J. Math. Kyoto Univ. (JMKYAZ)
47-3 (2007), 657–664

Geometric inequalities outside a convex set in a Riemannian manifold

By

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Abstract

Let M be an n-dimensional complete simply connected Riemannian manifold with nonpositive sectional curvature for $n = 2, 3$ and 4. We prove the following Faber-Krahn type inequality for the first eigenvalue λ_1 of the mixed boundary problem. A domain Ω outside a closed convex subset C in M satisfies

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*)$$

with equality if and only if Ω is isometric to the half ball Ω^* in $\mathbb{R}^n$, whose volume is equal to that of Ω. We also prove the Sobolev type inequality outside a closed convex set C in M.

1. Introduction

One of the most important inequalities in geometric analysis is the Faber-Krahn inequality. In the 1920's, for a bounded domain $\Omega \subset \mathbb{R}^n$, Faber and Krahn proved independently the following inequality

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*), \tag{1.1}$$

where equality holds if and only if Ω is a ball (See [1]). Here λ_1 denotes the first Dirichlet eigenvalue and Ω^* is a ball of the same n-dimensional volume as Ω. For the first Neumann eigenvalue μ_1, in 1954 Szegö[10] showed that for a simply connected domain $\Omega \subset \mathbb{R}^2$

$$\mu_1(\Omega) \leq \mu_1(\Omega^*),$$

where Ω^* is as above and equality holds if and only if Ω is a disk. It should be mentioned that μ_1 is the first positive eigenvalue of the Neumann boundary problem. Two years later Weinberger [11] generalized the inequality for $\Omega \subset \mathbb{R}^n$, $n \geq 2$. On the other hand, for the first eigenvalue λ_1 of the mixed boundary problem, Nehari [8, Theorem III] proved (1.1) for a simply connected bounded domain $\Omega \subset \mathbb{R}^2$ satisfying that a subarc $\alpha \subset \partial\Omega$ is concave with respect to

2000 *Mathematics Subject Classification*(*s*). 58J50, 35P15
Received April 16, 2007

Ω. In this case Ω^* is a half disk of the same area as Ω. Equality holds if and only if Ω is a half disk. In Section 2, we prove the Faber-Krahn type inequality (Theorem 2.1) extending Nehari's result to a Riemannian manifold case.

In [9], the author has proved the Sobolev type inquality outside a closed convex set in a nonpositively curved surface. In Section 3, we study Sobolev type inequality outside a closed convex set in a 3 and 4-dimensional Riemannian manifold with nonpositive sectional curvature.

The key ingredient in the proofs of our theorems is the following relative isoperimetric inequality.

Theorem 1 ([2], [3], [5], [9]). *Let M be an n-dimensional complete simply connected Riemannian manifold with nonpositive sectional curvature for n=2, 3 and 4, and let $C \subset M$ be a closed convex set with smooth boundary. Then for a domain $\Omega \subset M \sim C$ we have*

$$\frac{1}{2} n^n \omega_n \mathrm{Vol}(\Omega)^{n-1} \leq \mathrm{Vol}(\partial\Omega \sim \partial C)^n, \tag{1.2}$$

where equality holds if and only if Ω is a Euclidean half ball.

Recently Choe-Ghomi-Ritoré [4] have proved that this inequality holds for a domain in $\mathbb{R}^n$.

Theorem 2 ([4]). *Let $C \subset \mathbb{R}^n$ be a closed convex set with smooth boundary. Then for a domain $\Omega \subset \mathbb{R}^n \sim C$,* (1.2) *is still true and equality holds if and only if Ω is a Euclidean half ball.*

2. Faber-Krahn type inequality

Let Ω be a bounded domain outside a closed convex subset C with smooth boundary in an n-dimensional Riemannian manifold M. The Laplacian operator Δ acting on functions is locally given by

$$\Delta = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial x^j} \right),$$

where $(x^1, \ldots, x^n)$ is a local coordinate system, (g^{ij}) is the inverse of the metric tensor (g_{ij}), and $g = \det(g_{ij})$. We consider the mixed eigenvalue problem as follows :

$$\begin{aligned} \Delta u + \lambda u &= 0 \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \sim \partial C \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on} \quad \partial\Omega \cap \partial C, \end{aligned}$$

where ν is the outward unit normal to $\partial\Omega$ along $\partial\Omega \cap \partial C$ and $\sim$ denotes the set exclusion operator. Then, using the divergence theorem, we see that the first eigenvalue $\lambda_1(\Omega)$ of the mixed boundary problem satisfies

$$\lambda_1(\Omega) = \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2},$$

where $H_0^1(\Omega)$ is the Sobolev space such that $u \in H_0^1(\Omega)$ vanishes on $\partial\Omega \sim \partial C$. We note that $u \in H_0^1(\Omega)$ need not vanish on $\partial\Omega \cap \partial C$.

