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Positive continuous additive functionals of multidimensional Brownian motion and the Brownian local time

By

Hideaki Uemura

1. Introduction

The local time of multidimensional Brownian motion was first introduced by Imkeller and Weisz [7]. They showed the existence of the limit $L(t,x)$ of $\int_0^t p_N(\varepsilon, W_s - x)ds$ as a generalized Wiener functional unless $x = 0$, where W_s denotes the N-dimensional Brownian motion starting from the origin and $p_N(s,y)$ the Gaussian kernel:

$$p_N(s,y) = \left(\frac{1}{\sqrt{2\pi s}}\right)^N \mathrm{e}^{-|y|^2/2s}, \qquad (s > 0, y \in \mathbb{R}^N).$$

Let $\varphi \in \mathcal{D}$, a smooth function on $\mathbb{R}^N$ with compact support, satisfy $\int \varphi(y)dy = 1$. Put $\varphi_\varepsilon(y) = \varphi(y/\varepsilon)/\varepsilon^N$. Then, using the same technique we also find that $\int_0^t \varphi_\varepsilon(W_s - x)ds$ has the same limit. We note that the functions $p_N(\varepsilon, y - x)$ and $\varphi_\varepsilon(y - x)$ approximate the delta function at x. We call the limit $L(t,x)$ the local time of N-dimensional Brownian motion.

The local time is interpreted as a generalized Wiener functional corresponding to the delta function. Now we are interested in the existence of a generalized Wiener functional corresponding to another positive distribution T. To explain more rigorously, we determine the limit point of $\int_0^t T * \varphi_\varepsilon(W_s + x)ds$ under some conditions. Here $T * \varphi(y) = \int \varphi(y - x)\mu_T(dx)$, $\mu_T(dx)$ denoting the corresponding measure of T, i.e., $\langle T, \varphi\rangle = \int \varphi(x)\mu_T(dx)$. We also discuss on the integral representation of this functional using this measure μ_T and the Brownian local time. Details are discussed in Section 3.

Next we consider positive continuous additive functionals (PCAF in abbreviation) of N-dimensional Brownian motion. In the case where $N = 1$, one of the most typical additive functional is the local time, and every PCAF of the Brownian motion can be represented by the integral of the local time with respect to the Revuz measure associated to the PCAF(see, for instance,

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Revuz and Yor [12]). On the other hand, many authors investigated on the PCAF for given positive distribution T on $\mathbb{R}^N$. Appealing to Itô's formula, Fukushima [3] investigated on PCAF's corresponding to T under some conditions. Yamada [16] also studied on such PCAF's and showed that these PCAF's satisfy the occupation time formula. Moreover he obtained that these PCAF's can be represented by one-dimensional Brownian local times on hypersurface in the sense of distribution through Radon transform. It should be mentioned that Bass [1] obtained the same representation in almost sure sense under a little more restricted conditions. We also mention that, based on the occupation time formula, Nakajima [10] discussed on PCAF's corresponding to T through Fourier transform under milder conditions.

Now we are interested in the integral representations of PCAF with respect to the associated Revuz measure. In the case where $N \geq 2$, we cannot apply the argument developed in one-dimensional case, because there does not exist a Brownian local time as a random variable. There exists, however, the Brownian local time as a generalized Wiener functional, which had been introduced by Imkeller and Weisz [7] as mentioned above.

The second aim of this paper is to obtain the integral representation of square integrable PCAF using the associated Revuz measure and the Brownian local time. We also clarify that, under some conditions, square integrable PCAF of multidimensional Brownian motion is also identified with the generalized Wiener functional corresponding to the distribution defined by the Revuz measure which is determined in Section 3. Details are discussed in Section 4.

2. Preliminaries

In this section we prepare some notation. Let $(\mathbf{W}_0^N, \mathcal{F}_t, P)$ be the N-dimensional standard Wiener space, i.e.,

$$\begin{aligned}\mathbf{W}_0^N &= \{W_t = (W_t^1, W_t^2, \dots, W_t^N) : [0,\infty) \to \mathbb{R}^N;\\ &\qquad W_t \text{ is continuous and } W_0 = 0\},\\ \mathcal{F}_t &= \sigma\{W_s; 0 \leq s \leq t\}\end{aligned}$$

and

$$P = \text{ the standard Wiener measure.}$$

Definition 2.1. $A = \{A(t,x;W.); t \geq 0, x \in \mathbb{R}^N\}$ is called a positive continuous additive functional (PCAF) of the N-dimensional Brownian motion if and only if $A(t,x) = A(t,x;W.)$ is an $\mathcal{F}_t$-measurable continuous nondecreasing process satisfying $A(0,x) = 0$ such that, almost surely,

$$A(t+s,x;W.) - A(t,x;W.) = A(s,x+W_t;(\theta_t W).), \tag{2.1}$$

where $(\theta_t W)_s = W_{t+s} - W_t$.

Remark 1. In Revuz and Yor [12], positive additive functionals are defined as follows: Let $\{B_t; t \geq 0\}$ be the N-dimensional Brownian motion. $A = \{A(t); t \geq 0\}$ is called a positive additive functional of the N-dimensional Brownian motion if and only if (i) $A(t) = A(t; B.)$ is an $\mathcal{F}_t$-measurable non-decreasing process satisfying $A(0) = 0$ and (ii) $A(t+s; B.) - A(t; B.) = A(s; (\tau_t B).)$ is satisfied almost surely, where $(\tau_t B). = B_{t+\cdot}$.

Noting that $B_t = B_0 + W_t$, we set $A(t, x) = E[A(t)|B_0 = x]$. Then $A(t, x)$ $(t \geq 0, x \in \mathbb{R}^N)$ is a random variable of $W.$ and satisfies the conditions as in Definition 2.1.

We introduce the Revuz measure associated to the PCAF (see McKean and Tanaka [9], Fukushima, Oshima and Takeda [4] and Revuz and Yor [12]).

Definition 2.2. The Revuz measure ν_A associated to an additive functional A is the measure on $\mathbb{R}^N$ such that

$$\int_{\mathbb{R}^N} f(x)\nu_A(dx) = \int E\Big[\int_0^1 f(W_t + x) d_t A(t, x)\Big] dx \tag{2.2}$$

for all bounded Borel measurable function f.

Remark 2. Since dx is an invariant measure of the Brownian motion, (2.2) above is equivalent to the following:

$$\int_{\mathbb{R}^N} f(x)\nu_A(dx) = \frac{1}{h}\int E\Big[\int_a^{a+h} f(W_t + x) d_t A(t, x)\Big] dx$$

for any $a \geq 0$ and $h > 0$.

We prepare some notation of Wiener chaos as is the main tool of our consideration. For every $\boldsymbol{n} = (n_1, n_2, \ldots, n_N) \in \mathbb{Z}_+^N$, where $\mathbb{Z}_+$ denotes the totality of non-negative integers, we set $|\boldsymbol{n}| = n_1 + n_2 + \cdots + n_N$. We denote $\boldsymbol{n}$-ple Wiener-Itô integral with a kernel function $f_{\boldsymbol{n}}$ by $I_{\boldsymbol{n}}(f_{\boldsymbol{n}})$:

$$\begin{cases} f_{\boldsymbol{n}} = f_{\boldsymbol{n}}(t_1, t_2, \ldots, t_{|\boldsymbol{n}|}) = f_{\boldsymbol{n}}(t_1^{(1)}, \ldots, t_{n_1}^{(1)}, \ldots, t_1^{(N)}, \ldots, t_{n_N}^{(N)}), \\ I_{\boldsymbol{n}}(f_{\boldsymbol{n}}) = \displaystyle\int_0^\infty \cdots \int_0^\infty f_{\boldsymbol{n}}(t_1^{(1)}, \ldots, t_{n_1}^{(1)}, \ldots, t_1^{(N)}, \ldots, t_{n_N}^{(N)}), \\ \qquad\qquad dW^1_{t_1^{(1)}} \cdot \cdot dW^1_{t_{n_1}^{(1)}} \cdots dW^N_{t_1^{(N)}} \cdot \cdot dW^N_{t_{n_N}^{(N)}}, \end{cases}$$

where $(t_1, t_2, \ldots, t_{|\boldsymbol{n}|}) = (t_1^{(1)}, \ldots, t_{n_1}^{(1)}, \ldots, t_1^{(N)}, \ldots, t_{n_N}^{(N)})$ and $f_{\boldsymbol{n}} \in L^2([0, \infty)^{|\boldsymbol{n}|} \to \mathbb{R})$ is symmetric with respect to $t_1^{(j)}, \ldots, t_{n_j}^{(j)}$ for all fixed $j (j = 1, \ldots, N)$. We denote the totality of such functions by $L^2_{\boldsymbol{n}}$ or $L^2_{\boldsymbol{n}}(d\boldsymbol{t})$. $I_{\boldsymbol{0}}(f_{\boldsymbol{0}})$ represents a constant and we also use the notation $f_{\boldsymbol{0}}$ together with $I_{\boldsymbol{0}}(f_{\boldsymbol{0}})$. We remark that multiple Wiener integrals admit the following equality:

$$E[I_{\boldsymbol{n}}(f_{\boldsymbol{n}}) I_{\boldsymbol{m}}(g_{\boldsymbol{m}})] = \begin{cases} 0 & \text{if } \boldsymbol{n} \neq \boldsymbol{m}, \\ \boldsymbol{n}! \langle f_{\boldsymbol{n}}, g_{\boldsymbol{m}} \rangle_{L^2} & \text{if } \boldsymbol{n} = \boldsymbol{m}, \end{cases} \tag{2.3}$$

where $\boldsymbol{n}! = \prod n_i!$ and $\langle \cdot, * \rangle_{L^2}$ denoting the $L^2(dt_1 \dots dt_{|\boldsymbol{n}|})$-inner product.

