J. Math. Kyoto Univ. (JMKYAZ)
47-2 (2007), 247–267

Positive Toeplitz operators on pluriharmonic Bergman spaces

By

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Abstract

We study Toeplitz operators on the pluriharmonic Bergman spaces b^p for $1 < p < \infty$. We give characterizations of Toeplitz operators with positive symbols to be bounded, compact and in the Schatten classes. Also, we describe the essential spectra of Toeplitz operators with uniformly continuous symbols.

Acknowledgements. The author express gratitude to B. Choe for several helpful comments about the material in this paper. Also the author thank the referee for helpful comments.

1. Introduction

For fixed integer $n \geq 2$, let $B = B_n$ denote the open unit ball of $\mathbb{C}^n$. A function $u \in C^2(B)$ is said to be pluriharmonic if its restriction to an arbitrary complex line that intersects the ball is harmonic as a function of single complex variable. For $1 \leq p < \infty$, the pluriharmonic Bergman space $b^p = b^p(B)$ is the space of all complex-valued pluriharmonic functions u on B such that

$$\|u\|_p = \left\{ \int_B |u|^p dV \right\}^{1/p} < \infty,$$

where V denotes the normalized Lebesgue volume measure on B. It is well known that b^p is a closed subspace of L^p, and hence is a Banach space. In particular, b^2 is a Hilbert space. We will write Q for the Hilbert space orthogonal projection from L^2 onto b^2. Each point evaluation is a bounded linear functional on b^2. Hence, for each $z \in B$, there exists a unique function $R_z \in b^2$ which has the reproducing property

$$(1.1) \qquad u(z) = \int_B u\bar{R}_z \, dV \qquad (z \in B)$$

for all $u \in b^2$.

This research was supported by KOSEF(R01-2003-000-10243-0).
Received May 23, 2005

A function f in B is pluriharmonic if and only if it admits a decomposition $f = g + \bar{h}$, where g and h are holomorphic. Furthermore, if f is in L^2, then both g and h are in A^2 where $A^2 = A^2(B)$ denotes the holomorphic Bergman space; this is clearly a consequence of the boundedness of the Bergman projection P.

As a result of this observation we see that $b^2 = A^2 + \overline{A^2}$. In particular, there is a simple relation between the pluriharmonic Bergman kernel R_z and the well-known (holomorphic) Bergman kernel K_z;

$$R_z = K_z + \bar{K_z} - 1. \tag{1.2}$$

Thus, the explicit formula of R_z is given by

$$R_z(w) = \frac{1}{(1 - w \cdot \bar{z})^{n+1}} + \frac{1}{(1 - z \cdot \bar{w})^{n+1}} - 1 \qquad (w \in B).$$

Here and elsewhere, $z \cdot \bar{w} = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$ denotes the ordinary Hermitian inner product for points $z, w \in \mathbb{C}^n$. Moreover, using reproducing properties, we have

$$\|R_z(w)\|_2^2 = R(z,z) \approx \frac{1}{(1 - |z|)^{n+1}}. \tag{1.3}$$

For $\varphi \in L^2$ and for each $z \in B$, we have

$$Q\varphi(z) = \int_B \varphi(w) R_z(w) dV(w). \tag{1.4}$$

The formulas (1.1) and (1.2) lead us to the following representation of the projection Q;

$$Q(\varphi) = P(\varphi) + \overline{P(\bar{\varphi})} - P(\varphi)(0) \tag{1.5}$$

for functions $\varphi \in L^2$. From the explicit formula of $R_z(w) = R(z,w)$, one can see

$$|R(z,w)| \le \frac{C}{|1 - z \cdot \bar{w}|^{n+1}} \qquad (z, w \in B) \tag{1.6}$$

so that $R_z \in L^\infty$. Thus, the orthogonal projection Q extends to an integral operator, by means of (1.4), from L^1 into the space of all pluriharmonic functions on B. If $1 < p < \infty$, then Q is a bounded projection from L^p onto b^p. The integral transform Q even extends to $\mathcal{M}$, the space of all complex Borel measures on B. Namely, for each $\mu \in \mathcal{M}$, the integral

$$Q\mu(z) = \int_B R(z,w) d\mu(w) \qquad (z \in B)$$

defines a pluriharmonic function on B. For $\mu \in \mathcal{M}$, the Toeplitz operator T_μ with symbol μ is defined by

$$T_\mu u = Q(u d\mu)$$

for $u \in b^\infty$. In case $\mu = f dV$, we will write $T_\mu = T_f$. Note that T_μ is densely defined on b^p for each $1 < p < \infty$.

Toeplitz operators with positive symbols and uniformly continuous symbols on Bergman space were well studied. Especially, Miao [5] obtained the analogous results on the harmonic Bergman space of the ball. See [2] for results on the half-space and [3] for results on bounded smooth domains. In this paper, we obtain analogous results on the pluriharmonic Bergman space of the ball. In addition, we find the essential spectra of Toeplitz operator [Theorem 4.2]. Main idea used to prove Theorem 4.2 is motivated by [13].

In Section 2 some basic facts are collected. Section 3 is devoted to characterizations of Toeplitz operators with positive symbols to be bounded, compact and in the Schatten classes. In Section 4 we describe the essential spectra of corresponding Toeplitz operators with uniformly continuous symbols.

Notation. We use the notation $A \approx B$ if $A \lesssim B$ and $B \lesssim A$ by writing $A \lesssim B$ for positive quantities A and B if the ratio A/B has a positive upper bound. Constants will be explicitly denoted by the same letter C often with subscripts and indicating dependency, which may change at each occurrence. For $1 < p < \infty$, we use q to denote the conjugate exponent of p, i.e., $1/p + 1/q = 1$. We also use the usual inner product notation

$$\langle u, v \rangle = \int_B u\bar{v}\, dV$$

whenever $u\bar{v} \in L^1$.

2. Preliminaries

In this section we collect some basic facts which we need in later sections. Let $Aut(B)$ be the group of all biholomorphic maps of B onto B. It is known that $Aut(B)$ is generated by the unitary operator on $\mathbb{C}^n$ and the involutions ϕ_z of the form

$$\phi_z(w) = \frac{z - P_z(w) - (1 - |z|^2)^{\frac{1}{2}} Q_z(w)}{1 - w \cdot \bar{z}}$$

where $z \in B$, P_z is the orthogonal projection of $\mathbb{C}^n$ onto the subspace spanned by z, i.e.,

$$P_z(w) = \frac{w \cdot \bar{z}}{|z|^2}\, z \qquad if \quad z \neq 0$$

and $Q_z = I - P_z$. Recall that the well-known identity

$$1 - \phi_z(w) \cdot \overline{\phi_z(a)} = \frac{(1 - |z|^2)(1 - w \cdot \bar{a})}{(1 - w \cdot \bar{z})(1 - z \cdot \bar{a})} \tag{2.1}$$

holds for all $w, a \in \bar{B}$ (see Theorem 2.2.2 of [12]). The real Jacobian $J_R\phi_z$ of ϕ_z is given by

$$J_R\phi_z(w) = \left(\frac{1 - |z|^2}{|1 - w \cdot \bar{z}|^2} \right)^{n+1} \qquad (w \in B).$$

