0 \in H^s(\mathbb{R}^{2n})$. *Suppose that* u *is a regular solution of Cauchy problem* (1.1). *Then there exist* $C>0$ *such that for* $k=1,\cdots,n$:

$$\begin{aligned}&\|\Lambda^s_{x,y}(x_k u)\|^2_{L^\infty(]0,T[;L^2(\mathbb{R}^{2n}))} + \sigma_0\|\Lambda^\alpha_x\Lambda^s_{x,y}(x_k u)\|^2_{L^2(]0,T[;L^2(\mathbb{R}^{2n}))}\\ &\le C\Big\{\|\Lambda^s_{x,y}(x_k f)\|^2_{L^1(]0,T[;L^2(\mathbb{R}^{2n}))} + \|\Lambda^s_{x,y} u\|^2_{L^\infty(]0,T[;\,L^2(\mathbb{R}^{2n}))}\\ &\quad + \|\Lambda^s_{x,y}(x_k u_0)\|^2_{L^2(\mathbb{R}^{2n})}\Big\}.\end{aligned}\tag{2.6}$$

In fact, the combination of (2.3) and (2.6) give

$$\begin{aligned}&\|\Lambda^s_{x,y}(\langle x\rangle u)\|^2_{L^\infty(]0,T[;L^2(\mathbb{R}^{2n}))} + \sigma_0\|\Lambda^\alpha_x\Lambda^s_{x,y}(\langle x\rangle u)\|^2_{L^2(]0,T[;L^2(\mathbb{R}^{2n}))}\\ &\le C\Big\{\|\Lambda^s_{x,y}(\langle x\rangle f)\|^2_{L^1(]0,T[;L^2(\mathbb{R}^{2n}))} + \|\Lambda^s_{x,y} u\|^2_{L^\infty(]0,T[;\,L^2(\mathbb{R}^{2n}))}\\ &\quad + \|\Lambda^s_{x,y}(\langle x\rangle u_0)\|^2_{L^2(\mathbb{R}^{2n})}\Big\}.\end{aligned}\tag{2.7}$$

We will use this estimate in the proof of regularity. The proof of this proposition is similar to that of Proposition 2.1. We can use (2.2) to get

$$|([x_k,\ |\widetilde{D_x}|^{2\alpha}]u, \Lambda^{2s}_{x,y}(x_k u))| \\ \leq C\|\Lambda^s_{x,y}u(t)\|_{L^2(\mathbb{R}^{2n})}(\|\Lambda^\alpha_x\Lambda^s_{x,y}(x_k u)(t)\|_{L^2(\mathbb{R}^{2n})} + \|\Lambda^s_{x,y}u(t)\|_{L^2(\mathbb{R}^{2n})}).$$

2.2. Semi-linear Cauchy problems

We suppose now $s > n$ and $u_0 \in H^s(\mathbb{R}^{2n})$. We consider the following linearization problems of Cauchy problems (1.2) for $k = 1, 2, \cdots$ and $u_1 = S_1(u_0)$.

$$\begin{cases} \partial_t u_{k+1} + x\cdot\nabla_y u_{k+1} + \sigma(-\widetilde{\triangle}_x)^\alpha u_{k+1} = F(u_k) \\ u_{k+1}|_{t=0} = S_{k+1}(u_0), \end{cases} \tag{2.8}$$

where $S_k(u_0) = \psi(2^{-k}D_{x,y})(u_0)$ with $\psi \in C^\infty_0(B(0,2)), \psi = 1$ on $B(0,1)$. This is regularization of initial data, and we point out the following properties (for example, see [14] for more detail of this regularizations)

$$S_k(u_0) \in H^{+\infty};\quad \|S_k(u_0)\|_{H^s} \leq \|u_0\|_{H^s};\quad S_k(u_0) \longrightarrow u_0 \ \text{ in } H^s(\mathbb{R}^{2n}),$$

and

$$\|S_{k+1}(u_0) - S_k(u_0)\|_{L^2(\mathbb{R}^{2n})} \leq c_k 2^{-ks}, \ \text{ with } \ \|\{c_k\}\|_{\ell^2} \leq \|u_0\|_{H^s}.$$

By nonlinear microlocal analysis, we have the following nonlinear composition results (see for example [14]).

Lemma 2.2. *Let $F \in C^\infty(\mathbb{R}), F(0) = 0, s \geq 0$, if $u \in H^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, then $F(u) \in H^s(\mathbb{R}^N)$ with*

$$\|F(u)\|_{H^s} \leq C_M\|u\|_{H^s}$$

where the constant C_M depends only on $\|F^{(j)}\|_{L^\infty([-M,M])}, M^j, 0 \leq j \leq [s]+1$ with $\|u\|_{L^\infty} \leq M$. The same result is true for $u \in L^\infty(]0,T[; H^s(\mathbb{R}^N)) \cap L^\infty(]0,T[\times\mathbb{R}^N)$.

Theorem 1.1 gives the existence of this sequence $\{u_k\} \subset L^\infty$ $(]0,T[;$ H^m $(\mathbb{R}^{2n}))$ for any $T > 0$ and $m > n$, since $S_k(u_0) \in H^{+\infty}(\mathbb{R}^{2n})$.

We prove now the convergence of this sequence by the following two propositions.

Proposition 2.3. *For $s > n$, there exist $T_1 > 0$ and $M_1 > 0$ such that for any $k \geq 1$,*

$$\|u_k\|^2_{L^\infty(]0,T_1[;H^s(\mathbb{R}^{2n}))} + \|\Lambda^\alpha_x u_k\|^2_{L^2(]0,T_1[;H^s(\mathbb{R}^{2n}))} \leq M_1^2. \tag{2.9}$$

Proof. We denote by C_s the Sobolev embedding constant:

$$\|u\|_{L^\infty} \le C_s \|u\|_{H^s(\mathbb{R}^{2n})},$$

since $s > n$. Denote by $M_0 = \|u_0\|_{H^s}, M_s = C_s M_0$. We prove now (2.9) by induction.

For $k = 2$, we first use Lemma 2.2,

$$\|F(u_1)\|_{L^1(]0,T[;H^s(\mathbb{R}^{2n}))} \le C_{M_s} M_0.$$

Then Proposition 2.1 gives

$$\|\Lambda^s_{x,y} u_2\|^2_{L^\infty(]0,T[;L^2(\mathbb{R}^{2n}))} + \sigma_0 \|\Lambda^\alpha_x \Lambda^s_{x,y} u_2\|^2_{L^2(]0,T[;L^2(\mathbb{R}^{2n}))} \le C\{T^2 C^2_{M_s} + 1\} M_0^2.$$

It is enough to take M_1, T_1 such that $C\{T_1^2 C^2_{M_s} + 1\} M_0^2 \le M_1^2$.

Suppose now (2.9) is true for $2 \le j \le k$. We shall prove (2.9) for $k+1$. Since u_{k+1} is a regular solution of equation (2.8), (2.3) gives

$$\begin{aligned}\|\Lambda^s_{x,y} u_{k+1}\|^2_{L^\infty(]0,T[;L^2(\mathbb{R}^{2n}))} &+ \sigma_0 \|\Lambda^\alpha_x \Lambda^s_{x,y} u_{k+1}\|^2_{L^2(]0,T[;L^2(\mathbb{R}^{2n}))} \\ &\le C\left\{ \|\Lambda^s_{x,y} F(u_k)\|^2_{L^1(]0,T[;L^2(\mathbb{R}^{2n}))} + \|\Lambda^s_{x,y} S_{k+1}(u_0)\|^2_{L^2(\mathbb{R}^{2n})} \right\}.\end{aligned}$$

From the induction hypothesis,

$$\|u_k\|_{L^\infty(]0,T[\times\mathbb{R}^{2n})} \le C_s \|\Lambda^s_{x,y} u_k\|_{L^\infty(]0,T[;L^2(\mathbb{R}^{2n}))} \le C_s M_1 = \widetilde{M_s}$$

with $\widetilde{M_s} \ge M_s$ independent of k which implies $C_{\widetilde{M_s}} \ge C_{M_s}$. We get that, thanks to Lemma 2.2,

$$\begin{aligned}\|\Lambda^s_{x,y} F(u_k)\|_{L^1(]0,T[;L^2(\mathbb{R}^{2n}))} &\le C_{\widetilde{M_s}} \|\Lambda^s_{x,y}(u_k)\|_{L^1(]0,T[;L^2(\mathbb{R}^{2n}))} \\ &\le C_{\widetilde{M_s}} T \|\Lambda^s_{x,y}(u_k)\|_{L^\infty(]0,T[;L^2(\mathbb{R}^{2n}))} \le C_{\widetilde{M_s}} T M_1.\end{aligned}$$