With the notation above, we define Meyer-Watanabe's Sobolev spaces $\boldsymbol{D}_2^s$ of square integrable type. It is well-known that every $L^2(P)$ function F admits the Wiener chaos expansion (see Itô [8]):

$$F = \sum_{\boldsymbol{n} \in \mathbb{Z}_+^N} I_{\boldsymbol{n}}(f_{\boldsymbol{n}}).$$

For $s > 0$ we define $\boldsymbol{D}_2^s \subset L^2(P)$ as follows:

$$\boldsymbol{D}_2^s = \left\{ F = \sum_{\boldsymbol{n} \in \mathbb{Z}_+^N} I_{\boldsymbol{n}}(f_{\boldsymbol{n}}) \in L^2(P); \|F\|_s^2 = \sum (1 + |\boldsymbol{n}|)^s \boldsymbol{n}! \|f_{\boldsymbol{n}}\|^2 < \infty \right\}$$

where $\|f\|$ denotes the L^2-norm of f. We note that $\boldsymbol{D}_2^s$ endowed with the norm $\|\cdot\|_s$ forms a Banach space and that $\boldsymbol{D}_2^{-s}$ is the dual space of $\boldsymbol{D}_2^s$, which is considered as the totality of series of multiple Wiener integrals satisfying $\|F\|_{-s} < \infty$. $\boldsymbol{D}_2^s$ ($s \in \mathbb{R}$) is called Meyer-Watanabe's Sobolev space. Note that $\boldsymbol{D}_2^s$ above coincides with $\boldsymbol{D}_{2,s}$ in Ikeda and Watanabe [5] or $\mathbb{D}^{s,2}$ in Nualart [11].

We now introduce local times $L(t, x)$ of multidimensional Brownian motions (see Imkeller and Weisz [7]).

Proposition 2.1 ([7]). *Let $x (\neq 0) \in \mathbb{R}^N$ and $t > 0$ be given. Then there exists $L(t, x) \in \boldsymbol{D}_2^\alpha$ such that*

$$\int_0^t p_N(\varepsilon, W_s - x) ds \to L(t, x) \quad \text{as} \quad \varepsilon \to 0 \quad \text{in} \quad \boldsymbol{D}_2^\alpha$$

for all $\alpha < 1 - N/2$.

Remark 3. In the proposition above, $p_N(\varepsilon, \cdot - x)$ is used for a test function converging to δ_x. This can be replaced by $\varphi_\varepsilon(\cdot - x) = \varphi((\cdot - x)/\varepsilon)/\varepsilon^N$ where $\varphi \in \mathcal{D}$.

Remark 4. Proposition 2.1 is shown by the H-derivatives of $\int_0^t p_N(\varepsilon, W_s - x) ds$, and then Imkeller and Weisz also proved that $L(t, x)$ admits the following Itô-Wiener chaos expansions (cf. Imkeller and Weisz [7]):

$$L(t, x) = \sum \int_0^t \frac{1}{\boldsymbol{n}!} I_{\boldsymbol{n}} \left(\left(\frac{1}{\sqrt{s}} \right)^{|\boldsymbol{n}|} \mathbf{1}_{[0,s]}(t_1) \times \cdots \times \mathbf{1}_{[0,s]}(t_{|\boldsymbol{n}|}) \right) \times H_{\boldsymbol{n}} \left(\frac{x}{\sqrt{s}} \right) p_N(s, x) ds,$$

where, for $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ and $\boldsymbol{n} = (n_1, \dots, n_N) \in \mathbb{Z}_+^N$,

$$H_{\boldsymbol{n}}(x) = \prod H_{n_i}(x_i),$$

H_n denoting the Hermite polynomial;

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}, \quad x \in \mathbb{R}.$$

From (2.3), we find that the above expansion coincides with the following one (see Uemura [14], [15]);

$$\begin{aligned} L(t,x) &= \sum I_{\boldsymbol{n}}(f_{\boldsymbol{n}}(t,x)), \\ f_{\boldsymbol{n}}(t,x) &= \int_0^t g_{\boldsymbol{n}}(s,x)\,ds, \end{aligned}$$

(2.4)

$$g_{\boldsymbol{n}}(s,x) = \frac{1}{\boldsymbol{n}!}\left(\frac{1}{\sqrt{s}}\right)^{|\boldsymbol{n}|} H_{\boldsymbol{n}}\left(\frac{x}{\sqrt{s}}\right) p_N(s,x) \mathbf{1}_{[0,s]}(t_1) \times \cdots \times \mathbf{1}_{[0,s]}(t_{|\boldsymbol{n}|}).$$

Remark 5. $g_{\boldsymbol{n}}(s,x)(\boldsymbol{n} \in \mathbb{Z}_+^N)$ above are kernels in Itô-Wiener expansion of $\delta_x(W_s)$, i.e.,

$$\delta_x(W_s) = \sum I_{\boldsymbol{n}}(g_{\boldsymbol{n}}(s,x)).$$

This is also obtained by the same way as in Remark 4.

Proposition 2.2. *The local time $L(t,x)$ of N-dimensional Brownian motion is continuous with respect to $x (\neq 0)$ in $\boldsymbol{D}_2^\alpha$ where $\alpha < 1 - N/2$.*

Proof. $L(t,x) - L(t,y)$ admits the following chaos expansion:

$$L(t,x) - L(t,y) = \sum I_{\boldsymbol{n}}(f_{\boldsymbol{n}}(t,x) - f_{\boldsymbol{n}}(t,y)).$$

Let $B(x,r)$ be the closed ball centered at x with radius r. We set $r < |x|$. Note that for $\delta \in [1/4, 1/2]$

$$\sup_{\xi \in \mathbb{R}} \left| H_n(\xi) e^{-\delta\xi^2} \right| \leq C\sqrt{n!}\, n^{-(8\delta-1)/12} \tag{2.5}$$

and that, for every fixed $a > 0$, $s^{-a} e^{-(1-2\delta)|y|^2/2s}$ is bounded uniformly in $(0,t) \times B(x,r)$ (see Imkeller et al. [6] and Szegö [13]). Thus a slight computation gives that, if $y \in B(x,r)$, $\sum_{|\boldsymbol{n}|>M} (1+|\boldsymbol{n}|)^\alpha \boldsymbol{n}! \|f_{\boldsymbol{n}}(t,y)\|^2 < \varepsilon$ for all $\varepsilon > 0$ and M large enough. Therefore it suffices to show that $\lim_{y \to x} f_{\boldsymbol{n}}(t,y) = f_{\boldsymbol{n}}(t,x)$ in $L_{\boldsymbol{n}}^2$.

$$\begin{aligned} &\|f_{\boldsymbol{n}}(t,y) - f_{\boldsymbol{n}}(t,x)\|^2 \\ &= \left(\frac{1}{\boldsymbol{n}!}\right)^2 \int_0^\infty \cdots \int_0^\infty dt_1 \dots dt_{|\boldsymbol{n}|} \\ &\left(\int_0^t \left(\frac{1}{\sqrt{s}}\right)^{|\boldsymbol{n}|} \left\{ H_{\boldsymbol{n}}\left(\frac{y}{\sqrt{s}}\right) p_N(s,y) - H_{\boldsymbol{n}}\left(\frac{x}{\sqrt{s}}\right) p_N(s,x) \right\} \prod_{j=1}^{|\boldsymbol{n}|} \mathbf{1}_{[0,s]}(t_j)\,ds \right)^2 \end{aligned}$$

$$= \left(\frac{1}{\boldsymbol{n}!}\right)^2 \int_0^\infty \cdots \int_0^\infty dt_1 \ldots dt_{|\boldsymbol{n}|} \left(\int_0^t \left(\frac{1}{\sqrt{s}}\right)^{|\boldsymbol{n}|} \int_0^1 \left\{\frac{\partial}{\partial \xi} H_{\boldsymbol{n}}\left(\frac{x+\xi(y-x)}{\sqrt{s}}\right) p_N(s, x+\xi(y-x))\right\} d\xi \times \prod_{j=1}^{|\boldsymbol{n}|} \mathbf{1}_{[0,s]}(t_j)\, ds\right)^2$$