First we show that the first eigenvalue of the mixed boundary problem for a half ball in space form $\mathbb{M}^n(\kappa)$ is equal to that of Dirichlet boundary problem for a ball in $\mathbb{M}^n(\kappa)$, where $\mathbb{M}^n(\kappa)$ denotes an n-dimensional complete Riemannian manifold of constant sectional curvature κ.

Proposition 2.1. *Let $\lambda_1(B_+(r))$ be the first mixed eigenvalue of a half ball $B_+(r)$ with radius r in $\mathbb{M}^n(\kappa)$ and $\lambda_1(B(r))$ the first eigenvalue of the Dirichlet boundary problem of a ball $B(r)$ with the same radius r in $\mathbb{M}^n(\kappa)$. If $\kappa > 0$ assume $r < 1/\sqrt{\kappa}$. Then we have*

$$\lambda_1(B_+(r)) = \lambda_1(B(r))$$

Proof. First let ϕ be an eigenfunction of $B_+(r)$ associated with $\lambda_1(B_+(r))$. Then,

$$\begin{aligned} \Delta\phi + \lambda_1(B_+(r)) &= 0 \quad \text{in} \quad B_+(r) \\ \phi &= 0 \quad \text{on} \quad \partial B_+(r) \sim \partial\mathbb{H} \\ \frac{\partial\phi}{\partial\nu} &= 0 \quad \text{on} \quad \partial\mathbb{H}, \end{aligned}$$

where $\partial\mathbb{H}$ denotes the boundary of the half space, which has flat geodesic curvature. We can extend the eigenfunction ϕ to $\tilde{\phi}$ defined on $B(r)$ by reflecting ϕ across $\partial\mathbb{H}$.

Using $\lambda_1(B(r)) = \inf_{u \in H_0^1(B(r))} \dfrac{\int_{B(r)} |\nabla u|^2}{\int_{B(r)} u^2}$, we have

$$\lambda_1(B(r)) \le \frac{\int_{B(r)} |\nabla\tilde{\phi}|^2}{\int_{B(r)} \tilde{\phi}^2} = \frac{\int_{B_+(r)} |\nabla\phi|^2}{\int_{B_+(r)} \phi^2} = \lambda_1(B_+(r)), \tag{2.1}$$

where $H_0^1(B(r))$ is the Sobolev space on $B(r)$. Conversely let ψ be an eigenfunction of the Dirichlet problem in a ball B associated with $\lambda_1(B(r))$, that is,

$$\begin{aligned} \Delta\psi + \lambda_1(B(r)) &= 0 \quad \text{in} \quad B(r) \\ \psi &= 0 \quad \text{on} \quad \partial B(r). \end{aligned}$$

Since ψ is a radial function, $\dfrac{\partial\psi}{\partial\nu} = 0$ on $\partial\mathbb{H}$. Hence ψ satisfies the boundary condition for the mixed eigenvalue problem. We immediately get

$$\lambda_1(B_+(r)) \le \frac{\int_{B_+(r)} |\nabla\psi|^2}{\int_{B_+(r)} \psi^2} = \frac{\int_{B(r)} |\nabla\psi|^2}{\int_{B(r)} \psi^2} = \lambda_1(B(r)). \tag{2.2}$$

Therefore we have $\lambda_1(B_+(r)) = \lambda_1(B(r))$ by (2.1) and (2.2). □

We need the following well-known lemma before we prove our theorems.

Lemma 2.1. *Let Ω be a domain in an n-dimensional Riemannian manifold M and let f be any eigenfunction with the first eigenvalue λ_1 for mixed eigenvalue problem. Then f is strictly positive or strictly negative in Ω.*

Proof. Note that

$$\lambda_1(\Omega) = \frac{\int_\Omega |\nabla f|^2}{\int_\Omega f^2} = \frac{\int_\Omega |\nabla |f||^2}{\int_\Omega f^2}.$$

It follows that $|f|$ also is an eigenfunction associated with λ_1 and $|f| \in C^2(\Omega) \cap C^0(\overline{\Omega})$ by elliptic regularity theory[7]. We also have $\Delta |f| = -\lambda_1 |f| \leq 0$. Using maximum principle we have $|f| > 0$ in Ω and hence $f > 0$ or $f < 0$ in Ω. □

We now prove the following Faber-Krahn type inequality for the mixed eigenvalue problem using symmetrization and relative isoperimetric inequality.