We get finally

$$\begin{aligned}\|\Lambda^s_{x,y} u_{k+1}\|^2_{L^\infty(]0,T[;L^2(\mathbb{R}^{2n}))} &+ \sigma_0 \|\Lambda^\alpha_x \Lambda^s_{x,y} u_{k+1}\|^2_{L^2(]0,T[;L^2(\mathbb{R}^{2n}))} \\ &\le C\left\{ \left(C_{\widetilde{M_s}} T M_1\right)^2 + M_0^2 \right\},\end{aligned}$$

so that it is enough to take

$$M_1^2 = 2CM_0^2, \;\; 0 < T \le T_1 = \frac{1}{2} C^{-1}_{\widetilde{M_s}} C^{-1/2}.$$

We have proved Proposition 2.3. □

Proposition 2.4. *There exists a $0 < T_2 \le T_1$ such that $\{u_k\}$ is convergent in $L^\infty(]0,T_2[;L^2(\mathbb{R}^{2n}))$, $\{\Lambda^\alpha_x(u_k)\}$ is convergent in $L^2(]0,T_2[;L^2(\mathbb{R}^{2n}))$, and the limit $u \in L^\infty(]0,T_2[;H^s(\mathbb{R}^{2n}))$ is a solution of Cauchy problem (1.2) with*

$$\Lambda^\alpha_x(u) \in L^2(]0,T_2[;H^s(\mathbb{R}^{2n})).$$

Proof. We prove that there exists $0 < T_2 \leq T_1$ and $M_2 > 0$ such that for any $k \geq 1$

$$
\begin{aligned}
(2.10) \qquad & \|u_{k+1} - u_k\|^2_{L^\infty(]0,T_2[;L^2(\mathbb{R}^{2n}))} + \|\Lambda^\alpha_x(u_{k+1} - u_k)\|^2_{L^2(]0,T_2[;L^2(\mathbb{R}^{2n}))} \\
& \leq M_2^2 2^{-2ks}.
\end{aligned}
$$

We have in fact, from equation (2.8),

$$
\begin{cases}
\partial_t(u_{k+1} - u_k) + x \cdot \nabla_y(u_{k+1} - u_k) + \sigma(-\widetilde{\triangle}_x)^\alpha(u_{k+1} - u_k) \\
\qquad\qquad\qquad\qquad\qquad\qquad = F(u_k) - F(u_{k-1}) \\
(u_{k+1} - u_k)|_{t=0} = \Delta_{k+1}(u_0),
\end{cases}
$$

where $\Delta_{k+1}(u_0) = S_{k+1}(u_0) - S_k(u_0)$. From Proposition 2.3, we have that, for any $k \geq 1$,

$$
\|u_k\|_{L^\infty(]0,T_1[\times\mathbb{R}^{2n})} \leq C_s M_1 = \widetilde{M}_s.
$$

We have that for any $0 < T \leq T_1$,

$$
\|F(u_k) - F(u_{k-1})\|_{L^1(]0,T[;L^2(\mathbb{R}^{2n}))} \leq C_0 T \|(u_k - u_{k-1})\|_{L^\infty(]0,T[;L^2(\mathbb{R}^{2n}))}
$$

with $C_0 = \|F'\|_{L^\infty(]-\widetilde{M}_s,\widetilde{M}_s[)}$. Using again (2.3), we get, for any $0 < T \leq T_1$, and $k \geq 2$

$$
\begin{aligned}
& \|(u_{k+1} - u_k)\|^2_{L^\infty(]0,T[;L^2(\mathbb{R}^n))} + \sigma_0 \|\Lambda^\alpha_x(u_{k+1} - u_k)\|^2_{L^2(]0,T[;L^2(\mathbb{R}^n))} \\
& \qquad \leq C\left\{\left(C_0 T \|(u_k - u_{k-1})\|_{L^\infty(]0,T[;L^2(\mathbb{R}^{2n}))}\right)^2 + \|\Delta_{k+1}(u_0)\|^2_{L^2(\mathbb{R}^{2n}))}\right\}.
\end{aligned}
$$

On the other hand, by induction hypothesis,

$$
\|(u_k - u_{k-1})\|_{L^\infty(]0,T_2[;L^2(\mathbb{R}^{2n}))} \leq M_2 2^{-(k-1)s},
$$

and hypothesis on u_0,

$$
\|\Delta_{k+1}(u_0)\|_{L^2(\mathbb{R}^{2n}))} \leq \|u_0\|_{H^s} 2^{-ks}.
$$

Since it suffices to take

$$
M_2 = M_1, \;\; 0 < T_2 = 2^{-s-1} C_0^{-1} C^{-1/2} \leq T_1,
$$

we have proved Proposition 2.4. □

We have showed that there exists $0 < T_2 \leq T_1$ such that $\{u_k\}$ is bounded in $L^\infty(]0,T_2[,H^s(\mathbb{R}^{2n}))$ and $\{\Lambda^\alpha_x(u_k)\}$ is also bounded in $L^2(]0,T_2[,H^s(\mathbb{R}^{2n}))$. Hence $\{u_k\}$ is convergent in $L^\infty(]0,T_2[,L^2(\mathbb{R}^{2n}))$ and $\{\Lambda^\alpha_x(u_k)\}$ is convergent in $L^2(]0,T_2[,L^2(\mathbb{R}^{2n}))$ with the limit

$$
u = u_1 + \sum_{k=1}^{\infty}(u_{k+1} - u_k) \in L^\infty(]0,T_2[,L^2(\mathbb{R}^{2n})).
$$

By interpolation, the convergence is also in $L^\infty(]0,T_2[, H^{s'}(\mathbb{R}^{2n}))$ and $L^2(]0,T_2[, H^{s'}(\mathbb{R}^{2n}))$ for any $0 \leq s' < s$, and moreover

$$u \in L^\infty(]0,T_2[, H^s(\mathbb{R}^{2n})), \;\; \Lambda_x^\alpha u \in L^2(]0,T_2[, H^s(\mathbb{R}^{2n})).$$

We have proved the existence of solution for Theorem 1.3.

3. Sub-elliptic estimates

We study now the sub-elliptic estimates. Without loss of generality, we suppose in the following that $\sigma = \sigma_0 > 0$ is constant, and consider the operators

$$P = \partial_t + x \cdot \nabla_y + \sigma_0(-\widetilde{\triangle}_x)^\alpha$$

on an open domain $\Omega \subset \mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_y^n$. In the application, we will take $\Omega =]a,b[\times \mathbb{R}_x^n \times \mathbb{R}_y^n$, so that P are not pseudo-differential operators in Ω. We suppose now $1/3 < \alpha < 1$.

We put

$$\Lambda = \left(1 + |D_t|^2 + |D_x|^2 + |D_y|^2\right)^{1/2},$$

and

$$X_0 = \Lambda^{-1/3}(\partial_t + x \cdot \nabla_y), \;\; X_j = \Lambda^{\alpha-1}\partial_{x_j}, \quad j = 1, \cdots, n.$$

Then $X_j \in Op(S^\alpha_{1,0}(\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_y^n)), j = 1, \cdots, n$ is a family of pseudo-differential operators. But X_0 is not a pseudo-differential operator (of order $2/3$) on $]a,b[\times \mathbb{R}^{2n}$, since the coefficient x is not bounded on $]a,b[\times\mathbb{R}^{2n}$. We will pay more attention to treat this term (see [12], [13], [15]).

Proposition 3.1. *If Ω is an open domain of $\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_y^n$, $1/3 < \alpha < 1$, then there exists $C > 0$ such that*

$$\sum_{j=1}^n \|X_j v\|_{L^2}^2 \leq C\left\{Re(Pv,\, v) + \|v\|_{L^2}^2\right\}, \tag{3.1}$$

for any $v \in C_0^\infty(\Omega)$. For the X_0, we have

$$\|X_0 v\|_{L^2}^2 \leq C\left\{\sum_{k=1}^3 Re(Pv, A_k v) + \|v\|_{L^2}^2\right\}, \tag{3.2}$$

for any $v \in C_0^\infty(\Omega)$, where

$$A_1 \in Op(S^0_{1,0}(\mathbb{R}^{2n+1})), \;\; A_2 = (\Lambda^{-1/3} + \Lambda^{-1})X_0, \;\; A_3 = -(\partial_t + x \cdot \nabla_y)\Lambda^{-1}X_0.$$

Proof. For any $v \in C_0^\infty(\Omega)$, the integration by parts deduces immediately

$$Re(Pv, v) = Re(\sigma_0|\widetilde{D_x}|^\alpha v, |\widetilde{D_x}|^\alpha v) = \sigma_0\||\widetilde{D_x}|^\alpha v\|_{L^2}^2\,. \tag{3.3}$$