The pseudo-hyperbolic metric on B is defined as $\beta(z,w) = |\phi_z(w)|$. Note that β is Möbius invariant, i.e,

$$\beta(z,w) = \beta(\phi_a(z), \phi_a(w)) \ (z, w \in B) \tag{2.2}$$

for each $a \in B$. For $z \in B$ and r, $0 < r < 1$, let the pseudo-hyperbolic ball $E_r(z)$ with center z and radius r be defined by

$$E_r(z) = \{w \in B| \ \ \beta(z,w) < r\}.$$

Since ϕ_z is an involution, we have $E_r(z) = \phi_z(rB)$. Note that

$$V(E_r(z)) = V(\phi_z(rB)) = \int_{\phi_z(rB)} dV = \int_{rB} J_R\phi_z \, dV \approx (1-|z|)^{n+1}. \tag{2.3}$$

Lemma 2.1. *Let $r \in (0,1)$. For $z, w \in B$, we have $1 - |z| \approx 1 - |w|$ whenever $w \in E_r(z)$.*

Proof. If $w \in E_r(z)$, then $w = \phi_z(a)$ for some $|a| < r$. It follows that

$$1 - |z| < |1 - w\bar{z}| = \left|\frac{1-|z|^2}{1-a\bar{z}}\right| < \frac{2(1-|z|)}{1-r}.$$

Since the condition $w \in E_r(z)$ is symmetric, we have

$$1 - |w| < |1 - z\bar{w}| = \left|\frac{1-|w|^2}{1-a\bar{w}}\right| < \frac{2(1-|w|)}{1-r}.$$

Combining the above two estimates yields

$$1 - |z| \approx |1 - w\bar{z}| \approx 1 - |w|.$$

This completes the proof. $\square$

Lemma 2.2. *There exist some $r_0 \in (0,1)$ and a constant $C > 0$ such that*

$$C^{-1} \leq R(z,w)(1-|z|)^{n+1} \leq C$$

whenever $w \in E_{r_0}(z)$ and $z \in B$.

Proof. It follows from (1.6) that $R(z,w) \lesssim \dfrac{1}{(1-|z|)^{n+1}}$ for $w \in E_r(z)$. To show the lower estimate, for every $z \in B$, (2.1) with the explicit formula of $R(z,w)$ leads to

$$\begin{aligned} R(z,w)(1-|z|)^{n+1} &= (1-|z|)^{n+1}\left(\frac{1}{(1-w\cdot\bar{z})^{n+1}} + \frac{1}{(1-z\cdot\bar{w})^{n+1}} - 1\right) \\ &= (1-\phi_z(w)\cdot\bar{z})^{n+1} + (1-z\cdot\overline{\phi_z(w)})^{n+1} - (1-|z|)^{n+1}. \end{aligned}$$

Consider the function

$$F(a,z) = (1 - a\cdot\bar{z})^{n+1} + (1 - z\cdot\bar{a})^{n+1} - (1-|z|)^{n+1}.$$

Since F is uniformly continuous on $\bar{B}\times\bar{B}$, there exist some r_0 such that $|F(0,z) - F(a,z)| < \frac{1}{2}$ whenever $|a| \le r_0, z \in \bar{B}$. It follows that

$$F(a,z) \ge F(0,z) - \frac{1}{2} = \frac{3}{2} - (1-|z|)^{n+1} \ge \frac{1}{2}.$$

Now we conclude that

$$R(z,w)(1-|z|)^{n+1} = F(\phi_z(w), z) \ge \frac{1}{2}$$

as asserted. □

Lemma 2.3. *Let $1 < p < \infty$. Then there is a constant C such that*

$$C^{-1} \le \|R(z,\cdot)\|_p (1-|z|)^{(1-\frac{1}{p})(n+1)} \le C$$

for every $z \in B$.

Proof. Let $z \in B$. Fix r_0 provided by Lemma 2.2. Then, it follows from (2.3) that

$$\begin{aligned}\|R(z,\cdot)\|_p^p &\geqslant \int_{E_{r_0}(z)} |R(z,w)|^p \, dV(w)\\ &\gtrsim \frac{V(E_{r_0}(z))}{(1-|z|)^{p(n+1)}} \, dV(w)\\ &\approx (1-|z|)^{(1-p)(n+1)}.\end{aligned}$$

For the converse inequality, we have by Proposition 1.4.10 of [12]

$$\begin{aligned}\|R(z,\cdot)\|_p^p &= \int_B |R(z,w)|^p \, dV(w)\\ &\lesssim \int_B \frac{1}{|1 - z\cdot\bar{w}|^{p(n+1)}} \, dV(w)\\ &\lesssim \frac{1}{(1-|z|)^{(p-1)(n+1)}}.\end{aligned}$$

This completes the proof. □

3. Positive Toeplitz operators

In this section we give characterizations of Toeplitz operators with positive symbols to be bounded, compact and in the Schatten classes. For that purpose, we first characterize Carleson measures in terms of the averaging function and Berezin transform.

Let $1 \le p < \infty$. For $\mu \ge 0$, we say that μ is a Carleson measure on b^p if there exists a constant $C > 0$ such that

$$\int_B |u|^p \, d\mu \le C \int_B |u|^p \, dV$$

for all $u \in b^p$. So, μ is a Carleson measure on b^p if and only if the inclusion $i_p : b^p \to L^p(\mu)$ is bounded.

For a positive Borel measure μ on B (simply $\mu \ge 0$) and $r \in (0,1)$, the averaging function $\widehat{\mu}_r$ of μ over the pseudohyperbolic balls $E_r(z)$ is defined by

$$\widehat{\mu}_r(z) = \frac{\mu(E_r(z))}{V(E_r(z))} \qquad (z \in B).$$

Also, for $1 < p < \infty$, we define the Berezin p-transform $\tilde{\mu}_p$ on B by

$$\tilde{\mu}_p(z) = \int_B |r_{z,p}|^p \, d\mu \qquad (z \in B)$$

where

$$r_{z,p}(w) = \frac{R(z,w)}{||R(z,\cdot\,)||_p} \qquad (w \in B)$$

is the L^p-normalized reproducing kernel.

Measures and their averaging functions have the following submean value properties with respect to pseudohyperbolic balls.