$$= \left(\frac{1}{\boldsymbol{n}!}\right)^2 \int_0^\infty \cdots \int_0^\infty dt_1 \ldots dt_{|\boldsymbol{n}|} \left(\int_0^t \left(\frac{1}{\sqrt{s}}\right)^{|\boldsymbol{n}|} \cdot \left(\frac{1}{\sqrt{s}}\right) \times \int_0^1 \left\langle y-x, \boldsymbol{H}_{\boldsymbol{n}+\boldsymbol{e}}\left(\frac{x+\xi(y-x)}{\sqrt{s}}\right) p_N(s, x+\xi(y-x))\right\rangle d\xi \times \prod_{j=1}^{|\boldsymbol{n}|} \mathbf{1}_{[0,s]}(t_j)\, ds\right)^2$$

$$\leq |y-x|^2 \left(\frac{1}{\boldsymbol{n}!}\right)^2 \int_0^\infty \cdots \int_0^\infty dt_1 \ldots dt_{|\boldsymbol{n}|} \left(\int_0^t \left(\frac{1}{\sqrt{s}}\right)^{|\boldsymbol{n}|} \sup_{z \in B(x,r)} \left| \left(\frac{1}{\sqrt{s}}\right) \boldsymbol{H}_{\boldsymbol{n}+\boldsymbol{e}}\left(\frac{z}{\sqrt{s}}\right) p_N(s,z)\right| \prod_{j=1}^{|\boldsymbol{n}|} \mathbf{1}_{[0,s]}(t_j)\, ds\right)^2,$$

where $\boldsymbol{H}_{\boldsymbol{n}+\boldsymbol{e}}(x) = (H_{\boldsymbol{n}+\boldsymbol{e}_1}(x), \ldots, H_{\boldsymbol{n}+\boldsymbol{e}_N}(x))$ and $\boldsymbol{e}_k = (0, \ldots, 0, \overset{\overset{k}{\vee}}{1}, 0, \ldots, 0) \in \mathbb{Z}_+^N$ $(k = 1, 2, \ldots, N)$. Since the last term goes to 0 as y tends to x, the proof is completed. □

3. Generalized Wiener functionals corresponding to distributions

Let $T \in \mathcal{D}'$ be a positive distribution and $x \in \mathbb{R}^N$ be fixed, where $\mathcal{D}'$ denotes the dual space of $\mathcal{D}$. In this section we discuss on the existence of the generalized Wiener functional corresponding to T. We denote the Radon measure corresponding to T by μ_T, i.e., $\langle T, \varphi\rangle = \int_{\mathbb{R}^N} \varphi(y)\mu_T(dy)$ for all $\varphi \in \mathcal{D}$. To state our claims, we prepare three conditions on measure μ on $\mathbb{R}^N$.

Condition 3.1. $\sup_{y\in\mathbb{R}^N} \mu(B(y,r)) < \infty$ for all $r > 0$.

Condition 3.2. For all $\delta > 0$ and for all $\eta > 0, r > 0$ small enough,

$$\sup_{z\in B(x,r)} \int |y-z|^{2-N-\eta} e^{-\delta|y-z|^2} \mu(dy) < \infty.$$

Condition 3.3. For all $\delta > 0$ and for all $\eta > 0$ small enough,

$$\int |y-x|^{2-N-\eta} e^{-\delta|y-x|^2} \mu(dy) < \infty.$$

Our first assertion is as follows:

Theorem 3.1. *Let $T \in \mathcal{D}'$ be a positive distribution and μ_T be the corresponding Radon measure. Let $\alpha < 1 - N/2$. Suppose μ_T satisfies Conditions* 3.1 *and* 3.2. *Then it holds that*

$$\lim_{\varepsilon \to 0} \int_0^t T * \varphi_\varepsilon(W_s + x)ds = A_T(t,x) \quad \text{in} \quad \boldsymbol{D}_2^\alpha, \tag{3.1}$$

where

$$\begin{aligned} A_T(t,x) &= \sum I_{\boldsymbol{n}}(a^0_{\boldsymbol{n}}(t,x)), \\ a^0_{\boldsymbol{n}}(t,x) &= \int_0^t \int g_{\boldsymbol{n}}(s, z-x)\mu_T(dz)ds, \end{aligned} \tag{3.2}$$

$g_{\boldsymbol{n}}(s, z-x)$ is as in (2.4), *$\varphi_\varepsilon(y) = \varphi(y/\varepsilon)/\varepsilon^N$, $\varphi \in \mathcal{D}$ satisfying $\int_{\mathbb{R}^N} \varphi(y)dy = 1$ and $T * \varphi(y) = \int \varphi(y-z)\mu_T(dz)$.*

Remark 6. If we set $T = \delta_0$ and $x \neq 0$, then we obtain the local time $L(t,x)$ of N-dimensional Brownian motion as $A_T(t,-x)$. If $x = 0$, then Condition 3.2 is not satisfied and we do not have any limit point in (3.1).

Proof of Theorem 3.1. From Condition 3.1 we have $T * \varphi_\varepsilon \in C_b^\infty(\mathbb{R}^N)$, the totality of smooth functions which and whose derivatives of any orders are bounded. Thus $\int_0^t T * \varphi_\varepsilon(W_s + x)ds$ admits the following Itô-Wiener expansion:

$$\begin{aligned} \int_0^t T * \varphi_\varepsilon(W_s + x)ds &= \sum I_{\boldsymbol{n}}(a^\varepsilon_{\boldsymbol{n}}(t,x)), \\ a^\varepsilon_{\boldsymbol{n}}(t,x) &= \int_0^t \iint \varphi_\varepsilon(y-z)\mu_T(dz)g_{\boldsymbol{n}}(s, y-x)dyds. \end{aligned}$$

Let $y \in \mathbb{R}^N$, $\delta \in [1/4, 1/2)$ and $\eta > 0$. From (2.5) and that

$$\mathrm{e}^{-(1-2\delta)|y|^2/2s} \leq C(\sqrt{s})^{N-2+\eta}|y|^{-N+2-\eta}\mathrm{e}^{-(1-2\delta)|y|^2/4t},$$

we see that

$$\left| H_{\boldsymbol{n}}\left(\frac{y}{\sqrt{s}}\right) p_N(s,y) \right| \leq C\sqrt{\boldsymbol{n}!}\boldsymbol{n}^{-(8\delta-1)/12}s^{-1+\eta/2}|y|^{-N+2-\eta}\mathrm{e}^{-(1-2\delta)|y|^2/4t}, \tag{3.3}$$

and therefore

$$\begin{aligned} |\varphi_\varepsilon(y-z)g_{\boldsymbol{n}}(s, y-x)| &\leq C\sqrt{\boldsymbol{n}!}\boldsymbol{n}^{-(8\delta-1)/12}s^{(-|\boldsymbol{n}|-2+\eta)/2}|y-x|^{-N+2-\eta} \\ &\quad \times \mathrm{e}^{-(1-2\delta)|y-x|^2/4t} \times \mathbf{1}_{[0,s]}(t_1) \times \cdots \times \mathbf{1}_{[0,s]}(t_{|\boldsymbol{n}|})\mathbf{1}_{B(0,\varepsilon\rho)}(y-z), \end{aligned}$$

where $\boldsymbol{n}^a = \prod n_j^a$ and $\rho > 0$ is a constant such that the support of φ is included in $B(0,\rho)$. By taking $\eta < 2$, the right-hand side above belongs to $L^1(dsdy\mu_T(dz))$ from Condition 3.1. Therefore, from Fubini's theorem,

$$\begin{aligned}\int_0^t \iint \varphi_\varepsilon(y-z)\mu_T(dz)g_{\boldsymbol{n}}(s,y-x)dyds \\ &= \int_0^t \iint \varphi_\varepsilon(y-z)g_{\boldsymbol{n}}(s,y-x)dy\mu_T(dz)ds \\ &= \int_0^t \iint \varphi(w)g_{\boldsymbol{n}}(s,\varepsilon w+z-x)dw\mu_T(dz)ds.\end{aligned}$$

By a similar argument as above, we get

$$\begin{aligned}|\varphi(w)g_{\boldsymbol{n}}(s,\varepsilon w+z-x)| &\le C\sqrt{\boldsymbol{n}!}\boldsymbol{n}^{-(8\delta-1)/12}s^{(-|\boldsymbol{n}|-2+\eta)/2}|\varepsilon w+z-x|^{-N+2-\eta} \\ &\times \mathrm{e}^{-(1-2\delta)|\varepsilon w+z-x|^2/4t}\mathbf{1}_{[0,s]}(t_1)\times\cdots\times\mathbf{1}_{[0,s]}(t_{|\boldsymbol{n}|})\mathbf{1}_{B(0,\rho)}(w),\end{aligned}$$

which belongs to $L^1(dsdw\mu_T(dz))$ from Condition 3.2. Thus we have

$$a_{\boldsymbol{n}}^\varepsilon(t,x) = \int\int_0^t\int \varphi(w)g_{\boldsymbol{n}}(s,\varepsilon w+z-x)\mu_T(dz)dsdw.$$