Then a direct calculation gives

$$\|X_j v\|_{L^2} = \|\Lambda^{\alpha-1} D_{x_j} v\|_{L^2} \leq \||\widetilde{D_x}|^\alpha v\|_{L^2} + C\|v\|_{L^2}.$$

We have proved (3.1). In the future proof, we need also the estimate (3.3). Putting now $w = A_2 v = \Lambda^{-1/3} X_0 v$, we have

$$\|X_0 v\|_{L^2}^2 = Re(Pv, w) - Re(\sigma_0 |\widetilde{D_x}|^{2\alpha} v, w).$$

Since $|\widetilde{D_x}|^\alpha \sigma_0 |\widetilde{D_x}|^\alpha$ is a positive operator on L^2, it follows that

$$\left| Re(\sigma_0 |\widetilde{D_x}|^{2\alpha} v, w) \right| \leq Re(\sigma_0 |\widetilde{D_x}|^\alpha v, |\widetilde{D_x}|^\alpha v) + Re(\sigma_0 |\widetilde{D_x}|^\alpha w, |\widetilde{D_x}|^\alpha w)$$

We get

$$\|X_0 v\|_{L^2}^2 \leq |Re(Pv, w)| + |Re(\sigma_0 |\widetilde{D_x}|^\alpha w, |\widetilde{D_x}|^\alpha w)| + C\||\widetilde{D_x}|^\alpha v\|_{L^2}^2. \tag{3.4}$$

We study now the term

$$\begin{aligned} Re(\sigma_0 |\widetilde{D_x}|^\alpha w, |\widetilde{D_x}|^\alpha w) &= Re(\sigma_0 |\widetilde{D_x}|^{2\alpha} w, w) = Re(Pw, w) \\ &= Re(Pv, A_3 v) + Re([\Lambda^{-2/3},\, x] \cdot \partial_y (\partial_t + x \cdot \partial_y) v, w) \\ &\quad + Re(\sigma_0 [|\widetilde{D_x}|^{2\alpha},\, \Lambda^{-1/3} X_0] v, w). \end{aligned}$$

But we have

$$[\Lambda^{-2/3},\, x] = -\frac{2}{3} \Lambda^{-2/3-2} \partial_x,$$

and

$$[|\widetilde{D_x}|^{2\alpha},\, \Lambda^{-1/3} X_0] = \Lambda^{-2/3} [|\widetilde{D_x}|^{2\alpha},\, x] \cdot D_y.$$

We use now (2.2), to deduce that

$$\left| Re(\sigma_0 [|\widetilde{D_x}|^{2\alpha}, \Lambda^{-1/3} X_0] v, w) \right| \leq C(\||\widetilde{D_x}|^\alpha v\|_{L^2} + \|v\|_{L^2}) \|X_0 v\|_{L^2}.$$

It is easier for

$$|Re([\Lambda^{-2/3}, x] \cdot \partial_y (\partial_t + x \cdot \partial_y) v, w)| \leq C\|w\|_{L^2}^2$$

and

$$\begin{aligned} \|w\|_{L^2}^2 &= ((\partial_t + x \cdot \nabla_y)(v), \Lambda^{-1} X_0(v)) \\ &\leq |Re(Pv, \Lambda^{-1} X_0(v))| + |(\sigma_0 |\widetilde{D_x}|^{2\alpha} v, \Lambda^{-1} X_0(v))| \\ &\leq |Re(Pv, \Lambda^{-1} X_0(v))| + C\||\widetilde{D_x}|^\alpha v\|_{L^2}^2 + \frac{1}{16} \|X_0(v)\|_{L^2}^2. \end{aligned}$$

We get finally the desired estimate (3.2). $\square$

We study now the microlocal regularity,

$$[X_j, X_0] = \Lambda^{\alpha-1/3-1}\partial_{y_j} + (\alpha-1)\Lambda^{\alpha-1/3-3}\partial_x\cdot\partial_y\partial_{x_j}$$
$$= \Lambda^{\alpha-1/3}\Big(\Lambda^{-1}\partial_{y_j}\Big) + (\alpha-1)\Lambda^{-1/3}\widetilde{\Lambda}_0 X_j,$$

where $\widetilde{\Lambda}_0 = \Lambda^{-2}\partial_x\cdot\partial_y$ is a pseudo-differential operator of order 0. So that

$$\begin{aligned}&\Lambda^{\alpha/2-1/6}\Big(\Lambda^{-1}\partial_{y_j}\Big)\\ &\quad= \Lambda^{-\alpha/2+1/6}[X_j, X_0] - (\alpha-1)\Lambda^{-\alpha/2-1/6}\widetilde{\Lambda}_0 X_j,\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}&\Lambda^{\alpha/2-1/6}\Big(\Lambda^{-1}\partial_t\Big)\\ &\quad= \Lambda^{\alpha/2-1/6}\Lambda^{-1}\Big(\partial_t + x\cdot\nabla_y\Big) - \Lambda^{\alpha/2-1/6}\Lambda^{-1}\Big(x\cdot\nabla_y\Big)\\ &\quad= \Lambda^{\alpha/2-1/6-2/3}X_0\\ &\qquad - \sum_{j=1}^{n}\Big(\Lambda^{-\alpha/2+1/6}[X_j, X_0]x_j - (\alpha-1)\Lambda^{-\alpha/2-1/6}\widetilde{\Lambda}_0 X_j x_j\Big).\end{aligned} \tag{3.6}$$

We recall

$$\Lambda^{\alpha}\Big(\Lambda^{-1}\partial_{x_j}\Big) = X_j. \tag{3.7}$$

Now, if Ω is bounded, the family of pseudo-differential operators $X_0, X_1, \cdots, X_n$ satisfy the Hörmander-Kohn condition in the sense of [3]. When $\alpha = 2/3$, all X_j have the same order $2/3$, and the theorem of [3] shows the subelliptic regularity of order $2/3 - 1/2 = 1/6$, which coincides with $\alpha/2 - 1/6$ here. But the operators P and equation (1.1) is not (properly supported) pseudo-differential in the domain $\Omega =]a, b[\times\mathbb{R}^{2n}$. We have proved the following subelliptic estimates.

Proposition 3.2. *With same notations of Proposition* 3.1*, there exists* $C > 0$ *such that*

$$\sum_{j=1}^{n}\Big\|\Lambda^{\frac{\alpha}{2}-\frac{1}{6}-1}\partial_{y_j}v\Big\|^2_{L^2} \le C\left\{\sum_{k=1}^{3} Re(Pv, A_k v) + \|v\|^2_{L^2}\right\}. \tag{3.8}$$

For the differentiation with respect to variable t, we have

$$\Big\|\Lambda^{\frac{\alpha}{4}-\frac{1}{12}-1}\partial_t v\Big\|^2_{L^2} \le C\left\{\sum_{k=1}^{3} Re(Pv, A_k v) + \|\langle x\rangle v\|^2_{L^2}\right\}, \tag{3.9}$$

and

$$\Big\|\Lambda^{\frac{\alpha}{4}-\frac{1}{12}}v\Big\|^2_{L^2} \le C\left\{\sum_{k=1}^{3} Re(Pv, A_k v) + \|\langle x\rangle v\|^2_{L^2}\right\}. \tag{3.10}$$

for any $v \in C_0^\infty(\Omega)$, where $\langle x\rangle = (1+|x|^2)^{1/2}$.

Remark. By density, the estimates of above proposition are also true for any $v \in H_0^2(\Omega)$.

Proof. Since $\|(A+iB)v\|_{L^2}^2 \geq 0$, it follows that

$$(v, i[A,B]v) \leq \frac{3}{2}\Big(\|Av\|_{L^2}^2 + \|Bv\|_{L^2}^2\Big) + \frac{1}{2}\Big(\|(A-A^*)v\|_{L^2}^2 + \|(B-B^*)v\|_{L^2}^2\Big)$$

for any operators A, B.

Putting $A = -i[X_j, X_0]^*\Lambda^{-\alpha+1/3}X_j$ and $B = X_0$, we have $[X_j,\, X_0]^* = -[X_j,\, X_0]$ and

$$\begin{aligned} i[A,B] &= -[\,[X_j,\, X_0]\Lambda^{-\alpha+1/3}X_j,\, X_0] \\ &= [X_j,\, X_0]^*\Lambda^{-\alpha+1/3}[X_j,\, X_0] - [\,[X_j,\, X_0]\Lambda^{-\alpha+1/3},\, X_0]X_j \\ &= [X_j,\, X_0]^*\Lambda^{-\alpha+1/3}[X_j,\, X_0] + \widetilde{\Lambda}^{-1/3}X_j, \end{aligned}$$

where $\widetilde{\Lambda}^{-1/3}$ is a pseudo-differential operator of order $-1/3$.