Lemma 3.1. *Let $r, \varepsilon \in (0,1)$. Then there exist constants $C = C_{r,\varepsilon}$ such that the following hold for all $\mu \ge 0$ and $z \in B$.*

(1) $\mu(E_r(z)) \le \dfrac{C}{V(E_r(z))} \displaystyle\int_{E_r(z)} \mu(E_\varepsilon(w))\, dV(w).$

(2) $\widehat{\mu}_r(z) \le \dfrac{C}{V(E_r(z))} \displaystyle\int_{E_r(z)} \widehat{\mu}_\varepsilon(w)\, dV(w).$

Proof. Let $z \in B$, and $\mu \ge 0$. Here, χ_E denotes the characteristic function of E. It follows that

$$\begin{aligned}
\int_{E_r(z)} \mu(E_\varepsilon(w))\, dV(w) &= \int_{E_r(z)} \int_{E_\varepsilon(w)} d\mu(a) dV(w) \\
&= \int_B \int_{E_r(z)} \chi_{E_\varepsilon(w)}(a)\, dV(w) d\mu(a) \\
&\ge \int_{E_r(z)} \int_{E_r(z)} \chi_{E_\varepsilon(a)}(w)\, dV(w) d\mu(a) \\
&= \int_{E_r(z)} V[E_\varepsilon(a) \cap E_r(z)]\, d\mu(a) \\
&\ge \mu(E_r(z)) \inf_{a \in E_r(z)} V[E_\varepsilon(a) \cap E_r(z)].
\end{aligned}$$

Thus, it remains to show that

$$\inf_{a\in E_r(z)} V[E_\varepsilon(a)\cap E_r(z)] \gtrsim (1-|z|)^{n+1}. \tag{3.1}$$

To see this, let $a\in E_r(z)$ and $t\zeta=\phi_z(a)$ where $0\le t<r$ and $\zeta\in\partial B$. It is sufficient to consider the case $\varepsilon<\frac{r}{2}$. Then we need to consider only two cases i) $0\le t<r-\varepsilon$ and ii) $\varepsilon\le t<r$. If $0\le t<r-\varepsilon$, then $E_\varepsilon(a)\subset E_r(z)$. Thus, it follows from Lemma 2.1 that we have (3.1). If $\varepsilon\le t<r$, then let $s=t-\frac{\varepsilon}{N}$ where N is chosen so large that $N>\frac{1-r\varepsilon}{1-r^2}$ and put $b=\phi_z(s\zeta)$. It suffices to show that $E_\delta(b)\subset E_\varepsilon(a)\cap E_r(z)$ for some $\delta=\delta(r,\varepsilon)$ to be chosen later. Suppose $w\in E_\delta(b)$. Then we have, for $\delta\le\frac{\varepsilon}{N}$,

$$\beta(w,z)\le\beta(w,b)+\beta(b,z)<\delta+s=\delta+t-\frac{\varepsilon}{N}<r$$

and thus $E_\delta(b)\subset E_r(z)$. From (2.1), we can easily get the identity

$$|\phi_{t\zeta}(s\zeta)|^2=1-\frac{(1-t^2)(1-s^2)}{(1-st)^2}=\frac{(t-s)^2}{(1-st)^2}$$

for $0<s<1$. Hence, we get

$$|\phi_{t\zeta}(s\zeta)|=\frac{|t-s|}{1-st}. \tag{3.2}$$

It follows from (2.2) and (3.2) that we have

$$\begin{aligned}\beta(w,a)&\le\beta(w,b)+\beta(a,b)\\&=\delta+\beta(\phi_z(a),\phi_z(b))\\&=\delta+\beta(t\zeta,s\zeta)\\&=\delta+\frac{t-s}{1-st}\\&<\delta+\frac{\varepsilon}{N-(rN-\varepsilon)r}\\&\le\varepsilon\end{aligned}$$

if we choose $\delta\le\varepsilon(1-\frac{1}{N-(rN-\varepsilon)r})$. Thus we have $E_\delta(b)\subset E_\varepsilon(a)$. So, taking $\delta=\min\{\frac{\varepsilon}{N},\varepsilon(1-\frac{1}{N-(rN-\varepsilon)r})\}$, we obtain $E_\delta(b)\subset E_\varepsilon(a)\cap E_r(z)$. Since $b\in E_r(z)$, it follows from (2.3) and Lemma 2.1 that we have (3.1) as claimed. So, (1) holds. Also, (2) follows from (1) and Lemma 2.1. This completes the proof. $\square$

As an easy consequence of Lemma 3.1, we have the following.

Corollary 3.1. *Let $\mu\ge0$. If $\widehat{\mu}_\varepsilon$ is bounded for some $\varepsilon\in(0,1)$, then so is $\widehat{\mu}_r$ for all $r\in(0,1)$.*

Lemma 3.2. *Let $r\in(0,1)$ and $1<p<\infty$, there exists a constant C such that $\widehat{\mu}_r\le C\widetilde{\mu}_p$ for any $\mu\ge0$.*

Proof. Let $z \in B$. Assume $r = r_0$ where r_0 is the number provided by Lemma 2.2. Then, by Lemma 2.2 and Lemma 2.3, we have

$$\int_{E_r(z)} |r_{z,p}|^p d\mu \approx \frac{\mu(E_r(z))}{(1-|z|)^{n+1}} \approx \widehat{\mu}_r(z)$$

so that

$$\widehat{\mu}_r(z) \lesssim \int_{E_r(z)} |r_{z,p}|^p d\mu \leq \widetilde{\mu}_p(z).$$

It follows from Corollary 3.1 that $\widehat{\mu}_r \leq C\widetilde{\mu}_p$ for a given r. This completes the proof. $\square$

We also will need a decomposition of B whose proof is essentially the same as the ball version of that for covering Lemma of [7]. So, we omit the details.

Lemma 3.3. *Let $r \in (0,1)$. Then there exists a sequence $\{z_i\}$ in B such that*

(1) $\cup_{i=1}^{\infty} E_{\frac{r}{3}}(z_i) = B$.

(2) *There is a positive integer N such that each $E_r(z_i)$ intersects at most N of the balls $E_r(z_i)$.*

Note that $|z_i| \to 1$ as $i \to \infty$. In what follows, the sequence $\{z_i\}$=$\{z_i(r)\}$ will always refer to the sequence chosen in Lemma 3.3.

Now, we characterize Carleson measure on b^p in terms of averaging functions and Berezin transforms.

Theorem 3.1. *Let $1 < p < \infty$ and $r \in (0,1)$. For $\mu \geq 0$, The following conditions are all equivalent.*

(1) *μ is a Carleson measure on b^p.*

(2) *$\widetilde{\mu}_p$ is a bounded on B.*

(3) *$\widehat{\mu}_r$ is a bounded on B.*

(4) *The sequence $\{\widehat{\mu}_r(z_i)\}$ is bounded.*

Note that conditions (1) and (2) are independent of r, while conditions (3) and (4) are independent of p. Thus, the notion of Carleson measures on b^p is independent of $1 < p < \infty$. So, we will simply say that $\mu \geq 0$ is a Carleson measure if one of the four conditions above holds for μ.

Proof. Since $||r_{z,p}||_p = 1$ and μ is a Carleson measure, the implication (1) $\Rightarrow$ (2) follows immediately.

Next, the implication (2) $\Rightarrow$ (3) follows from Lemma 3.2.

Clearly, we have (3) $\Rightarrow$ (4).