Therefore

$$\begin{aligned}&\|a_{\boldsymbol{n}}^\varepsilon(t,x)-a_{\boldsymbol{n}}^0(t,x)\|^2 \\ &= \left(\frac{1}{\boldsymbol{n}!}\right)^2 \Bigg\|\int\int_0^t \varphi(w)\left(\frac{1}{\sqrt{s}}\right)^{|\boldsymbol{n}|}\prod_{j=1}^{|\boldsymbol{n}|}\mathbf{1}_{[0,s]}(t_j) \\ &\qquad \left\{\int H_{\boldsymbol{n}}\left(\frac{\varepsilon w+z-x}{\sqrt{s}}\right)p_N(s,\varepsilon w+z-x)\mu_T(dz)\right. \\ &\qquad\qquad \left.-\int H_{\boldsymbol{n}}\left(\frac{z-x}{\sqrt{s}}\right)p_N(s,z-x)\mu_T(dz)\right\}dsdw\Bigg\|^2 \\ &= \left(\frac{1}{\boldsymbol{n}!}\right)^2\{I(\varepsilon,\varepsilon)+I(0,0)-2I(\varepsilon,0)\},\end{aligned}$$

where

$$\begin{aligned}&I(\varepsilon_1,\varepsilon_2) \\ &= \int_0^\infty\cdots\int_0^\infty\left(\int\int_0^t\varphi(w_1)\left(\frac{1}{\sqrt{s}}\right)^{|\boldsymbol{n}|}\prod_{j=1}^{|\boldsymbol{n}|}\mathbf{1}_{[0,s]}(t_j)\right. \\ &\qquad \left.\int H_{\boldsymbol{n}}\left(\frac{\varepsilon_1 w_1+z_1-x}{\sqrt{s}}\right)p_N(s,\varepsilon_1 w_1+z_1-x)\mu_T(dz_1)dsdw_1\right) \\ &\quad \times\left(\int\int_0^t\varphi(w_2)\left(\frac{1}{\sqrt{u}}\right)^{|\boldsymbol{n}|}\prod_{j=1}^{|\boldsymbol{n}|}\mathbf{1}_{[0,u]}(t_j)\right.\end{aligned}$$

$$\int H_{\boldsymbol{n}}\left(\frac{\varepsilon_2 w_2 + z_2 - x}{\sqrt{u}}\right) p_N(u, \varepsilon_2 w_2 + z_2 - x)\mu_T(dz_2)du dw_2\Bigg) dt_1 \cdots dt_{|\boldsymbol{n}|}$$
$$= 2\iiint_0^t\int_0^1\iint \varphi(w_1)\varphi(w_2) s v^{|\boldsymbol{n}|/2} H_{\boldsymbol{n}}\left(\frac{\varepsilon_1 w_1 + z_1 - x}{\sqrt{s}}\right) p_N(s, \varepsilon_1 w_1 + z_1 - x)$$
$$H_{\boldsymbol{n}}\left(\frac{\varepsilon_2 w_2 + z_2 - x}{\sqrt{sv}}\right) p_N(sv, \varepsilon_2 w_2 + z_2 - x)\mu_T(dz_1)\mu_T(dz_2) dv ds dw_1 dw_2.$$

From (2.5) and (3.3) it holds that $\sup_{\varepsilon_1,\varepsilon_2}\sum_{|\boldsymbol{n}|=n} 1/\boldsymbol{n}! I(\varepsilon_1,\varepsilon_2) < C(1+n)^{N(1-(8\delta-1)/6)-2}$, where C is a constant independent of n (see Imkeller and Weisz [7]). Thus, by setting $\alpha < 1 - N(1-(8\delta-1)/6)$, $\|\sum_{|\boldsymbol{n}|\geq K} I_{\boldsymbol{n}}(a^{\varepsilon}_{\boldsymbol{n}}(t,x) - a^0_{\boldsymbol{n}}(t,x))\|_\alpha$ is small enough uniformly in ε if K is large enough. Therefore it suffices to show that $\lim_{\varepsilon\to 0} I(\varepsilon,\varepsilon) = \lim_{\varepsilon\to 0} I(\varepsilon, 0) = I(0,0)$.

Set $A_{00} = \{(z_1,z_2); |z_1 - x| \leq M, |z_2 - x| \leq M\}$, $A_{01} = \{(z_1,z_2); |z_1 - x| \leq M, |z_2 - x| > M\}$ and $A_1 = \{(z_1,z_2); |z_1 - x| > M\}$ for $M > 1$. From (3.3) we have

$$\int_{|z-x|>M}\left|H_{\boldsymbol{n}}\left(\frac{\varepsilon w + z - x}{\sqrt{s}}\right)\right| p_N(s, \varepsilon w + z - x)\mu_T(dz)$$
$$\leq C s^{-1+\eta/2}\int_{|z-x|>M} |\varepsilon w + z - x|^{-N+2-\eta} \mathrm{e}^{-(1-\delta)|\varepsilon w + z - x|^2/4t}\mu_T(dz)$$
$$\leq C s^{-1+\eta/2}\int_{|\varepsilon w + z - x|>M-1} |\varepsilon w + z - x|^{-N+2-\eta} \mathrm{e}^{-(1-\delta)|\varepsilon w + z - x|^2/4t}\mu_T(dz)$$
$$\leq C(M-1)^{-\eta/2} s^{-1+\eta/2}\sup_{y\in B(x,1)}\int |z-y|^{-N+2-\eta/2}\mathrm{e}^{-(1-\delta)|z-y|^2/4t}\mu_T(dz)$$

for any $\varepsilon < 1/\rho$. Therefore

$$\Bigg|\iiint_0^t\int_0^1\iint_{A_1}\varphi(w_1)\varphi(w_2) s v^{|\boldsymbol{n}|/2} H_{\boldsymbol{n}}\left(\frac{\varepsilon_1 w_1 + z_1 - x}{\sqrt{s}}\right) p_N(s, \varepsilon_1 w_1 + z_1 - x)$$
$$\times H_{\boldsymbol{n}}\left(\frac{\varepsilon_2 w_2 + z_2 - x}{\sqrt{sv}}\right) p_N(sv, \varepsilon_2 w_2 + z_2 - x)\mu_T(dz_1)\mu_T(dz_2) dv ds dw_1 dw_2\Bigg|$$
$$\leq C(M-1)^{-\eta/2}\iint\int_0^t\int_0^1 \varphi(w_1)\varphi(w_2) s^{-1+\eta} v^{|\boldsymbol{n}|/2-1+\eta/2}$$
$$\times \sup_{y_1\in B(x,1)}\int |z_1 - y_1|^{-N+2-\eta/2}\mathrm{e}^{-(1-\delta)|z_1-y_1|^2/4t}\mu_T(dz_1)$$
$$\times \int |\varepsilon_2 w_2 + z_2 - x|^{-N+2-\eta}\mathrm{e}^{-(1-\delta)|\varepsilon_2 w + z_2 - x|^2/4t}\mu_T(dz_2) dv ds dw_1 dw_2$$
$$\leq C(M-1)^{-\eta/2}\sup_{y_1\in B(x,1)}\int |z_1 - y_1|^{-N+2-\eta/2}\mathrm{e}^{-(1-\delta)|z_1-y_1|^2/4t}\mu_T(dz_1)$$
$$\times \sup_{y_2\in B(x,1)}\int |z_2 - y_2|^{-N+2-\eta}\mathrm{e}^{-(1-\delta)|z_2-y_2|^2/4t}\mu_T(dz_2),$$

which converges to 0 uniformly as M tends to infinity. We also see that the integral on A_{01} converges to 0 uniformly as M tends to infinity by the same computation. We finally consider the integral on A_{00}. Note that

$$\mathbf{1}_{A_{00}}(z_1,z_2)\mathbf{1}_{[0,t]}(s)\mathbf{1}_{[0,1]}(v)\varphi(w_1)\varphi(w_2)\mu_T(dz_1)\mu_T(dz_2)dvdsdw_1dw_2$$

is a finite measure. From (3.3) and Condition 3.2, it is easy to see that

$$sv^{|\boldsymbol{n}|/2}H_{\boldsymbol{n}}\left(\frac{\varepsilon_1 w_1+z_1-x}{\sqrt{s}}\right)p_N(s,\varepsilon_1 w_1+z_1-x) \times H_{\boldsymbol{n}}\left(\frac{\varepsilon_2 w_2+z_2-x}{\sqrt{sv}}\right)p_N(sv,\varepsilon_2 w_2+z_2-x)$$

is uniformly integrable with respect to this measure. Therefore

$$\begin{aligned}\lim_{\varepsilon_1,\varepsilon_2\to 0}&\iint\int_0^t\int_0^1\iint_{A_{00}}\varphi(w_1)\varphi(w_2)sv^{|\boldsymbol{n}|/2}\\&\times H_{\boldsymbol{n}}\left(\frac{\varepsilon_1 w_1+z_1-x}{\sqrt{s}}\right)p_N(s,\varepsilon_1 w_1+z_1-x)\\&\times H_{\boldsymbol{n}}\left(\frac{\varepsilon_2 w_2+z_2-x}{\sqrt{sv}}\right)p_N(sv,\varepsilon_2 w_2+z_2-x)\mu_T(dz_1)\mu_T(dz_2)dvdsdw_1dw_2\\=&\int_0^t\int_0^1\iint_{A_{00}}sv^{|\boldsymbol{n}|/2}H_{\boldsymbol{n}}\left(\frac{z_1-x}{\sqrt{s}}\right)p_N(s,z_1-x)\\&\times H_{\boldsymbol{n}}\left(\frac{z_2-x}{\sqrt{sv}}\right)p_N(sv,z_2-x)\mu_T(dz_1)\mu_T(dz_2)dvds.\end{aligned}$$