We obtain, from (3.5),

$$\begin{aligned} \Big\|\Lambda^{\frac{\alpha}{2}-\frac{1}{6}-1}(\partial_{y_j}v)\Big\|_{L^2}^2 &\leq \Big\|\Lambda^{-\frac{\alpha}{2}+\frac{1}{6}}[X_j,\, X_0]v\Big\|_{L^2}^2 + C\|X_jv\|_{L^2}^2 \\ &\leq C\Big(\|X_jv\|_{L^2}^2 + \|X_0v\|_{L^2}^2 + \|v\|_{L^2}^2\Big). \end{aligned}$$

Those estimates for $j = 1, \cdots, n$ together with (3.1) and (3.2) show (3.8).

To prove (3.9), we take $w = \Lambda^{\frac{\alpha}{2}-\frac{1}{6}-2}(\partial_t v)$. Then we have that

$$\begin{aligned} &\|\Lambda^{\frac{\alpha}{4}-\frac{1}{12}-1}(\partial_t v)\|_{L^2}^2 \\ &\quad = Re(Pv,\, \widetilde{\Lambda_0}(v)) - Re(\sigma_0|\widetilde{D_x}|^{2\alpha}v, \widetilde{\Lambda_0}(v)) - Re(x\cdot\nabla_y v,\, \Lambda^{\frac{\alpha}{2}-\frac{1}{6}-2}(\partial_t v)), \end{aligned}$$

where $\widetilde{\Lambda_0} = \Lambda^{\frac{\alpha}{2}-\frac{1}{6}-2}\partial_t \in Op(S^0_{1,0}(\mathbb{R}^{2n+1}))$. Then

$$|Re(\sigma_0|\widetilde{D_x}|^{2\alpha}v, \widetilde{\Lambda_0}(v))| \leq C\{\||\widetilde{D_x}|^{\alpha}v\|_{L^2}^2 + \|v\|_{L^2}^2\}.$$

For the last term, we have

$$\begin{aligned} |Re(x\cdot\nabla_y v,\, \Lambda^{\frac{\alpha}{2}-\frac{1}{6}-2}(\partial_t v))| &= \left|\sum_{k=1}^n Re(\Lambda^{-1}\partial_t(x_k v),\, \Lambda^{\frac{\alpha}{2}-\frac{1}{6}-1}(\partial_{y_k}v))\right| \\ &\leq C\sum_{k=1}^n \|x_k v\|_{L^2}\|\Lambda^{\frac{\alpha}{2}-\frac{1}{6}-1}(\partial_{y_k}v)\|_{L^2}, \end{aligned}$$

so that we get the desired estimate (3.9) by using (3.8) and (3.3).

Finally, combination of (3.1), (3.8) and (3.9) give (3.10). We have proved Proposition 3.2. $\square$

We can use directly (3.6) to get

$$\|\Lambda^{\frac{\alpha}{2}-\frac{1}{6}-1}(\partial_t v)\|_{L^2}^2 \le \|\Lambda^{\frac{\alpha}{2}-\frac{1}{6}-2/3}X_0 v\|_{L^2}^2 + \sum_{j=1}^n \|\Lambda^{\frac{\alpha}{2}-\frac{1}{6}-1}(\partial_{y_j}(x_j v))\|_{L^2}^2,$$

then (3.8) with test function $(x_k v)$ and (3.2) deduce the following estimate

$$\begin{aligned}\left\|\Lambda^{\frac{\alpha}{2}-\frac{1}{6}-1}\partial_t v\right\|_{L^2}^2 \le C\Bigg\{&\sum_{k=1}^3 Re(Pv, A_k v)\\ &+\sum_{j=1}^n\sum_{k=1}^3 Re(P(x_j v), A_k(x_j v)) + \|\langle x\rangle v\|_{L^2}^2\Bigg\},\end{aligned} \tag{3.11}$$

which gives also the following hypoelliptic estimate with weight

$$\begin{aligned}\left\|\Lambda^{\frac{\alpha}{2}-\frac{1}{6}} v\right\|_{L^2}^2 \le C\Bigg\{&\sum_{k=1}^3 Re(Pv, A_k v)\\ &+\sum_{j=1}^n\sum_{k=1}^3 Re(P(x_j v), A_k(x_j v)) + \|\langle x\rangle v\|_{L^2}^2\Bigg\},\end{aligned} \tag{3.12}$$

The difference between (3.12) and (3.10) is that, the gain of regularity of (3.10) is one half of (3.12), but we suppress the weight x_j in Pv. For high order regularity, we will study this scale between the gain of regularity and the power of weight $\langle x\rangle$.

4. Regularity of weak solutions

We prove now the regularity of weak solutions of Theorem 1.1. By combination the estimate(3.1), (3.2) and (3.10), we have obtained the following sub-elliptic estimate :

$$\left\|\Lambda^{\frac{\alpha}{4}-\frac{1}{12}} v\right\|_{L^2(\mathbb{R}^{2n+1})}^2 \le C\Big\{\|\langle x\rangle Pv\|_{L^2(\mathbb{R}^{2n+1})}^2 + \|\langle x\rangle v\|_{L^2(\mathbb{R}^{2n+1})}^2\Big\}, \tag{4.1}$$

for any $v \in H_0^2(]0,T[\times\mathbb{R}^{2n})$. Here we used the fact

$$\|A_1 v\|_{L^2} \le C\|v\|_{L^2},\ \ \|(A_2 + \langle x\rangle^{-1}A_3)v\|_{L^2} \le C\|X_0 v\|_{L^2}.$$

For $\delta > 0$, we set

$$\Lambda_\delta = \big(1 + \delta(|D_t|^2 + |D_x|^2 + |D_y|^2)\big)^{1/2}.$$

We will use the following notations : for $\varphi, \psi \in C_0^\infty$, we say $\varphi \subset\subset \psi$ if $\psi = 1$ in a neighborhood of supp φ.

We prove the following results

Proposition 4.1. *Let $1/3 < \alpha < 1$ and $s \geq 0$. Assume that f, $\langle x\rangle f \in H^s(]a,b[\times\mathbb{R}^{2n})$. Let $u \in H^s(]a,b[\times\mathbb{R}^{2n})$ be a weak solution of equation $Pu = f$ on $]a,b[\times\mathbb{R}^{2n}$ such that $\langle x\rangle u \in H^s(]a,b[\times\mathbb{R}^{2n})$. Then for any $\varphi, \psi \in C_0^\infty(]a,b[), \varphi \subset\subset \psi$ and $0 < \delta < 1$, there exists a constant $C > 0$ independent of δ such that*

$$\begin{aligned}\left\|\Lambda^{\frac{\alpha}{4}-\frac{1}{12}}\psi(\Lambda_\delta)^{-2}\Lambda^s(\varphi u)\right\|^2_{L^2(\mathbb{R}^{2n+1})} &\leq C\Big\{\|\widetilde{\psi}\Lambda^s(\langle x\rangle f)\|^2_{L^2(\mathbb{R}^{2n+1})} \\ &+ \|\widetilde{\psi}\Lambda^s(\langle x\rangle u)\|^2_{L^2(\mathbb{R}^{2n+1})} + \|\widetilde{\psi}\Lambda^s f\|^2_{L^2(\mathbb{R}^{2n+1})} + \|\widetilde{\psi}\Lambda^s u\|^2_{L^2(\mathbb{R}^{2n+1})}\Big\},\end{aligned} \tag{4.2}$$

with some $\widetilde{\psi} \in C_0^\infty(]a,b[), \psi \subset\subset \widetilde{\psi}$. Here the cut-off function are only for t variable.

Take the limit $\delta \to 0$ in (4.2). Then it deduces

$$\psi\Lambda^s(\varphi u) \in H^{\frac{\alpha}{4}-\frac{1}{12}}(\mathbb{R}^{2n+1}),$$

because the commutator $[\psi,\ \Lambda^s]$ is a pseudo-differential operator of order $s-1$. We have obtained a gain of regularity of order $\frac{1}{4}(\alpha-\frac{1}{3})$ for (weak) solution with a supplement condition $\langle x\rangle f \in H^s(]a,b[\times\mathbb{R}^{2n})$ and $\langle x\rangle u \in H^s(]a,b[\times\mathbb{R}^{2n})$.