Theorem 2.1. *Let M be an n-dimensional complete simply connected Riemannian manifold with nonpositive sectional curvature for n=2, 3 and 4, and let $C \subset M$ be a closed convex set with smooth boundary. Then for a domain $\Omega \subset M \sim C$, we have*

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*), \tag{2.3}$$

where Ω^* is a half ball in $\mathbb{R}^n$, whose volume is equal to that of the domain Ω. Equality holds if and only if the domain Ω is isometric to the half ball Ω^* in $\mathbb{R}^n$.

Proof. Let f be the first eigenfunction of Ω, that is,

$$\begin{aligned} \Delta f + \lambda_1(\Omega) f = 0 &\text{ in } \Omega \\ f = 0 &\text{ on } \partial\Omega \sim \partial C \\ \frac{\partial f}{\partial \nu} = 0 &\text{ on } \partial\Omega \cap \partial C. \end{aligned}$$

We may assume that f is nonnegative by lemma 2.1. Consider the set $\Omega_t = \{x \in \Omega : f(x) > t\}$ and $\Gamma_t = \{x \in \Omega : f(x) = t\}$. Using a symmetrization procedure, we construct the concentric geodesic half ball Ω_t^* in $\mathbb{R}^n$ such that $\mathrm{Vol}(\Omega_t^*) = \mathrm{Vol}(\Omega_t)$ for each t, and $\Omega_0^* = \Omega^*$. We define a function $F : \Omega^* \to \mathbb{R}_+$ such that F is a radially decreasing function and $\partial\Omega_t^* \sim \partial\mathbb{H} = \{x \in \Omega^* : F(x) = t\}$.
Then it suffices to prove

$$\int_\Omega f^2 dv = \int_{\Omega^*} F^2 dv, \tag{2.4}$$

$$\int_\Omega |\nabla f|^2 dv \geq \int_{\Omega^*} |\nabla F|^2 dv. \tag{2.5}$$

For (2.4), using the co-area formula [6],

$$\int_\Omega f^2 dv = \int_0^\infty \int_{\Gamma_t} \frac{f^2}{|\nabla f|} dA_t dt = \int_0^\infty t^2 \Big(\int_{\Gamma_t} \frac{dA_t}{|\nabla f|} \Big) dt$$
$$= -\int_0^\infty t^2 \frac{d}{dt} \mathrm{Vol}(\Omega_t) dt = -\int_0^\infty t^2 \frac{d}{dt} \mathrm{Vol}(\Omega_t^*) dt = \int_{\Omega^*} F^2 dv,$$

where dA_t is the $(n-1)$-dimensional volume element on Γ_t. Here we have used the identity

$$\frac{d}{dt}\mathrm{Vol}(\Omega_t) = -\int_{\Gamma_t} |\nabla f|^{-1} dA_t.$$

For (2.5), using Hölder inequality we have

$$\int_{\Gamma_t} dA_t = \int_{\Gamma_t} |\nabla f|^{1/2} |\nabla f|^{-1/2} dA_t$$
$$\leq \Big(\int_{\Gamma_t} |\nabla f| \Big)^{1/2} \Big(\int_{\Gamma_t} |\nabla f|^{-1} \Big)^{1/2}$$
$$= \Big(\int_{\Gamma_t} |\nabla f| \Big)^{1/2} \Big(-\frac{d}{dt} \mathrm{Vol}(\Omega_t) \Big)^{1/2}.$$

From the relative isoperimetric inequality (1.2) as mentioned in the introduction, we see that

$$\text{(2.6)} \qquad \int_{\Gamma_t} |\nabla f| dA_t \geq \frac{\mathrm{Vol}(\Gamma_t)^2}{-\frac{d}{dt}\mathrm{Vol}(\Omega_t)}$$
$$\geq \frac{\mathrm{Vol}(\Gamma_t^*)^2}{\int_{\Gamma_t^*} |\nabla F|^{-1} dA_t^*} = \int_{\Gamma_t^*} |\nabla F| dA_t^*,$$

where $\Gamma_t^* = \{x \in \Omega^* : F(x) = t\}$, and dA_t^* is the $(n-1)$-dimensional volume element on Γ_t^*. Integrating in t, we get (2.5). To have equality, the second inequality in (2.6) should become equality. Since equality in the relative isoperimetric inequality holds if and only if Ω is isometric to a half ball in $\mathbb{R}^n$, we get the conclusion. □

Using [4], we can also prove the following.