Hence we conclude that $\lim_{\varepsilon\to 0}I(\varepsilon,\varepsilon)=\lim_{\varepsilon\to 0}I(\varepsilon,0)=I(0,0)$, which completes the proof. □

Remark 7. $A_T(t,x)$ admits the Itô-Wiener expansion as in (3.2). Since this chaos expansion belongs to $\boldsymbol{D}_2^\alpha$ under Condition 3.3, $A_T(t,x)$ exists as an element of $\boldsymbol{D}_2^\alpha$ under Condition 3.3.

By the same estimate as above, we obtain that

$$\|L(t,y-x)\|_\alpha\le Ct^{\eta/2}|y-x|^{-N+2-\eta}\mathrm{e}^{-(1-2\delta)|y-x|^2/4t}, \tag{3.4}$$

where $7/8-3(1-\alpha)/(4N)<\delta<1/2$. Since $L(t,y-x)$ is continuous in $\boldsymbol{D}_2^\alpha$ with respect to y by Proposition 2.2, $L(t,y-x)$ is Bochner integrable with respect to $\mu_T(dy)$ in $\boldsymbol{D}_2^\alpha$, and we obtain the following theorem.

Theorem 3.2. *Under Condition* 3.3*, it holds that*

$$A_T(t,x)=\int L(t,y-x)\mu_T(dy).$$

4. Brownian local time representation of PCAF

In this section we first consider a PCAF $A=\{A(t,x;W.);t\ge 0,x\in\mathbb{R}^N\}$. We denote the associated Revuz measure by ν_A. We assume the following conditions:

Condition 4.1. For all (t,x), $E[A(t,x)^2] < \infty$.

Condition 4.2. For all $R > 0$ and $0 < r < t$,

$$\int E\left[\left(\int_r^t \chi_R(W_u + x) d_u A(u,x)\right)^2\right]^{1/2} dx < \infty,$$

where $\chi_R(x)$ is a $[0,1]$-valued smooth function such that

$$\chi_R(x) = \begin{cases} 1, & \text{if } |x| \le R, \\ 0, & \text{if } |x| \ge R+1. \end{cases}$$

Our assertion is as follows:

Theorem 4.1. *Let $A = \{A(t,x;W.); t \ge 0, x \in \mathbb{R}^N\}$ be a PCAF satisfying Conditions* 4.1 *and* 4.2. *Suppose ν_A satisfies Condition* 3.3. *Then $\int L(t, y-x)\nu_A(dy)$ exists in $\boldsymbol{D}_2^\alpha$ $(\alpha < 1 - N/2)$ and moreover*

$$A(t,x) = \int L(t, y-x)\nu_A(dy).$$

Combined Theorem 3.2 and the theorem above, we obtain the following corollary.

Corollary 4.1. *Let $A = \{A(t,x;W.); t \ge 0, x \in \mathbb{R}^N\}$ be a PCAF satisfying Conditions* 4.1 *and* 4.2. *Let T be a positive distribution corresponding to the Revuz measure ν_A. Assume ν_A satisfies Condition* 3.3. *Then it holds that $A(t,x) = A_T(t,x)$.*

In the rest of this section, we prove Theorem 4.1 with several lemmas. Without loss of generality we assume $t = 1$ for the simplicity of the proof.

Lemma 4.1. *Assume Condition* 3.3. *Then the following holds in $\boldsymbol{D}_2^\alpha$ $(\alpha < 1 - N/2)$:*

$$\lim_{r \to 0} \int L(r, y-x)\nu_A(dy) = 0.$$

Proof. From (3.4) and Condition 3.3, it follows that

$$\begin{aligned} \lim_{r\to 0}\left\| \int L(r,y-x)\nu_A(dy)\right\|_\alpha &\le \lim_{r\to 0}\int \|L(r,y-x)\|_\alpha \, \nu_A(dy) \\ &\le \lim_{r\to 0} C r^{\eta/2} \int |y-x|^{-N+2-\eta} \mathrm{e}^{-(1-\delta)|y-x|^2/4} \nu_A(dy) = 0, \end{aligned}$$

which completes the proof. □

Thus our target turns to $\int \{L(1, y-x) - L(r, y-x)\}\nu_A(dy)$ for r small enough.

Lemma 4.2. *Assume Condition* 3.3. *Set* $s_i = r + (i-1)(1-r)/m$ $(i = 1, 2, \dots, m+1)$. *Then the following approximation holds in* $\boldsymbol{D}_2^{\alpha-2}$ $(\alpha < 1 - N/2)$:

$$\lim_{m\to\infty}\left\| \int (\{L(1, y-x) - L(r, y-x)\}\nu_A(dy) - \sum_{i=1}^{m} \frac{1-r}{m} \int \delta_{y-x}(W_{s_i})\nu_A(dy) \right\|_{\alpha-2} = 0.$$

Proof. By the same computation as in (3.4) we get

$$\|\delta_{y-x}(W_s)\|_{\alpha-1} < C|y-x|^{-N+2-\eta}\mathrm{e}^{-(1-2\delta)|y-x|^2/4} \tag{4.1}$$

where $7/8 - 3(1-\alpha)/(4N) < \delta < 1/2$, and therefore $\int \delta_{y-x}(W_{s_i})\nu_A(dy)$ exists in $\boldsymbol{D}_2^{\alpha-1}$. For the later use we remark that the constant C above is independent of s. Moreover $\int\{L(1, y-x) - L(r, y-x)\}\nu_A(dy) - \sum_{i=1}^m (1-r)/m \int \delta_{y-x}(W_{s_i})\nu_A(dy)$ admits the following Itô-Wiener expansion:

$$\begin{aligned}&\int\{L(1, y-x) - L(r, y-x)\}\nu_A(dy) - \sum_{i=1}^{m} \frac{1-r}{m}\int \delta_{y-x}(W_{s_i})\nu_A(dy)\\ &= \sum I_{\boldsymbol{n}}\left(\int\left\{\int_r^1 g_{\boldsymbol{n}}(s, y-x)ds - \sum_{i=1}^{m}\frac{1-r}{m}g_{\boldsymbol{n}}(s_i, y-x)\right\}\nu_A(dy)\right).\end{aligned}$$

Note that

$$\begin{aligned}&\int_r^1 g_{\boldsymbol{n}}(s, y-x)ds - \sum_{i=1}^{m}\frac{1-r}{m}g_{\boldsymbol{n}}(s_i, y-x)\\ &= \frac{1}{\boldsymbol{n}!}\sum_{i=1}^{m}\int_{s_i}^{s_{i+1}}\left(\frac{1}{\sqrt{s}}\right)^{|\boldsymbol{n}|}H_{\boldsymbol{n}}\left(\frac{y-x}{\sqrt{s}}\right)p_N(s, y-x)\\ &\qquad\qquad \times\left(\prod \mathbf{1}_{[0,s]}(t_j) - \prod \mathbf{1}_{[0,s_i]}(t_j)\right)ds\\ &\quad + \frac{1}{\boldsymbol{n}!}\sum_{i=1}^{m}\int_{s_i}^{s_{i+1}}\left\{\left(\frac{1}{\sqrt{s}}\right)^{|\boldsymbol{n}|}H_{\boldsymbol{n}}\left(\frac{y-x}{\sqrt{s}}\right)p_N(s, y-x)\right.\\ &\qquad\qquad \left. - \left(\frac{1}{\sqrt{s_i}}\right)^{|\boldsymbol{n}|}H_{\boldsymbol{n}}\left(\frac{y-x}{\sqrt{s_i}}\right)p_N(s_i, y-x)\right\}\prod \mathbf{1}_{[0,s_i]}(t_j)ds\\ &= \tilde{g}_{\boldsymbol{n}}^{(1)}(m, y-x) + \tilde{g}_{\boldsymbol{n}}^{(2)}(m, y-x)\end{aligned}$$

and set

$$A_\Delta^{(1)}(m) = \sum I_{\boldsymbol{n}}\left(\int \tilde{g}_{\boldsymbol{n}}^{(1)}(m, y-x)\nu_A(dy)\right)$$

and

$$A_\Delta^{(2)}(m) = \sum I_{\boldsymbol{n}}\left(\int \tilde{g}_{\boldsymbol{n}}^{(2)}(m, y-x)\nu_A(dy)\right).$$