Proof of Proposition 4.1. We will choose $v = \psi\Lambda_\delta^{-2}\Lambda^s(\varphi u)$ as test function in (3.2) and (3.10), (if we consider the partial Sobolev space, we take $v = \psi\Lambda_\delta^{-2}\Lambda^s_{x,y}(\varphi u)$ as test function)

$$\begin{aligned}&\left\|\Lambda^{\frac{\alpha}{4}-\frac{1}{12}}\psi(\Lambda_\delta)^{-2}\Lambda^s(\varphi u)\right\|^2_{L^2} + \left\|X_0\psi(\Lambda_\delta)^{-2}\Lambda^s(\varphi u)\right\|^2_{L^2} \\ &\quad\leq C\left\{\sum_{k=1}^{3} Re(P\psi(\Lambda_\delta)^{-2}\Lambda^s(\varphi u), A_k\psi(\Lambda_\delta)^{-2}\Lambda^s(\varphi u))\right. \\ &\qquad\left. + \|\langle x\rangle\psi(\Lambda_\delta)^{-2}\Lambda^s(\varphi u)\|^2_{L^2}\right\}.\end{aligned}$$

We calculate the commutator terms

$$[P,\ \psi\Lambda_\delta^{-2}\Lambda^s\varphi]u = \partial_t(\psi\Lambda_\delta^{-2}\Lambda^s\varphi)\psi u + \psi[x,\ \Lambda_\delta^{-2}\Lambda^s]\cdot\nabla_y(\varphi u)$$

which is a pseudo-differential operator of order s for (x,y) variables. And moreover

$$\begin{aligned}&\langle x\rangle[P,\ \psi\Lambda_\delta^{-2}\Lambda^s\varphi]\langle x\rangle^{-1} \\ &\quad= [P,\ \psi\Lambda_\delta^{-2}\Lambda^s\varphi] + \langle x\rangle[[P,\ \psi\Lambda_\delta^{-2}\Lambda^s\varphi],\ \langle x\rangle^{-1}],\end{aligned}$$

where the second term is also a pseudo-differential operator of order s. There exists $C > 0$ independent of δ such that

$$\|\langle x\rangle[P,\ \psi\Lambda_\delta^{-2}\Lambda^s\varphi]u\|_{L^2} \leq C\|\widetilde{\psi}\Lambda^s\langle x\rangle\psi u\|_{L^2},$$

and

$$\|\langle x\rangle\psi\Lambda_\delta^{-2}\Lambda^s\varphi Pu\|_{L^2} \le \|\widetilde{\psi}\Lambda^s\langle x\rangle\psi Pu\|_{L^2}.$$

We get the estimate (4.2) directly from (4.1). □

We study now the regularity of $\langle x\rangle^{-1}u$.

Theorem 4.1. *Let $1/3 < \alpha < 1$ and $s \ge 0$. We suppose that $f \in H^s(]a,b[\times\mathbb{R}^{2n}))$ and $u \in H^s(]a,b[\times\mathbb{R}^{2n})$ is a weak solution of equation $Pu = f$ on $]a,b[\times\mathbb{R}^{2n}$. Then we have*

$$\langle x\rangle^{-1}u \in H^{s+\frac{\alpha}{4}-\frac{1}{12}}(]a',\, b'[\times\mathbb{R}^{2n}),$$

for any $a < a' < b' < b$.

The proof of this theorem uses the following hypoelliptic estimate with weight $\langle x\rangle^{-1}$.

Proposition 4.2. *Suppose that $1/3 < \alpha < 1$. Then there exists $C > 0$ such that*

$$\begin{aligned}&\left\|\Lambda^{\frac{\alpha}{4}-\frac{1}{12}}(\langle x\rangle^{-1}v)\right\|^2_{L^2} + \left\|X_0(\langle x\rangle^{-1}v)\right\|^2_{L^2}\\ &\quad\le C\left\{\sum_{k=1}^{4} Re(\langle x\rangle^{-1}Pv,\, A_k(\langle x\rangle^{-1}v) + \|v\|^2_{L^2}\right\}\end{aligned}\tag{4.3}$$

for any $v \in C_0^\infty(]a,b[\times\mathbb{R}^{2n})$, where A_1, A_2, A_3 are the same operators as in Proposition 3.1 and $A_4 = \langle x\rangle^2$.

Proof. Putting the test function $\langle x\rangle^{-1}v$ in (3.2) and (3.10), we have that

$$\begin{aligned}&\left\|\Lambda^{\frac{\alpha}{4}-\frac{1}{12}}(\langle x\rangle^{-1}v)\right\|^2_{L^2} + \left\|X_0(\langle x\rangle^{-1}v)\right\|^2_{L^2}\\ &\qquad\qquad\le C\left\{\sum_{k=1}^{3} Re(P(\langle x\rangle^{-1}v),\, A_k(\langle x\rangle^{-1}v) + \|v\|^2_{L^2}\right\}.\end{aligned}$$

We need to estimate the commutator term

$$\sum_{k=1}^{3}|([P,\ \langle x\rangle^{-1}]v,\ A_k(\langle x\rangle^{-1}v))|$$

by th right hand side of (4.3). Since $\langle x\rangle^{-1} \in C_b^\infty(\mathbb{R}^n)$, Lemma 2.1 implies

$$\|[P,\ \langle x\rangle^{-1}]v\|^2_{L^2} = \|\sigma_0[|D_x|^{2\alpha},\ \langle x\rangle^{-1}]v\|^2_{L^2} \le C\Big(\||\widetilde{D_x}|^\alpha v\|^2_{L^2} + \|v\|^2_{L^2}\Big).$$

By using Cauchy-Schwarz inequality,

$$\begin{aligned}
&\sum_{k=1}^{2}|(([P,\,\langle x\rangle^{-1}]v,\,A_k(\langle x\rangle^{-1}v))| \\
(4.4)\qquad &\quad\le C\Big(\||\widetilde{D_x}|^{\alpha}v\|^2_{L^2}+\|v\|^2_{L^2}\Big)+\frac{1}{1000}\|X_0(\langle x\rangle^{-1}v)\|^2_{L^2} \\
&\quad\le C\Big(Re(Pv,v)+\|v\|^2_{L^2}\Big)+\frac{1}{1000}\|X_0(\langle x\rangle^{-1}v)\|^2_{L^2}.
\end{aligned}$$

For the last term, we have

$$\begin{aligned}
&([P,\,\langle x\rangle^{-1}]v,\,A_3(\langle x\rangle^{-1}v)) \\
&\quad=-\sigma_0([|\widetilde{D_x}|^{2\alpha},\,\langle x\rangle^{-1}]v,\,\partial_t\Lambda^{-1}X_0(\langle x\rangle^{-1}v)) \\
&\qquad-\sum_{j=1}^{n}\sigma_0(x_j[|\widetilde{D_x}|^{2\alpha},\,\langle x\rangle^{-1}]v,\,\partial_{y_j}\Lambda^{-1}X_0(\langle x\rangle^{-1}v)).
\end{aligned}$$

The estimation of $\sigma_0([|\widetilde{D_x}|^{2\alpha},\,\langle x\rangle^{-1}]v,\,\partial_t\Lambda^{-1}X_0(\langle x\rangle^{-1}v))$ is the same as (4.4). On the other hand, for $j=1,\cdots,n$,

$$\begin{aligned}
&x_j[|\widetilde{D_x}|^{2\alpha},\,\langle x\rangle^{-1}]=x_j|\widetilde{D_x}|^{2\alpha}\langle x\rangle^{-1}-x_j\langle x\rangle^{-1}|\widetilde{D_x}|^{2\alpha} \\
&\quad=[x_j,\;|\widetilde{D_x}|^{2\alpha}]\langle x\rangle^{-1}+[|\widetilde{D_x}|^{2\alpha},x_j\langle x\rangle^{-1}] \\
&\quad=\langle x\rangle^{-1}[x_j,\;|\widetilde{D_x}|^{2\alpha}]+[[x_j,\;|\widetilde{D_x}|^{2\alpha}],\;\langle x\rangle^{-1}]+[|\widetilde{D_x}|^{2\alpha},x_j\langle x\rangle^{-1}].
\end{aligned}$$

Since $[x_j,\;|\widetilde{D_x}|^{2\alpha}]$ is a Fourier multiplier, we use Lemma 2.1 with the functions $\langle x\rangle^{-1},x_j\langle x\rangle^{-1}\in C_b^{\infty}(\mathbb{R}^n)$,

$$\begin{aligned}
&|(x_j[|\widetilde{D_x}|^{2\alpha},\,\langle x\rangle^{-1}]v,\,\partial_{y_j}\Lambda^{-1}X_0(\langle x\rangle^{-1}v))| \\
&\quad\le C\Big(\||\widetilde{D_x}|^{\alpha}v\|^2_{L^2}+\|v\|^2_{L^2}\Big)+\frac{1}{1000}\|X_0(\langle x\rangle^{-1}v)\|^2_{L^2} \\
&\quad\le C\Big(Re(Pv,v)+\|v\|^2_{L^2}\Big)+\frac{1}{1000}\|X_0(\langle x\rangle^{-1}v)\|^2_{L^2}.
\end{aligned}$$