Finally, suppose (4) and show (1). Let $u \in b^p$. Since $|u|^p$ is plurisubharmonic, we have

$$|u(w)|^p \lesssim \frac{1}{(1-|w|)^{n+1}} \int_{E_{\frac{r}{3}}(w)} |u|^p dV$$

for $w \in B$. This, together with Lemma 2.1, yields

$$\begin{aligned}\sup_{w\in E_{\frac{r}{3}}(z)} |u(w)|^p &\lesssim \sup_{w\in E_{\frac{r}{3}}(z)} \frac{1}{(1-|w|)^{n+1}} \int_{E_{\frac{r}{3}}(w)} |u|^p dV \\ &\lesssim \frac{1}{(1-|z|)^{n+1}} \int_{E_r(z)} |u|^p dV \\ &\lesssim \frac{1}{V(E_r(z))} \int_{E_r(z)} |u|^p dV\end{aligned}$$

for $z \in B$. Now by Lemma 3.3, we have

$$\begin{aligned}\int_B |u|^p d\mu &\le \sum_{i=1}^{\infty} \int_{E_{\frac{r}{3}}(z_i)} |u|^p d\mu \\ &\le \sum_{i=1}^{\infty} \mu(E_{\frac{r}{3}}(z_i)) \sup_{w\in E_{\frac{r}{3}}(z_i)} |u(w)|^p \\ &\lesssim \sum_{i=1}^{\infty} \frac{\mu(E_{\frac{r}{3}}(z_i))}{V(E_r(z_i))} \int_{E_r(z_i)} |u|^p dV \qquad (3.3) \\ &\le N(\sup_i \widehat{\mu}_r(z_i)) \int_B |u|^p dV \\ &\le C \int_B |u|^p dV.\end{aligned}$$

Hence, μ is a Carleson measure on b^p. The proof is complete. ☐

The above proof shows that the implications (3) ⇒ (4) ⇒ (1) holds for $p = 1$. So, we have the following.

Corollary 3.2. *If $\mu \ge 0$ is a Carleson measure, μ is a Carleson measure on b^1.*

Also, by carefully examining the proof above, one can see that the following equivalences between various quantities.

Corollary 3.3. *Let $1 < p < \infty$ and $r \in (0, 1)$. For $\mu \ge 0$, we have*

$$\sup_{0\ne u\in b^p} \frac{\int_B |u|^p \, d\mu}{\int_B |u|^p \, dV} \approx \sup_{z\in B} \tilde{\mu}_p(z) \approx \sup_{z\in B} \widehat{\mu}_r(z) \approx \sup_i \widehat{\mu}_r(z_i).$$

Having Theorem 3.1, we now turn to the characterizations of bounded positive Toeplitz operators on b^p. For $\mu \ge 0$, recall that the Toeplitz operator T_μ densely defined on b^p is given by

$$T_\mu u(z) = \int_B R(z, w) u(w) \, d\mu(w)$$

for functions $u \in b^\infty$.

Lemma 3.4. *Let $\mu \geq 0$ be a Carleson measure. Then we have*

$$\langle T_\mu u, v\rangle = \int_B u\bar{v}\, d\mu$$

for $u, v \in b^\infty$.

Proof. Since μ is a Carleson measure on b^1 by Corollary 3.2, we have

$$\int_B |R(z,w)|\, d\mu(w) \lesssim \int_B \frac{1}{|1 - z \cdot \bar{w}|^{n+1}}\, dV(w)$$

for $z \in B$. It follows from Fubini's theorem that

$$\int_B \int_B |R(z,w)|\, d\mu(w)\, dV(z) \lesssim \int_B \int_B \frac{1}{|1 - z \cdot \bar{w}|^{n+1}}\, dV(w)\, dV(z) < \infty$$

for $z \in B$. This justifies interchanging the order of integrations below. Now, for $u, v \in b^\infty$, we have

$$\begin{aligned} \langle T_\mu u, v\rangle &= \int_B \overline{v(z)} \int_B R(z,w) u(w)\, d\mu(w)\, dV(z) \\ &= \int_B u(w) \int_B R(z,w)\overline{v(z)}\, dV(z)\, d\mu(w) \\ &= \int_B u(w)\overline{v(w)}\, d\mu(w). \end{aligned}$$

The proof is complete. □

We now give characterization of positive bounded Toeplitz operators in terms of Carleson measures.

Theorem 3.2. *Let $\mu \geq 0$ and $1 < p < \infty$. Then, the following two conditions are equivalent.*
(1) *$T_\mu : b^p \to b^p$ is bounded.*
(2) *μ is a Carleson measure.*
Moreover, $\|T_\mu\|$ is equivalent to any of quantities in Corollary 3.3.

Proof. First, assume that T_μ is bounded on b^p. Let $z \in B$. We have by Lemma 2.2, Lemma 2.3 and Lemma 3.4

$$|\langle T_\mu r_{z,p}, r_{z,q}\rangle| \geq (1 - |z|)^{n+1} \int_{E_r(z)} |R(z,w)|^2\, d\mu(w) \approx \widehat{\mu}_r(z)$$

for $r = r_0$ where r_0 is the number provided by Lemma 2.2. On the other hand, since $\|r_{z,q}\|_q = 1$, it follows from Hölder's inequality that

$$\widehat{\mu}_r(z) \lesssim |\langle T_\mu r_{z,p}, r_{z,q}\rangle| \leq \|T_\mu r_{z,p}\|_p \|r_{z,q}\|_q \leq \|T_\mu\| \tag{3.4}$$

for all $z \in B$, where $\|T_\mu\|$ denotes the operator norm of $T_\mu : b^p \to b^p$. Thus μ is a Carleson measure by Theorem 3.1.

Conversely, suppose that μ is a Carleson measure. Let $u, v \in b^\infty$. By Lemma 3.4 and Corollary 3.3, we have

$$\begin{aligned} |\langle T_\mu u, v\rangle| &= \left|\int_B u\bar{v}\, d\mu\right| \\ &\leq \left(\int_B |u|^p\, d\mu\right)^{1/p} \left(\int_B |v|^q\, d\mu\right)^{1/q} \\ &\lesssim \sup_{z\in B} \widehat{\mu}_r(z) \left(\int_B |u|^p\, dV\right)^{1/p} \left(\int_B |v|^q\, dV\right)^{1/q} \end{aligned} \tag{3.5}$$

where the last inequality holds, because μ is a Carleson measure. Since the set of pluriharmonic polynomials is dense in b^p and b^q, the duality argument shows that T_μ is bounded on b^p and $||T_\mu|| \lesssim \sup_{z\in B} \widehat{\mu}_r(z)$. The proof is complete. □

Next, we give the corresponding characterization for compact positive Toeplitz operators. In order to do so, we introduce the notion of vanishing Carleson measures. For $\mu \geq 0$ and $1 < p < \infty$, we say that μ is a vanishing Carleson measure on b^p if the inclusion $i_p : b^p \to L^p(\mu)$ is compact, or equivalently, if

$$\int_B |u_n|^p\, d\mu \to 0$$

whenever $u_n \to 0$ weakly in b^p. To characterize vanishing Carleson measures on b^p, we first need the following.