Theorem 2.2. *Let $C \subset \mathbb{R}^n$ be a closed convex set with smooth boundary. Then for a domain $\Omega \subset \mathbb{R}^n \sim C$, we have*

$$\text{(2.7)} \qquad \lambda_1(\Omega) \geq \lambda_1(\Omega^*),$$

where Ω^* is a half ball in $\mathbb{R}^n$, whose volume is equal to that of the domain Ω. Equality holds if and only if the domain Ω is isometric to the half ball Ω^* in $\mathbb{R}^n$.

3. Sobolev type inequality

In this section we prove Sobolev type inequality outside a closed convex set in a Riemannian manifold.

Theorem 3.1. *Let M be an n-dimensional complete simply connected Riemannian manifold with nonpositive sectional curvature for n=2, 3 and 4. Let $C \subset M$ be a closed convex set. Then we have*

$$\frac{1}{2}n^n w_n\Big(\int_{M\sim C}|f|^{\frac{n}{n-1}}dA\Big)^{n-1} \leq \Big(\int_{M\sim C}|\nabla f|dA\Big)^n, f\in W_0^{1,1}(M\sim C).$$

Equality holds if and only if up to a set of measure zero, $f = c\chi_D$ where c is a constant and D is a half ball in $\mathbb{R}^n$.

Proof. For simplicity, we assume $f \geq 0$. By the co-area formula

$$\int_M |\nabla f|dv = \int_0^\infty \text{Area}(f=\sigma)d\sigma.$$

We apply the relative isoperimetric inequality (1.2) to obtain

$$\int_M |\nabla f|dv = \int_0^\infty \text{Area}(f=\sigma)d\sigma \geq n\Big(\frac{\omega_n}{2}\Big)^{\frac{1}{n}}\int_0^\infty \text{Vol}(f>\sigma)^{\frac{n-1}{n}}d\sigma.$$

Since we have

$$\int_M |f|^{\frac{n}{n-1}}dv = \int_0^\infty \text{Vol}(f^{\frac{n}{n-1}}>\rho)d\rho = \frac{n}{n-1}\int_0^\infty \text{Vol}(f>\sigma)\sigma^{\frac{1}{n-1}}d\sigma,$$

it suffices to show that

$$\int_0^\infty \text{Vol}(f>\sigma)^{\frac{n-1}{n}}d\sigma \geq \Big(\frac{n}{n-1}\Big)^{\frac{n-1}{n}}\Big(\int_0^\infty \text{Vol}(f>\sigma)\sigma^{\frac{1}{n-1}}d\sigma\Big)^{\frac{n-1}{n}}.$$

Define

$$\begin{aligned} F(\sigma) &:= \text{Vol}(f>\sigma),\\ \varphi(t) &:= \int_0^t F(\sigma)^{\frac{n-1}{n}}d\sigma,\\ \psi(t) &:= \Big(\int_0^t F(\sigma)\sigma^{\frac{1}{n-1}}d\sigma\Big)^{\frac{n-1}{n}}. \end{aligned}$$

Then we can see that $\varphi(0) = \psi(0) = 0$. Since $F(\sigma)$ is monotone decreasing, we obtain

$$\varphi'(t) \geq \Big(\frac{n}{n-1}\Big)^{\frac{n-1}{n}}\psi'(t).$$

It follows that

$$\varphi(\infty) \geq \Big(\frac{n}{n-1}\Big)^{\frac{n-1}{n}}\psi(\infty).$$

Moreover it is easy to see that quality holds if and only if f is $c\chi_D$ where c is a constant and D is a half ball in $\mathbb{R}^n$. $\square$

Applying the same arguments as in the proof of the above theorem and the relative isoperimetric inequality (1.2), we also have the following theorem.

Theorem 3.2. *Let $C \subset \mathbb{R}^n$ be a closed convex set with smooth boundary. Then we have*

$$\frac{1}{2}n^n w_n \Big(\int_{\mathbb{R}^n \sim C} |f|^{\frac{n}{n-1}} dA \Big)^{n-1} \leq \Big(\int_{\mathbb{R}^n \sim C} |\nabla f| dA \Big)^n, \quad f \in W_0^{1,1}(\mathbb{R}^n \sim C).$$

Equality holds if and only if up to a set of measure zero, $f = c\chi_D$ where c is a constant and D is a half ball in $\mathbb{R}^n$.

Remark. In our Theorem 3.1 and 3.2, the function f may not vanish on ∂C. It is sufficient that f is compactly supported in the relative topology on $S \sim C$ for a closed convex set $C \subset S$.

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