We proved finally (4.3). □

Similarly to Proposition 4.1, we have the following :

Proposition 4.3. *Let $1/3<\alpha<1$ and $s\ge 0$. Assume that $f\in H^s(]a,b[\times\mathbb{R}^{2n})$. If $u\in H^s(]a,b[\times\mathbb{R}^{2n})$ is a (weak) solution of equation $Pu=f$ in $]a,b[\times\mathbb{R}^{2n}$, then for any $\varphi,\psi\in C_0^{\infty}(]a,b[),\varphi\subset\subset\psi$ and $0<\delta<1$, there exists a $C>0$ independent of δ such that*

$$\begin{aligned}
&\Big\|\Lambda^{\frac{\alpha}{4}-\frac{1}{12}}\langle x\rangle^{-1}\psi(\Lambda_\delta)^{-2}\Lambda^s(\varphi u)\Big\|^2_{L^2(\mathbb{R}^{2n+1})} \\
(4.5)\qquad &\qquad\le C\Big\{\|\Lambda^s\widetilde{\psi}f\|^2_{L^2(\mathbb{R}^{2n+1})}+\|\Lambda^s\widetilde{\psi}u\|^2_{L^2(\mathbb{R}^{2n+1})}\Big\},
\end{aligned}$$

with some $\widetilde{\psi}\in C_0^{\infty}(]a,T[),\psi\subset\subset\widetilde{\psi}$.

We just choose $v = \psi\Lambda_\delta^{-2}\Lambda^s(\varphi u)$ as test function in (4.3). The estimation of commutator term

$$[P, \psi\Lambda_\delta^{-2}\Lambda^s\varphi]u = \partial_t(\psi\Lambda_\delta^{-2}\Lambda^s\varphi)\psi u + \psi[x, \Lambda_\delta^{-2}\Lambda^s]\cdot\nabla_y(\varphi u)$$

is the same as in the proof of Proposition 4.1. We get also

$$\begin{aligned}(4.6)\qquad &\Big\||\widetilde{D_x}|^\alpha\langle x\rangle^{-1}\psi(\Lambda_\delta)^{-2}\Lambda^s(\varphi u)\Big\|_{L^2}^2\\ &\qquad\le C\Big\{\|\langle x\rangle^{-1}\Lambda^s(\psi f)\|_{L^2}^2 + \|\Lambda^s\psi u\|_{L^2}^2\Big\},\end{aligned}$$

and

$$(4.7)\qquad \big\|X_0\langle x\rangle^{-1}\psi(\Lambda_\delta)^{-2}\Lambda^s(\varphi u)\big\|_{L^2}^2 \le C\Big\{\|\Lambda^s\psi f\|_{L^2}^2 + \|\Lambda^s\psi u\|_{L^2}^2\Big\}.$$

Proof of Theorem 4.1. Take the limit $\delta \to 0$ in (4.5). Then it deduces

$$\langle x\rangle^{-1}\psi\Lambda^s(\varphi u) \in H^{\frac{\alpha}{4}-\frac{1}{12}}(\mathbb{R}^{2n+1}),$$

because the commutator $[\psi,\ \Lambda^s]$ is a pseudo-differential operator of order $s-1$. We obtain a gain of regularity for $\langle x\rangle^{-1}u$. □

We have also proved that

$$u \in H_{loc}^{s+\frac{\alpha}{4}-\frac{1}{12}}(]a,b[\times\mathbb{R}^{2n})$$

since for any $\varphi \in C_0^\infty(]a,b[),\ \psi \in C_0^\infty(\mathbb{R}_x^n)$, we have

$$\begin{aligned}&\|\Lambda^{s+\frac{\alpha}{4}-\frac{1}{12}}(\varphi(t)\psi(x)u)\|_{L^2}\\ &\quad\le \|\psi(x)\Lambda^{s+\frac{\alpha}{4}-\frac{1}{12}}(\varphi(t)u)\|_{L^2} + C\|\Lambda^s(\varphi(t)u)\|_{L^2}\\ &\quad\le C\Big\{\|\langle x\rangle^{-1}\Lambda^{s+\frac{\alpha}{4}-\frac{1}{12}}(\varphi(t)u)\|_{L^2} + \|\Lambda^s(\varphi u)\|_{L^2}\Big\}\\ &\quad\le C\Big\{\|\Lambda^{s+\frac{\alpha}{4}-\frac{1}{12}}(\langle x\rangle^{-1}\varphi(t)u)\|_{L^2} + \|\Lambda^s(\varphi u)\|_{L^2}\Big\}\,.\end{aligned}$$

5. Linear and nonlinear hypoellipticity

We can not use Proposition 4.1 to gain the high order regularity by induction, i.e. the hypoellipticity of the usual sense. Since we get only the regularity of φu, but in the right hand side of (4.2) , we need some regularity of $\langle x\rangle\varphi u$. We consider now the function space with the weight $\langle x\rangle^{-k}$.

We have to study the commutator of $\langle x\rangle^{-k}$ with $|\widetilde{D_x}|^{2\alpha}$ and Λ^s. We give firstly the following lemma.

Lemma 5.1. *Let $B \in Op(S_{1,0}^m(\mathbb{R}^{2n+1}))$, then, for any $k \in \mathbb{N}$,*

$$(5.1)\qquad [\langle x\rangle^{-k}, B] = \langle x\rangle^{-k-1}B_1 + \langle x\rangle^{-k-2}B_2,$$

where $B_1 \in Op(S^{m-1}_{1,0}(\mathbb{R}^{2n+1}))$ *with symbol*

$$\langle x\rangle^{-1}x\cdot\partial_\xi B(x,\xi),\quad and\quad B_2\in Op(S^{m-2}_{1,0}(\mathbb{R}^{2n+1})).$$

For the commutator with Fourier multiplier $|\widetilde{D_x}|^{2\alpha}$*, we have that, for* $k = 2\ell,\ \ell\in\mathbb{N}$*,*

$$[\langle x\rangle^{-k},|\widetilde{D_x}|^{2\alpha}] = \langle x\rangle^{-1}F_1\langle x\rangle^{-k} + \langle x\rangle^{-2}F_2\langle x\rangle^{-k}, \tag{5.2}$$

where F_1 *is an operators of form*

$$F_1 = \sum_{j=1}^n a_jA_j(D_x),\quad with\ a_j\in C_b^\infty(\mathbb{R}^n),\ \ A_j(\xi) = D_{\xi_j}\big(|\xi|^{2\alpha}\chi^2(\xi)\big),$$

and $F_2\in\mathcal{L}(L^2,L^2)$ *is a finite sum of form* $\widetilde{a}(x)\widetilde{b}(D_x)$ *with* $\widetilde{a},\widetilde{b}\in C_b^\infty(\mathbb{R}^n)$*.*

Remark. In the application, if we take $B=\Lambda_\delta^{-m}, 0\le m, 0<\delta<1$ an uniformly bounded family in $Op(S^0_{1,0}(\mathbb{R}^{2n+1}))$, then $B_1\in Op(S^{-1}_{1,0}(\mathbb{R}^{2n+1}))$, $B_2\in Op(S^{-2}_{1,0}(\mathbb{R}^{2n+1}))$ is also uniformly bounded. We remark also

$$\|F_1w\|_{L^2}\le C\||\widetilde{D_x}|^\alpha w\|_{L^2}\le C\{\|\Lambda_x^\alpha w\|_{L^2}+\|w\|_{L^2}\}.$$

Proof. (5.1) is just precise pseudo-differential calculus. For (5.2), we can also use the classical pseudo-differential calculus, we have

$$[\langle x\rangle^{-k},|\widetilde{D_x}|^{2\alpha}] = \langle x\rangle^{-k}[|\widetilde{D_x}|^{2\alpha},\ \langle x\rangle^{k}]\langle x\rangle^{-k},$$

and

$$\langle x\rangle^{2\ell} = \sum_{|\lambda|\le\ell}C_\lambda x^{2\lambda}$$

where $\lambda=(\lambda_1,\cdots,\lambda_n), x^{2\lambda}=x_1^{2\lambda_1}\cdots x_n^{2\lambda_n}$. For $0<|\lambda|\le\ell$, we have that

$$\begin{aligned}x^{2\lambda}|\widetilde{D_x}|^{2\alpha}v &= \mathcal{F}^{-1}\Big(D_\xi^{2\lambda}\big(|\xi|^{2\alpha}\chi^2(\xi)\hat{v}\big)\Big)\\ &= \sum_{0\le\lambda'<2\lambda}C^{2\lambda}_{\lambda'}\mathcal{F}^{-1}\Big(D_\xi^{2\lambda-\lambda'}\big(|\xi|^{2\alpha}\chi^2(\xi)\big)D_\xi^{\lambda'}\hat{v}\Big)+|\widetilde{D_x}|^{2\alpha}\big(x^{2\lambda}v\big),\end{aligned}$$

and

$$\mathcal{F}^{-1}\Big(D_\xi^{2\lambda-\lambda'}\big(|\xi|^{2\alpha}\chi^2(\xi)\big)D_\xi^{\lambda'}\hat{v}\Big) = \sum_{0\le\mu\le\lambda'}C^{\lambda'}_\mu x^\mu\mathcal{F}^{-1}\Big(D_\xi^{2\lambda-\mu}\big(|\xi|^{2\alpha}\chi^2(\xi)\big)\hat{v}\Big).$$