Lemma 3.5. *Let $1 < p < \infty$. Then $r_{z,p} \to 0$ weakly in b^p as $|z| \to 1$.*

Proof. If $u \in b^q$, then by Lemma 2.3

$$|\langle u, r_{z,p}\rangle| \leq C(1 - |z|)^{(1-\frac{1}{p})(n+1)}|u(z)|.$$

Thus if u is a bounded function in b^q, then $|\langle u, r_{z,p}\rangle| \to 0$ as $|z| \to 1$. Since polynomials are dense in b^q, this implies that $|\langle u, r_{z,p}\rangle| \to 0$ as $|z| \to 1$ for all $u \in b^q$. The proof is complete. □

Now, we have a characterization for vanishing Carleson measures.

Theorem 3.3. *Let $1 < p < \infty$ and $r \in (0, 1)$. For $\mu \geq 0$, the following conditions are all equivalent.*

(1) *μ is a vanishing Carleson measure on b^p.*

(2) $\lim_{|z|\to 1} \tilde{\mu}_p(z) = 0$.

(3) $\lim_{|z|\to 1} \widehat{\mu}_r(z) = 0$.

(4) $\lim_{i\to\infty} \widehat{\mu}_r(z_i) = 0$.

One can see from the theorem above that the notion of vanishing Carleson measures on b^p is also independent of $1 < p < \infty$. So, we will simply say that $\mu \geq 0$ is a vanishing Carleson measure if one of the four conditions above holds for μ.

Proof. Since $r_{z,p} \to 0$ weakly in b^p as $|z| \to 1$ by Lemma 3.5, we clearly have (1) $\Rightarrow$ (2).

The implication (2) $\Rightarrow$ (3) can be easily seen from Lemma 3.2.

Since $|z_i| \to 1$, we have (3) $\Rightarrow$ (4).

Finally, we assume (4) and show (1). Let $\{u_n\}$ be a sequence converging to 0 weakly in b^p and $\{z_i\}$ be the sequence from Lemma 3.3. By (3.3), we have

$$\int_B |u_n|^p \, d\mu \lesssim \sum_{i<j} \int_{E_{r/3}(z_i)} |u_n|^p \, d\mu + N \sup_{i \geq j} \widehat{\mu}_r(z_i) \int_B |u_n|^p \, dV \tag{3.6}$$

for any i, j. Here, N is the positive integer provided by Lemma 3.3. Since $u_n \to 0$ weakly in b^p, one can easily see that $u_n \to 0$ uniformly on compact subsets of B and $\{u_n\}$ is bounded in L^p-norm. Thus, fixing j and taking the limit $n \to \infty$ in (3.6), we obtain

$$\limsup_n \int_B |u_n|^p \, d\mu \lesssim \sup_{i \geq j} \widehat{\mu}_r(z_i)$$

for each j. By assumption, taking the limit $j \to \infty$, we conclude

$$\limsup_n \int_B |u_n|^p \, d\mu = 0.$$

Namely, μ is a vanishing Carleson measure, as desired. The proof is complete. □

As a result corresponding to Theorem 3.2, we characterize compact positive Toeplitz operators in terms of vanishing Carleson measures.

Theorem 3.4. *Let $\mu \geq 0$ and $1 < p < \infty$. Then, the following two conditions are equivalent.*

(1) *$T_\mu : b^p \to b^p$ is compact.*

(2) *μ is a vanishing Carleson measure.*

Proof. First, suppose that T_μ is compact. By (3.4), we have

$$\widehat{\mu}_r(z) \lesssim |\langle T_\mu r_{z,p}, r_{z,q} \rangle| \leq \|T_\mu r_{z,p}\|_p$$

for all $z \in B$. Since $r_{z,p} \to 0$ weakly in b^p by Lemma 3.5, we have

$$\widehat{\mu}_r(z) \lesssim ||T_\mu r_{z,p}||_p \to 0 \qquad (|z| \to 1).$$

Thus, by Theorem 3.3, we conclude that μ is a vanishing Carleson measure.

Now, suppose that μ is a vanishing Carleson measure. Let $\{u_n\}$ be a sequence converging to 0 weakly in b^p and $v \in b^\infty$. By (3.5) and a duality argument, we obtain

$$\|T_\mu u_n\|_p \le C \left(\int_B |u_n|^p \, d\mu \right)^{1/p}.$$

Then by assumption, we see that $\|T_\mu u_n\|_p \to 0$ as $n \to \infty$. The proof is complete. □

We now turn to the characterization of positive Schatten class Toeplitz operators. Before proceeding, let's review briefly some basic facts about Schatten class operators.

For a compact operator T on a separable Hilbert space X, let $\{s_m(T)\}$ be the nonzero eigenvalues (listed by multiplicity) of $|T| = (T^*T)^{1/2}$ arranged so that the sequence is non-increasing, where T^* denotes the Hilbert space adjoint of T. This sequence is called the singular value sequence of T. For $1 \le p < \infty$, we say T is a Schatten p-class operator if

$$||T||_{S_p(X)} = \left(\sum_m |s_m(T)|^p \right)^{1/p} < \infty.$$

Let $S_p(X)$ be the space of all Schatten p-class operators on X. As is well known, $S_p(X)$ is a Banach space with the above norm and is a two-sided ideal in the space of all bounded linear operators on X.

Also, for $T \in S_1(X)$ and an orthonormal basis $\{e_m\}$ for X, the sum

$$\operatorname{tr}(T) = \sum_m \langle Te_m, e_m \rangle$$

is absolutely convergent and independent of the choice of $\{e_m\}$. The sum above is called the trace of T. If $T \in S_p(X)$ and $T \ge 0$, we have

$$||T||_{S_p(X)} = [\operatorname{tr}(T^p)]^{1/p}$$

for $1 \le p < \infty$. See [7], for example, for more information and related facts.

In the rest of this section we use the notations $S_p = S_p(b^2)$, $\tilde{\mu} = \tilde{\mu}_2$ and $r_z = r_{z,2}$ for simplicity. Also, the measure $d\lambda$ is defined on B by

$$d\lambda(z) = R(z,z) dV(z).$$

Note that $||R(z,\cdot)||_2^2 = R(z,z)$. Hence, using the same arguments of Lemma 13 in [9], we have

$$\operatorname{tr}(T) = \int_B \langle TR(z,\ \cdot), R(z,\ \cdot) \rangle \, dV(z) = \int_B \langle Tr_z, r_z \rangle \, d\lambda(z) \tag{3.7}$$

for every $T \in S_1$.

We now give a characterization of positive Toeplitz operators belonging to S_p in terms of L^p-behavior of the averaging function and Berezin transform of symbol measures.