We first estimate $\|A_\Delta^{(1)}(m)\|_{\alpha-1}$. Since

$$\begin{aligned}\|A_\Delta^{(1)}(m)\|_{\alpha-1}^2 &= \sum_{n=0}^\infty (1+n)^{\alpha-1} \sum_{|\boldsymbol{n}|=n} \boldsymbol{n}! \left\| \int \tilde{g}_{\boldsymbol{n}}^{(1)}(m, y-x)\nu_A(dy) \right\|^2 \\ &\le \sum_{n=0}^\infty (1+n)^{\alpha-1} \sum_{|\boldsymbol{n}|=n} \boldsymbol{n}! \left(\int \left\| \tilde{g}_{\boldsymbol{n}}^{(1)}(m, y-x) \right\| \nu_A(dy) \right)^2,\end{aligned}$$

it suffices to estimate $\|\tilde{g}_{\boldsymbol{n}}^{(1)}(m, y-x)\|$. From (2.5) we have

$$\begin{aligned}&\left\| \left(\frac{1}{\sqrt{s}}\right)^{|\boldsymbol{n}|} H_{\boldsymbol{n}}\left(\frac{y-x}{\sqrt{s}}\right) p_N(s, y-x) \left(\prod \mathbf{1}_{[0,s]}(t_j) - \prod \mathbf{1}_{[0,s_i]}(t_j) \right) \right\|^2 \\ &= \left\{ \left(\frac{1}{\sqrt{s}}\right)^{|\boldsymbol{n}|} H_{\boldsymbol{n}}\left(\frac{y-x}{\sqrt{s}}\right) p_N(s, y-x) \right\}^2 (s^{|\boldsymbol{n}|} - s_i^{|\boldsymbol{n}|}) \\ &\le C s^{-|\boldsymbol{n}|} \boldsymbol{n}! \boldsymbol{n}^{-(8\delta-1)/6} s^{-N} \mathrm{e}^{-(1-2\delta)|y-x|^2/s} (s-s_i) |\boldsymbol{n}| s^{|\boldsymbol{n}|-1} \\ &\le C \boldsymbol{n}! \boldsymbol{n}^{-(8\delta-1)/6} |\boldsymbol{n}| \frac{1}{m} r^{-N-1} \mathrm{e}^{-(1-2\delta)|y-x|^2}.\end{aligned}$$

Thus it holds that

$$\|\tilde{g}_{\boldsymbol{n}}^{(1)}(m, y-x)\| \le C \frac{1}{\sqrt{\boldsymbol{n}!}} \boldsymbol{n}^{-(8\delta-1)/12} \sqrt{|\boldsymbol{n}|} \sqrt{\frac{1}{m}} r^{-(N+1)/2} (1-r) \mathrm{e}^{-(1-2\delta)|y-x|^2/2}$$

and that

$$\begin{aligned}&\left\| \int \tilde{g}_{\boldsymbol{n}}^{(1)}(m, y-x)\nu_A(dy) \right\| \\ &\qquad \le C \frac{1}{\sqrt{\boldsymbol{n}!}} \boldsymbol{n}^{-(8\delta-1)/12} \sqrt{|\boldsymbol{n}|} \sqrt{\frac{1}{m}} r^{-(N+1)/2} (1-r) \int \mathrm{e}^{-(1-2\delta)|y-x|^2/2} \nu_A(dy),\end{aligned}$$

which is finite under Condition 3.3. Hence it holds that

$$\text{(4.2)} \quad \|A_\Delta^{(1)}(m)\|_{\alpha-2}^2 \le \|A_\Delta^{(1)}(m)\|_{\alpha-1}^2 \le C \frac{1}{m} \left(\int \mathrm{e}^{-(1-2\delta)|y-x|^2/2} \nu_A(dy) \right)^2.$$

We next estimate $\|A_\Delta^{(2)}(m)\|_{\alpha-2}$. Note that

$$\begin{aligned}&\left(\frac{1}{\sqrt{s}}\right)^{|\boldsymbol{n}|} H_{\boldsymbol{n}}\left(\frac{y-x}{\sqrt{s}}\right) p_N(s, y-x) - \left(\frac{1}{\sqrt{s_i}}\right)^{|\boldsymbol{n}|} H_{\boldsymbol{n}}\left(\frac{y-x}{\sqrt{s_i}}\right) p_N(s_i, y-x) \\ &= \int_{s_i}^s \frac{\partial}{\partial u} \partial_y^{\boldsymbol{n}} p_N(u, y-x) du = \int_{s_i}^s \frac{1}{2} \Delta \partial_y^{\boldsymbol{n}} p_N(u, y-x) du \\ &= \int_{s_i}^s \frac{1}{2} \sum_{k=1}^N \left(\frac{1}{\sqrt{u}}\right)^{|\boldsymbol{n}|+2} H_{\boldsymbol{n}+2\boldsymbol{e}_k}\left(\frac{y-x}{\sqrt{u}}\right) p_N(u, y-x) du,\end{aligned}$$

where $\partial_y^{\boldsymbol{n}} = \partial^{n_1}/\partial y_1^{n_1} \cdots \partial^{n_N}/\partial y_N^{n_N}$ and Δ denotes the Laplacian. Therefore it holds that

$$\begin{aligned}
&\Bigg\| \Bigg\{ \left(\frac{1}{\sqrt{s}}\right)^{|\boldsymbol{n}|} H_{\boldsymbol{n}}\left(\frac{y-x}{\sqrt{s}}\right) p_N(s, y-x) \\
&\qquad - \left(\frac{1}{\sqrt{s_i}}\right)^{|\boldsymbol{n}|} H_{\boldsymbol{n}}\left(\frac{y-x}{\sqrt{s_i}}\right) p_N(s_i, y-x) \Bigg\} \prod \mathbf{1}_{[0,s_i]}(t_j) \Bigg\| \\
&\le C \int_{s_i}^{s} \sum_{k=1}^{N} \sqrt{(\boldsymbol{n}+2\boldsymbol{e}_k)!}(\boldsymbol{n}+2\boldsymbol{e}_k)^{-(8\delta-1)/12} \left(\frac{1}{\sqrt{u}}\right)^{|\boldsymbol{n}|+2} u^{-N/2} \\
&\qquad \times \mathrm{e}^{-(1-2\delta)|y-x|^2/2u} du \times s_i^{|\boldsymbol{n}|/2} \\
&\le C \sum_{k=1}^{N} \sqrt{(\boldsymbol{n}+2\boldsymbol{e}_k)!}(\boldsymbol{n}+2\boldsymbol{e}_k)^{-(8\delta-1)/12} \int_{s_i}^{s} u^{-(N+2)/2} \mathrm{e}^{-(1-2\delta)|y-x|^2/2u} du
\end{aligned}$$

and that

$$\begin{aligned}
&\|\tilde{g}_{\boldsymbol{n}}^{(2)}(m, y-x)\| \\
&\le C \sum_{k=1}^{N} \frac{\sqrt{(\boldsymbol{n}+2\boldsymbol{e}_k)!}}{\boldsymbol{n}!}(\boldsymbol{n}+2\boldsymbol{e}_k)^{-(8\delta-1)/12} \sum_{i=1}^{m} \int_{s_i}^{s_{i+1}} \int_{s_i}^{s} u^{-(N+2)/2} \\
&\qquad \times \mathrm{e}^{-(1-2\delta)|y-x|^2/2u} du ds \\
&\le C \sum_{k=1}^{N} \frac{\sqrt{(\boldsymbol{n}+2\boldsymbol{e}_k)!}}{\boldsymbol{n}!}(\boldsymbol{n}+2\boldsymbol{e}_k)^{-(8\delta-1)/12} \frac{1}{m} \int_{r}^{1} u^{-(N+2)/2} \mathrm{e}^{-(1-2\delta)|y-x|^2/2u} du \\
&\le C \sum_{k=1}^{N} \frac{\sqrt{(\boldsymbol{n}+2\boldsymbol{e}_k)!}}{\boldsymbol{n}!}(\boldsymbol{n}+2\boldsymbol{e}_k)^{-(8\delta-1)/12} \frac{1}{m} r^{-(N+2)/2}(1-r) \mathrm{e}^{-(1-2\delta)|y-x|^2/2}.
\end{aligned}$$