Now if $|2\lambda-\mu|=1$ we obtain a term in F_1, and if $|2\lambda-\mu|\ge2$ we obtain a term in F_2. We have proved Lemma 5.1. $\square$

Proposition 5.1. *Suppose that $1/3 < \alpha < 1$ and $k \in \mathbb{N}$, there exists $C > 0$ such that*

$$
\begin{aligned}
&\left\|\Lambda^{\frac{\alpha}{4}-\frac{1}{12}}(\langle x\rangle^{-k-1}v)\right\|^2_{L^2} + \left\|X_0(\langle x\rangle^{-k-1}v)\right\|^2_{L^2} \\
&\quad \le C\left\{\sum_{k=1}^{4} Re(\langle x\rangle^{-k-1}Pv, A_k(\langle x\rangle^{-k-1}v) + \|\langle x\rangle^{-k}v\|^2_{L^2}\right\}
\end{aligned} \tag{5.3}
$$

for any $v \in C_0^\infty(]a,b[\times\mathbb{R}^{2n})$, where $A_1, \cdots, A_4$ are the same as in Proposition 4.2 .

Remark. By density, (5.3) is true for any $v \in H^2_0(]a,b[\times\mathbb{R}^{2n})$.

Proof. If $k = 2\ell+1$ is odd, we choose the test function $\langle x\rangle^{-k-1}v = \langle x\rangle^{-2\ell-2}v$ in (3.2) and (3.10), we have that

$$
\begin{aligned}
&\left\|\Lambda^{\frac{\alpha}{4}-\frac{1}{12}}(\langle x\rangle^{-k-1}v)\right\|^2_{L^2} + \left\|X_0(\langle x\rangle^{-k-1}v)\right\|^2_{L^2} \\
&\qquad \le C\left\{\sum_{k=1}^{3} Re(P(\langle x\rangle^{-2(\ell+1)}v),\ A_k(\langle x\rangle^{-k-1}v) + \|\langle x\rangle^{-k}v\|^2_{L^2}\right\}.
\end{aligned}
$$

By using the Lemma 5.1, we have that

$$
\begin{aligned}
([P,\ \langle x\rangle^{-2\ell-2}]v &= \sigma_0[|\widetilde{D_x}|^{2\alpha}, \langle x\rangle^{-2\ell-2}]v \\
&= \langle x\rangle^{-1}F_1\langle x\rangle^{-2\ell-2}v + \langle x\rangle^{-2}F_2\langle x\rangle^{-2\ell-2}v,
\end{aligned}
$$

then for $k+1 = 2\ell+2$, by using Cauchy-Schwarz inequality,

$$
\begin{aligned}
&\sum_{k=1}^{3} |Re([P,\ \langle x\rangle^{-k-1}]v,\ A_k(\langle x\rangle^{-k-1}v)| \\
&\qquad \le C\Big\{\||\widetilde{D_x}|^{\alpha}\langle x\rangle^{-2(\ell+1)}v\|^2_{L^2} + \|\langle x\rangle^{-k-1}v\|^2_{L^2}\Big\} \\
&\qquad\quad + \frac{1}{1000}\|X_0\langle x\rangle^{-k-1}v\|^2_{L^2}.
\end{aligned}
$$

Here we have used the fact $\|\langle x\rangle^{-1}A_3w\|_{L^2} \le C\|X_0w\|_{L^2}$.

If $k = 2\ell$ is even, we choose the test function $\langle x\rangle^{-k}v = \langle x\rangle^{-2\ell}v$ in (4.3), we have that

$$
\begin{aligned}
&\left\|\Lambda^{\frac{\alpha}{4}-\frac{1}{12}}(\langle x\rangle^{-k-1}v)\right\|^2_{L^2} + \left\|X_0(\langle x\rangle^{-k-1}v)\right\|^2_{L^2} \\
&\quad \le C\left\{\sum_{k=1}^{4} Re(\langle x\rangle^{-1}P(\langle x\rangle^{-2\ell}v), A_k(\langle x\rangle^{-k-1}v) \right. \\
&\qquad \left. + \|\langle x\rangle^{-k}v\|^2_{L^2}\right\}.
\end{aligned}
$$

Using again Lemma 5.1, we obtain

$$([P,\ \langle x\rangle^{-2\ell}]v = \sigma_0[|\widetilde{D_x}|^{2\alpha}, \langle x\rangle^{-2\ell}]v = \langle x\rangle^{-1}F_1\langle x\rangle^{-2\ell}v + \langle x\rangle^{-2}F_2\langle x\rangle^{-2\ell}v,$$

which deduce that, with $k = 2\ell$,

$$\begin{aligned}\sum_{k=1}^{4} &|Re(\langle x\rangle^{-1}[P,\ \langle x\rangle^{-k}]v, A_k(\langle x\rangle^{-k-1}v)| \\ &\leq C\Big\{\||\widetilde{D_x}|^\alpha\langle x\rangle^{-2\ell}v\|_{L^2}^2 + \|\langle x\rangle^{-k}v\|_{L^2}^2\Big\} \\ &\qquad + \frac{1}{1000}\|X_0\langle x\rangle^{-k-1}v\|_{L^2}^2.\end{aligned}$$

Finally, the same calculus for the commutator term shows, for $k = 2\ell$,

$$\begin{aligned}\||\widetilde{D_x}|^\alpha\langle x\rangle^{-2\ell}v\|_{L^2}^2 &= Re(P\langle x\rangle^{-2\ell}v, \langle x\rangle^{-2\ell}v) \\ &\leq C\Big\{Re(\langle x\rangle^{-2\ell}\widetilde{P}v, \langle x\rangle^{-2\ell}v) + \|\langle x\rangle^{-2\ell}v\|_{L^2}^2\Big\} \\ &\leq C\Big\{Re(\langle x\rangle^{-k-1}\widetilde{P}v, A_4\langle x\rangle^{-k-1}v) + \|\langle x\rangle^{-2\ell}v\|_{L^2}^2\Big\}.\end{aligned}$$

It is easier for the case of $k = 2\ell + 1$

$$\||\widetilde{D_x}|^\alpha\langle x\rangle^{-2(\ell+1)}v\|_{L^2}^2 \leq C\Big\{Re(\langle x\rangle^{-k-1}Pv, \langle x\rangle^{-k-1}v) + \|\langle x\rangle^{-2\ell-2}v\|_{L^2}^2\Big\}.$$

We have proved Proposition 5.1. □

Proof of Theorem 1.2. Recall that the hypothesis of Theorem 1.2 is that : $u \in L^2(]a,b[\times\mathbb{R}^{2n})$ and for some $s \geq 0$, $f \in H^s(]a,b[\times\mathbb{R}^{2n})$. Using the remark at the end of Section 3, we can obtain without modifying the results of Proposition 4.3, the estimation (4.5) with $s = 0$. Take the limit $\delta \to 0$ in (4.5). Then it deduces that the solution of Theorem 1.2 has the following regularity

$$\langle x\rangle^{-1}(\varphi u) \in H^{\frac{\alpha}{4}-\frac{1}{12}}(\mathbb{R}^{2n+1}).$$

We have proved the Theorem 1.2 if $s = 0$.

We shall prove higher order regularity by induction.

Proposition 5.2. *Let* $\varepsilon_0 = \frac{\alpha}{4} - \frac{1}{12} > 0$ *and* $u \in L^2(]a,b[\times\mathbb{R}^{2n})$. *Suppose that for some* $k_0 \in \mathbb{N}$ *we have*

$$\begin{aligned}\Lambda^{k\varepsilon_0}(\langle x\rangle^{-k}\varphi(t)u) &\in L^2(\mathbb{R}^{2n+1}) \quad \textit{and} \\ \Lambda^{k\varepsilon_0}(\langle x\rangle^{-k}\varphi(t)Pu) &\in L^2(\mathbb{R}^{2n+1})\end{aligned} \tag{5.4}$$

for any $\varphi \in C_0^\infty(]a,b[)$ *and* $0 \leq k \leq k_0$. *Then we have that*

$$\Lambda^{(k_0+1)\varepsilon_0}(\langle x\rangle^{-k_0-1}\varphi(t)u) \in L^2(\mathbb{R}^{2n+1}) \tag{5.5}$$

for any $\varphi \in C_0^\infty(]a,b[)$.