Theorem 3.5. *Let $1 \leq p < \infty$ and $r \in (0,1)$. For $\mu \geq 0$, the following conditions are all equivalent.*
(1) $T_\mu \in S_p$.
(2) $\tilde{\mu} \in L^p(\lambda)$.
(3) $\widehat{\mu}_r \in L^p(\lambda)$.
(4) $\sum_i \widehat{\mu}_r(z_i)^p < \infty$.
Moreover, we have

$$||T_\mu||_{S_p} \approx \left(\int_B |\tilde{\mu}|^p \, d\lambda\right)^{1/p} \approx \left(\int_B |\widehat{\mu}_r|^p \, d\lambda\right)^{1/p} \approx \left(\sum_i \widehat{\mu}_r(z_i)^p\right)^{1/p}.$$

Proof. First, suppose (1) and show (2). Since $T_\mu \geq 0$, it follows from (3.7) that

$$||T_\mu||_{S_p}^p = \mathrm{tr}(T_\mu^p) = \int_B \langle T_\mu^p r_z, r_z \rangle \, d\lambda(z).$$

By Proposition 6.3.3 of [7] and Lemma 3.4, we have

$$||T_\mu||_{S_p}^p \geq \int_B \langle T_\mu r_z, r_z \rangle^p \, d\lambda(z) = \int_B \tilde{\mu}(z)^p \, d\lambda(z).$$

So, we have (2).

It follows from Lemma 3.2 that

$$\int_B \widehat{\mu}_r(z)^p \, d\lambda(z) \lesssim \int_B \tilde{\mu}(z)^p \, d\lambda(z).$$

So, we have (2) $\Rightarrow$ (3).

Now, suppose (3) and show (4). By Lemma 3.1 and Jensen's inequality, we have

$$\widehat{\mu}_r(z_i)^p \lesssim \frac{1}{(1-|z_i|)^{n+1}} \int_{E_r(z_i)} \widehat{\mu}_r(z)^p \, dV(z) \approx \int_{E_r(z_i)} \widehat{\mu}_r(z)^p \, d\lambda(z)$$

for all i. Summing up all these together, we have

$$\sum_i \widehat{\mu}_r(z_i)^p \lesssim \sum_i \int_{E_r(z_i)} \widehat{\mu}_r(z)^p \, d\lambda(z) \lesssim N \int_B \widehat{\mu}_r(z)^p \, d\lambda(z) < \infty$$

where N is the positive integer provided by Lemma 3.3. Thus, we have (4).

Finally, suppose (4) and show (1). First, consider the case $p = 1$. By (3.7) and Lemma 3.4, we have

$$\mathrm{tr}(T_\mu) = \int_B \int_B |R(z,w)|^2 \, d\mu(w) \, dx = \int_B R(w,w) \, d\mu(w).$$

Thus, by Lemma 2.1 and Lemma 3.3, we have

$$\mathrm{tr}(T_\mu) \leq \sum_i \int_{E_r(z_i)} R(z,z) \, d\mu(z) \approx \sum_i \widehat{\mu}_r(z_i) < \infty.$$

So, we have $T_\mu \in S_1$. Now, consider the case $1 < p < \infty$. For any $\zeta \in \mathbb{C}$ with $0 \le \operatorname{Re}\zeta \le 1$, define a complex Borel measure μ_ζ on B by

$$d\mu_\zeta(z) = \sum_i [\widehat{\mu}_r(z_i)]^{p\zeta - 1} \chi_{E_r(z_i)}(z)\, d\mu(z)$$

and consider corresponding Toeplitz operators T_{μ_ζ} acting on b^2. Note $T_\mu \le T_{\mu_{\frac{1}{p}}}$. Thus, the complex interpolation (see Theorem 2.2.7 of [7]) gives

$$||T_\mu||_{S_p} \le ||T_{\mu_{\frac{1}{p}}}||_{S_p} \le M_0^{1-\frac{1}{p}} M_1^{\frac{1}{p}},$$

where $M_0 = \sup\{||T_{\mu_\zeta}|| : \operatorname{Re}\zeta = 0\}$ and $M_1 = \sup\{||T_{\mu_\zeta}||_{S_1} : \operatorname{Re}\zeta = 1\}$. One can see $M_0 < \infty$ by the same argument as in the proof of Theorem 12 of [9]. To estimate M_1, let $\{u_n\}$ and $\{v_n\}$ be two orthonormal bases for b^2. Then, for $\operatorname{Re}\zeta = 1$, it follows from the same way as in the proof of Theorem 12 of [9] that

$$\sum_k |\langle T_{\mu_\zeta} u_n, v_n\rangle| \le \sum_n \int_B |u_n||v_n|\, d|\mu_\zeta| \le \int_B R(z,w)\, d|\mu_\zeta|(z)$$

and thus

$$||T_{\mu_\zeta}||_{S_1} \lesssim \sum_n [\widehat{\mu}_r(z_i)]^{p-1} \int_{E_r(z_i)} R(z,z)\, d\mu(z) \approx \sum_n \widehat{\mu}_r(z_i)^p.$$

So, we have

$$||T_\mu||_{S_p}^p \lesssim M_1 \lesssim \sum_i (\widehat{\mu}_r(z_i))^p < \infty.$$

This completes the proof. □

4. Toeplitz operators with continuous symbols

In this section, we describe the essential spectra of Toeplitz operators with uniformly continuous symbols. Let's recall the notion of the essential spectrum. Fix p with $1 < p < \infty$. Let $\mathcal{L}_p$ be the algebra of all bounded linear operators on b^p and $\mathcal{K}_p$ be the two sided compact ideal of $\mathcal{L}_p$. For an operator $T \in \mathcal{L}_p$ and a complex number λ, we say that $\lambda \in \sigma_e(T; b^p)$, the essential spectrum of T, if $(T - \lambda) + \mathcal{K}_p$ is not invertible in the Calkin algebra $\mathcal{L}_p/\mathcal{K}_p$. In other words, $\lambda \in \sigma_e(T; b^p)$ if and only if $T - \lambda$ is not Fredholm.

For $f \in L^\infty(B)$, the Hankel operators acting on b^p with symbol f is defined by

$$H_f u = (I - Q)(fu)$$

for all $u \in b^p$. The operator H_f is clearly bounded on b^p;

$$\|H_f\| \le \|f\|_\infty.$$

In what follows, we use $A^p = A^p(B)$, $0 < p < \infty$ to denote the holomorphic Bergman space, i.e, $A^p = H(B) \cap L^p$ where $H(B)$ is the space of all holomorphic function on B.

Lemma 4.1. *If $v \in A^1$ and $v(0) = 0$, then we have*

$$\overline{P(w_i \bar{v})} = P(\bar{w}_i v)(0)$$

for all $1 \le i \le n$ where w_i is the i-th coordinate function.

Proof. See Lemma 5 of [10]. □

Recall that the Bloch space $\mathcal{B} = \mathcal{B}(B)$ is the space of all holomorphic functions f on B with the property that the function $(1-|z|^2)|\partial f(z)|$ is bounded on B where $\partial = (\frac{\partial}{\partial_{z_1}}, \dots, \frac{\partial}{\partial_{z_n}})$. The little Bloch space $\mathcal{B}_0 = \mathcal{B}_0(B)$ is the subspace of $\mathcal{B}$ consisting of functions f such that

$$(1 - |z|^2)|\partial f(z)| \to 0 \qquad \text{as} \qquad |z| \to 1^-.$$

Let h_f denote the Hankel operator acting on A^p with symbol $f \in L^\infty(B)$ defined by

$$h_f u = (I - P)(fu)$$

for all $u \in A^p$.