Since $\mathrm{e}^{-(1-2\delta)|y-x|^2/2}$ is integrable with respect to $\nu_A(dy)$ under Condition 3.3 and $(\boldsymbol{n}+2\boldsymbol{e}_k)!/\boldsymbol{n}! \times (\boldsymbol{n}+2\boldsymbol{e}_k)^{-(8\delta-1)/12} \le (|\boldsymbol{n}|+1)(|\boldsymbol{n}|+2)\boldsymbol{n}^{-(8\delta-1)/12}$, we easily see that

$$\|A_{\Delta}^{(2)}(m)\|_{\alpha-2} \le C \frac{1}{m} \int \mathrm{e}^{-(1-2\delta)|y-x|^2/2} \nu_A(dy). \tag{4.3}$$

From (4.2) and (4.3) above, we obtain the desired result. □

We consider $(1-r)/m \times \sum_{i=1}^{m} \int \delta_{y-x}(W_{s_i})\nu_A(dy)$. We claim the following:

Lemma 4.3. *Assume Condition* 3.3. *Let* $\alpha < 1 - N/2$. *Then it holds that*

$$\begin{aligned}
\lim_{R\to\infty} \sup_{m} \Bigg\| \frac{1-r}{m} \sum_{i=1}^{m} \Bigg(&\int \delta_{y-x}(W_{s_i})\nu_A(dy) \\
&- \int \chi_R(y)\delta_{y-x}(W_{s_i})\nu_A(dy) \Bigg) \Bigg\|_{\alpha-1} = 0.
\end{aligned}$$

Proof. From (4.1) we easily have

$$\left\|\frac{1-r}{m}\sum_{i=1}^{m}\left(\int \delta_{y-x}(W_{s_i})\nu_A(dy) - \int \chi_R(y)\delta_{y-x}(W_{s_i})\nu_A(dy)\right)\right\|_{\alpha-1}$$
$$\leq \frac{1-r}{m}\sum_{i=1}^{m}\int_{|y|\geq R}\|\delta_{y-x}(W_{s_i})\|_{\alpha-1}\nu_A(dy)$$
$$\leq C(1-r)\int_{|y-x|\geq R-|x|}|y-x|^{-N+2-\eta}\mathrm{e}^{-(1-2\delta)|y-x|^2/4}\nu_A(dy)$$

where $7/8-3(1-\alpha)/(4N)<\delta<1/2$. The right-hand side above vanishes as R tends to infinity by Condition 3.3, which completes the proof. □

Thus we consider $(1-r)/m\times\sum_{i=1}^{m}\int\chi_R(y)\delta_{y-x}(W_{s_i})\nu_A(dy)$, which admits the Itô-Wiener expansion

$$\sum I_{\boldsymbol{n}}\left(\frac{1-r}{m}\sum_{i=1}^{m}\int\chi_R(y)g_{\boldsymbol{n}}(s_i,y-x)\nu_A(dy)\right).$$

From the definition of Revuz measure, we have

$$\frac{1-r}{m}\sum_{i=1}^{m}\int\chi_R(y)g_{\boldsymbol{n}}(s_i,y-x)\nu_A(dy)$$
$$=\sum_{i=1}^{m}\int\hat{E}\left[\int_{s_i}^{s_{i+1}}\chi_R(\hat{W}_u+z)g_{\boldsymbol{n}}(s_i,\hat{W}_u+z-x)d_u\hat{A}(u,z)\right]dz,$$

where $(\hat{W}.,\hat{A},\hat{P})$ denotes the independent copy of $(W.,A,P)$.

Lemma 4.4. *Assume Condition* 4.2. *Then it holds that*

$$\lim_{m\to\infty}\left\|\sum_{i=1}^{m}\int\hat{E}\left[\int_{s_i}^{s_{i+1}}\{g_{\boldsymbol{n}}(s_i,\hat{W}_u+z-x)-g_{\boldsymbol{n}}(s_i,\hat{W}_{s_i}+z-x)\}\right.\right.$$
$$\left.\left.\times\chi_R(\hat{W}_u+z)d_u\hat{A}(u,z)\right]dz\right\|=0.$$

Proof. We first note that

$$|g_{\boldsymbol{n}}(s_i,y)-g_{\boldsymbol{n}}(s_i,z)|$$
$$=\left|\frac{1}{\boldsymbol{n}!}\left(\frac{1}{\sqrt{s_i}}\right)^{|\boldsymbol{n}|}\left\{H_{\boldsymbol{n}}\left(\frac{y}{\sqrt{s_i}}\right)p_N(s_i,y)-H_{\boldsymbol{n}}\left(\frac{z}{\sqrt{s_i}}\right)p_N(s_i,z)\right\}\right|\prod\mathbf{1}_{[0,s_i]}(t_j)$$
$$\leq\frac{1}{\boldsymbol{n}!}\sup_w\left(\sum_{k=1}^{N}|\partial_w^{\boldsymbol{n}+\boldsymbol{e}_k}p_N(s_i,w)|^2\right)^{1/2}|y-z|\prod\mathbf{1}_{[0,s_i]}(t_j)$$

$$= \frac{1}{\boldsymbol{n}!} \sup_w \left(\sum_{k=1}^N \left| \left(\frac{1}{\sqrt{s_i}} \right)^{|\boldsymbol{n}|+1} H_{\boldsymbol{n}+\boldsymbol{e}_k} \left(\frac{w}{\sqrt{s_i}} \right) p_N(s_i, w) \right|^2 \right)^{1/2} |y-z| \prod \mathbf{1}_{[0,s_i]}(t_j)$$

$$\leq C \frac{1}{\sqrt{\boldsymbol{n}!}} (|\boldsymbol{n}|+1) \boldsymbol{n}^{-1/4} \left(\frac{1}{\sqrt{r}} \right)^{N+1} |y-z| \left(\frac{1}{\sqrt{s_i}} \right)^{|\boldsymbol{n}|} \prod \mathbf{1}_{[0,s_i]}(t_j).$$

Thus we can write

$$\left\| \sum_{i=1}^m \int \hat{E} \left[\int_{s_i}^{s_{i+1}} \{ g_{\boldsymbol{n}}(s_i, \hat{W}_u + z - x) - g_{\boldsymbol{n}}(s_i, \hat{W}_{s_i} + z - x) \} \right. \right.$$
$$\left. \left. \times \chi_R(\hat{W}_u + z) d_u \hat{A}(u,z) \right] dz \right\|$$
$$\leq \sum_{i=1}^m \int \hat{E} \left[\int_{s_i}^{s_{i+1}} \| g_{\boldsymbol{n}}(s_i, \hat{W}_u + z - x) - g_{\boldsymbol{n}}(s_i, \hat{W}_{s_i} + z - x) \| \right.$$
$$\left. \times \chi_R(\hat{W}_u + z) d_u \hat{A}(u,z) \right] dz$$
$$\leq C \frac{1}{\sqrt{\boldsymbol{n}!}} (|\boldsymbol{n}|+1) \boldsymbol{n}^{-1/4} \left(\frac{1}{\sqrt{r}} \right)^{N+1}$$
$$\times \sum_{i=1}^m \int \hat{E} \left[\sup_{s_i < u < s_{i+1}} |\hat{W}_u - \hat{W}_{s_i}| \int_{s_i}^{s_{i+1}} \chi_R(\hat{W}_u + z) d_u \hat{A}(u,z) \right] dz$$
$$\leq C \frac{1}{\sqrt{\boldsymbol{n}!}} (|\boldsymbol{n}|+1) \boldsymbol{n}^{-1/4} \left(\frac{1}{\sqrt{r}} \right)^{N+1}$$
$$\times \int \hat{E} \left[\max_{1 \leq i \leq m} \sup_{s_i < u < s_{i+1}} |\hat{W}_u - \hat{W}_{s_i}| \int_r^1 \chi_R(\hat{W}_u + z) d_u \hat{A}(u,z) \right] dz$$
$$\leq C \frac{1}{\sqrt{\boldsymbol{n}!}} (|\boldsymbol{n}|+1) \boldsymbol{n}^{-1/4} \left(\frac{1}{\sqrt{r}} \right)^{N+1} \hat{E} \left[\max_{1 \leq i \leq m} \sup_{s_i < u < s_{i+1}} |\hat{W}_u - \hat{W}_{s_i}|^2 \right]^{1/2}$$
$$\times \int \hat{E} \left[\left(\int_r^1 \chi_R(\hat{W}_u + z) d_u \hat{A}(u,z) \right)^2 \right]^{1/2} dz.$$

Set

$$M_k = \left\{ \sup_{s_k < u < s_{k+1}} |\hat{W}_u - \hat{W}_{s_k}|^2 = \max_{1 \leq i \leq m} \sup_{s_i < u < s_{i+1}} |\hat{W}_u - \hat{W}_{s_i}|^2 \right\} \ (k = 1, \ldots, m).$$