Consider now $s > 0$ in Theorem 1.2 and take $k_0 \in \mathbb{N}$ such that $k_0\varepsilon_0 \leq s$. Then we have by hypothesis of Theorem 1.2 that $f \in H^{k_0\varepsilon_0}(]a, b[\times\mathbb{R}^{2n}) \subset H^s(]a, b[\times\mathbb{R}^{2n})$. Since $\langle x\rangle^{-k_0} \in C_b^\infty(\mathbb{R}^n_x)$ we have $\langle x\rangle^{-k_0} f \in H^{k_0\varepsilon_0}\,(]a, b[\times\mathbb{R}^{2n})$. We prove finally, by induction results of Proposition 5.2, that

$$\langle x\rangle^{-k_0-1}\varphi(t)u \in H^{s+\varepsilon_0}(\mathbb{R}^{2n+1})$$

with $k_0 = [s\varepsilon_0^{-1}] + 1$. We have proved Theorem 1.2.

Proof of Proposition 5.2. The proof is similar to that of Proposition 4.3. We choose $v = \psi\Lambda_\delta^{-2-k_0\varepsilon_0}\Lambda^{k_0\varepsilon_0}(\varphi u) \in H_0^2(]a, b[\times\mathbb{R}^{2n})$ as test function in (5.3). We have

$$\begin{aligned}
&\left\|\Lambda^{\varepsilon_0}(\langle x\rangle^{-k_0-1}\psi\Lambda_\delta^{-2-k_0\varepsilon_0}\Lambda^{k_0\varepsilon_0}(\varphi u))\right\|_{L^2}^2 + \left\|X_0(\langle x\rangle^{-k_0-1}v)\right\|_{L^2}^2 \\
&\quad\leq C\left\{\sum_{j=1}^{4} Re\Big(\langle x\rangle^{-k_0-1}P\psi\Lambda_\delta^{-2-k_0\varepsilon_0}\Lambda^{k_0\varepsilon_0}(\varphi u),\ A_j(\langle x\rangle^{-k_0-1}v)\Big)\right. \\
&\qquad\left. + \|\langle x\rangle^{-k_0}v\|_{L^2}^2\right\}.
\end{aligned}$$

For the commutator terms,

$$[P,\ \psi\Lambda_\delta^{-2-k_0\varepsilon_0}\Lambda^{k_0\varepsilon_0}\varphi]u = \partial_t(\psi\Lambda_\delta^{-2}\Lambda^{k_0\varepsilon_0}\varphi)\psi u + \psi[x,\ \Lambda_\delta^{-2-k_0\varepsilon_0}\Lambda^{k_0\varepsilon_0}]\cdot\nabla_y(\varphi u),$$

we have immediately

$$\begin{aligned}
&\sum_{j=1}^{4}\left|\big(\langle x\rangle^{-k_0-1}[P, \psi\Lambda_\delta^{-2-k_0\varepsilon_0}\Lambda^{k_0\varepsilon_0}\varphi]u,\ A_j(\langle x\rangle^{-k_0-1}v)\big)\right| \\
&\qquad\leq C\Big\{\|\Lambda^{k_0\varepsilon_0}\langle x\rangle^{-k_0}(\psi u)\|_{L^2}^2 + \|\psi u\|_{L^2}^2\Big\} + \frac{1}{1000}\|X_0(\langle x\rangle^{-k_0-1}v))\|_{L^2}^2.
\end{aligned}$$

Finally we prove,

$$\begin{aligned}
&\left\|\Lambda^{\varepsilon_0}(\langle x\rangle^{-k_0-1}\psi\Lambda_\delta^{-2-k_0\varepsilon_0}\Lambda^{k_0\varepsilon_0}(\varphi u))\right\|_{L^2}^2 \\
&\quad\leq C\Big\{\|\Lambda^{k_0\varepsilon_0}\langle x\rangle^{-k_0}\varphi Pu\|_{L^2(\mathbb{R}^{2n+1})}^2 \\
&\qquad + \|\Lambda^{k_0\varepsilon_0}(\langle x\rangle^{-k_0}\varphi u)\|_{L^2}^2 + \|\varphi u\|_{L^2}^2\Big\}.
\end{aligned}$$

Taking $\delta \to 0$, we have proved Proposition 5.2, since $[\langle x\rangle^{-k_0-1}\psi,\ \Lambda^{k_0\varepsilon_0}]$ is a pseudo-differential operator of order $k_0\varepsilon_0 - 1$. □

Proof of Theorem 1.3. We have proved, in Proposition 2.4, that the Cauchy problem (1.2) admits a weak solution $u \in L^\infty(]0, T[; H^s(\mathbb{R}^{2n}))$ if $u_0 \in H^s(\mathbb{R}^{2n}))$ and $s > n$. By using Sobolev embedding theorem, the condition

$s > n$ implies that $u \in L^\infty(]0,T[; H^s(\mathbb{R}^{2n})) \cap L^\infty(]0,T[\times\mathbb{R}^{2n})$. Now Lemma 2.2 ensures the stability in Sobolev space by nonlinear composition.

We prove the following proposition for nonlinear hypoellipticity. It deduces immediately Theorem 1.3.

Proposition 5.3. *Suppose that* $1/3 < \alpha < 1$ *and* $F \in C^\infty(\mathbb{R}), F(0) = 0$. *Let* $u \in L^2(]a,b[\times\mathbb{R}^{2n}) \cap L^\infty(]a,b[\times\mathbb{R}^{2n})$ *be a weak solution of equation* $Pu = F(u)$ *in* $]a,b[\times\mathbb{R}^{2n}$. *Then for any* $m \in \mathbb{N}$, *there exists* $m_0 \in \mathbb{N}$ *such that*

$$\langle x\rangle^{-m_0} u \in H^m(]a',b'[\times\mathbb{R}^{2n}),$$

for any $a < a' < b' < b$. *In particular, we have that* $u \in C^\infty(]a,b[\times\mathbb{R}^{2n})$.

Proof. We prove also this proposition by induction. By hypothesis, we have that $u \in L^2(]a,b[\times\mathbb{R}^{2n}) \cap L^\infty(]a,b[\times\mathbb{R}^{2n})$, then $\widetilde{F}(t,x,y) = F(u(t,x,y)) \in L^2(]a,b[\times\mathbb{R}^{2n})$. Proposition 5.2 with $k = 0$ deduces that for any $\varphi \in C_0^\infty(]0,T[)$, there exists a constant $C > 0$ and $\psi \in C_0^\infty(]0,T[)$ with $\varphi \subset\subset \psi$ such that

$$\left\|\langle x\rangle^{-1}\varphi u\right\|^2_{H^{a/4-1/12}} \le C\left\{\|\psi\widetilde{F}\|^2_{L^2} + \|\psi u\|^2_{L^2}\right\}. \tag{5.6}$$

We suppose now for some $k \in \mathbb{N}$ and any $\varphi \in C_0^\infty(]0,T[)$,

$$\langle x\rangle^{-k}\varphi u \in H^{k\varepsilon_0}(\mathbb{R}^{2n+1}),$$

here $\varepsilon_0 = \frac{1}{4}\left(\alpha - \frac{1}{3}\right) > 0$. We want to prove that

$$\langle x\rangle^{-k-1}\varphi u \in H^{(k+1)\varepsilon_0}(\mathbb{R}^{2n+1}).$$

But from Proposition 5.2, we need only to prove that

$$\|\Lambda^{k\varepsilon_0}\langle x\rangle^{-k}\varphi F(u)\|^2_{L^2(\mathbb{R}^{2n+1})} \le C\Big\{\|\Lambda^{k\varepsilon_0}\langle x\rangle^{-k}\psi u\|^2_{L^2(\mathbb{R}^{2n+1})} + \|\psi u\|^2_{L^2}\Big\},$$

with the constant C as in Lemma 2.2. The proof of this estimate is also the same as that of Lemma 2.2. We just remark that, for the nonlinear function

$$\widetilde{F}(x,v) = \langle x\rangle^{-k}F(\langle x\rangle^k v),$$

if $v \in H^{k_0\varepsilon_0}$ and $\langle x\rangle^k v \in L^\infty$, then for any $\lambda \in \mathbb{N}^n$,

$$|\partial_x^\lambda \widetilde{F}(x,v)| \le C_\lambda,$$

and $\widetilde{F}((x,0) = 0$. We omit the detail of this modification and sending to [14] for example. □

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