Lemma 4.2. *Let $1 < p < \infty$ and $v \in H(B)$. Then $h_{\bar{v}}$ is compact on A^p if and only if $v \in \mathcal{B}_0$.*

Proof. See Corollary 24 of [8]. □

Lemma 4.3. *Let $1 < p < \infty$. If $f \in C(\bar{B})$, then h_f is compact on A^p.*

Proof. It is clear that $h_f = 0$ for $f \in H^\infty$. Moreover, it is known that for $f \in H^\infty$, $h_{\bar{f}}$ is compact on A^p if and only if $f \in \mathcal{B}_0$ by Lemma 4.2. Since $H^\infty \cap C(\bar{B}) \subset \mathcal{B}_0$, we have $h_{\bar{f}}$ is compact on A^p for $f \in H^\infty \cap C(\bar{B})$. Since holomorphic and antiholomorphic monomials span a uniformly dense subset of $C(\bar{B})$, h_f is compact on A^p. This completes the proof. □

Remark. For holomorphic function f and g, we will use the fact that $f + \bar{g} \in b^p$ for $1 < p < \infty$ implies $f, g \in A^p$ by the boundedness of the Bergman projection P. As a result of this observation with $A^p \cap \overline{A_0^p} = \{0\}$, we see that $b^p = A^p \oplus \overline{A_0^p}$ where $A_0^p = \{v \in A^p : v(0) = 0\}$.

Let $1 < p < \infty$. Recall that q is the conjugate exponent of p. The annihilator of A^q is defined by

$$(A^q)^\perp = \{u \in b^p : \langle u, v\rangle = 0 \;\text{ for all } v \in A^q\}.$$

Lemma 4.4. *For $1 < p < \infty$, we have $\overline{A_0^p} = (A^q)^\perp$.*

Proof. For a given function $\bar{u} \in \overline{A_0^p}$, we have

$$\langle \bar{u}, v\rangle = \int \bar{u}\bar{v}\, dV = \overline{u(0)v(0)} = 0$$

for all $v \in A^q$. Therefore $\bar{u} \in (A^q)^{\perp}$. Conversely, if we choose $u \in (A^q)^{\perp}$, then by the remark, there is a unique decomposition $u = u' + \bar{u''}$ where $u' \in A^p$ and $\bar{u''} \in \overline{A_0^p}$. Note from $(A^q)^* \cong A^p$ that $A^p \cap (A^q)^{\perp} = \{0\}$. Thus, with the previous inclusion $\overline{A_0^p} \subset (A^q)^{\perp}$, we have $u' = 0$ and $u = \bar{u''}$. Consequently $u \in \overline{A_0^p}$. This completes the proof. □

Theorem 4.1. *Let $1 < p < \infty$. If $f \in C(\bar{B})$, Then H_f is compact on b^p.*

Proof. Fix $1 < p < \infty$ and let $\phi \in b^p$. Then, by the remark and Lemma 4.4, we have $b^p = A^p \oplus (A^q)^{\perp}$. Thus there are functions $u \in A^p$ and $v \in (A^q)^{\perp}$ such that $\phi = u + \bar{v}$. Let

$$E = \{f \in C(\bar{B}) | H_f \in \mathcal{K}_p\}.$$

We will show $E = C(\bar{B})$. First, we need to show $z_i \in E$ where z_i is the i-th coordinate function. Since $Q(z_i u) = z_i u$, we have

$$H_{z_i}(u) = (I - Q)(z_i u) = 0.$$

Also, by Lemma 4.1, we have

$$\begin{aligned} H_{z_i}(\bar{v}) &= (I - Q)(z_i \bar{v}) \\ &= \overline{(I - P)(\bar{z}_i v)} - P(z_i \bar{v}) + \overline{P(\bar{z}_i v)(0)} \\ &= \overline{(I - P)(\bar{z}_i v)} \\ &= \overline{h_{\bar{z}_i}(v)}. \end{aligned}$$

Let $M : b^p \longrightarrow (A^q)^{\perp}$ be the projection defined by

$$M(\phi) = \bar{v}. \tag{4.1}$$

Let $u = \phi - \bar{v}$. Then $u \in A^p$ and $\phi = u + \bar{v}$. Because

$$\|\overline{M(\phi)}\|_p = \|v\|_p = \|P(\bar{\phi})\|_p \leq \|\bar{\phi}\|_p,$$

M is bounded linear operator. It follows from (4.1) that

$$H_{z_i}(\phi) = \overline{h_{\bar{z}_i}(v)} = \overline{h_{\bar{z}_i}} \circ \bar{M}(\phi).$$

By Lemma 4.3, we have $H_{z_i} \in \mathcal{K}_p$. Next, we show $\bar{z}_i \in E$. Because $Q(\bar{z}_i \bar{v}) = \bar{z}_i \bar{v}$, we have

$$H_{\bar{z}_i}(\bar{v}) = (I - Q)(\bar{z}_i \bar{v}) = 0.$$

Next, by (1.5) and Lemma 4.1, we have

$$\begin{aligned} H_{\bar{z}_i}(u) &= H_{\bar{z}_i}(\tilde{u}) \\ &= (I - Q)(\bar{z}_i \tilde{u}) \\ &= (I - P)(\bar{z}_i \tilde{u}) - \overline{P(z_i \bar{\tilde{u}})} + P(\bar{z}_i \tilde{u})(0) \\ &= (I - P)(\bar{z}_i \tilde{u}) \\ &= h_{\bar{z}_i}(u - u(0)) \end{aligned}$$

where $\tilde{u} = u - u(0)$. It follows that

$$H_{\bar{z}_i}(\phi) = h_{\bar{z}_i}(u - u(0)).$$

Letting $E_0\phi = \phi(0)$, we obtain

$$H_{\bar{z}_i} = h_{\bar{z}_i} \circ (I - M - E_0).$$

Clearly, E is a closed subspace of $C(\bar{B})$. Also, we know that $1, z_i, \bar{z}_j \in E$. Now, for any $f \in C(\bar{B})$, straightforward calculation shows that

$$H_{fg} = H'_f H_g + H_f T_g$$

where H'_f denotes the Hankel operator extended to the whole L^p. Thus, it follows that E is a closed subalgebra of $C(\bar{B})$ containing all the polynomials in z_i and $\bar{z}_j$ where $i, j = 1, \ldots, n$. By the Stone-Weierstrass Theorem, $E = C(\bar{B})$. Therefore H_f is compact for every $f \in C(\bar{B})$. This completes the proof. □

As a consequence of Theorem 4.1, we have the following which will be used in the proof of Theorem 4.2 below.