Then it is easy to see that

$$\hat{E} \left[\max_{1 \leq i \leq m} \sup_{s_i < u < s_{i+1}} |\hat{W}_u - \hat{W}_{s_i}|^2 \right] = \sum_{k=1}^m \hat{E} \left[\mathbf{1}_{M_k} \sup_{s_k < u < s_{k+1}} |\hat{W}_u - \hat{W}_{s_k}|^2 \right]$$
$$\leq \sum_{k=1}^m \hat{P}[M_k]^{1/2} \hat{E} \left[\sup_{s_k < u < s_{k+1}} |\hat{W}_u - \hat{W}_{s_k}|^4 \right]^{1/2} \leq C \sqrt{\frac{1}{m}}.$$

Hence, from Condition 4.2, we conclude the assertion. □

Remark 8. By setting $A_R(t,x) = \int_0^t \chi_R(W_s + x) d_s A(s,x)$, clearly A_R is also a PCAF. Since dx is an invariant measure, we have

$$\sum_{i=1}^{m} \int \hat{E}\left[\int_{s_i}^{s_{i+1}} g_{\boldsymbol{n}}(s_i, \hat{W}_{s_i} + z - x)\chi_R(\hat{W}_u + z) d_u \hat{A}(u,z)\right] dz$$
$$= \sum_{i=1}^{m} \int g_{\boldsymbol{n}}(s_i, z)\hat{E}[\hat{A}_R(s_{i+1} - s_i, z + x)] dz.$$

Therefore the proof above gives

$$\lim_{m\to\infty} \left\| \frac{1-r}{m} \sum_{i=1}^{m} \int \chi_R(y)\delta_{y-x}(W_{s_i})\nu_A(dy) \right.$$
$$\left. - \sum_{i=1}^{m} \sum I_{\boldsymbol{n}} \left(\int g_{\boldsymbol{n}}(s_i, z)\hat{E}[\hat{A}_R(s_{i+1} - s_i, z + x)] dz \right) \right\|_{\alpha-1} = 0.$$

Proposition 4.1. *Let $F(y)$ be a measurable function on $\mathbb{R}^N$ such that $\int |F(y)| \mathrm{e}^{-\delta |y|^2} dy < \infty$ for all $\delta > 0$ and $E[F(W_t)^2] < \infty$. Then $F(W_t)$ admits the following Itô-Wiener expansion*:

$$F(W_t) = \sum I_{\boldsymbol{n}} \left(\int F(y) g_{\boldsymbol{n}}(t,y) dy \right). \tag{4.4}$$

Proof. From (2.5) it is easy to see that

$$|g_{\boldsymbol{n}}(t,y)| \leq C\sqrt{\boldsymbol{n}!}\boldsymbol{n}^{-(8\delta-1)/12} \mathrm{e}^{-(1-2\delta)|y|^2/2t} \left(\frac{1}{\sqrt{t}}\right)^{|\boldsymbol{n}|+N} \prod \mathbf{1}_{[0,t]}(t_j)$$

for $\delta \in [1/4, 1/2)$, and therefore

$$\sum I_{\boldsymbol{n}} \left(\int F(y) g_{\boldsymbol{n}}(t,y) dy \right) \in \boldsymbol{D}_2^{\alpha-1}$$

for $\alpha < 1 - N(1 - (8\delta - 1)/6)$, which is identified by the Itô-Wiener expansion of $\int F(y)\delta_y(W_t)dy$. On the other hand, since $F(W_t) \in L^2(P) \subset \boldsymbol{D}_2^{\alpha-1}$,

$$\langle F(W_t), G\rangle = E[F(W_t)G] = \int E[F(y)G | W_t = y] p_N(t,y) dy$$
$$= \int F(y)\langle \delta_y(W_t), G\rangle dy = \left\langle \int F(y)\delta_y(W_t)dy, G \right\rangle$$

holds for any $G \in \boldsymbol{D}_2^{1-\alpha}$, where $\langle F, G\rangle$ denotes the coupling of $F \in \boldsymbol{D}_2^{\alpha-1}$ and $G \in \boldsymbol{D}_2^{1-\alpha}$. Therefore $F(W_t) = \int F(y)\delta_y(W_t)dy$ in $\boldsymbol{D}_2^{\alpha-1}$, which completes the proof. □

Remark 9. Since $F(W_t) \in L^2(P)$, the right-hand side of (4.4) converges in $L^2(P)$.

Under Condition 3.3, it is easy to see that

$$\int \hat{E}[\hat{A}_R(s_{i+1}-s_i, z+x)]dz = m \int \chi_R(z+x)\nu_A(dz) < \infty.$$

Since A_R is a PCAF and $(W_{s_i}, \hat{W}) = (W_{s_i}, \theta_{s_i} W)$ in law,

$$\hat{E}[\hat{A}_R(s_{i+1}-s_i, W_{s_i}+x)] = E[A_R(s_{i+1}, x) - A_R(s_i, x)|\mathcal{F}_{s_i}]$$

and therefore

$$E[\hat{E}[\hat{A}_R(s_{i+1}-s_i, W_{s_i}+x)]^2] \le E[(A_R(s_{i+1}, x) - A_R(s_i, x))^2] < \infty.$$

Thus, applying Proposition 4.1 to $\hat{E}[\hat{A}_R(s_{i+1}-s_i, z+x)]$, we obtain that

$$\begin{aligned}
&\sum_{i=1}^{m} \sum I_{\boldsymbol{n}} \left(\int g_{\boldsymbol{n}}(s_i, z-x) \hat{E}[\hat{A}_R(s_{i+1}-s_i, z)]dz \right) \\
&= \sum_{i=1}^{m} \hat{E}[\hat{A}_R(s_{i+1}-s_i, W_{s_i}+x)] \\
&= \sum_{i=1}^{m} E[A_R(s_{i+1}, x) - A_R(s_i, x)|\mathcal{F}_{s_i}].
\end{aligned}$$

Proposition 4.2. *Let A be a PCAF satisfying Condition* 4.1. *Let $\Delta = \{0 = s_0 < s_1 < \cdots < s_n < s_{n+1} = 1\}$ be a partition of $[0,1]$, and put $|\Delta| = \max |s_{i+1} - s_i|$. Then we obtain*

$$\lim_{|\Delta| \to 0} E\left[\left\{ A(1,x) - \sum_{i=1}^{n} E[A(s_{i+1}, x) - A(s_i, x)|\mathcal{F}_{s_i}] \right\}^2 \right] = 0.$$

Proof. Since

$$E[A(s_{i+1}, x) - A(s_i, x)|\mathcal{F}_{s_i}] = E[A(s_{i+1}-s_i, x+y)]\Big|_{y=W_{s_i}},$$

this proposition immediately follows from Dynkin [2, Theorem 6.3]. □

Proof of Theorem 4.1. From (3.4) and Condition 3.3, it is clear that $\int L(r, y-x)\nu_A(dy)$ exists as an element of $\boldsymbol{D}_2^{\alpha}$.

Let $\varepsilon > 0$ be arbitrary. From Lemma 4.1 there exists an $r > 0$ such that

$$\left\| \int L(r, y-x)\nu_A(dy) \right\|_{\alpha} < \varepsilon/7.$$

Simultaneously this r can be taken such that

$$E[A(r,x)^2] < \varepsilon/7.$$

Let R be taken such that

$$E[([A_R(1,x) - A_R(r,x)] - [A(1,x) - A(r,x)])^2] < \varepsilon/7$$

and that

$$\sup_m \left\| \frac{1-r}{m} \sum_{i=1}^{m} \left(\int \delta_{y-x}(W_{s_i}) \nu_A(dy) - \int \chi_R(y) \delta_{y-x}(W_{s_i}) \nu_A(dy) \right) \right\|_{\alpha-1} < \varepsilon/7,$$

which is ensured by Lemma 4.3. If m is large enough,

$$\left\| \int \{L(1, y-x) - L(r, y-x)\} \nu_A(dy) - \sum_{i=1}^{m} \frac{1-r}{m} \int \delta_{y-x}(W_{s_i}) \nu_A(dy) \right\|_{\alpha-2} < \varepsilon/7$$

from Lemma 4.2,

$$\left\| \frac{1-r}{m} \sum_{i=1}^{m} \int \chi_R(y) \delta_{y-x}(W_{s_i}) \nu_A(dy) - \sum_{i=1}^{m} \sum I_{\boldsymbol{n}} \left(\int g_{\boldsymbol{n}}(s_i, z) \hat{E}[\hat{A}_R(s_{i+1} - s_i, z+x)] dz \right) \right\|_{\alpha-1} < \varepsilon/7$$

from Remark 8 and

$$E\left[\left(A_R(1,x) - A_R(r,x) - \sum_{i=1}^{m} E[A_R(s_{i+1}, x) - A_R(s_i, x) | \mathcal{F}_{s_i}] \right)^2 \right] < \varepsilon/7$$

from Proposition 4.2. Therefore

$$\left\| A(1,x) - \int L(1, y-x) \nu_A(dy) \right\|_{\alpha-2} < \varepsilon,$$

which completes the proof. □

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Department of Mathematics Education
Aichi University of Education
Kariya, Aichi 448-8542
Japan
e-mail: huemura@auecc.aichi-edu.ac.jp

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