Corollary 4.1. *Let $1 < p < \infty$. If $f, g \in C(\bar{B})$, Then $T_{fg} - T_g T_f$ and $T_f T_g - T_g T_f$ are compact on b^p.*

Proof. For any $f, g \in L^\infty$, a straightforward calculation yields

$$T_{fg} - T_g T_f = T'_g H_f \tag{4.2}$$

where T'_g denotes the Toeplitz operator extended to the whole L^p. Thus, it follows from (4.2) and Theorem 4.1 that $T_{fg} - T_g T_f$ is compact on b^p if $f \in C(\bar{B})$. Hence $T_f T_g - T_g T_f$ is compact on b^p for $f, g \in C(\bar{B})$. The proof is complete. □

Proposition 4.1. *Let $1 < p < \infty$. If $f \in L^\infty(B)$ has compact support, then T_f is compact on b^p.*

Proof. Let K be compact subset of B on which f is supported. Suppose $\{u_n\}$ is a bounded sequence in b^p. Then $\{u_n\}$ is uniformly bounded on each compact subset of B. So, $\{u_n\}$ is a normal family. Hence there exists pluriharmonic function u on B such that some subsequence $\{u_{n_j}\}$ of $\{u_n\}$ converges uniformly on K to u. Thus $\{u_{n_j} f\}$ converges in b^p to uf. Hence

$$T_f(u_{n_j}) \longrightarrow T_f(u)$$

in b^p. Therefore T_f is compact b^p. This completes the proof. □

We are now ready to give a characterization of $\sigma_e(T_f; b^p)$ for $1 < p < \infty$.

Theorem 4.2. *Let $1 < p < \infty$ and $f \in C(\bar{B})$. Then $\sigma_e(T_f; b^p) = f(\partial B)$.*

Proof. Fix p. First, we show $f(\partial B) \subset \sigma_e(T_f; b^p)$. If we choose $\zeta \in f(\partial B)$, then there is a $\eta \in \partial B$ such that $f(\eta) = \zeta$. We claim that $T_{f-\zeta}$ is not a Fredholm operator, so that $\zeta \in \sigma_e(T_f; b^p)$. We will prove this claim by contradiction. Suppose that $T_{f-\zeta}$ is a Fredholm operator. Then, there exists $\Phi \in \mathcal{L}_p$ such that $\Phi T_{f-\zeta} - I \in \mathcal{K}_p$. By Lemma 3.5, it follows that

$$\|(I - \Phi T_{f-\zeta})(r_{z,p})\|_p \to 0 \quad as \quad |z| \to 1.$$

Therefore

$$\|(\Phi T_{f-\zeta})(r_{z,p})\|_p \to 1 \quad as \quad |z| \to 1. \tag{4.3}$$

On the other hand, we have

$$\begin{aligned}
\|\Phi T_{f-\zeta}(r_{z,p})\|_p^p &\le \|\Phi\| \|T_{f-\zeta}(r_{z,p})\|_p^p \\
&\lesssim \|(f-\zeta) r_{z,p}\|_p^p \\
&= \int_{A_\delta} |(f-\zeta) r_{z,p}|^p dV + \int_{B\setminus A_\delta} |(f-\zeta) r_{z,p}|^p dV \\
&= I_1(z) + I_2(z)
\end{aligned}$$

for $\delta > 0$. We can easily see that $|w - \eta|$ decrease to 0 as $|1 - w \cdot \bar{\eta}|$ decrease to 0. So, given $\varepsilon > 0$, we can find $\delta > 0$ such that $w \in B$ and $|1 - w \cdot \bar{\eta}| < \delta$ implies $|f(w) - \zeta| < \varepsilon$ since $f \in C(\bar{B})$. Therefore we have

$$\begin{aligned}
I_1(z) &= \int_{A_\delta} |(f-\zeta) r_{z,p}|^p dV \\
&\le \varepsilon^p \int_{A_\delta} |r_{z,p}|^p dV \\
&\le \varepsilon^p.
\end{aligned}$$

Also, we know that

$$\begin{aligned}
I_2(z) &= \int_{B\setminus A_\delta} |(f-\zeta) r_{z,p}|^p dV \\
&\le \frac{\|f-\zeta\|_{L^\infty}^p}{\|R_z\|_p^p} \int_{B\setminus A_\delta} |R_z|^p dV \\
&\le C(1-|z|)^{(p-1)(n+1)} \int_{|1-w\cdot\bar{\eta}|\ge\delta} \frac{1}{|1 - z\cdot\bar{w}|^{\frac{2p(n+1)}{2}}} dV(w).
\end{aligned}$$

Letting $z \to \eta$, we obtain $I_2(z) \to 0$. Consequently,

$$\limsup_{z\to\eta} \|\Phi T_{f-\zeta}(r_{z,p})\|_p \le \varepsilon$$

for each $0 < \varepsilon < 1$. Since $\varepsilon > 0$ is arbitrary, we have

$$\|\Phi T_{f-\zeta}(r_{z,p})\|_p \to 0$$

as $z \to \eta$. This gives a contradiction to (4.3).

Next, we show $\sigma_e(T_f; b^p) \subset f(\partial B)$. We need to show that if $\zeta \in \mathbb{C}\backslash f(\partial B)$, then $T_{f-\zeta}$ is invertible in the Calkin algebra. Suppose $\zeta \notin f(\partial B)$. Then there are $0 < r < 1$ and $g \in C(\bar{B})$ satisfying $(f-\zeta)g = 1$ on $\bar{B}\backslash B(0,r)$. The function $h = 1 - (f-\zeta)g$ is compactly supported, so by Proposition 4.1, T_h is compact on $b^p(B)$. Using (4.2) we have

$$T_{f-\zeta}T_g = T_{(f-\zeta)g} - T'_{f-\zeta}H_g = I - T_h - T'_{f-\zeta}H_g.$$

By Lemma 4.1, the operator $T'_{f-\zeta}H_g$ is compact on b^p, thus $T_h + T'_{f-\zeta}H_g$ is compact on b^p, and consequently $T_{f-\zeta}$ is right-invertible in the Calkin algebra. Similarly, we can also see that $T_{f-\zeta}$ is left-invertible in the Calkin algebra. The proof is complete. □

Let $||T_f||_{p,e}$ denote the norm of $T_f : b^p \to b^p$ in the Calkin algebra of b^p. Then, we have the following consequence. So, $||T_f||_{p,e}$ is the distance from T_f to $\mathcal{K}_p$.

Corollary 4.2. *Let $1 < p < \infty$ and $f \in C(\bar{B})$. Then we have*

$$||f||_{L^\infty(\partial B)} \leq \|T_f\|_{p,e} \tag{4.4}$$

and the equality holds for $p = 2$.

Proof. The assertion (4.4) follows from Theorem 4.2 and the spectral radius formula. It is easily seen that $T_f^* = T_{\bar{f}}$. Thus, Corollary 4.1 shows that T_f is normal in the Calkin algebra, so that $\|T_f\|_e = \rho(T_f)$ (See [4], Theorem 7.12). Hence, the equality holds in (4.4) for $p = 2$. This completes the proof. □

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